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PROBLEMS FOR JUNIORS

JP.406. If $a, b, c > 0; a+b+c = 3$ then: $(a^3+2)(b^3+2)(c^3+2) \geq 27$

Proposed by Daniel Sitaru - Romania

JP.407. In ΔABC the following relationship holds:

$$\left(\frac{R}{2r}\right)^3 \geq \frac{(a+b+c)(a^2+b^2+c^2)(a^3+b^3+c^3)}{27a^2b^2c^2}$$

Proposed by Alex Szoros-Romania

JP.408. In ΔABC the following relationship holds:

$$\left(\frac{R}{r}\right)^2 + 4 \geq \frac{(r_a+r_b)(r_b+r_c)(r_c+r_a)}{r_a r_b r_c} \geq \frac{3R+2r}{r} \geq 2\left(\frac{a}{b}+\frac{b}{a}\right)+4$$

Proposed by Alex Szoros-Romania

JP.409. If $a, b, c > 1$ and $0 \leq \lambda \leq 1$ then:

$$\frac{\log_b a}{\lambda + \log_a b + \log_a c} + \frac{\log_c b}{\lambda + \log_b a + \log_b c} + \frac{\log_a c}{\lambda + \log_c a + \log_c b} \geq \frac{3}{\lambda + 2}$$

Proposed by Marin Chirciu-Romania

JP.410. If $x, y, z > 0$ and $n \in \mathbb{N}, n \geq 2$ then:

$$\sum_{cyc} \sqrt[n]{x^{2n-1}(y+z)} \geq \left(1+\frac{1}{2^n}\right)(xy+yz+zx)$$

Proposed by Marin Chirciu-Romania

JP.411. In ΔABC the following relationship holds:

$$\frac{ab(a+b)}{\sqrt{2(a^2+b^2)}} + \frac{bc(b+c)}{\sqrt{2(b^2+c^2)}} + \frac{ca(c+a)}{\sqrt{2(c^2+a^2)}} \geq 4\sqrt{3}F$$

Proposed by Marian Ursărescu - Romania

JP.412. In $\Delta ABC, I$ - incenter, the following relationship holds:

$$AI^6 + BI^6 + CI^6 \leq 64[(R^2 - Rr + r^2)^3 - 24r^6]$$

Proposed by Marian Ursărescu - Romania

JP.413. If $(a_n)_{n\geq 1}$ be increasing sequence with $a_i > 0, \forall i = \overline{1,n}$ and $k \in \mathbb{N}, k \geq 2$ solve for real numbers:

$$\sqrt[k]{\frac{a_1^x+1}{a_2^x+1}} + \sqrt[k]{\frac{a_2^x+1}{a_3^x+1}} + \sqrt[k]{\frac{a_{n-1}^x+1}{a_n^x+1}} = n - 2 + \sqrt[k]{\frac{a_1^x+1}{a_n^x+1}}$$

Proposed by Florică Anastase - Romania

JP.414. Solve for real numbers:

$$\begin{cases} x, y, z > 0 \\ x^4 + 1 = 5(y^2 + z^2) \\ x + y + z = 2\sqrt[3]{xyz} + \frac{3xyz}{xy+yz+zx} \end{cases}$$

Proposed by Daniel Sitaru - Romania

JP.415. In any quadrilateral with the sides' lengths a, b, c, d:

$$\frac{1}{a(b+c+d-a)} + \frac{1}{b(a+c+d-b)} + \frac{1}{c(a+b+d-c)} +$$
$$+ \frac{1}{d(a+b+c-d)} \geq \frac{32}{(a+b+c+d)^2}$$

Proposed by Florentin Vişescu - Romania

JP.416. Solve in $\mathbb{R}_+$ the equation:

$$\sqrt{3n-1+\sqrt{8n^2-4n}} \cdot \sqrt{7n-5+\sqrt{48n^2-68n+24}} \cdot$$
$$\cdot \sqrt{5n-3+\sqrt{24n^2-28n+8}} \cdot \sqrt{5n-3+\sqrt{16n^2-12n}} = (10n-6)^2$$

Proposed by George - Florin Şerban - Romania

JP.417. Prove that in any ΔABC the following inequality holds:

$$\sum \sin^3 \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} \geq \frac{3}{16} \sum \cos A$$

Proposed by Gheorghe Alexe and George Florin Şerban - Romania

JP.418. Let be $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n > 0$,

$$\left(\sum_{k=1}^{n} x_k\right)^2 > 2\prod_{k=1}^{n} x_k, \sum_{k=1}^{n} x_k < \sum_{k=1}^{n} y_k$$

$$\left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n} y_k\right) < \prod_{k=1}^{n} x_k + \prod_{k=1}^{n} y_k$$

Prove that:

$$\left(\sum_{k=1}^{n} x_k\right) \cdot \left(\prod_{k=1}^{n} y_k\right) > \left(\sum_{k=1}^{n} y_k\right) \cdot \left(\prod_{k=1}^{n} x_k\right)$$

Proposed by George Florin Şerban - Romania

JP.419. Find all $a, b \in \mathbb{Z}$ such that:

$$\sqrt[3]{1+\sqrt{2019-ab}} + \sqrt[3]{1-\sqrt{2019-ab}} \in \mathbb{Z}$$

Proposed by Pedro Pantoja - Brazil

JP.420. Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 = 3$. Prove that:

$$\frac{a^3+b^3+c^2+1}{b^3(c^2+1)} + \frac{b^3+c^3+a^2+1}{c^3(a^2+1)} + \frac{c^3+a^3+b^2+1}{a^3(b^2+1)} \geq 6$$

Proposed by Pedro Pantoja - Brazil

PROBLEMS FOR SENIORS

SP.406. Let a, b, c, d be positive real numbers. Find the maximum value of the expression:

$$\frac{\sqrt[4]{\frac{abc}{ab+ac+bc}} + \sqrt[4]{\frac{abd}{ab+ad+bd}} + \sqrt[4]{\frac{acd}{ac+ad+cd}} + \sqrt[4]{\frac{bcd}{bc+bd+cd}}}{\sqrt[16]{a^4+b^4+c^4+d^4}}$$

Proposed by Kunihiko Chikaya-Japan

SP.407. If $X, Y, Z \in M_{11}(\mathbb{C}); X^3 = Y^5 = Z^7 = I_{11}$, $XY = YX; YZ = ZY; ZX = XZ$;

$$\Omega = 2XYZ + X^2(Y+Z) + Y^2(Z+X) + Z^2(X+Y)$$

then $\det(\Omega) \neq 0$.

Proposed by Daniel Sitaru - Romania

SP.408.Let ABC be an equilateral triangle such that $|z_A| = |z_B| = |z_C|$. Find $z \in \mathbb{C}$ such that:

$$\begin{cases} |z - z_A| \le |z_B + z_C| \\ |z - z_B| \le |z_C + z_A| \\ |z - z_C| \le |z_A + z_B| \end{cases}$$

Proposed by Ionuţ Florin Voinea - Romania

SP.409.Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that:

$$f(x - y) = f(x) - f(y) - xy(x - y), \forall x, y \in \mathbb{Q}$$

Proposed by Ionuţ Florin Voinea - Romania

SP.410.Let $z_1, z_2, z_3 \in \mathbb{C}^*$ different in pairs such that $|z_1| = |z_2| = |z_3| = 1, A(z_1), B(z_2), C(z_3)$. Prove that:

$$\sum_{cyc} |2z_1 - z_2 - z_3|^4 = 243 \Rightarrow AB = BC = CA.$$

Proposed by Marian Ursărescu - Romania

SP.411.Let $z_1, z_2, z_3 \in \mathbb{C}^*$ different in pairs such that $|z_1| = |z_2| = |z_3|, A(z_1), B(z_2), C(z_3)$.

$$\sum_{cyc} \frac{1}{8z_1z_2z_3 - (z_1^2 + z_2z_3)(z_2 + z_3)} = \frac{3}{10z_1z_2z_3} \Rightarrow AB = BC = CA$$

Proposed by Marian Ursărescu - Romania

SP.412.Let $A \in M_n(\mathbb{R})$ such that $A^{2021} = I_n + A + A^2 + \ldots + A^{2019}$. Prove that:

$$\det(A^3 + I_n) \ge 0$$

Proposed by Marian Ursărescu - Romania

SP.413.Let $\alpha > 1$ fixed. For $\forall n \in \mathbb{N}^*$ denote $k(n) = \min\{k \in \mathbb{N} | (n+1)^k \ge \alpha \cdot n^k\}$ and $(x_n)_{n\ge1}, x_{n+1} = x_n + \frac{1}{e^{x_n}}$. Find

$$\Omega = \lim_{n\to\infty} \frac{k(n) \cdot \log \sqrt[n]{n}}{x_n}$$

Proposed by Florică Anastase - Romania

SP.414. Solve for real numbers:

$$\begin{cases} x^4 = \sqrt{y^4 + 8} - \sqrt{y^4 + 3} \\ y^4 = \sqrt{z^4 + 8} - \sqrt{z^4 + 3} \\ z^4 = \sqrt{t^4 + 8} - \sqrt{t^4 + 3} \\ t^4 = \sqrt{x^4 + 8} - \sqrt{x^4 + 3} \end{cases}$$

Proposed by Daniel Sitaru - Romania

SP.415. Solve for real numbers:

$$\tan x + 2\tan 2x + 4\tan 4x + 8\cot 8x = 1$$

Proposed by Daniel Sitaru - Romania

SP.416. If $-3 < x, y, z < 3, x + y + z = 0$ then:

$$\left|\frac{xyz}{9 + xy + yz + zx}\right| < 3$$

Proposed by Daniel Sitaru - Romania

SP.417. Let $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ be sequences of real numbers such that:

$$x_n = \sum_{k=3}^{n} \tan\left(\frac{\pi}{k}\right) - \pi \log n, y_n = \sum_{k=1}^{n} 2^{k-1} \cdot \left[\frac{k^2}{k+1}\right], [*] \text{ - GIF.}$$

Find:

$$\Omega = \lim_{n\to\infty} \frac{2^n \cdot x_n}{y_n}$$

Proposed by Florică Anastase - Romania

SP.418. Solve for real numbers:

$$\begin{cases} \sin^3 x + \cos^3 y + z^3 + 3z = 3z^2 + 2 \\ \sin^2 x + \cos^y + z^2 = 2z + 2 \\ \sin x + \cos y + z = 2 \end{cases}$$

Proposed by Daniel Sitaru - Romania

SP.419. If $a, b, c \in \mathbb{R}; \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = 1$, then solve for real numbers:

$$\sin x \cdot \sin y \cdot \sin z = \frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b}$$

Proposed by Daniel Sitaru - Romania

SP.420. If $x, y, z \in \mathbb{R}, 32(x^5 + y^5 + z^5) = 3$, then:

$$\sum_{cyc}(2x^6 + x^4 + x^3 + x^2) + \frac{51}{32} \geq 2(x + y + z)$$

Proposed by Daniel Sitaru - Romania

UNDERGRADUATE PROBLEMS

UP.406. If $0 < a \leq b$ then:

$$\left(\int_a^b \frac{x^2+1}{x^3+1}dx\right)\left(\int_a^b \frac{\sqrt{x}}{x^3+1}dx\right) \leq \frac{(b-a)^2}{\sqrt{a}(1+a^2)}$$

Proposed by Daniel Sitaru - Romania

UP.407. If $0 < a \leq b$ then:

$$2\int_a^b \int_a^b \sqrt{x^2+xy+y^2}dxdy \geq \sqrt{3}(b+a)(b-a)^2$$

Proposed by Daniel Sitaru - Romania

UP.408. If $f, g : [a, b] \to (0, \infty); 0 < a \leq b; f, g$ - continuous, then:

$$6\int_a^b \frac{f(x)g(x)}{f(x)+g(x)}dx \leq \int_a^b (f(x)+g(x))dx + \int_a^b \sqrt{f(x)g(x)}dx$$

Proposed by Daniel Sitaru - Romania

UP.409. If $a, b, c, d \in (0, \frac{4\pi}{\pi^2-4})$ then:

$$\int_a^b \frac{\tan^{-1} x}{x}dx + \int_c^d \frac{\tan^{-1} x}{x}dx \geq \frac{\pi}{2} \cdot \log\left(\frac{4\pi\sqrt{bd}}{(2a+\pi)(2c+\pi)}\right)$$

Proposed by Daniel Sitaru - Romania

UP.410. Let $(x_n)_{n\geq 1}$ be sequence of real numbers such that

$$x_n = \sum_{k=1}^{n} \sin\frac{\pi}{k} - \pi \log n$$

Find:

$$\Omega = \lim_{n\to\infty} x_n \cdot \sum_{k=1}^{n} \frac{1}{n + \sqrt[3]{(k+1)^2(k^2+1)^2}}$$

Proposed by Florică Anastase - Romania

UP.411. Let $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ be sequences of real numbers such that:

$$x_n = \sum_{k=1}^{n} \sin\frac{1}{k} + \log\left(\sin\frac{1}{n}\right), y_n = \sum_{k=1}^{n^2+n}\left[\sqrt{k} + \frac{1}{2}\right], [*] \text{ - GIF.}$$

Find:

$$\Omega = \lim_{n\to\infty} \frac{x_n}{y_n}$$

Proposed by Florică Anastase - Romania

UP.412.Find:

$$\Omega=\lim_{n\to\infty}\frac{\left[\frac{1}{\sqrt{1-\frac{1}{2}}}+\frac{1}{\sqrt{1-\frac{1}{2^2}}}+\ldots+\frac{1}{\sqrt{1-\frac{1}{2^n}}}\right]^{\alpha}}{[\sqrt[3]{1}]+[\sqrt[3]{2}]+[\sqrt[3]{3}]+\ldots+[\sqrt[3]{n^3-1}]}, [*] \text{ -GIF}, \alpha\in\mathbb{R}$$

Proposed by Florică Anastase - Romania

UP.413. Find:

$$\Omega=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{\tan^{-1}k}{k}\cdot\tan^{-1}(n-k+1)$$

Proposed by Daniel Sitaru - Romania

UP.414. If $0<a\le b$ then:

$$\int_a^b\int_a^b\int_a^b\frac{z\cdot\min(x,\frac{1}{y},y+\frac{1}{x})dxdydz}{z^2+1}\le\frac{\sqrt{2}}{2}(b-a)^2\log\left(\frac{b^2+1}{a^2+1}\right)$$

Proposed by Daniel Sitaru - Romania

UP.415. Let ABC denote a triangle and H its orthocenter. Let point M be the middle of the segment AH. Prove that: (a) angle BMC is acute. (b) area $\Delta BMC=\frac{1}{8}\cdot AH^2\cdot\tan\widehat{BMC}$.

Proposed by George Apostolopoulos - Greece

UP.416. Let ABC denote a triangle with circumradius R. Let D,E,F be chosen on sides BC,CA,AB, respectively, so that AD,BE and CF bisect the angles of ABC. Prove:

$R\ge 2R'$, where R' denotes the circumradius of triangle DEF.

Proposed by George Apostolopoulos - Greece

UP.417. Find:

$$\Omega(a)=\lim_{x\to\infty}\left((x+a)\sqrt[x+1]{\Gamma(x+2)}\sin\frac{1}{x+a}-x\sqrt[x]{\Gamma(x+1)}\sin\frac{1}{x}\right); a>0$$

Proposed by D.M. Bătineţu - Giurgiu, Neculai Stanciu - Romania

UP.418. In ΔABC the following relationship holds:

$$\frac{3}{2}\cdot\sqrt[6]{\frac{4F}{R^2}}\le\sum_{cyc}\sqrt{\frac{r_a}{b+c}}\le\frac{1}{2}\left(1+\frac{4R}{r}\right)\sqrt{\frac{Rr}{2F}}$$

Proposed by Marin Chirciu - Romania

UP.419. If $n \in \mathbb{N}; n \geq 3$ then:

$$n^{\frac{1}{n}+\frac{1}{n^2}+\dots+\frac{1}{n^n}} > (n+1)^{\sqrt[n]{(n+1)^{n+1}}}$$

Proposed by Daniel Sitaru - Romania

UP.420. If $x \geq 0$ then:

$$\frac{3\cosh(4x) + 5\cosh(3x)}{\cosh x(3+5e^{-x})(3+5e^{x})} \geq \frac{\operatorname{sech}^5 x}{3 + 5\operatorname{sech} x}$$

Proposed by Daniel Sitaru - Romania

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