Proposed by N. Bhandari, Prove that

$$\int_0^1 \frac{x}{x^2+1}\ln^2\left(\frac{x}{1-x}\right)dx = \int_0^\infty \frac{\ln^2(x)}{(x+1)(1+(x+1)^2)}dx = \frac{\ln^3(2)}{24} + \frac{13\pi^2}{96}\ln(2)$$

Solution: Choose $f(z) = \dfrac{z}{z^2+1}\left(\overbrace{\log(z)}^{+} - \overbrace{\log(1-z)}^{-}\right)^3$ with $0 \leq \arg(1-z) < 2\pi$

and -$\pi \leq \arg(z) < \pi$, Lets discuss continuity of $f(z)$. We have two different logaritiums

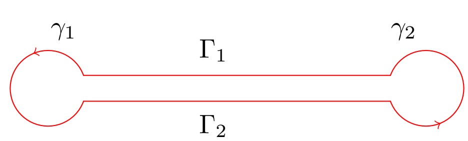
$\overbrace{\log(z)}^{+}$ having argument $-\pi \leq \arg(z) < \pi$, Branch cut will be $(-\infty, 0\,]$. this logarithm is Continuous in $\mathbb{C} \setminus (-\infty, 0\,]$

$\overbrace{\log(1-z)}^{-}$ with argument $0 \leq \arg(1-z) < 2\pi$ Branch cut will be $(-\infty, 1\,]$

Now we will prove $f(z)$ is single valued and continuous in $\mathbb{C} \setminus [0,1]$ check out real axis

$z =$	**From Top**		**From Bottom**		$\arg(z) - \arg(1-z)$
$z = -r$	$\arg(z) \to \pi$	$\arg(1-z) \to 2\pi$	$\arg(z) \to -\pi$	$\arg(1-z) \to 0$	Both are same
$1 > z = r > 0$	$\arg(z) \to 0$	$\arg(1-z) \to 2\pi$	$\arg(z) \to 0$	$\arg(1-z) \to 0$	Phase factor $= 2\pi$

Our table show $f(z)$ is continuous on $\mathbb{C} \setminus [\,0\,,1\,]$. Now we will integrate $f(z)$ on dogbone shaped contour



Here $\gamma_1 : |z| = \epsilon$ and $\gamma_2 : |1-z| = \delta$ These ϵ and δ are small

$$\oint f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz + \int_{\Gamma_1} f(z)dz + \int_{\Gamma_2} f(z)dz$$

$$\left|\int_{\gamma_1} f(z)dz\right| \leq \frac{2\epsilon^2}{1-\epsilon}\left(\log(\epsilon)+\frac{\epsilon}{1-\epsilon}+\pi\right)\pi \Rightarrow \lim_{\epsilon\to 0^+}\int_{\gamma_1} f(z)dz = 0$$

Similarly $\lim_{\delta\to 0^+}\int_{\gamma_2} f(z)dz = 0$

$$\int_{\Gamma_1} f(z)dz+\int_{\Gamma_2} f(z)dz = \int_0^1 \frac{x}{x^2+1}\left(\log^3\left(\frac{x}{1-x}\right)-\left(\log\left(\frac{x}{1-x}\right)-2\pi\iota\right)^3\right)dx$$

$$= 6\pi\iota\int_0^1 \frac{x}{x^2+1}\ln^2\left(\frac{x}{1-x}\right)dx - 4\pi^3\iota\ln(2) + 12\pi^2\int_0^1 \frac{x}{x^2+1}\ln\left(\frac{x}{1-x}\right)dx$$

We know that $\oint f(z)dz = -2\pi\iota\sum_i \mathbf{Res}_{z=z_i} f(z)$

$$\mathbf{Res}_{z=\infty} f(z) = -\mathbf{Res}_{z=0}\frac{1}{z^2}f\left(\frac{1}{z}\right) = \mathbf{Res}_{z=0}\frac{1}{z(z^2+1)}\left(\log(1-z)+\iota\pi\right)^3 = -\iota\pi^3$$

$$\mathbf{Res}_{z=\iota} f(z) = \frac{1}{2}\left(\frac{\iota\pi}{2}-\frac{\ln(2)}{2}-\frac{7\pi\iota}{4}\right)^3 = \frac{125\iota\pi^3}{128}-\frac{\ln^3(2)}{16}-\frac{15\pi\iota\ln^2(2)}{32}$$

$$+\frac{75}{64}\pi^2\ln(2)$$

$$\mathbf{Res}_{z=-\iota} f(z) = \frac{27\iota\pi^3}{128}-\frac{\ln^3(2)}{16}-\frac{9\pi\iota\ln^2(2)}{32}+\frac{27}{64}\pi^2\ln(2)$$

Sum of these residues $= \frac{3\iota\pi^3}{16}+\frac{19\pi^3\iota}{16}-\frac{\ln^3(2)}{8}-\frac{3\iota\pi\ln^2(2)}{4}+\frac{51}{32}\pi^2\ln(2)$

Comparing Real and Imaginary parts we get

$$6\pi\int_0^1 \frac{x}{x^2+1}\ln^2\left(\frac{x}{1-x}\right)dx - 4\pi^3\ln(2) = -\frac{51}{16}\pi^3\ln(2)+\frac{\pi\ln^3(2)}{4}$$

and $12\pi^2\int_0^1 \frac{x}{x^2+1}\ln\left(\frac{x}{1-x}\right)dx = \frac{3\pi^4}{8}-\frac{3}{2}\pi^2\ln^2(2)$

$$\int_0^1 \frac{x}{x^2+1}\ln\left(\frac{x}{1-x}\right)dx = \frac{\pi^2}{32}-\frac{1}{8}\ln^2(2)$$

$$\boxed{\int_0^1 \frac{x}{x^2+1}\ln^2\left(\frac{x}{1-x}\right)dx = \frac{\ln^3(2)}{24}+\frac{13\pi^2}{96}\ln(2)}$$

Also $\int_0^1 \frac{x}{x^2+1}\ln^2\left(\frac{x}{1-x}\right)dx \overbrace{=}^{x/(1-x)\to x} \int_0^\infty \frac{\ln^2(x)}{(x+1)((x+1)^2+1)}dx$ Done !

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