CRUX MATHEMATICORUM CHALLENGES-(II)

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4165. Prove that for all real numbers x_1, x_2, x_3 and x_4, we have,

$$|x_1 + x_2 + x_3 + x_4| + 2(|x_1| + |x_2| + |x_3| + |x_4|) \geq 6 \sqrt[6]{\prod_{1 \leq i < j \leq 4} |x_i + x_j|}$$

Proposed by Daniel Sitaru - Romania

Solution with generalization by Michel Bataille.
We prove the stronger result that for any complex numbers x_1, x_2, x_3 and x_4, we have

$$(1) \qquad |x_1 + x_2 + x_3 + x_4| + 2(|x_1| + |x_2| + |x_3| + |x_4|) \geq \sum_{1 \leq i < j \leq 4} |x_i + x_j|$$

The proposed inequality then follows from (1) by the AM-GM Inequality. To prove (1), we will make use of Hlawka's inequality which states that

$$(2) \qquad |a + b + c| + |a| + |b| + |c| \geq |a + b| + |b + c| + |c + a|$$

for all complex numbers a, b, c.
Setting $a = x_1, b = x_2$ and $c = x_3 + x_4$, then from (2) we have

$$(3) \quad |x_1+x_2+x_3+x_4|+|x_1|+|x_2|+|x_3+x_4| \geq |x_1+x_2|+|x_2+x_3+x_4|+|x_1+x_3+x_4|$$

Applying (2) again, we obtain:

$$(4) \qquad |x_2 + x_3 + x_4| \geq |x_2 + x_3| + |x_3 + x_4| + |x_2 + x_4| - |x_2| - |x_3| - |x_4|$$

and

$$(5) \qquad |x_1 + x_3 + x_4| \geq |x_1 + x_3| + |x_3 + x_4| + |x_1 + x_4| - |x_1| - |x_3| - |x_4|$$

Adding (4) and (5) and denoting the right side of (3) by R, then we have:

$$(6) \qquad R \geq |x_3 + x_4| - |x_1| - |x_2| - 2|x_3| - 2|x_4| + \sum_{1 \leq i < j \leq 4} |x_i + x_j|$$

From (3) and (6), we deduce that

$$|x_1 + x_2 + x_3 + x_4| + |x_1| + |x_2| + |x_3 + x_4| \geq$$
$$\geq |x_3 + x_4| - |x_1| - |x_2| - 2|x_3| - 2|x_4| + \sum_{1 \leq i < j \leq 4} |x_i + x_j|$$

from which (1) follows immediately. □

4205. Prove that for $0 < a < c < b, a, b, c \in \mathbb{R}$, we have:

$$\frac{1}{c\sqrt{ab}} \int_a^b x \arctan x dx > \frac{(c-a)\arctan\sqrt{ac}}{\sqrt{bc}} + \frac{(b-c)\arctan\sqrt{bc}}{\sqrt{ac}}$$

Proposed by Daniel Sitaru - Romania

Solution by Paul Braken.
Let $f(x) = x \arctan x$ for $x > 0$. Since $f(0) = f'(0) = 0$,
$f'(x) = \arctan x + x(1+x^2)^{-1}$ and $f''(x) = 2(1+x^2)^{-2}$, then f is positive, strictly increasing and strictly convex. By the Mean Value Theorem, we have that:

$$f(p) + f'(p)(x-p) < f(x)$$

for distinct positive x and p. Hence

$$\begin{aligned}(c-a)f(\sqrt{ac}) &< (c-a)f(\sqrt{ac}) + \frac{1}{2}f'(\sqrt{ac})(c-a)(\sqrt{c}-\sqrt{a})^2 \\ &= (c-a)f(\sqrt{ac}) + f'(\sqrt{ac})\int_a^c (x-\sqrt{ac})dx \\ &< \int_a^c f(x)dx,\end{aligned}$$

and

$$(b-c)f(\sqrt{bc}) < \int_c^b f(x)dx$$

Therefore

$$(c-a)\sqrt{ac}\arctan\sqrt{ac} + (b-c)\sqrt{bc}\arctan\sqrt{bc} < \int_a^b x\arctan x dx$$

Dividing by $(\sqrt{ac})(\sqrt{bc})$ yields the desired inequality. □

4226. Prove that if $0 < a < b$ then:

$$\left(\int_a^b \frac{\sqrt{1+x^2}}{x}dx\right)^2 > (b-a)^2 + \ln^2\left(\frac{b}{a}\right)$$

Proposed by Daniel Sitaru - Romania

Solutions by – a composite of virtually the same solutions by Arkady Alt; Michel Bataille; M. Bello, M. Benito, O. Ciaurri, E. Fernandez, and L. Roncal (jointly); and Digby Smith

Note first that

$$\left(\int_a^b \frac{\sqrt{1+x^2}}{x}dx\right)^2 > (b-a)^2 + \ln^2\frac{b}{a}$$

$$\Leftrightarrow \left(\int_a^b \frac{\sqrt{1+x^2}}{x}dx\right)^2 - \left(\int_a^b \frac{1}{x}dx\right)^2 > (b-a)^2$$

$$\Leftrightarrow \int_a^b \frac{\sqrt{1+x^2}+1}{x}dx \cdot \int_a^b \frac{\sqrt{1+x^2}-1}{x}dx > (b-a)^2 \tag{1}$$

Let $f(x) = \frac{\sqrt{1+x^2}+1}{x}, d \in [a,b]$. Then $f(x) > 0$ and $\frac{1}{f(x)} = \frac{\sqrt{1+x^2}-1}{x}$. By the integral form of the Cauchy - Schwarz Inequality, we have:

$$\left(\int_a^b f(x)dx\right)\left(\int_a^b \frac{1}{f(x)}dx\right) = \left(\int_a^b (\sqrt{f(x)})^2 dx\right)\left(\int_a^b \left(\sqrt{\frac{1}{f(x)}}\right)^2 dx\right)$$

$$\geq \left(\int_a^b 1 dx \right)^2$$

(2) $$= (b-a)^2$$

But equality cannot hold in (2) as f is not a constant on $[a, b]$. Hence, from (1) and (2) the result follows.

4256. Let $a, b, c \in \mathbb{R}$ such that $a + b + c = 1$. Prove that:

$$\frac{e^b - e^a}{b - a} + \frac{e^c - e^b}{c - b} + \frac{e^a - e^c}{a - c} > 4$$

Proposed by Daniel Sitaru - Romania

Solution 1 by Dionne Bailey, Elsie Campbell, and Charles Diminnie.
We will prove the slight improvement that

$$\frac{e^b - e^a}{b - a} + \frac{e^c - e^b}{c - b} + \frac{e^a - e^c}{a - c} > 3e^{\frac{1}{3}} > 4$$

for distinct $a, b, c \in \mathbb{R}$, which satisfy the condition $a + b + c = 1$.
Note first that the last inequality follows from the fact that

(1) $$\left(\frac{4}{3}\right)^3 = \frac{64}{27} = 2.\overline{370} < e$$

For the remainder of our solution, we will utilize Hadamard's Inequality which states that if $f(x)$ is continuous and convex on $[p, q]$, then

(2) $$\frac{1}{q-p} \int_p^q f(x) dx \geq f\left(\frac{p+q}{2}\right)$$

A proof of this result can be found in R. P. Boas, Jr., *A Primer of Real Functions (3rd ed.)*, Carus Mathematical Monograph No. 13, The Mathematical Association of America, 1981, pg. 174.
Since a and b must be distinct and

$$\frac{e^b - e^a}{b - a} = \frac{e^a - e^b}{a - b},$$

we may assume without loss of generality that $a < b$. Then, since $f(x) = e^x$ is continuous and convex on $\mathbb{R}$, (2) implies that

(3) $$\frac{e^b - e^a}{b - a} = \frac{1}{b - a} \int_a^b e^x dx \geq e^{\frac{a+b}{2}}$$

Similar arguments show that

(4) $$\frac{e^c - e^b}{c - b} \geq e^{\frac{b+c}{2}} \text{ and } \frac{e^a - e^c}{a - c} \geq e^{\frac{a+c}{2}}$$

Further, because $f(x) = e^x$ is strictly convex on $\mathbb{R}$, Jensen's Theorem and the distinct values of a, b and c imply that

(5) $$e^{\frac{a+b}{2}} + e^{\frac{b+c}{2}} + e^{\frac{a+c}{2}} > 3e^{\frac{1}{3}\left(\frac{a+b}{2} + \frac{b+c}{2} + \frac{a+c}{2}\right)} = 3e^{\frac{a+b+c}{3}} = 3e^{\frac{1}{3}}$$

Finally, it follows from (1), (3), (4), and (5) that

$$\frac{e^b - e^a}{b - a} + \frac{e^c - e^b}{c - b} + \frac{e^a - e^c}{a - c} \geq e^{\frac{a+b}{2}} + e^{\frac{b+c}{2}} + e^{\frac{a+c}{2}} > 3e^{\frac{1}{3}} > 3\left(\frac{4}{3}\right) = 4.$$

□

Solution 2 by M. Bello, M. Benito, O. Ciaurri, E. Fernandez, and L. Roncal.
We prove a more general result.
Let $a_1, a_2, \dots, a_n \in \mathbb{R}$ such that $a_1 + a_2 + \dots + a_n = 1$, then

$$\sum_{k=1}^{n} \frac{e^{a_{k+1}} - e^{a_k}}{a_{k+1} - a_k} \geq n e^{\frac{1}{n}} \tag{1}$$

with $a_{n+1} = a_1$. Moreover, the equality holds if and only if $a_i = \frac{1}{n}$, for $i = 1, \dots, n$ (in this case the left hand side has to be understood as a limit).
The proposed inequality follows taking $n = 3, a_1 = a, a_2 = b$, and $a_3 = c$ and using that $3e^{\frac{1}{3}} = 4.186837 > 4$.
Let us prove (1). From the inequality $\frac{\sinh x}{x} \geq 1$, for $x \in \mathbb{R}$, with equality for $x = 0$ only, taking $x = \frac{x_{k+1} - a_k}{2}$, we deduce that

$$\frac{e^{a_{k+1}} - e^{a_k}}{a_{k+1} - a_k} \geq e^{\frac{a_{k+1} + a_k}{2}},$$

with equality when $a_{k+1} = a_k$. In this way, applying the AM-GM inequality, we have

$$\sum_{k=1}^{n} \frac{e^{a_{k+1}} - e^{a_k}}{a_{k+1} - a_k} \geq \sum_{k=1}^{n} e^{\frac{a_{k+1} + a_k}{2}} \geq n e^{\frac{a_1 + \dots + a_n}{n}} = n e^{\frac{1}{n}}$$

and the equality holds when $a_i = \frac{1}{n}$, for $i = 1, \dots, n$, only. □

Solution 3 by Paul Braken.
By Taylor's theorem, we have the expansion with remainder

$$e^b = e^a + e^a(b-a) + \frac{1}{2}e^a(b-a)^2 + \frac{e^{\tau_1}}{6}(b-a)^3,$$

where τ_1 in the remainder is between a and b. This implies that

$$\frac{e^b - e^a}{b-a} = e^a + \frac{1}{2}e^a(b-a) + \frac{e^{\tau_1}}{6}(b-a)^2 \geq e^a + \frac{1}{2}e^a(b-a),$$

since $e^{\tau_1} > 0$ and $(b-a)^2 \geq 0$ always holds. In exactly the same way, we obtain the inequalities

$$\frac{e^c - e^b}{c-b} = e^b + \frac{1}{2}e^b(c-b) + \frac{e^{\tau_2}}{6}(c-b)^2 \geq e^b + \frac{1}{2}e^b(c-b),$$
$$\frac{e^a - e^c}{a-c} = e^c + \frac{1}{2}e^c(a-c) + \frac{e^{\tau_3}}{6}(a-c)^2 \geq e^c + \frac{1}{2}e^c(a-c).$$

Adding these three results, the following lower bound for the function in (1) is obtained,
(2)

$$h(a,b,c) = \frac{e^b - e^a}{b-a} + \frac{e^c - e^b}{c-b} + \frac{e^a - e^c}{a-c} \geq e^a + e^b + e^c + \frac{1}{2}\Big(e^a(b-a) + e^b(c-b) + e^c(a-c)\Big)$$

This result holds for all a, b, c and is independent of the constraint which has not been used.
Let us minimize the function on the right of (2),

$$f(a,b,c) = e^a + e^b + e^c + \frac{1}{2}\Big(e^b(c-b) + e^b(c-b) + e^c(a-c)\Big),$$

by introducing a Lagrange multiplier λ

$$\mathcal{L} = f(a,b,c) - \lambda(a+b+c-1).$$

Differentiating $\mathcal{L}$ with respect to a, b, c and λ, the following nonlinear system results,

$$\begin{aligned} e^a + e^c + e^a(b-a) - 2\lambda = 0, \\ e^b + e^a + e^b(c-b) - 2\lambda = 0, \\ e^c + e^b + e^c(a-c) - 2\lambda = 0, \\ a+b+c-1 = 0. \end{aligned}$$

This set of equations maps into itself under a cyclic permutation of the variables. The first three equations of (3) can be put in the form,

$$1+b-a+e^{c-a} = 2\lambda e^{-a}, 1+c-b+e^{a-b} = 2\lambda e^{-b}, 1+a-c+e^{c-b} = 2\lambda e^{-c}$$

For example, adding these three equations, and expression for λ results,

$$\lambda = \frac{e^{a-b}+e^{c-a}+e^{c-b}+3}{2(e^{-a}+e^{-b}+e^{-c})}.$$

In fact, the solution to the system (3) is given by

$$a=b=c=\frac{1}{3}, \quad \lambda = e^{\frac{1}{3}}.$$

The minimum value of f is found to be

$$f\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right) = 3e^{\frac{1}{3}} > 4 \tag{4}$$

This will correspond to a minimum since a maximum is not expected. Take for example $a = N, b = -N+1$ and $c = 0$, then $e^N \to \infty$ as $N \to \infty$, so h can be made as large as we please. Combining (2) and (4), these imply (1).
Letting $c \to b$ and then $b \to a$ in h and the constraint, or using Taylor's formula, it can be seen that h reduces to $3e^{\frac{1}{3}}$ which matches the minimum (4). Thus the absolute minimum of h under the constraint is $3e^{\frac{1}{3}}$. □

4265. Consider real numbers $a, b, c \in (0,1)$ such that $a+b+c=1$. Show that:

$$\frac{4}{\pi}(\arctan a + \arctan b + \arctan c) > \frac{1}{2-(ab+bc+ca)}$$

Proposed by Daniel Sitaru - Romania

Solution by the team D. Bailey, E. Campbell, and C. Diminnie.
Since $\frac{4}{\pi}\arctan x$ is concave for $x \geq 0$ and is equal to x for $x=0$ and $x=1$,

$$\frac{4}{\pi}\arctan x \geq x$$

for $0 \leq x \leq 1$. Therefore the left side of the inequality is not less than $a+b+c=1$. Since

$$\begin{aligned} 2(ab+bc+ca) &= (a+b+c)^2 - (a^2+b^2+c^2) \\ &= 1-(a^2+b^2+c^2) \\ &\leq 1-(ab+bc+ca), \end{aligned}$$

then $ab+bc+ca \le \frac{1}{3}$ and

$$\frac{1}{2-(ab+bc+ca)} \le \frac{3}{5} < 1$$

The result follows. □

4276. Let P be a point on the interior of a triangle ABC and let $PA = x, PB = y$ and $PC = z$. Prove that:

$$27(ax+by-cz)(by+cz-ax)(cz+ax-by) \le (ax+by+cz)^3$$

Proposed by Daniel Sitaru - Romania

Solution by Digby Smith.
Let $p = ax, q = by$ and $r = cz$. Substituting, expanding, then applying Schur's inequality before applying the AM-GM inequality gives

$$\begin{aligned}
&(ax+by-cz)(by+cz-ax)(cz+ax-by)\\
&= (p+q-r)(q+r-p)(r+p-q)\\
&= pq(p+q)+qr(q+r)+rp(r+p)-p^3-q^3-r^3-2pqr\\
&\le pqr\\
&\le \left(\frac{p+q+r}{3}\right)^3
\end{aligned}$$

making

$$27(ax+by-cz)(by+cz-ax)(cz+ax-by) \le (ax+by+cz)^3$$

with equality if and only if $ax = by = cz$. □

4298. Compute:

$$L = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{n+k}{2+\sin(n+k)+(n+k)^2}$$

Proposed by Daniel Sitaru - Romania

Solution by Missouri State University Problem Solving Group.
Define

$$f(n,k) = \frac{n+k}{2+\sin(n+k)+(n+k)^2} \text{ and } g(n,k) = \frac{1}{n+k}$$

Since $1 \le 2+\sin(n+k) \le 3$, then for $1 \le k \le n$, we have

$$\begin{aligned}
|g(n,k)-f(n,k)| &= \frac{2+\sin(n+k)}{(n+k)(2+\sin(n+k)+(n+k)^2)}\\
&\le \frac{3}{(n+k)(1+(n+k)^2)} \le \frac{3}{n^3}.
\end{aligned}$$

Therefore

$$\lim_{n\to\infty}\left|\sum_{k=1}^{n} g(n,k)-f(n,k)\right| \le \lim_{n\to\infty}\sum_{k=1}^{n}\frac{3}{n^3} = \lim_{n\to\infty}\frac{3}{n^2} = 0$$

In particular, we now have

$$L = \lim_{n\to\infty} \sum_{k=1}^{n} f(n,k) = \lim_{n\to\infty} \sum_{k=1}^{n} g(n,k) = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k}.$$

Let $h(x) = \frac{1}{x}$. Since h is continuous on $[1,2]$, it is integrable on $[1,2]$. Therefore

$$L = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n(1+\frac{k}{n})} = \lim_{n\to\infty} \sum_{k=1}^{n} h\Big(1+\frac{k}{n}\Big)\Big(\frac{1}{n}\Big) = \int_1^2 h(x)dx = \ln 2.$$

□

4309. Let a, b and c be real numbers such that $a+b+c=3$. Prove that:

$$2(a^4+b^4+c^4) \geq ab(ab+1)+bc(bc+1)+ca(ca+1)$$

Proposed by Daniel Sitaru - Romania

Solution 1 by Sefket Arslanagic.
Using the inequality

$$3(x^2+y^2+z^2) \geq (x+y+z)^2,$$

we have that

$$a^4+b^4+c^4 \geq \frac{1}{3}(a^2+b^2+c^2)(a^2+b^2+c^2)$$

$$\geq \frac{1}{9}(a+b+c)^2(a^2+b^2+c^2) = a^2+b^2+c^2$$

$$\geq ab+bc+ca$$

Also $a^4+b^4+c^4 \geq a^2b^2+b^2c^2+c^2a^2$, so that the desired inequality holds. □

Solution 2 by AN-anduud Problem Solving Group.
Using the inequality

$$x^2+y^2+z^2 \geq xy+yz+zx,$$

we have that

$$a^4+b^4+c^4 \geq (ab)^2+(bc)^2+(ca)^2.$$

Since

$$x^4-4x+3 = (x-1)^2[(x+1)^2+2] \geq 0,$$

we find that

$$a^4+b^4+c^4 \geq 4(a+b+c)-9 = 3 = \frac{1}{3}(a+b+c)^2$$

$$= \frac{1}{6}[(a-b)^2+(b-c)^2+(c-a)^2+6(ab+bc+ca)]$$

$$\geq ab+bc+ca$$

Adding these two inequalities for $a^4+b^4+c^4$ yields the desired result. Equality holds iff $a=b=c=1$. □

Solution 3 by Paolo Perfetti and Angel Plaza, independently.
Recall the Muirhead Inequalities for three variables. For $a, b, c > 0$ and $p \geq q \geq r$, let

$$[p, q, r] = \frac{1}{6}(a^p b^q c^r + a^p b^r c^q + a^q b^p c^r + a^q b^r c^p + a^r b^p c^q + a^r b^q c^p)$$

Then,

$$p \geq u, p + q \geq u + v \text{ and } p + q + r = u + v + w$$

together imply that $[p, q, r] \geq [u, v, w]$.
Make the given inequality homogeneous by replacing each 1 by $\frac{1}{9}(a+b+c)^2$. Thus we have to prove that

$$18(a^4 + b^4 + c^4) \geq 11(a^2b^2 + b^2c^2 + a^2c^2) + (a^3b + ab^3 + b^3c + bc^3 + a^3c + ac^3) + \\ +5(a^2bc + ab^2c + abc^2)$$

or, equivalently,

$$9[4, 0, 0] \geq \frac{11}{2}[2, 2, 0] + [3, 1, 1] + \frac{5}{2}[2, 1, 1].$$

This is true since

$$[4, 0, 0] \geq [2, 2, 0],$$
$$[4, 0, 0] \geq [3, 1, 1],$$
$$[4, 0, 0] \geq [2, 1, 1].$$

□

B.72. Prove that in triangle ABC, the following relationship holds:

$$\frac{\sin A}{\sin \frac{B}{2} \sin \frac{C}{2}} + \frac{\sin B}{\sin \frac{C}{2} \sin \frac{A}{2}} + \frac{\sin C}{\sin \frac{A}{2} \sin \frac{B}{2}} \geq \frac{2s}{r}$$

Proposed by Daniel Sitaru - Romania

Solution by Nguyen Van Canh-Ben Tre-Vietnam.

$$\prod \sin \frac{A}{2} = \frac{r}{4R}, \prod \cos \frac{A}{2} = \frac{s}{4R}$$

$$\frac{\sin A}{\sin \frac{B}{2} \sin \frac{C}{2}} + \frac{\sin B}{\sin \frac{C}{2} \sin \frac{A}{2}} + \frac{\sin C}{\sin \frac{A}{2} \sin \frac{B}{2}} = \sum \frac{\sin A}{\sin \frac{B}{2} \sin \frac{C}{2}} =$$

$$\frac{2}{\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}} \cdot \sum \frac{(\sin \frac{A}{2})^2}{\frac{1}{\cos \frac{A}{2}}} \overset{\text{Cauchy-Schwartz}}{\geq} \frac{2s}{r} \cdot \frac{(\sum \sin \frac{A}{2})^2}{\sum \cos \frac{B}{2} \cos \frac{C}{2}}$$

We need to prove that:

$$\frac{(\sum \sin \frac{A}{2})^2}{\sum \cos \frac{B}{2} \cos \frac{C}{2}} \geq 1;$$

$$(1) \qquad \leftrightarrow \left(\sum \sin \frac{A}{2}\right)^2 \geq \sum \cos \frac{B}{2} \cos \frac{C}{2}$$

$$(\exists \Delta A'B'C' \text{ such that: } A = \pi - 2A', B = \pi - 2B', C = \pi - 2C')$$

Now,

$$(1) \leftrightarrow \left(\sum \sin \frac{\pi - 2A'}{2}\right)^2 \geq \sum \cos \frac{\pi - 2B'}{2} \cos \frac{\pi - 2C'}{2};$$

$$\leftrightarrow \left(\sum \cos A'\right)^2 \geq \sum \sin B' \sin C'$$
$$\leftrightarrow \left(1 + \frac{r'}{R'}\right)^2 \geq \frac{p'^2 + 4R'r' + r'^2}{4R'^2};$$
$$\leftrightarrow 4(R' + r')^2 \geq p'^2 + 4R'r' + r'^2;$$
$$\leftrightarrow p'^2 \leq 4R'^2 + 4R'r' + 3r'^2$$

(Which is clearly true by Gerretsen's Inequality). So, (1) is true. Proved. □

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