CRUX MATHEMATICORUM CHALLENGES-(III)

DANIEL SITARU - ROMANIA

4316. Let $f : [0, 11] \to \mathbb{R}$ be an integrable and convex function. Prove that:

$$\int_3^5 f(x)dx + \int_6^8 f(x)dx \leq \int_0^2 f(x)dx + \int_8^{11} f(x)dx.$$

Proposed by Daniel Sitaru - Romania

Solution 1 by Roy Barbara.
Let $g(x) = ax + b$ be the linear function that satisfies $g(3) = f(3)$ and $g(8) = f(8)$. Because $f(x)$ is convex, $f(x) \geq g(x)$ when $0 \leq x \leq 3$ or $8 \leq x \leq 10$, and $f(x) \leq g(x)$ when $3 \leq x \leq 8$. The left side does not exceed

$$\int_3^5 g(x)dx + \int_6^8 g(x)dx = 22a + 4b,$$

and the right side is not less than

$$\int_0^2 g(x)dx + \int_9^{11} g(x)dx = 22a + 4b.$$

The result follows. □

Solution 2 by Editorial Board.
Since $f(x)$ is convex,

$$f(3 + x) \leq \frac{2}{3}f(x) + \frac{1}{3}f(9 + x) \text{ and } f(6 + x) \leq \frac{1}{3}f(x) + \frac{2}{3}f(9 + x).$$

Therefore,

$$\int_3^5 f(x)dx + \int_6^8 f(x)dx = \int_0^2 [f(3 + x) + f(6 + x)]dx$$

$$\leq \int_0^2 [f(x) + f(9 + x)]dx = \int_0^2 f(x)dx + \int_9^{11} f(x)dx.$$

□

Solution 3 by Oliver Geupel.
Recall the Hermite-Hadamard Inequality for convex functions:

$$(b - a)f\Big(\frac{a + b}{2}\Big) \leq \int_a^b f(x)dx \leq \frac{1}{2}(b - a)(f(a) + f(b)).$$

Therefore

$$\int_3^5 f(x)dx \leq f(3) + f(5) \leq \Big(\frac{7}{9}f(1) + \frac{2}{9}f(10)\Big) + \Big(\frac{5}{9}f(1) + \frac{4}{9}f(10)\Big)$$

$$= \frac{4}{3}f(1) + \frac{2}{3}f(10),$$

and, similarly,

$$\int_6^8 f(x)dx \le f(6) + f(8) \le \frac{2}{3}f(1) + \frac{4}{3}f(10).$$

Therefore

$$\int_3^5 f(x)dx + \int_6^8 f(x)dx \le 2f(1) + 2f(10) \le \int_0^2 f(x)dx + \int_9^{11} f(x)dx.$$

□

4346. Find all $x, y, z \in (0, \infty)$ such that

$$\begin{cases} 64(x+y+z)^2 = 27(x^2+1)(y^2+1)(z^2+1) \\ x+y+z = xyz \end{cases}$$

Proposed by Daniel Sitaru - Romania

Solution 1 by Paul Bracken.
Let

$$(x, y, z) = (\tan A, \tan B, \tan C),$$

where $0 < A, B, C < \frac{\pi}{2}$. Then the two equations become

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C$$

and

$$64(\tan A \tan B \tan C)^2 = 27(\sec^2 A)(\sec^2 B)(\sec^2 C).$$

These are equivalent to $A + B + C = \pi$ (expand $\tan(A + B + C)$) and

$$\sin^2 A \sin^2 B \sin^2 C = \frac{27}{64}.$$

Since $2 \ln \sin t$ is a strictly concave function of t on $(0, \frac{\pi}{2})$, by Jensen's inequality we get

$$\ln \sin^2 A + \ln \sin^2 B + \ln \sin^2 C \le 3 \ln \sin^2\left(\frac{A+B+C}{3}\right) = 3 \ln \sin^2\left(\frac{\pi}{3}\right).$$

Hence

$$\sin^2 A \sin^2 B \sin^2 C \le \left(\frac{3}{4}\right)^3 = \frac{27}{64},$$

with equality if and only if $A = B = C = \frac{\pi}{3}$.
Therefore the equations are satisfied if and only if $(x, y, z) = (\sqrt{3}, \sqrt{3}, \sqrt{3})$. □

Solution 2 by Nghia Doan.
Let

$$p = x + y + z = xyz \text{ and } q = xy + yz + zx.$$

Since

$$x^2 + y^2 + z^2 = p^2 - 2q$$

and

$$x^2y^2 + y^2z^2 + z^2x^2 = q^2 - 2p^2,$$

the first equation becomes

$$64p^2 = 27[p^2 + (q^2 - 2p^2) + (p^2 - 2q) + 1] = 27(q-1)^2.$$

Since
$$q = \Big(\frac{1}{xy} + \frac{1}{yz} + \frac{1}{zx}\Big)(xy + yz + zx) \geq 9,$$
then $8p = 3\sqrt{3}(q-1)$. By the AM-GM inequality,
$$p^3 = (x+y+z)^3 \geq 27xzy = p,$$
so that $p \geq 3\sqrt{3}$, with equality if and only if $x = y = z = \sqrt{3}$.
On the other hand: $(xy + yz + zx)^2 \geq 3((xy)(yz) + (yz)(zx) + (zx)(xy))$
$$= 3xyz(x+y+z) = 3p^2,$$
so that $q \geq \sqrt{3}p$ with equality if and only if $xy = yz = zx$. Hence
$$8p \geq 3\sqrt{3}(\sqrt{3}p - 1) = 9p - 3\sqrt{3},$$
so that $p \leq 3\sqrt{3}$.
It follows that the only solution of the given system of equations is
$x = y = z = \sqrt{3}$. □

Solution 3 by Madhav Modak.
From the previous solution, we have that
$$p^2 = \Big(\frac{27}{65}\Big)(q-1)^2, p^2 \geq 3q \text{ and } q^2 \geq 3p^2.$$
Substituting for p^2 in these two inequalities yield respectively
$$0 \leq 9q^2 - 82q + 9 = (9q-1)(q-9)$$
and
$$0 \leq -(17q^2 - 162q + 81) = (9-q)(17q-9).$$
The only value of the pair (p,q) that allows both inequalities to holds is $(\sqrt{3}, 9)$ and this in turn forces $x = y = z = 3\sqrt{3}$ as the unique solution of the system. □

4359. Let a, b and c be positive real numbers. Prove that:
$$3\ln(a^b + b^c + c^a) + \frac{a}{c} + \frac{b}{a} + \frac{c}{b} \geq a + b + c + \ln 27.$$
Proposed by Daniel Sitaru - Romania

Solution by Richard B. Eden and Ramanujan Srihari.
Let $f(x) = \ln x, x > 0$. Then $f''(x) = -\frac{1}{x^2} < 0$ so f is concave. By Jensen's Inequality we then have

(1) $$3\ln\Big(\frac{a^b + b^c + c^a}{3}\Big) \geq b\ln a + c\ln b + a\ln c$$

with equality if and only if $a = b = c$.
Next, consider $g(x) = \ln x + \frac{1}{x} - 1, x > 0$. Then $g'(x) = \frac{x-1}{x^2}$ which implies $g(x) \geq g(1) = 0$ so $\ln x \geq 1 - \frac{1}{x}$ for all $x > 0$. Hence,

(2) $$b\ln a \geq b\Big(1 - \frac{1}{a}\Big), \quad c\ln b \geq c\Big(1 - \frac{1}{b}\Big), \quad a\ln c \geq a\Big(1 - \frac{1}{c}\Big)$$

From (1) and (2), we obtain:
$$3\ln(a^b + b^c + c^a) \geq b\Big(1 - \frac{1}{a}\Big) + c\Big(1 - \frac{1}{b}\Big) + a\Big(1 - \frac{1}{c}\Big) + \ln 27$$

so
$$3\ln(a^b+b^c+c^a)+\frac{a}{c}+\frac{b}{a}+\frac{c}{b}\geq a+b+c+\ln 27$$
follows, completing the proof. $\square$

4366. Let x_n be the base angle of a right triangle with base n and altitude 1. Find:
$$\sum_{k=1}^{\infty} x_{k^2+k+1}.$$
Proposed by Daniel Sitaru - Romania

Solution 1.
The arms of the right triangle have lengths 1 and n, and x_n is the angle adjacent to the latter arm. Thus, $x_n = \arctan\frac{1}{n}$. Observe that
$$\tan(x_k - x_{k+1}) = \frac{\frac{1}{k}-\frac{1}{k+1}}{1+\frac{1}{k(k+1)}} = \frac{1}{k^2+k+1} = \tan(x_{k^2+k+1}).$$
Therefore
$$\sum_{k=1}^{\infty} x_{k^2+k+1} = \sum_{k=1}^{n}(x_k - x_{k+1}) = x_1 - x_{n+1} = \frac{\pi}{4} - \arctan\frac{1}{n+1},$$
so that
$$\sum_{k=1}^{\infty} x_{k^2+k+1} = \frac{\pi}{4}.$$
$\square$

Solution 2.
Let
$$u_n = \tan\left(\sum_{k=1}^{\infty} x_{k^2+k+1}\right).$$
Checking the values of u_n for small values of n, we are led to the conjecture that $u_n = n(n+2)$. Suppose that this holds for $n = m-1$. Then
$$u_n = \tan\left(x_{m^2+m+1} + \sum_{k=1}^{m-1} x_{k^2+k+1}\right)$$
$$= \frac{\frac{1}{m^2+m+1}+\frac{m-1}{m+1}}{1-\frac{m-1}{(m+1)(m^2+m+1)}} = \frac{m(m^2+1)}{(m+2)(m^2+1)} = \frac{m}{m+2}.$$
Thus, an induction argument, along with $u_1 = \frac{1}{3}$, establishes that $u_n = \frac{n}{n+2}$ for each positive integer n. Since the limit as n tends to infinity of u_n is 1, the sum of the given series is $\frac{\pi}{4}$. $\square$

4418. Consider a convex cyclic quadrilateral with sides a, b, c, d and area S. Prove that:
$$\frac{(a+b)^5}{c+d}+\frac{(b+c)^5}{d+a}+\frac{(c+d)^5}{a+b}+\frac{(d+a)^5}{b+c}\geq 64S^2.$$
Proposed by Daniel Sitaru - Romania

We make some preliminary remarks. The formula for the area S of a quadrilateral with sides a, b, c, d and perimeter $2s = a+b+c+d$ is

$$S = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2\theta},$$

where θ is half the sum of two opposite angles. This is dominated by the area of a cyclic quadrilateral with the same sides, namely

$$\begin{aligned}&\sqrt{(s-a)(s-b)(s-c)(s-d)}\\ &= \frac{1}{4}\sqrt{(b+c+d-a)(c+d+a-b)(d+a+b-c)(a+b+c-d)}\\ &= \frac{1}{4}\sqrt{[(a+b)^2-(c-d)^2][(c+d)^2-(a-b)^2]}\\ &= \frac{1}{4}\sqrt{[(a+c)^2-(b-d)^2][(b+d)^2-(a-c)^2]}\end{aligned}$$

Solution 1 by Oliver Geupel.
Let

$$(w, x, y, z) = (s-a, s-b, s-c, s-d).$$

Then

$$(a+b, b+c, c+d, d+a) = (y+z, z+w, w+x, x+y).$$

Applying the arithmetic – geometric means inequality twice, we find that

$$\begin{aligned}&\frac{(a+b)^5}{c+d} + \frac{(b+c)^5}{d+a} + \frac{(c+d)^5}{a+b} + \frac{(d+a)^5}{b+c}\\ &= \frac{(y+z)^5}{w+x} + \frac{(z+w)^5}{x+y} + \frac{(w+x)^5}{y+z} + \frac{(x+y)^5}{z+w}\\ &\geq 4(y+z)(z+w)(w+x)(x+y)\\ &\geq 4(2\sqrt{yz})(2\sqrt{zw})(2\sqrt{wx})(2\sqrt{xy})\\ &= 64xyzw \geq 64S^2\end{aligned}$$

Equality holds if and only if the quadrilateral is a square. □

Solution 2 by Sefket Arslanagic.
By the arithmetic-geometric means inequality,

$$\begin{aligned}S &\leq \sqrt{(s-a)(s-b)}\sqrt{(s-c)(s-d)}\\ &\leq \frac{1}{4}(2s-a-b)(2s-c-d) = \frac{1}{4}(c+d)(a+b).\end{aligned}$$

Similarly, $S \leq \frac{1}{4}(b+c)(a+d)$. Therefore

$$\begin{aligned}64S^2 &= 4(16S^2)\\ &\leq 4(a+b)(b+c)(c+a)(d+a)\\ &= 4\Big[\frac{(a+b)^5}{c+d}\cdot\frac{(b+c)^5}{d+a}\cdot\frac{(c+d)^5}{a+b}\cdot\frac{(d+a)^5}{b+c}\Big]^{\frac{1}{4}}\\ &\leq \frac{(a+b)^5}{c+d} + \frac{(b+c)^5}{d+a} + \frac{(c+d)^5}{a+b} + \frac{(d+a)^5}{b+c}.\end{aligned}$$

□

Solution 3 by C.R. Pranesachar.
By the arithmetic-geometric means inequality,

$$\frac{(a+b)^5}{c+d}+\frac{(c+d)^2}{a+b}\geq 2[(a+b)^2(c+d)^2]$$
$$\geq 2[(a+b)^2-(c-d)^2][(c+d)^2-(a-b)^2]$$
$$\geq 32S^2.$$

A similar inequality holds for the other two terms of the left side and the result follows. □

Solution 4 by Digby Smith.

$$64S^2=64(s-a)(s-b)(s-c)(s-d)$$
$$\leq 64\Big[\frac{(s-a)+(s-b)+(s-c)+(s-d)}{4}\Big]^4=64\Big(\frac{2s}{4}\Big)^4=4s^4.$$

From an instance of the Holder inequality, for positive x,y,z,t,m,n,p,q,

$$\Big(\frac{x^5}{m}+\frac{y^5}{n}+\frac{z^5}{p}+\frac{t^5}{q}\Big)(m+n+p+q)(1+1+1+1)^3\geq(x+y+z+t)^5$$

applied to

$$(x,y,z,t;m,n,p,q)=(a+b,b+c,c+d,d+a,c+d,d+a,a+b,b+c),$$

we find that the left side is not less than

$$\frac{2^5(a+b+c+d)^5}{4^4\cdot 2(a+b+c+d)}=\frac{2^{10}s^5}{2^8s}=4s^4\geq 64S^2.$$

□

Solution 5 by Walther Janous.
We prove a more general result: *Let* $p>q>0$ *and* $p+q\geq 1$. *Then*

$$\frac{(a+b)^p}{(c+d)^q}+\frac{(b+c)^p}{(d+a)^q}+\frac{(c+d)^p}{(a+b)^q}+\frac{(d+a)^p}{(b+c)^q}\geq 2^{p-q+2}S^{\frac{p-q}{2}}.$$

Applying the arithmetic-geometric means inequality to the denominator yields

$$\frac{(a+b)^p}{(c+d)^q}+\frac{(c+d)^p}{(a+b)^q}=\frac{(a+b)^{p+q}+(c+d)^{p+q}}{[(a+b)(c+d)]^q}$$
$$\geq 2^{2q}\cdot\frac{[(a+b)^{p+q}+(c+d)^{p+q}}{(a+b+c+d)^{2q}},$$

with an analogous inequality for the other two terms on the left side. Using the convexity of x^{p+q}, we see that the left side is not less than

$$2^{2q}\Big[\frac{(a+b)^{p+q}+(b+c)^{p+q}+(c+d)^{p+q}+(d+a)^{p+q}}{(a+b+c+d)^{2q}}\Big]$$
$$\geq\frac{2^{2q}\cdot 4}{(a+b+c+d)^{2q}}\Big[\frac{(a+b)+(b+c)+(c+d)+(d+a)}{4}\Big]^{p+q}$$
$$=\frac{2^{2q+2}}{(a+b+c+d)^{2q}}\Big[\frac{a+b+c+d}{2}\Big]^{p+q}=2^{q-p+2}(a+b+c+d)^{p-q}$$

On the other hand, from the AM-GM inequality [as in Solution 4],

$$S\leq\frac{(a+b+c+d)^2}{4}$$

whereupon

$$2^{p-q+2}S^{\frac{p-q}{2}} \leq 2^{p-q+2}\Big[\frac{(a+b+c+d)^{p-q}}{2^{2(p-q)}}\Big]$$
$$= 2^{q-p+2}(a+b+c+d)^{p-q}.$$

The result follows. □

4389. Consider the real numbers a, b, c and d. Prove that:

$$a(c+d) - b(c-d) \leq \sqrt{2(a^2+b^2)(c^2+d^2)}$$

Proposed by Daniel Sitaru - Romania

Solution by Michel Bataille.
The inequality certainly holds if $a(c+d) - b(c-d) < 0$ and otherwise is equivalent to

$$(ac + ad - bc + bd)^2 \leq 2(a^2+b^2)(c^2+d^2).$$

Now, a simple calculation shows that

$$2(a^2+b^2)(c^2+d^2) - (ac+ad-bc+bd)^2 = (ac+bd-ad+bc)^2 \geq 0$$

so we are done. □

4398. Prove that for $n \in \mathbb{N}^*$, we have

$$\frac{1}{2n-1}\int_0^1 \sin^2(x^n)dx \geq \frac{2}{n}(1-\cos 1).$$

Proposed by Daniel Sitaru - Romania

Since $a^2 + b^2 \geq 2ab$ for all $a, b \in \mathbb{R}$, we have

$$\frac{1}{2n-1} + \int_0^1 \sin^2(x^n)dx = \int_0^1 (x^{2n-2} + \sin^2(x^n))dx$$
$$\geq 2\int_0^1 x^{n-1}\sin(x^n)dx = -\frac{1}{n}\cos(x^n)\Big|_0^1 = -\frac{2}{n}(\cos 1 - \cos 0) = -\frac{2}{n}(1-\cos 1).$$

4410. Prove that:

$$\int_0^{\frac{\pi}{4}} \sqrt{\sin 2x}dx < \sqrt{2} - \frac{\pi}{4}.$$

Proposed by Daniel Sitaru - Romania

Solution 1 by Michel Bataille and Angel Plaza (independently).
The substitution $u = \frac{\pi}{4} - x$ leads to

$$\int_0^{\frac{\pi}{4}} \sqrt{\sin 2x}dx = \int_0^{\frac{\pi}{4}} \sqrt{\cos 2u}du.$$

From the Cauchy-Schwarz Inequality,

$$1 + \sqrt{\cos 2x} < \sqrt{2}(1+\cos 2x)^{\frac{1}{2}} = 2\cos x.$$

Therefore

$$\frac{\pi}{4} + \int_0^{\frac{\pi}{4}} \sqrt{\sin 2x}dx = \int_0^{\frac{\pi}{4}} (1+\sqrt{\cos 2x})dx < 2\int_0^{\frac{\pi}{4}} \cos x dx = \sqrt{2}.$$

The result follows. □

Solution 2 by Brian Bradie and Daniel Văcaru (independently).
By the Root-Mean-Square (or the Jensen) Inequality,

$$\frac{1+\sqrt{\sin 2x}}{2} < \sqrt{\frac{1+\sin 2x}{2}} = \frac{\cos x + \sin x}{\sqrt{2}} = \sin\Big(x+\frac{\pi}{4}\Big).$$

Hence

$$\frac{\pi}{4} + \int_0^{\frac{\pi}{4}} \sqrt{\sin 2x}dx = \int_0^{\frac{\pi}{4}} (1+\sqrt{\sin 2x})dx = 2\Big[-\cos\Big(x+\frac{\pi}{4}\Big)\Big]_0^{\frac{\pi}{4}} = \sqrt{2},$$

from which the result follows. □

4553. Find:

$$\lim_{n\to\infty}\left(\frac{\int_0^1 x^2(x+n)^n dx}{(n+1)^n}\right)$$

Proposed by Daniel Sitaru - Romania

Solution by Devis Alvarado, lightly edited.
The limit is equivalent to

$$\lim_{n\to\infty}\frac{\int_0^1 x^2(x+n)^n dx}{(n+1)^n} = \lim_{n\to\infty}\int_0^1 \frac{x^2(x+n)^n}{(n+1)^n}dx = \lim_{n\to\infty}\int_0^1 x^2\frac{(1+\frac{x}{n})^n}{(1+\frac{1}{n})}dx$$

For $n \geq 1$ define $f_n : [0,1] \to \mathbb{R}$ by $f_n(x) = x^2\frac{(1+\frac{x}{n})^n}{(1+\frac{1}{n})^n}$. We have

$$|f_n(x)| = f_n(x) = x^2\frac{(1+\frac{x}{n})^n}{(1+\frac{1}{n)^n}} \leq x^2\Big(1+\frac{x}{n}\Big)^n.$$

It is well known that $\{(1+\frac{x}{n})^n\}_{n\geq 1}$ converges to e^x. Since $x \in [0,1]$ it is also an increasing sequence. We conclude that $|f_n(x)| \leq x^2e^x \leq e$.
Apply the Bounded Convergence Theorem to get

$$\lim_{n\to\infty}\int_0^1 x^2\frac{(1+\frac{x}{n})^n}{(1+\frac{1}{n})^n}dx = \int_0^1 \lim_{n\to\infty}\left(x^2\frac{(1+\frac{x}{n})^n}{(1+\frac{1}{n})^n}\right)dx = \int_0^1 x^2\frac{\lim_{n\to\infty}(1+\frac{x}{n})^n}{\lim_{n\to\infty}(1+\frac{1}{n})^n}dx$$

$$= \int_0^1 x^2\cdot\frac{e^x}{e}dx = \frac{1}{e}[x^2e^x - 2xe^x + 2e^x]_0^1 = 1 - \frac{2}{e}.$$

Therefore

$$\lim_{n\to\infty}\left(\frac{\int_0^1 x^2(x+n)^n dx}{(n+1)^n}\right) = 1 - \frac{2}{e}.$$

□

4565. Let m_a, m_b and m_c be the lengths of the medians of a triangle ABC. Prove that

$$4(am_bm_c + bm_cm_a + cm_am_b) \geq 9abc.$$

Proposed by Daniel Sitaru - Romania

Solution by Sergey Sadov.
Consider the triangle in the complex plane. Let the origin (complex zero) be at the center of mass of the triangle and u, v, w be the complex coordinates of the midpoints of the sides a, b and c, respectively. Then

$$m_a = 3|u|, \quad m_b = 3|v|, \quad m_c = 3|w|,$$

and

$$a = 2|u - w|, \quad b = 2|w - u|, \quad c = 2|u - v|$$

Put

$$\xi = \frac{4}{9} \cdot \frac{m_a}{a} \cdot \frac{m_b}{b} = \frac{u}{v-w} \cdot \frac{v}{w-u},$$
$$\eta = \frac{4}{9} \cdot \frac{m_b}{b} \cdot \frac{m_c}{c} = \frac{v}{w-u} \cdot \frac{w}{u-v},$$
$$\zeta = \frac{4}{9} \cdot \frac{m_c}{c} \cdot \frac{m_a}{a} = \frac{w}{u-v} \cdot \frac{u}{v-w}.$$

The required identity takes the form $|\xi|+|\eta|+|\zeta| \geq 1$, and it follows, by the triangle inequality, from the identity $\xi + \eta + \zeta = -1$, which we are about to prove.
Equivalently, we want to prove that

$$(u-v)(v-w)(w-u) + uv(u-v) + vw(v-w) + wu(w-u) = 0.$$

Consider the coefficients at powers of u:

$$u^2 : (w-v) + v - w = 0,$$
$$u^1 : (v-w)(v+w) - v^2 + w^2 = 0,$$
$$u^0 : vw(w-v) + vw(v-w) = 0.$$

The proof is finished.
A generalization. In the above proof we did not use the relation $u + v + w = 0$. Therefore we have in fact proved a more general fact:
Let D be any point in the plane of triangle ABC. Then

$$AD \cdot BD \cdot c + BD \cdot CD \cdot a + CD \cdot AD \cdot b \geq abc.$$

The given problem is equivalent to the particular case of this proposition with D being the center of mass.
Case of equality. A natural question to ask is: when, in the described generalization, does the inequality turn to equality. I will show that this happens if and only if D is the orthocenter. As a corollary, in the original problem the equality takes place only for the equilateral triangle.For the equality

$$|-1| = |\xi + \eta + \zeta| = |\xi| + \eta| + \zeta|$$

to hold, it is necessary and sufficient that ξ, η, ζ be real and nonpositive. At least on of them is nonzero. Suppose $\xi \neq 0$ and consider the condition $\xi < 0$. It means that

$$\frac{w-v}{v} \cdot \frac{w-u}{u} > 0.$$

Hence the arguments of the complex numbers $\frac{w-v}{v}$ and $\frac{w-u}{u}$ have equal magnitudes and opposite signs. Geometrically it means that the signed magnitudes of the angles DBA and ACD (considering the counterclockwise direction as positive) are equal.
Denote the unsigned magnitude of the angles as $\angle DBA = \angle DCA = \alpha'$,
$\angle DAB = \angle DCB = \beta'$ and $\angle DAC = \angle DBC = \gamma'$. Then

$$\beta' + \gamma' = \alpha (= \angle A), \alpha' + \beta' = \gamma, \gamma' + \alpha' = \beta, 2(\alpha' + \beta' + \gamma') = \pi.$$

It follows that $\alpha = \frac{\pi}{2} - \alpha$ etc. This condition defines the orthocenter. $\square$

4583. Let

$$A = \begin{pmatrix} \frac{a^2}{(a+b)^2} & \frac{2ab}{(a+b)^2} & \frac{b^2}{(a+b)^2} \\ \frac{c^2}{(b+c)^2} & \frac{b^2}{(b+c)^2} & \frac{2bc}{(b+c)^2} \\ \frac{2ca}{(c+a)^2} & \frac{a^2}{(c+a)^2} & \frac{c^2}{(c+a)^2} \end{pmatrix},$$

where a, b and c are positive real numbers. Find the value of the sum of all the entries of A^n, where n is a natural number $n \geq 2$.

Proposed by Daniel Sitaru - Romania

Solution by Michel Bataille.
First, we remark that the entries of each row of A sum to 1, that is,

$$A \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{1}$$

Second, the sum $s(M)$ of all the entries of any 3×3 matrix M is

$$s(M) = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} M \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

It follows that

$$s(A) = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} A \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 3.$$

Now, assume that for some integer $n \geq 1$, we have $s(A^n) = 3$. Then, using (1), we obtain

$$s(A^{n+1}) = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} A^n \cdot A \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} A^n \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = s(A^n) = 3.$$

By induction, we have $s(A^n) = 3$ for all positive integers n. $\square$

B68. Prove that for $\frac{\sqrt{3}}{3} \leq a, b, c \leq 1$, we have:

$$\sqrt[3]{abc} \cdot \tan^{-1}\left(\sqrt{\frac{ab+bc+ca}{3}}\right) \leq \sqrt{\frac{ab+bc+ca}{3}} \cdot \tan^{-1}(\sqrt[3]{abc})$$

When does equality occur?

Proposed by Daniel Sitaru - Romania

Solution 1 by Ravi Prakash – New Delhi – India.
Let $g(x) = \frac{x}{1+x^2} - \tan^{-1} x, 0 \leq x \leq 1$,

$$g'(x) = \frac{1}{1+x^2} - \frac{2x^2}{(1+x^2)^2} - \frac{1}{1+x^2} = -\frac{2x^2}{(1+x^2)^2} < 0 \text{ for } 0 < x < 1$$

$$\Rightarrow g \text{ is strictly decreansig on } [0,1] \Rightarrow g(x) < g(0) \text{ for } 0 < x \leq 1$$

$$\Rightarrow \frac{x}{1+x^2} - \tan^{-1} x < 0 \text{ for } 0 < x \leq 1$$

Let $f(x) = \begin{cases} \frac{\tan^{-1} x}{x}, 0 < x \leq 1 \\ 1 \text{ if } x = 0 \end{cases}$

$$f'(x) = \frac{\frac{x}{1+x^2} - \tan^{-1} x}{x^2} < 0 \text{ for } 0 < x < 1 \Rightarrow f \text{ is decreasing on } [0,1]$$

$$\text{For } a,b,c > 0, \frac{ab+bc+ca}{3} \geq (abbcca)^{\frac{1}{3}} \Rightarrow A = \sqrt{\frac{ab+bc+ca}{3}} \geq (abc)^{\frac{1}{3}} = G$$

Equality when $a = b = c$. Thus, $\frac{\tan^{-1} A}{A} \leq \frac{\tan^{-1} G}{G} \Rightarrow G \tan^{-1} A \leq A \tan^{-1} G$
Equality when $a = b = c$. □

Solution 2 by Vince Kong – Hong Kong.
Consider: $AM \geq GM$ over ab, bc and ca:

$$\frac{ab+bc+ca}{3} \geq \sqrt[3]{ab \cdot bc \cdot ca} = \sqrt[3]{(abc)^2}$$

(1)
$$\sqrt{\frac{ab+bc+ca}{3}} = \sqrt[3]{abc}$$

Consider:

$$f(x) = \frac{\tan^{-1} x}{x} \text{ for } x \in \left[\frac{\sqrt{3}}{3}, 1\right]$$

$$f'(x) = \frac{d}{dx} \cdot \frac{\tan^{-1} x}{x} = \frac{1}{x}\frac{d}{dx} \tan^{-1} x + \tan^{-1} x \left(\frac{d}{dx}\frac{1}{x}\right)$$

Let

$$y = \tan^{-1} x$$
$$\tan y = x$$
$$\frac{dy}{dx} \sec^2 y = 1$$
$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$
$$= \frac{1}{x} \cdot \frac{1}{1+x^2} + \tan^{-1} x \left(-\frac{1}{x^2}\right)$$

Put $f'(x) = 0 : \frac{1}{x(1+x^2)} - \frac{\tan^{-1} x}{x^2} = 0$

$$\frac{1}{1+x^2} = \tan^{-1} x, x = 0. \text{ Only 1 extremum point and outside } \left[\frac{\sqrt{3}}{3}, 1\right]$$

$$\begin{cases} f\left(\frac{\sqrt{3}}{3}\right) = \frac{\frac{\pi}{6}}{\frac{\sqrt{3}}{3}} = \frac{\pi}{2\sqrt{3}} \\ f(1) = \frac{\frac{\pi}{4}}{1} = \frac{\pi}{4} < \frac{\pi}{2\sqrt{3}} \end{cases}$$

$\therefore f(x)$ is decreasing in $x \in [\frac{\sqrt{3}}{3}, 1]$

$$\text{By (1): } f(\sqrt[3]{abc}) \geq f\left(\sqrt{\frac{ab+bc+ca}{3}}\right)$$

$$\frac{\tan^{-1}(\sqrt[3]{abc})}{\sqrt[3]{abc}} \geq \frac{\tan^{-1}(\sqrt{\frac{ab+bc+ca}{3}})}{\sqrt{\frac{ab+bc+ca}{3}}}$$

$$\therefore \sqrt{\frac{ab+bc+ca}{3}} \cdot \tan^{-1}(\sqrt[3]{abc}) \geq \sqrt[3]{abc} \cdot \tan^{-1}\left(\sqrt{\frac{ab+bc+ca}{3}}\right)$$

□

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