CRUX MATHEMATICORUM CHALLENGES-(IV)

DANIEL SITARU - ROMANIA

4055. Prove that if $x, y > 0, x \neq y$ and $0 < a < b < \frac{1}{2} < c < d < 1$ then:

$$x\left[\left(\frac{y}{x}\right)^a + \left(\frac{y}{x}\right)^d - \left(\frac{y}{x}\right)^b - \left(\frac{y}{x}\right)^c\right] > y\left[\left(\frac{x}{y}\right)^b + \left(\frac{x}{y}\right)^c - \left(\frac{x}{y}\right)^a - \left(\frac{x}{y}\right)^d\right]$$

Proposed by Daniel Sitaru - Romania

Proof.
Let be $f : [0,1] \to \mathbb{R}, f(\alpha) = \frac{x^{1-\alpha}y^\alpha + x^\alpha y^{1-\alpha}}{2}, x, y \in (0, \infty), x \neq y$ fixed values.

$$\lim_{\substack{\alpha \to 0 \\ \alpha > 0}} f(\alpha) = \frac{x+y}{2}, \quad \lim_{\substack{\alpha \to 1 \\ \alpha < 1}} f(a) = \frac{x+y}{2}$$

$$f'(a) = \frac{1}{2}(\ln y - \ln x)(x^{1-\alpha}y^\alpha - x^\alpha y^{1-\alpha})$$

$$f'(a) = 0 \Rightarrow x^{1-\alpha}y^\alpha = x^\alpha y^{1-\alpha} \Rightarrow \left(\frac{x}{y}\right)^{1-2\alpha} = 0 \Rightarrow a = \frac{1}{2}$$

$$\min f(\alpha) = f\left(\frac{1}{2}\right) = \sqrt{xy}$$

$$f(a) > f(b) > f\left(\frac{1}{2}\right), \quad f\left(\frac{1}{2}\right) < f(c) < f(d)$$

By adding:

$$f(a) + f(d) > f(b) + f(c)$$

$$\frac{x^{1-a}y^a + x^a y^{1-a}}{2} + \frac{x^{1-d}y^d + x^d y^{1-d}}{2} > \frac{x^{1-b}y^b + x^b y^{1-b}}{2} > \frac{x^{1-b}y^b + x^b y^{1-b}}{2} + \frac{x^{1-c}y^c + x^c y^{1-c}}{2}$$

$$x\left[\left(\frac{y}{x}\right)^a + \left(\frac{y}{x}\right)^d - \left(\frac{y}{x}\right)^b - \left(\frac{y}{x}\right)^c\right] > y\left[\left(\frac{x}{y}\right)^b + \left(\frac{x}{y}\right)^c - \left(\frac{x}{y}\right)^a - \left(\frac{x}{y}\right)^d\right]$$

□

4100. Prove that in $\Delta ABC; m(\hat{A}) < 90°$ the following relationship holds:

$$\frac{c \cos B}{ac + 2S} + \frac{b \cos C}{ab + 2S} < \frac{a}{2S}$$

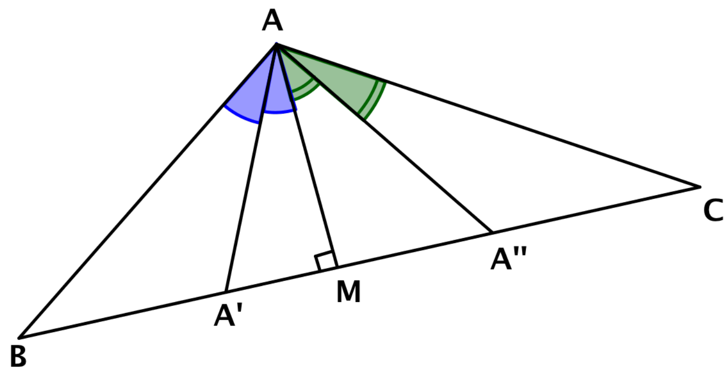
Proposed by Daniel Sitaru - Romania

Proof.
Let be $AM \perp BC; AA'$ bisector of $\sphericalangle BAM; AA''$ bisector of $\sphericalangle MAC$;
$A' \in (BM); A'' \in (MC)$

$$\frac{A'B}{A'B} = \frac{c}{h_a} \Rightarrow \frac{c \cos B}{A'M} = \frac{c + h_a}{h_a} \Rightarrow A'M = \frac{h_a c \cos B}{c + h_a}$$

$$\frac{A''M}{A''C} = \frac{h_a}{b} \Rightarrow \frac{A''M}{b \cos C} = \frac{h_a}{b} \Rightarrow A''M = \frac{h_a b \cos C}{b + h_a}$$

$$A'A'' < BC = a$$



$$A'A'' = A'M + MA'' = \frac{c\cos B \cdot \frac{2S}{a}}{c + \cdot\frac{2S}{a}} + \frac{b\cos C \cdot \frac{2S}{a}}{b + \frac{2S}{a}} < a$$

$$\frac{2Sc\cos B}{ac+2S} + \frac{2Sb\cos C}{ab+2S} < a$$

$$\frac{c\cos B}{ac+2S} + \frac{b\cos C}{ab+2S} < \frac{a}{2S}$$

□

4115. Prove that for all natural numbers $n \geq 2$, we have:

$$n^{\ln 2} \leq \sqrt[3]{3} \cdot \sqrt[n+1]{n} \cdot \sqrt[n+2]{n} \cdot \ldots \cdot \sqrt[2n]{n}$$

Proposed by Daniel Sitaru - Romania

Proof.
Let be $a_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}$

$$a_{n+1} - a_n = \frac{1}{2n+1} - \frac{1}{2n+2} = \frac{1}{(2n+1)(2n+2)} > 0$$

Let be $b_n = \frac{1}{n} + \frac{1}{n+1} + \ldots + \frac{1}{2n+1} + \frac{1}{2n+2}$

$$b_{n+1} - b_n = \frac{-3n-2}{n(2n+1)(2n+2)} < 0$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \ln 2$$

The sequence $(a_n)_{n\geq 1}$ is increasing and $(b_n)_{n\geq 1}$ is decreasing

$$a_n < \ln 2 < b_n \Rightarrow n^{a_n} < n^{\ln 2} < n^{b_n}$$

$$1 < \frac{n^{\ln 2}}{n^{a_n}} < n^{b_n - a_n} = n^{\frac{1}{n}}$$

On the other hand $n^{\frac{1}{n}} = \sqrt[n]{n} \in (1, \sqrt[3]{3}]; (\forall) n \in N^*; n \geq 2$. It follows:

$$\frac{n^{\ln 2}}{n^{a_n}} < \sqrt[n]{n} \leq \sqrt[3]{3}$$

$$n^{\ln 2} < \sqrt[3]{3} \cdot n^{a_n}$$

$$n^{\ln 2} < \sqrt[3]{3} \cdot n^{\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}}$$

$$n^{\ln 2} < \sqrt[3]{3} \cdot \sqrt[n+1]{n} \cdot \sqrt[n+2]{n} \cdot \ldots \cdot \sqrt[2n]{n}$$

□

4122. Prove that for $n \in \mathbb{N}$, the following holds

$$\left(\frac{e^n-1}{n}\right)^{2n+1} \leq \frac{(e-1)(e^2-1)(e^3-1)\cdot\ldots\cdot(e^{2n}-1)}{(2n)!}$$

Proposed by Daniel Sitaru - Romania

Proof.
Let be $f:[0,1]\to\mathbb{R}, f(x)=e^x$ and $I_n=\int_0^1 e^{nx}dx$.

$$I_n^2=\left(\int_0^1 e^{nx}dx\right)^2=\left(\int_0^1 \sqrt{e^{(n-k)x}}\cdot\sqrt{e^{(n+k)x}}dx\right)^2\leq$$
$$\leq\left(\int_0^1 e^{(n-k)x}dx\right)\left(\int_0^1 e^{(n+k)x}dx\right); 0\leq k\leq n; k\in\mathbb{N}$$

(Cauchy - Schwarz - integral form)

$$I_n^2\leq I_{n-k}\cdot I_{n+k};\quad 0\leq k\leq n$$
$$I_n^2\leq I_{n-1}\cdot I_{n+1}$$
$$I_n^2\leq I_{n-2}\cdot I_{n+2}$$
$$-------------$$
$$I_n^2\leq I_0\cdot I_{2n}$$

By multiplying:

$$I_n^{2n}\leq I_0 I_1\cdots I_{n-1}I_{n+1}\cdots I_{2n}$$
$$I_n^{2n+1}\leq I_0 I_1\cdot\ldots\cdot I_{2n}$$
$$\left(\frac{e^{nx}}{n}\Big|_0^1\right)^{2n+1}\leq\left(e^x\Big|_0^1\right)\cdot\left(\frac{e^2x}{2}\Big|_0^1\right)\cdot\ldots\cdot\left(\frac{e^{2nx}}{2n}\Big|_0^1\right)$$
$$\left(\frac{e^n-1}{n}\right)^{2n+1}\leq\frac{(e-1)(e^2-1)(e^3-1)\cdot\ldots\cdot(e^{2n}-1)}{(2n)!}$$

□

4188. Let $0<x<y<z<\frac{\pi}{2}$. Prove that:

$$(x+y)\sin z+(x-z)\sin y<(y+z)\sin x$$

Proposed by Daniel Sitaru - Romania

Proof.

$$\frac{\sin y}{y}-\frac{\sin x}{x}=\frac{x\sin y-y\sin x}{xy}=$$
$$=\frac{x\sin y-x\sin x-y\sin x+x\sin x}{xy}=$$
$$=\frac{x(\sin y-\sin x)-(y-x)\sin x}{xy}=$$
$$=\frac{y-x}{xy}\left[x\frac{\sin y-\sin x}{x-y}-\sin x\right]=$$

$$= \frac{y-x}{xy}\left[2x\frac{\sin\frac{y-x}{2}\cos\frac{y+x}{2}}{y-x} - \sin x\right] <$$
$$< \frac{y-x}{xy}\left[x\cos\frac{y+x}{2} - \sin x\right] <$$
$$< \frac{y-x}{xy}(x\cos x - \sin x) = \frac{y-x}{xy}\cos x(x - \tan x) < 0$$
because $y > x; x < \tan x$.
$$\frac{\sin y}{y} - \frac{\sin x}{x} < 0 \Rightarrow x\sin y < y\sin x$$
Analogous:
$$y\sin z < z\sin y$$
$$x\sin z < z\sin x$$
By adding:
$$x(\sin y + \sin z) + y\sin z < y\sin x + z(\sin y + \sin x)$$
$$(x+y)\sin z + x\sin y < (x+z)\sin x + z\sin y$$
$$(x+y)\sin z + (x-z)\sin y < (y+z)\sin x$$

□

4205. Prove that for $0 < a < c < b, a, b, c \in \mathbb{R}$, we have
$$\frac{1}{c\sqrt{ab}}\int_a^b x\arctan x dx > \frac{(c-a)\arctan\sqrt{ac}}{\sqrt{bc}} + \frac{(b-c)\arctan\sqrt{bc}}{\sqrt{ac}}$$
Proposed by Daniel Sitaru - Romania

Proof.
Let be $f : [a, c] \to \mathbb{R}; f(x) = x\arctan x$
$$f'(x) = \arctan x + \frac{x}{1+x^2}; f''(x) = \frac{2}{(1+x^2)^2} > 0$$
It follows f convex on $[a, c]$ and $f'(x) \geq \lim_{x\to 0} f'(x) = 0$
It follows f increasing on $[a, c]$.
$$I = \int_a^c x\arctan x dx = \int_a^c f(x)dx = \int_a^c f(a+c-x)dx$$
with the change of variable $y = a + c - x$
$$2I = \int_a^c \Big(f(x) + f(a+c-x)\Big)dx$$
$$I = \int_a^c \frac{f(x) + f(a+c-x)}{2}dx > \int_a^c f\Big(\frac{x+a+c-x}{2}\Big)dx =$$
$$= \int_a^c f\Big(\frac{a+c}{2}\Big)dx = f\Big(\frac{a+c}{2}\Big)\cdot x\Big|_a^c =$$
(1) $$= (c-a)f\Big(\frac{a+c}{2}\Big) > (c-a)f(\sqrt{ac})$$
Analogous, if $f : [c, b] \to \mathbb{R}$ then:
(2) $$\int_c^b f(x)dx > (b-c)f(\sqrt{bc})$$

From (1); (2):

$$\int_a^b f(x) = \int_a^c f(x)dx + \int_c^b f(x)dx > (c-a)f(\sqrt{ac}) + (b-c)f(\sqrt{ac})$$

$$\int_a^b x\arctan x dx > (c-a)\sqrt{ac}\arctan\sqrt{ac} + (b-c)\sqrt{bc}\cdot\arctan\sqrt{bc}$$

$$\frac{1}{c\sqrt{ab}}\int_a^b x\arctan x dx > \frac{(c-a)\arctan\sqrt{ac}}{\sqrt{bc}} + \frac{(b-c)\arctan\sqrt{bc}}{\sqrt{ac}}$$

□

4218. Prove that if $a, b, c \in (0, \infty); n \in \mathbb{N}; n \geq 3$ then:

$$\frac{1}{n}\sqrt[n]{a+b+c} \geq \frac{3\sqrt[3]{abc}}{(a+b+c)^{n-1}+n-1}$$

Proposed by Daniel Sitaru - Romania

Proof.

$$\sqrt[n]{a+b+c} = \frac{\sqrt[n]{a+b+c}\Big((a+b+c)^{n-1}+n-1\Big)}{(a+b+c)^{n-1}+n-1}$$

$$= \frac{\sqrt[n]{a+b+c}(a+b+c)^{n-1} + \underbrace{\sqrt[n]{a+b+c} + \ldots + \sqrt[n]{a+b+c}}_{\text{"}n-1\text{" times}}}{(a+b+c)^{n-1}+n-1} \geq$$

$$\overbrace{\geq}^{\text{AM-GM}} \frac{n\sqrt[n]{(a+b+c)^n}}{(a+b+c)^{n-1}+n-1} = \frac{n(a+b+c)}{(a+b+c)^{n-1}+n-1}$$

$$\sqrt[n]{a+b+c} \geq \frac{n(a+b+c)}{(a+b+c)^{n-1}+n-1} \geq \frac{3n\sqrt[3]{abc}}{(a+b+c)^{n-1}+n-1}$$

$$\frac{1}{n}\sqrt[n]{a+b+c} \geq \frac{3\sqrt[3]{abc}}{(a+b+c)^{n-1}+n-1}$$

□

4226. Prove that if $0 < a < b$ then:

$$\left(\int_a^b \frac{\sqrt{1+x^2}}{x}dx\right)^2 > (b-a)^2 + \ln^2\left(\frac{b}{a}\right)$$

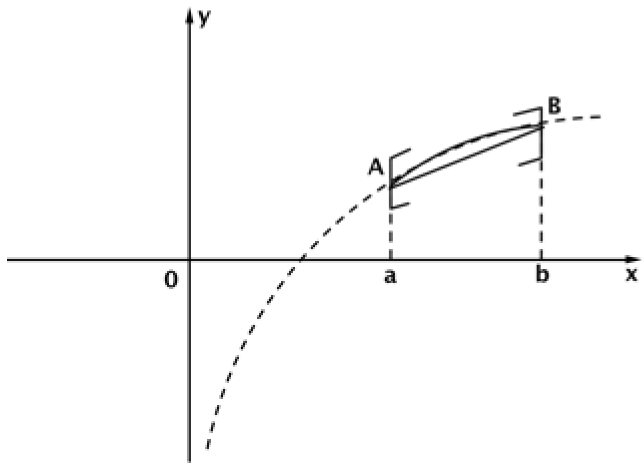
Proposed by Daniel Sitaru - Romania

Proof.
Let be $f : (0, \infty) \to \mathbb{R}; f(x) = \ln x$

$$f'(x) = \frac{1}{x}; 1 + (f')^2(x) = 1 + \frac{1}{x^2} = \frac{1+x^2}{x^2}$$

The length of the f graphic for $x \in [a, b]$ is:

$$L_f = \int_a^b \sqrt{t(f')^2(x)}dx = \int_a^b \frac{\sqrt{1+x^2}}{x}dx$$



$$[AB] = \sqrt{(b-a)^2 + (\ln b - \ln a)^2} = \sqrt{(b-a)^2 + \ln^2\left(\frac{b}{a}\right)}$$

The length of the graphic is bigger then the rope:

$$L_f > [AB]$$

$$\int_a^b \frac{\sqrt{1+x^2}}{x} dx > \sqrt{(b-a)^2 + \ln^2\left(\frac{b}{a}\right)}$$

$$\left(\int_a^b \frac{\sqrt{1+x^2}}{x} dx\right)^2 > (b-a)^2 + \ln^2\left(\frac{b}{a}\right)$$

□

4249. Let a, b, c be real numbers with at most of them equal to zero. Prove that

$$\frac{(a-b)^2(b-c)^2(c-a)^2}{a^2b^2 + b^2c^2 + c^2a^2} \leq 2(a^2 + b^2 + c^2 - ab - bc - ca)$$

Proposed by Daniel Sitaru - Romania

Proof.

Let be $A = \begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \end{pmatrix} = \begin{pmatrix} B \\ C \end{pmatrix}$

$$B = \begin{pmatrix} 1 & 1 & 1 \\ a & b & c \end{pmatrix}; B^T = \begin{pmatrix} 1 & a \\ 1 & b \\ 1 & c \end{pmatrix}$$

$$B \cdot B^T = \begin{pmatrix} 1 & 1 & 1 \\ a & b & c \end{pmatrix} \begin{pmatrix} 1 & a \\ 1 & b \\ 1 & c \end{pmatrix} = \begin{pmatrix} 3 & a+b+c \\ a+b+c & a^2+b^2+c^2 \end{pmatrix}$$

$$\det(B \cdot B^T) = 3(a^2 + b^2 + c^2) - (a+b+c)^2 =$$
$$= 2(a^2 + b^2 + c^2 - ab - bc - ca)$$

$$C = \begin{pmatrix} \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \end{pmatrix}; C^T = \begin{pmatrix} \frac{1}{a} \\ \frac{1}{b} \\ \frac{1}{c} \end{pmatrix}$$

$$C \cdot C^T = \begin{pmatrix} \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \end{pmatrix} \begin{pmatrix} \frac{1}{a} \\ \frac{1}{b} \\ \frac{1}{c} \end{pmatrix} = \begin{pmatrix} \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \end{pmatrix}$$

$$\det(C \cdot C^T) = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$$

$$\det(B \cdot B^T) \cdot \det(C \cdot C^T) =$$

$$= 2(a^2 + b^2 + c^2 - ab - bc - ca)\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) \tag{1}$$

$$\det A = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \end{vmatrix} \overset{\substack{C_2 - C_1 \\ C_3 - C_1}}{=} \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ \frac{1}{a} & \frac{1}{b} - \frac{1}{a} & \frac{1}{c} - \frac{1}{a} \end{vmatrix} =$$

$$= \begin{vmatrix} b-a & c-a \\ \frac{1}{b} - \frac{1}{a} & \frac{1}{c} - \frac{1}{a} \end{vmatrix} = \begin{vmatrix} b-a & c-a \\ \frac{a-b}{ba} & \frac{a-c}{ca} \end{vmatrix} = \frac{-(b-a)(c-a)}{a} \begin{vmatrix} 1 & 1 \\ \frac{1}{b} & \frac{1}{c} \end{vmatrix} =$$

$$= \frac{-(b-a)(c-a)}{a}\left(\frac{1}{c} - \frac{1}{b}\right) = \frac{-(b-a)(c-a)(b-c)}{abc} =$$

$$= \frac{(b-a)(c-a)(c-b)}{abc} \tag{2}$$

From Laplace rule for $A = \begin{pmatrix} B \\ C \end{pmatrix}$:

$$|\det A| = \left| \sum_{\substack{1 \le j_1 < j_2 < j_3 \le 3 \\ 1 \le j_1' \le 3}} (-1)^{1+2+3+j_1+j_2+j_3} \det(B \cdot j_1, j_2, j_3) \cdot \det(C \cdot j_1') \right| \le$$

$$\underbrace{\le}_{\text{Cauchy - Schwarz}} \sqrt{\sum_{\substack{1 \le j_1 < j_2 < j_3 \le 3 \\ 1 \le j_1' \le 3}} \left|(-1)^{1+2+3+j_1+j_2+j_3} \det(B \cdot j_1, j_2, j_3)\right| \cdot \left| \sum_{1 \le j_1' \le 3} \det(C \cdot j_1') \right|} =$$

$$= \sqrt{\det(B \cdot B^T) \cdot \det(C \cdot C^T)}$$

$$|\det A|^2 \le \det(B \cdot B^T) \cdot \det(C \cdot C^T)$$

From (1) and (2):

$$\left(\frac{(b-a)(c-a)(c-b)}{abc}\right)^2 \le 2(a^2 + b^2 + c^2 - ab - bc - ca)\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)$$

$$\frac{(b-a)^2(c-a)^2(c-b)^2}{a^2b^2c^2} \le \frac{2(a^2 + b^2 + c^2 - ab - bc - ca)(b^2c^2 + a^2c^2 + a^2b^2)}{a^2b^2c^2}$$

$$\frac{(b-a)^2(c-a)^2(c-b)^2}{b^2c^2 + a^2c^2 + a^2b^2} \le 2(a^2 + b^2 + c^2 - ab - bc - ca)$$

□

Mathematics Department, National Economic College "Theodor Costescu", Drobeta Turnu - Severin, Romania

Email address: dansitaru63@yahoo.com