# CRUX MATHEMATICORUM CHALLENGES-(V)

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4256. Let $a, b, c \in \mathbb{R}$ such that $a + b + c = 1$. Prove that

$$\frac{e^b - e^a}{b - a} + \frac{e^c - e^b}{c - b} + \frac{e^a - e^c}{a - c} > 4$$

*Proposed by Daniel Sitaru - Romania*

*Proof.*
From MacLaurin's formula applied to the function $f : \mathbb{R} \to \mathbb{R}; f(x) = e^x$ we obtain:

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + \int_0^x \frac{(x-t)^{n-1}}{(n-1)!}f^{(n)}(t)dt$$

$$f(x) > f(0) + \frac{x}{1!}f'(0)$$

$$\int_a^b f(x)dx > f(0) \cdot x\Big|_a^b + f'(0) \cdot \frac{x^2}{2}\Big|_a^b$$

$$\int_a^b f(x)dx > (b-a)f(0) + \frac{b^2 - a^2}{2}f'(0)$$

$$\int_a^b e^x dx > b - a + \frac{b^2 - a^2}{2}$$

$$\frac{1}{b-a}e^x\Big|_a^b > 1 + \frac{b+a}{2}$$

$$\frac{e^b - e^a}{b - a} > 1 + \frac{b+a}{2}$$

Analogous:

$$\frac{e^c - e^b}{c - b} > 1 + \frac{c+b}{2}$$

$$\frac{e^a - e^c}{a - c} > 1 + \frac{a+c}{2}$$

By adding:

$$\frac{e^b - e^a}{b - a} + \frac{e^c - e^b}{c - b} + \frac{e^a - e^c}{a - c} > 3 + a + b + c = 4$$

□

4265. Prove that if $a, b, c \in (0, 1); a + b + c = 1$ then:

$$\frac{4}{\pi}(\arctan a + \arctan b + \arctan c) > \frac{1}{2 - (ab + bc + ca)}$$

*Proposed by Daniel Sitaru - Romania*

*Proof.*
Let be $f:[0,1]\to\mathbb{R}; f(x)=\arctan x-\frac{\pi x^2}{2(x^2+1)}$

$$f'(x)=\frac{1}{1+x^2}-\frac{\pi}{2}\cdot\frac{(x^2)'(x^2+1)-x^2(x^2+1)'}{(x^2+1)^2}$$

$$f'(x)=\frac{2(x^2-\pi x+1)}{2(x^2+1)^2}; f'(x)=0\Rightarrow x_1=\frac{\pi-\sqrt{\pi^2-4}}{2}\in[0,1]$$

$$f(0)=f(1)=0; f(x_1)>0\Rightarrow \min_{x\in[0,1]} f(x)=0\Rightarrow$$

$$\Rightarrow f(x)>0;(\forall)x\in(0,1)$$

(1) $$\arctan x-\frac{\pi x^2}{2(x^2+1)}>0$$

Let be $x=a; x=b$, respectively $x=c$ in (1)
By adding:

$$\sum\arctan a-\frac{\pi}{2}\sum\frac{a^2}{a^2+1}>0$$

$$\frac{2}{\pi}\sum\arctan a>\sum\frac{a^2}{a^2+1}\geq \overbrace{\geq}^{\text{Cauchy-Schwarz}}$$

$$\geq\frac{(a+b+c)^2}{a^2+b^2+c^2+3}=\frac{1}{(a+b+c)^2-2(ab+bc+ca)+3}=$$

$$\geq\frac{1}{4-2(ab+bc+ca)}$$

By multiplying with 2:

$$\frac{4}{\pi}(\arctan a+\arctan b+\arctan c)>\frac{1}{2-(ab+bc+ca)}$$

□

4276. Let $P$ be a point on the interior of a triangle $ABC$ and let $PA=x, PB=y$ and $PC=z$. Prove that

$$27(by+cz-ax)(ax+cz-by)(ax+by-cz)\leq(ax+by+cz)^3$$

*Proposed by Daniel Sitaru - Romania*

*Proof.*

(1) Lemma (Duality Principle of Murray - Klamkin)

If $P\in Int(\Delta ABC); PA=x; PB=y; PC=z; AB=c; BC=a; CA=b$ then $ax; by; cz$ represents the sides lengths of a triangle.
Proof:
Let be $z_1,z_2,z_3,z_4\in\mathbb{C}; A(z_1); B(z_2); C(z_3); P(z_4)$

$$(z_1-z_4)(z_2-z_3)+(z_2-z_4)(z_3-z_1)+(z_3-z_4)(z_1-z_2)=0$$

$$-(z_3-z_4)(z_1-z_2)=(z_1-z_4)(z_2-z_3)+(z_2-z_4)(z_3-z_1)$$

$$|-(z_3-z_4)(z_1-z_2)|=|(z_1-z_4)(z_2-z_3)+(z_3-z_4)(z_3-z_1)|\leq$$

$$\leq|z_1-z_4|\cdot|z_2-z_3|+|z_2-z_4||z_3-z_1|$$

But: $|z_2 - z_3| = a; |z_3 - z_1| = b; |z_1 - z_2| = c$

$$|z_1 - z_4| = x; |z_2 - z_4| = y; |z_3 - z_4| = z$$

It follows:

$$cz < ax + by \text{ and analogous:}$$

$$ax < by + cz; by < ax + cz$$

From Mitrinovic's inequality:

$$3\sqrt{3} \leq p \Rightarrow 3\sqrt{3}\frac{S}{p} \leq p \Rightarrow$$

$$\Rightarrow 3\sqrt{3}S \leq p^2$$

$$3\sqrt{3p(p-a)(p-b)(p-c)} \leq p^2$$

$$27p(p-a)(p-b)(p-c) \leq p^4$$

$$27(p-a)(p-b)(p-c) \leq p^3$$

We apply the inequality for the triangle of sides: $ax, by, cz$:

$$27\Big(\frac{by+cz-ax}{2}\Big)\Big(\frac{ax+cz-by}{2}\Big)\Big(\frac{ax+by-cz}{2}\Big) \leq \frac{(ax+by+cz)^3}{8}$$

$$27(by+cz-ax)(ax+cz-by)(ax+by-cz) \leq (ax+by+cz)^3$$

□

4298. Compute:

$$L = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{n+k}{2+\sin(n+k)+(n+k)^2}$$

*Proposed by Daniel Sitaru - Romania*

*Proof.*
Let be $a > 0$. We prove that:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\Big(\frac{n+k}{a+(n+k)^2-\frac{1}{n+k}}\Big) = 0 \Leftrightarrow$$

(1) $$\Leftrightarrow \lim_{n\to\infty}\sum_{k=1}^{n}\frac{-a}{(n+k)[a+(n+k)^2]} = 0$$

From:$1 \leq k \leq n$ it follows

$$(n+1)[a+(n+1)^2] \leq (n+k)[a+(n+k)^2] \leq (n+n)(a+2n)^2$$

$$\frac{1}{2(a+(2n)^2)} \leq \sum_{k=1}^{n}\frac{1}{(n+k)[a+(n+k)^2]} \leq \frac{1}{(1+\frac{1}{n})[a+(n+1)^2]}$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(n+k)[a+(n+k)^2]} = 0$$

$$-1 \leq \sin(n+k) \leq 1 \Rightarrow 1 \leq 2+\sin(n+k) \leq 3$$

$$\frac{n+k}{3(n+k)^2} \leq \frac{n+k}{2+\sin(n+k)+(n+k)^2} \leq \frac{n+k}{1+(n+k)^2}$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n+k}{3+(n+k)^2}\le\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n+k}{2+\sin(n+k)+(n+k)^2}\le$$

$$\le\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n+k}{1+(n+k)^2}$$

From (1) $\Rightarrow \lim_{n\to\infty}\sum_{k=1}^{n}\frac{n+k}{a+(n+k)^2}=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+k}=$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+\frac{k}{n}}=\int_0^1\frac{dx}{1+x}=\ln 2$$

$$\lim_{n\to\infty}\frac{1}{n+k}\le L\le\lim_{n\to\infty}\frac{1}{n+k}$$

$$L=\ln 2$$

□

4309. Let $a, b$ and $c$ be real numbers such that $a+b+c=3$. Prove that

$$2(a^4+b^4+c^4)\ge ab(ab+1)+bc(bc+1)+ca(ca+1)$$

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*Proof.*
We prove by induction after $n$ that:

$$a^{2^n}+b^{2^n}+n\ge(ab)^{2^{n-1}}+(ab)^{2^{n-2}}+\ldots+ab+a+b$$

Checking:

$$n=1\Rightarrow a^2+b^2+1\ge(ab)+a+b$$

$$2a^2+2b^2+2\ge 2ab+2a+2b$$

$$(a-b)^2+(a-1)^2+(b-1)^2\ge 0 \text{ checks}$$

$$P(k): a^{2^k}+b^{2^k}+k\ge(ab)^{2^{k-1}}+(ab)^{2^{k-2}}+\ldots+ab+a+b$$

we assume is true

$$P(k+1): a^{2^{k+1}}+b^{2^{k+1}}+k+1\ge(ab)^{2^k}+(ab)^{2^{k-1}}+\ldots+ab+a+b$$

we have to prove

$$a^{2^{k+1}}+b^{2^{k+1}}+1+k=(a^{2^k})^2+(b^{2^k})^2+1+k\ge$$

$$\ge(ab)^{2^k}+a^{2^k}+b^{2^k}+k+1\ge$$

$$\ge(ab)^{2^k}+(ab)^{2^{k-1}}+\ldots+ab+a+b+1$$

$$P(k)\to P(k+1)$$

For $k=2$:

$$a^4+b^4+2\ge(ab)^2+ab+a+b$$

Analogous:

$$b^4+c^4+2\ge(bc)^2+bc+b+c$$

$$c^4+a^4+2\ge(ca)^2+ca+c+b$$

$$2(a^4+b^4+c^4)+6\ge(ab)(ab+1)+ac(ac+1)+bc(bc+1)+2(a+b+c)$$

$$2(a^4+b^4+c^4)+6\ge\sum ab(ab+1)+2\cdot 3$$

$$2(a^4+b^4+c^4)\ge ab(ab+1)+bc(bc+1)+ca(ca+1)$$

□

4331. Let $S$ be a unit sphere. Suppose that the surface of $S$ is coloured with 4 distinct colours. Prove that there exists two points $X, Y \in S$ of the same colour with $|XY| \in \{\sqrt{3}, \sqrt{\frac{3}{2}}\}$.

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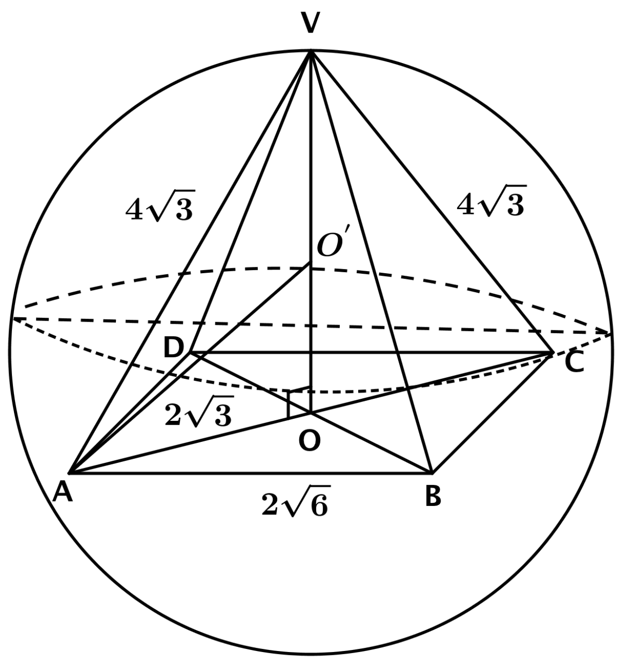
*Proof.*




Let $VABCD$ be a regular quadrilateral pyramid inscribed in a sphere, having the diagonal section equilateral triangle of side $4\sqrt{3}$ cm.
Because the radius of the sphere $R = 4$ cm, the side of the base of the pyramid is given by the relation:

$$R^2 = (h - R)^2 + (2\sqrt{3})^2 \Rightarrow l = 2\sqrt{6}$$

The side edge $VA = 4\sqrt{3}$ cm.
Walking the pyramid inside the sphere we find an infinity of segments equal to $VA, AC$ or $AB$. □

4346. Find all $x, y, z \in (0, \infty)$ such that:

$$\begin{cases} 64(x+y+z)^2 = 27(x^2+1)(y^2+1)(z^2+1) \\ x+y+z = xyz \end{cases}$$

*Proposed by Daniel Sitaru - Romania*

*Proof.*

$$x,y,z\in(0,\infty)\Rightarrow(\exists)A,B,C\in\left(0,\frac{\pi}{2}\right)\text{ such that:}$$
$$x=\tan A;y=\tan B;z=\tan C$$

$$\tan A\tan B\tan C=\tan A+\tan B+\tan C \tag{1}$$

$$\tan(A+B+C)=\frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}\overbrace{=}^{(1)}0$$

Hence: $A+B+C=m\pi;m\in\mathbb{Z}$

$$A,B,C\in\left(0,\frac{\pi}{2}\right)\Rightarrow A+B+C=\pi$$

$$\sin A=\frac{\tan A}{\sqrt{1+\tan^2 A}}=\frac{x}{\sqrt{1+x^2}};\sin B=\frac{y}{\sqrt{1+y^2}};\sin C=\frac{z}{\sqrt{1+z^2}}$$

By first equation of the system:

$$64(x+y+z)^2=27(x^2+1)(y^2+1)(z^2+1)$$
$$8(x+y+z)=3\sqrt{3(x^2+1)(y^2+1)(z^2+1)}$$
$$8xyz=3\sqrt{3(x^2+1)(y^2+1)(z^2+1)}$$
$$\frac{x}{\sqrt{1+x^2}}\cdot\frac{y}{\sqrt{1+y^2}}\cdot\frac{z}{\sqrt{1+z^2}}=\frac{3\sqrt{3}}{8}$$
$$\sin A\sin B\sin C=\frac{3\sqrt{3}}{8}\Rightarrow A=B=C=\frac{\pi}{3}$$
$$\Rightarrow x=\tan A=\tan\frac{\pi}{3}=\sqrt{3};y=z=\sqrt{3}$$

□

4359. Let $a,b$ and $c$ be positive real numbers. Prove that

$$3\ln(a^b+b^c+c^a)+\frac{a}{c}+\frac{b}{a}+\frac{c}{b}\geq a+b+c+\ln 27$$

*Proposed by Daniel Sitaru - Romania*

*Proof.*

Let be $f:(0,\infty)\to\mathbb{R};f(x)=\ln x-1+\frac{1}{x}$

$$f'(x)=\frac{1}{x}-\frac{1}{x^2}=\frac{x-1}{x^2};\min f(x)=f(1)=0\Rightarrow$$
$$\Rightarrow f(x)\geq 0;(\forall)x>0$$
$$\ln x\geq 1-\frac{1}{x}.\text{ For }x=a\Rightarrow\ln a\geq 1-\frac{1}{a}$$

We multiply the relationship with $b$:

$$b\ln a\geq b-\frac{b}{a}.\text{ Analogous: }c\ln b\geq c-\frac{c}{b};$$
$$a\ln c\geq a-\frac{a}{c}.\text{ By adding:}$$
$$b\ln a+c\ln b+a\ln c\geq a+b+c-\left(\frac{a}{c}+\frac{b}{a}+\frac{c}{b}\right)$$

$$\ln\Big(a^b \cdot b^c \cdot c^a\Big)+\Big(\frac{a}{c}+\frac{b}{a}+\frac{c}{b}\Big) \geq a+b+c \tag{1}$$

From means inequality:

$$\Big(\frac{a^b+b^c+c^a}{3}\Big)^3 \geq a^b b^c c^a$$

$$3\ln\Big(a^b+b^c+c^a\Big)-\ln 27 \geq \ln\Big(a^b \cdot b^c \cdot c^a\Big) \tag{2}$$

From (1); (2):

$$3\ln\Big(a^b+b^c+c^a\Big)-\ln 27+\Big(\frac{a}{c}+\frac{b}{a}+\frac{c}{b}\Big) \geq$$

$$\geq \ln\Big(a^b \cdot b^c \cdot c^a\Big)-\ln 27+\Big(\frac{a}{c}+\frac{b}{a}+\frac{c}{b}\Big) \geq a+b+c$$

$$3\ln\Big(a^b+b^c+c^a\Big)+\frac{a}{c}+\frac{b}{a}+\frac{c}{b} \geq a+b+c+\ln 27$$

□

4389. Consider the real numbers $a, b, c$ and $d$. Prove that

$$a(c+d)-b(c-d) \leq \sqrt{2(a^2+b^2)(c^2+d^2)}$$

*Proposed by Daniel Sitaru - Romania*

*Proof.*

Let be $\vec{x}=a\vec{i}+b\vec{j}; \vec{y}=c\vec{i}+d\vec{j}$

$$\sin(\widehat{\vec{x},\vec{y}})=\frac{ad-bc}{\sqrt{(a^2+b^2)(c^2+d^2)}}; \cos(\widehat{\vec{x};\vec{y}})=\frac{ac+bd}{\sqrt{(a^2+b^2)(c^2+d^2)}}$$

Let be $f:\mathbb{R}\to\mathbb{R}; f(x)=\sin x+\cos x$

$$f'(x)=\cos x-\sin x; f'(x)=0 \Rightarrow \cos x=\sin x \Rightarrow$$

$$\Rightarrow x=\frac{\pi}{4}+k\pi; k\in\mathbb{Z}$$

$$\max f=f\Big(\frac{\pi}{4}\Big)=\sqrt{2} \Rightarrow \sin x+\cos x \leq \sqrt{2}; (\forall)x\in\mathbb{R}$$

$$\sin(\widehat{\vec{x},\vec{y}})+\cos(\widehat{\vec{x},\vec{y}}) \leq \sqrt{2}$$

$$\frac{ad-bc+ac+bd}{\sqrt{(a^2+b^2)(c^2+d^2)}} \leq \sqrt{2}$$

$$a(c+d)-b(c-d) \leq \sqrt{2(a^2+b^2)(c^2+d^2)}$$

□

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