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Prove that

$$\sum_{n=1}^{\infty} \frac{((n-1)!)^2 H_{n-1}^{(2)} F_{2n}}{(2n)!} = \frac{12\sqrt{5}}{125}\zeta(4)$$

$$\sum_{n=1}^{\infty} \frac{((n-1)!)^2 F_{2n}}{(2n)!} = \frac{24\sqrt{5}}{125}\zeta(2)$$

where $H_m^{(n)}$ is generalized harmonic number and F_n is nth fibonacci number.

Proposed by Amrit Awasthi, India

Solution proposed by Narendra Bhandari,Bajura,Nepal

We start with latter summation and since

$$\sum_{n=1}^{\infty} \frac{((n-1)!)^2 F_{2n}}{(2n)!} = \sum_{n=1}^{\infty} \frac{F_{2n}}{n^2}\binom{2n}{n}^{-1} = \frac{2}{\sqrt{5}}\left(\arcsin^2\left(\frac{\phi}{2}\right) - \arcsin^2\left(-\frac{1}{2\phi}\right)\right)$$

where we used Lehmer identity and Binet nth Fibonacci formula

$$\sum_{m=1}^{\infty} \frac{(2x)^{2m}}{m^2}\binom{2m}{m}^{-1} = 2\arcsin^2(x)\ ,\ F_{2n} = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{2n} - \left(\frac{1-\sqrt{5}}{2}\right)^{2n}\right)$$

respectively. Further we recall $\sin\left(\frac{\pi}{10}\right) = \frac{-1+\sqrt{5}}{4}$ and $\sin\left(\frac{3\pi}{10}\right) = \frac{1+\sqrt{5}}{4}$ and the last relation it simplifies to

$$\frac{2}{\sqrt{5}}\left(\frac{9\pi^2}{100} - \frac{\pi^2}{100}\right) = \frac{4\pi^2}{25\sqrt{5}} = \frac{24\sqrt{5}}{125}\zeta(2)$$

Now for the first identity we can use the following less-known identity

$$\frac{2}{3}\arcsin^4\left(\frac{x}{2}\right) = \sum_{n=1}^{\infty} \frac{x^{2n}}{n^2}\binom{2n}{n}^{-1}\sum_{k=1}^{n-1}\frac{1}{k} = \sum_{n=1}^{\infty} \frac{x^{2n} H_{n-1}^{(2)}}{n^2}\binom{2n}{n}^{-1}$$

which directly follows

$$\sum_{n=1}^{\infty} \frac{H_{n-1}^{(2)} F_{2n}}{n^2}\binom{2n}{n}^{-1} = \frac{2}{3\sqrt{5}}\left(\arcsin^4\left(\frac{\phi}{2}\right) - \arcsin^4\left(-\frac{1}{2\phi}\right)\right)$$

$$= \frac{2\pi^4}{375\sqrt{5}} = \frac{12\sqrt{5}}{125}\zeta(4)$$