Inspired by Prof. Dan Sitaru

Prove that

$$\lim_{n\to\infty} \frac{8^n}{n\binom{2n-2}{n-1}^2}\left(\sum_{k=0}^{n}\left(1+\frac{2k}{n}\binom{n}{k}\right)^2 - 4\binom{2n-2}{n-1}\right) = 32\pi$$

$$\lim_{n\to\infty} \frac{32^n}{\sqrt{n^3}\binom{2n-2}{n-1}^3}\left(\sum_{k=0}^{n}\left(1+\frac{2k}{n}\binom{n}{k}\right)^2 - 4\binom{2n-2}{n-1}\right) = 128\sqrt{\pi^3}$$

Proposed by Narendra Bhandari,Bajura,Nepal
Solved by: Surjeet Singhania ,Himachal Pradesh ,India

Solutions: Denote $X_n = \sum_{k=0}^{n}\left(1+\frac{2k}{n}\binom{n}{k}\right)^2 = 1+\sum_{k=1}^{n}\left(1+2\binom{n-1}{k-1}\right)^2$

$$X_n = n+1+4\underbrace{\sum_{k=1}^{n}\binom{n-1}{k-1}^2}_{X} +4\sum_{k=1}^{n}\binom{n-1}{k-1} = n+1+4X+4\times 2^{n-1}$$

$X = \sum_{k=1}^{n}\binom{n-1}{k-1}^2 = \sum_{k=0}^{n-1}\binom{n-1}{k}^2$ cauchy integral formula can help us here

$$\frac{1}{2\pi\iota}\oint_{|z|=1}\frac{(1+z)^n}{z^{k+1}}dz = \frac{n(n-1)(n-2)\dots(n-k+1)}{k!} = \binom{n}{k}$$

$$X = \frac{1}{2\pi\iota}\sum_{k=0}^{n-1}\binom{n-1}{k}\oint_{|z|=1}\frac{(1+z)^{n-1}}{z^{k+1}}dz = \frac{1}{2\pi\iota}\oint_{|z|=1}\frac{(1+z)^{n-1}}{z}\sum_{k=0}^{n-1}\binom{n-1}{k}\frac{1}{z^k}dz$$

$$\boxed{X = \frac{1}{2\pi\iota}\oint_{|z|=1}\frac{(1+z)^{2n-1}}{z^n}dz = \binom{2n-2}{n-1}}$$

Hence $X_n = \sum_{k=0}^{n}\left(1+\frac{2k}{n}\binom{n}{k}\right)^2 = n+1+4\binom{2m-2}{n-1}+2^{n+1}$

First limit becomes $\lim_{n\to\infty}\frac{8^n}{n\binom{2n-2}{n-1}^2}\left(\sum_{k=0}^{n}\left(1+\frac{2k}{n}\binom{n}{k}\right)^2 - 4\binom{2n-2}{n-1}\right)$

$$= \lim_{n\to\infty}\frac{8^n\left(n+1+2^{n+1}\right)}{n\binom{2n-2}{n-1}^2} = \lim_{n\to\infty}\frac{8^n(n+1+2^{n+1})}{n\left(\frac{4^{2n-2}}{\pi(n-1)}\right)} = \pi\lim_{n\to\infty}\frac{n+1+2^{n+1}}{2^{n-4}} = 32\pi$$

I used stirling's approximation i.e $\binom{2n}{n}\sim\frac{4^n}{\sqrt{\pi n}}$ in previous steps and will use in next steps

Second limit $\lim_{n\to\infty} \frac{32^n}{\sqrt{n^3}\binom{2n-2}{n-1}^3} \left(\sum_{k=0}^{n} \left(1 + \frac{2k}{n}\binom{n}{k}\right)^2 - 4\binom{2n-2}{n-1} \right)$

$$= \lim_{n\to\infty} \frac{32^n(n+1+2^{n+1})}{\frac{4^{3n-3}}{\sqrt{\pi^3}}} = \sqrt{\pi^3} \lim_{n\to\infty} \frac{n+1+2^{n+1}}{2^{n-6}} = 128\sqrt{\pi^3}$$