Two Classes of Alternating Harmonic Series

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Prove that

$$\sum_{n=1}^{\infty}(-1)^{n+1}H_n\left(\frac{1}{n+1}-\frac{1}{n+3}+\frac{1}{n+5}-\cdots\right)=\frac{\pi}{16}\log(2)+\frac{3}{16}\log^2(2)-\frac{\pi^2}{192}$$

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{H_n}{n+1}\left(\frac{1}{n+1}-\frac{1}{n+3}+\frac{1}{n+5}-\cdots\right)=\frac{\zeta(3)}{8}-\frac{\log^3(2)}{48}-\frac{\pi^2}{192}\log(2)$$

where H_n is nth harmonic number and $\zeta(z)$ is Riemann zeta function.

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Solution by proposer

Proof of first series

Since

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{n+2k+1}=\int_0^1\left(\sum_{k=0}^{\infty}(-1)^k x^{n+2k}\right)dx=\int_0^1\frac{x^n}{1+x^2}dx \tag{1}$$

We know that the generating function of harmonic number is given by $\sum_{n=1}^{\infty}H_nx^n=-\frac{\log(1-x)}{1-x}$. Now replacing x by $-x$ and by (1) we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^{n+1}H_n\int_0^1\frac{x^n}{1+x^2}dx&=\int_0^1\frac{\log(1+x)}{(1+x)(1+x^2)}dx=\frac{1}{2}\int_0^1\frac{\log(1+x)}{1+x}dx\\&-\frac{1}{2}\int_0^1\frac{x\log(1+x)}{1+x^2}dx+\frac{1}{2}\int_0^1\frac{\log(1+x)}{1+x^2}dx\\&=\frac{\log^2(2)}{4}-\mathcal{I}_1+\mathcal{I}_2\end{aligned}$$

Integral $\mathcal{I}_1$ can be easily evaluated via harmonic numbers by making the use of the identity $\int_0^1 x^{2k-1}\log(1+x)dx=\frac{H_{2k}-H_k}{2k}$. However, we will make a different approache which uses the notion of double integral, ie

$$\begin{aligned}\mathcal{I}_1=-\frac{1}{2}\int_0^1\int_0^1\frac{x^2}{(1+x^2)(1+xy)}dydx&\overset{\text{PFD}}{=}-\frac{1}{2}\int_0^1\int_0^1\frac{dxdy}{(1+x^2)(1+xy)}\\+\frac{1}{2}\int_0^1\int_0^1\frac{dxdy}{(1+x^2)(1+y^2)}&-\frac{1}{2}\int_0^1\int_0^1\frac{xydxdy}{(1+x^2)(1+y^2)}\end{aligned}$$

We interchange integrals (justified by Fubini theorem) and integrating we have

$$\mathcal{I}_1 = -\frac{1}{2}\int_0^1 \frac{\log(1+y)dy}{y(1+y^2)} + \frac{\pi^2}{32} - \frac{\log^2(2)}{8} = -\frac{\pi^2}{96} - \frac{\log^2(2)}{8} - \mathcal{I}_1$$

$$\mathcal{I}_1 = -\frac{\pi^2}{192} - \frac{\log^2(2)}{16}$$

Here the integrals $\int_0^1 \frac{\log(1+y)}{y}dy = \frac{\pi^2}{12}$, $\int\int_{[0,1]^2} \frac{dxdy}{(1+x^2)(1+y^2)} = \frac{\pi^2}{16}$ and $\int\int_{[0,1]^2} \frac{xy\,dxdy}{((1+x^2)(1+y^2)} = \frac{\log^2(2)}{4}$ are easily doable. To evaluate we may follow the same technique used for calculating $\mathcal{I}_2$ but here we use the trigonometry substitution $x = \tan y$ giving us

$$\mathcal{I}_2 = \frac{1}{2}\int_0^{\frac{\pi}{4}} \log(1+\tan y)\,dy = \frac{1}{2}\int_0^{\frac{\pi}{4}} \log\left(\frac{2}{1+\tan y}\right) dy = \frac{\pi}{16}\log(2)$$

Above we used reflection property of integral $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$ and $\tan\left(\frac{\pi}{4} - y\right) = \frac{1-\tan y}{1+\tan y}$. Collecting the values of $\mathcal{I}_1$ and $\mathcal{I}_2$ we get

$$\sum_{n=1}^{\infty}(-1)^{n+1}H_n\left(\frac{1}{n+1} - \frac{1}{n+3} + \frac{1}{n+5} - \cdots\right) = \frac{\pi}{16}\log(2) + \frac{3}{16}\log^2(2) - \frac{\pi^2}{192}$$

which prove the announced result.

Note: The integral $2\mathcal{I}_2 = \int_0^1 \frac{\log(1+y)}{1+y^2}dy$ is quite well know integral, called Putnam Integral which appear in 2005 and is equal to $\frac{\pi}{8}\log(2)$ and the next integral $2\mathcal{I}_1$ is Integral due to Cornel loan Vălean which appears in American Mathematical Monthly, Problem 11966, Vol.124, March 2017 which we deduce by the means of double integral technique moreover, it can be done by the mean differentiation under integral sign and harmonic number.

Proof of second series

To evaluate the second proposed series, we again make the use of generating function of harmonic number and since it is easy to see that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1} H_n x^n =$

$\dfrac{\log^2(1+x)}{2x} \cdots (*)$ and by (1) it follows that

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{H_n}{n+1}\int_0^1 \frac{x^n}{1+x^2}dx \overset{(*)}{=} \frac{1}{2}\int_0^1 \frac{\log^2(1+x)}{x(1+x^2)} = \frac{1}{2}\int_0^1 \frac{\log^2(1+x)}{x}dx$$
$$-\frac{1}{2}\int_0^1 \frac{x\log^2(1+x)}{1+x^2}dx = \mathcal{I}_3 - \mathcal{I}_4$$

The former integral $\mathcal{I}_3$, we evaluate by substituting $1+x=\dfrac{1}{u}$ giving us

$$2\mathcal{I}_3 = -\int_1^{\frac{1}{2}} \frac{\log^2(u)}{u(1-u)}du \overset{\text{PFD}}{=} \frac{1}{3}\log^3\left(\frac{1}{2}\right) - \sum_{n=0}^{\infty}\int_1^{\frac{1}{2}} u\log^2(u)du$$
$$= \frac{1}{3}\log^2\left(\frac{1}{2}\right) - \sum_{n=0}^{\infty}\frac{1}{2^{n+1}}\left(\frac{2}{(n+1)^3} - \frac{2\log\left(\frac{1}{2}\right)}{(n+1)^2} + \frac{\log^2\left(\frac{1}{2}\right)}{n+1}\right) - 2\zeta(3)$$

The last expression above is obtained by integrating by parts (twice) for the integral,

$$\sum_{n=0}^{\infty}\int_1^{\frac{1}{2}} u\log^2(u)du = \sum_{n=0}^{\infty} u^{n+1}\left(\frac{2}{(n+1)^3} - \frac{2\log(u)}{(n+1)^2} + \frac{\log^2(u)}{n+1}\right)\Bigg|_1^{\frac{1}{2}}$$

and it is easy to see that the series attains the polylogarithm form of order $3, 2,$ and 1 which can be expressed as

$$\operatorname{Li}_3\left(\frac{1}{2}\right) - \log\left(\frac{1}{2}\right)\operatorname{Li}_2\left(\frac{1}{2}\right) + \log^3\left(\frac{1}{2}\right) - 2\zeta(3) = \frac{1}{3}\log^3\left(\frac{1}{2}\right) - \frac{\zeta(3)}{4}$$

We simplify by using the well known results, namely $\operatorname{Li}_3\left(\dfrac{1}{2}\right) = \dfrac{7}{8}\zeta(3) + \dfrac{\log^3(2)}{6} - \dfrac{\pi^2}{12}\log(2)$ and $\operatorname{Li}_2\left(\dfrac{1}{2}\right) = \dfrac{\pi^2}{12} - \dfrac{\log^2(2)}{3}$. So finally, we have that

$$\mathcal{I}_3 = \frac{1}{2}\left(\frac{1}{3}\log^3\left(\frac{1}{2}\right) + \frac{\zeta(3)}{4} - \frac{1}{3}\log^3\left(\frac{1}{2}\right)\right) = \frac{\zeta(3)}{8}$$

And for the latter integral we note that $x^2+1=(x+i)(x-i)$ where i is imaginary unit and by partial fraction decomposition we have

$$
\begin{aligned}
2\mathcal{I}_4 &= \int_0^1 \frac{x\log^2(1+x)}{(x+i)(x-i)}dx = \frac{1}{2i}\int_0^1 x\log^2(1+x)\left(\frac{1}{x-i}-\frac{1}{x+i}\right)dx \\
&= \frac{1}{2i}\int_1^2\left(\frac{1}{x-1-i}-\frac{1}{x-1+i}\right)(x-1)\log^2(x)dx \\
&= 2\Re\left(\operatorname{Li}_3\left(\frac{1\pm i}{2}\right)\right) - 2\Re\left(\operatorname{Li}_3(1\pm i)\right) + 2\log(2)\Re\left(\operatorname{Li}_2(1\pm i)\right) \\
&\overset{\text{LF}}{=} 2\left(\frac{\log^3(2)}{48}-\frac{11\pi^2}{192}\log(2)\right) + 2\log(2)\left(\frac{\pi^2}{16}\right) = \frac{\log^3(2)}{24}+\frac{\pi^2}{96}\log(2)
\end{aligned}
$$

Dividing both sides by 2 gives us the desired result of $\mathcal{I}_4$. Here notation **LF** stands for Landen's dilogarithm (see [1]) and trilogarithm identities (see [2]) and $\Re(z)$ denotes the real part of z.

Alternative Solution: We can also prove the result of $\mathcal{I}_4$ alternatively by developing the connection with $\mathcal{I}_1$.

$$
2\mathcal{I}_4 = \int_0^1 \frac{x\log^2(1+x)}{1+x^2}dx \overset{\text{IBP}}{=} \frac{\log^3(2)}{2} - \underbrace{\int_0^1 \frac{\log(1+x)\log(1+x^2)}{1+x}dx}_{\mathcal{I}_5}
$$

If we substitute $x = \frac{1-y}{1+y}$, then $\mathcal{I}_5$ integral boils down

$$
\begin{aligned}
\mathcal{I}_5 &= \int_0^1 \frac{\log\left(\frac{2}{1+y}\right)\log\left(\frac{2(1+y^2)}{(1+y)^2}\right)}{1+y}dy = \log^3(2)+\log(2)\int_0^1\frac{\log(1+y^2)}{1+y}dy - \mathcal{I}_5 \\
&+ 2\int_0^1\frac{\log^2(1+y)}{1+y}dy - \frac{3}{2}\log^3(2) = \frac{\log(2)}{2}\int_0^1\frac{\log(1+x^2)}{1+x}dx + \frac{\log^3(2)}{12} \\
&\overset{\text{IBP}}{=} \frac{\log(2)}{2}\left[\log^2(2) - 2\int_0^1\frac{y\log(1+y)}{1+y^2}dy\right] + \frac{\log^3(2)}{12} = \frac{7}{12}\log^3(2) - \log(2)\mathcal{I}_1
\end{aligned}
$$

Using the deduced value of $\mathcal{I}_1$, we get

$$
\mathcal{I}_4 = -\frac{1}{24}\log^3(2) + \frac{\pi^2}{192}\log(2) + \frac{\log^3(2)}{16} = \frac{\pi^2}{192}\log(2) + \frac{\log^3(2)}{48}
$$

Subtracting $\mathcal{I}_3$ and $\mathcal{I}_4$ proves our announced result, namely

$$
\sum_{n=1}^{\infty}(-1)^{n+1}\frac{H_n}{n+1}\left(\frac{1}{n+1}-\frac{1}{n+3}+\frac{1}{n+5}-\cdots\right) = \frac{\zeta(3)}{8} - \frac{\log^3(2)}{48} - \frac{\pi^2}{192}\log(2)
$$

References

[1] Weisstein, Eric W. "*Landen's Identity.*", MathWorld,
https://mathworld.wolfram.com/LandensIdentity.html .

[2] Weisstein, Eric W. "*Trilogarithmic.*", MathWorld,
https://mathworld.wolfram.com/Trilogarithm.html .