Madhava-Leibniz Formula

The Leibniz formula for π, named after Gottfried Leibniz, states that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots = \frac{\pi}{4}, \tag{1}$$

,an alternating series. It is also called the Leibniz-Madhava series as it is a special case of a more general series expansion for the inverse tangent function, first discovered by the Indian mathematician Madhava of Sangamagrama in the 14th century, the specific case first published by Leibniz around 1676.

Proof

We start with the integral

$$I_{4n} = \int_0^{\pi/4} \tan^{4n}(x)\, dx$$

Now in order to evaluate we establish the reduction formula as follows :

$$I_{4n} = \int_0^{\pi/4} \tan^{4n-2}(x) \tan^2(x)\, dx$$

$$= \int_0^{\pi/4} \tan^{4n-2}(x)(\sec^2(x) - 1)\, dx$$

$$= \int_0^{\pi/4} \tan^{4n-2}(x) \sec^2(x)\, dx - \int_0^{\pi/4} \tan^{4n-2}(x)\, dx$$

$$I_{4n} = \int_0^{1} t^{4n-2}\, dt - I_{4n-2}$$

where for first integral on right hand side we made the substitution $\tan(x) = t$ where as the second integral is nothing but I_{4n-2}.Hence we get ,

$$I_{4n} = \frac{1}{4n-1} - I_{4n-2} \tag{i}$$

Similarly we have

$$I_{4n-2} = \frac{1}{4n-3} - I_{4n-4}$$

Plugging back in (i) yields

$$I_{4n} = \frac{1}{4n-1} - \frac{1}{4n-3} + I_{4n-4}$$

Proceeding in a similar way gives

$$I_{4n} = \frac{1}{4n-1} - \frac{1}{4n-3} + \frac{1}{4n-5} - .. + .. + \frac{1}{3} - \int\limits_0^{\pi/4} \tan^2(x)\ \mathrm{dx}$$

$$= \frac{1}{4n-1} - \frac{1}{4n-3} + \frac{1}{4n-5} - .. + .. + \frac{1}{3} - (\tan(x) - x)\big|_0^{\pi/4}$$

$$I_{4n} = \frac{1}{4n-1} - \frac{1}{4n-3} + \frac{1}{4n-5} - .. + .. + \frac{1}{3} - 1 + \frac{\pi}{4} \qquad \text{(ii)}$$

Now , it's easy to follow that $\forall\ 0 \leq x < \pi/4$, $0 \leq \tan(x) < 1$. Henceforth , we must have

$$\lim_{n\to\infty} \int\limits_0^{\pi/4} \tan^{4n}(x)\, dx = 0$$

Using (ii) gives

$$\lim_{n\to\infty} \frac{1}{4n-1} - \frac{1}{4n-3} + \frac{1}{4n-5} - .. + .. + \frac{1}{3} - 1 + \frac{\pi}{4} = 0$$

Transposing the terms gives :

$$\lim_{n\to\infty} 1 - \frac{1}{3} + \frac{1}{5} - .. + .. - \ldots - \frac{1}{4n-1} = \frac{\pi}{4}$$

Therefore we get :

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots = \frac{\pi}{4}, \qquad (2)$$

This alternative proof is dedicated to Gottfried Leibniz on his birthday that is 1st July,1646.

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