The Infimum and Supremum of a set defined by a rational function of two integer or real variables

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1 Applying the Calculus of Minima and Maxima of a function of two real variables

Let $\mathcal{A} := \left\{ \frac{mn}{(am)^2 + n^2},\ n, m \in \mathbb{Z},\ a \in \mathbb{R} \setminus \{0\} \right\}$

$$f(m,n) := \frac{mn}{(am)^2 + n^2}$$

To obtain the stationary points of $f(m,n)$, we take the partial derivatives of f wrt. to m and the partial derivatives of f wrt. to n, equate both results to 0 and solve simultaneously.[2]

$$\begin{aligned}\frac{\partial f}{\partial m} &= \frac{mn}{(am)^2+n^2}\left(\frac{\partial}{\partial m}(\ln m) - \frac{\partial}{\partial m}\left(\ln\left((am)^2+n^2\right)\right)\right)\\ &= \frac{mn}{(am)^2+n^2}\left(\frac{1}{m} - \frac{2a^2m}{(am)^2+n^2}\right)\\ &= \frac{mn}{(am)^2+n^2}\left(\frac{(am)^2+n^2-2a^2m(m)}{m((am)^2+n^2)}\right)\\ &= \frac{mn}{(am)^2+n^2}\left(\frac{n^2-(am)^2}{m((am)^2+n^2)}\right)\\ &= \frac{n(n^2-(am)^2)}{((am)^2+n^2)^2}\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial n} &= \frac{mn}{(am)^2+n^2}\left(\frac{\partial}{\partial n}(\ln n) - \frac{\partial}{\partial n}\left(\ln\left((am)^2+n^2\right)\right)\right)\\ &= \frac{mn}{(am)^2+n^2}\left(\frac{1}{n} - \frac{2n}{(am)^2+n^2}\right)\\ &= \frac{mn}{(am)^2+n^2}\left(\frac{(am)^2+n^2-2n(n)}{n((am)^2+n^2)}\right)\\ &= \frac{mn}{(am)^2+n^2}\left(\frac{(am)^2-n^2}{n((am)^2+n^2)}\right)\\ &= \frac{m((am)^2-n^2)}{((am)^2+n^2)^2}\end{aligned}$$

If there is a stationary point in $f(m,n)$,

$$\frac{\partial f}{\partial m} = 0,\ \frac{\partial f}{\partial n} = 0$$

$$\begin{aligned}&\implies \frac{mn(n^2-(am)^2)}{((am)^2+n^2)^2} = 0,\ \frac{mn((am)^2-n^2)}{((am)^2+n^2)^2} = 0.\\ &\implies n^2-(am)^2 = 0,\ (am)^2-n^2 = 0\end{aligned}$$

Both implies, $n^2-(am)^2 = 0$

$$\implies n = \pm am$$

To conclude if the function has a true maximum or minimum, we apply the second derivative test.

$$\frac{\partial^2 f}{\partial m^2} = \frac{n^2-(am)^2}{(am)^2+n^2}\frac{\partial}{\partial m}\left(\frac{n}{(am)^2+n^2}\right) + \frac{n}{(am)^2+n^2}\cdot\frac{n^2-(am)^2}{(am)^2+n^2}\left(\frac{\partial}{\partial m}\left(\ln\left(n^2-(am)^2\right)\right) - \frac{\partial}{\partial m}\left(\ln\left((am)^2+n^2\right)\right)\right)$$
$$= \frac{n^2-(am)^2}{(am)^2+n^2}\cdot\frac{n(-2a^2m)}{((am)^2+n^2)^2} + \frac{n}{(am)^2+n^2}\cdot\frac{n^2-(am)^2}{(am)^2+n^2}\left(-\frac{2a^2m}{n^2-(am)^2}-\frac{2a^2m}{(am)^2+n^2}\right)$$
$$= \frac{-2a^2mn(n^2-(am)^2)}{((am)^2+n^2)^3} + \frac{n(n^2-(am)^2)}{((am)^2+n^2)^2}\left(\frac{-2a^2m((am)^2+n^2)-2a^2m(n^2-(am)^2)}{(n^2-(am)^2)((am)^2+n^2)}\right)$$
$$= \frac{-2a^2mn(n^2-(am)^2)}{((am)^2+n^2)^3} + \frac{n(n^2-(am)^2)}{((am)^2+n^2)^2}\left(\frac{-4a^2mn^2}{(n^2-(am)^2)((am)^2+n^2)}\right)$$
$$= \frac{-2a^2mn(n^2-(am)^2)}{((am)^2+n^2)^3} - \frac{4a^2mn^3}{((am)^2+n^2)^3}$$
$$= \frac{-2a^2mn(n^2-(am)^2)-4a^2mn^3}{((am)^2+n^2)^3} = \frac{2a^2mn(am)^2-6a^2mn^3}{((am)^2+n^2)^3}$$
$$= \frac{mn\left(2a^2(am)^2-6a^2n^2\right)}{((am)^2+n^2)^3}$$

$$\frac{\partial^2 f}{\partial n^2} = \frac{(am)^2-n^2}{(am)^2+n^2}\frac{\partial}{\partial n}\left(\frac{m}{(am)^2+n^2}\right) + \frac{m}{(am)^2+n^2}\cdot\frac{(am)^2-n^2}{(am)^2+n^2}\left(\frac{\partial}{\partial n}\left(\ln\left((am)^2-n^2\right)\right) - \frac{\partial}{\partial n}\left(\ln\left((am)^2+n^2\right)\right)\right)$$
$$= \frac{(am)^2-n^2}{(am)^2+n^2}\cdot\frac{m(-2n)}{((am)^2+n^2)^2} + \frac{m}{(am)^2+n^2}\cdot\frac{(am)^2-n^2}{(am)^2+n^2}\left(-\frac{2n}{(am)^2-n^2}-\frac{2n}{(am)^2+n^2}\right)$$
$$= \frac{-2mn((am)^2-n^2)}{((am)^2+n^2)^3} + \frac{m((am)^2-n^2)}{((am)^2+n^2)^2}\left(\frac{-2n((am)^2+n^2)-2n((am)^2-n^2)}{((am)^2-n^2)((am)^2+n^2)}\right)$$
$$= \frac{-2mn((am)^2-n^2)}{((am)^2+n^2)^3} + \frac{m((am)^2-n^2)}{((am)^2+n^2)^2}\left(\frac{-4n(am)^2}{((am)^2-n^2)((am)^2+n^2)}\right)$$
$$= \frac{-2mn((am)^2-n^2)}{((am)^2+n^2)^3} - \frac{4mn(am)^2}{((am)^2+n^2)^3}$$
$$= \frac{-2mn((am)^2-n^2)-4mn(am)^2}{((am)^2+n^2)^3} = \frac{-6mn(am)^2+2mn^3}{((am)^2+n^2)^3}$$
$$= \frac{mn\left(-6(am)^2+2n^2\right)}{((am)^2+n^2)^3}$$

$$\begin{aligned}
\frac{\partial}{\partial n}\left(\frac{\partial f}{\partial m}\right) = \frac{\partial^2 f}{\partial m \partial n} &= \frac{\partial}{\partial n}\left(\frac{n(n^2-(am)^2)}{((am)^2+n^2)^2}\right)\\
&= \frac{n^2-(am)^2}{(am)^2+n^2}\frac{\partial}{\partial n}\left(\frac{n}{(am)^2+n^2}\right) + \frac{n}{(am)^2+n^2}\cdot\frac{n^2-(am)^2}{(am)^2+n^2}\left(\frac{\partial}{\partial n}\left(\ln\left(n^2-(am)^2\right)\right)\right.\\
&\left.-\frac{\partial}{\partial n}\left(\ln\left((am)^2+n^2\right)\right)\right)\\
&= \frac{n^2-(am)^2}{(am)^2+n^2}\cdot\frac{(am)^2)-n^2}{((am)^2+n^2)^2} + \frac{n}{(am)^2+n^2}\cdot\frac{n^2-(am)^2}{(am)^2+n^2}\left(\frac{2n}{n^2-(am)^2}\right.\\
&\left.-\frac{2n}{(am)^2+n^2}\right)\\
&= \frac{-(n^2-(am)^2)^2}{((am)^2+n^2)^3} + \frac{n(n^2-(am)^2)}{((am)^2+n^2)^2}\left(\frac{2n((am)^2+n^2)-2n(n^2-(am)^2)}{(n^2-(am)^2)((am)^2+n^2)}\right)\\
&= \frac{-(n^2-(am)^2)^2}{((am)^2+n^2)^3} + \frac{n(n^2-(am)^2)}{((am)^2+n^2)^2}\left(\frac{4n(am)^2}{(n^2-(am)^2)((am)^2+n^2)}\right)\\
&= \frac{-(n^2-(am)^2)^2}{((am)^2+n^2)^3} + \frac{4n^2(am)^2}{((am)^2+n^2)^3}\\
&= \frac{-(n^2-(am)^2)^2+4n^2(am)^2}{((am)^2+n^2)^3}
\end{aligned}$$

$$\begin{aligned}
\left.\frac{\partial^2 f}{\partial m^2}\right|_{n=\pm am} &= \frac{-4a(am)^4}{8(am)^6} = \frac{-(\pm a)}{2(am)^2} = \frac{\mp a}{2(am)^2}\\
\left.\frac{\partial^2 f}{\partial n^2}\right|_{n=\pm am} &= \frac{-4(am)^4}{8a(am)^6} = \frac{-(\pm 1)}{2a(am)^2} = \frac{\mp 1}{2a(am)^2}\\
\left.\left(\frac{\partial^2 f}{\partial m \partial n}\right)^2\right|_{n=\pm am} &= \left(\frac{4(am)^4}{8(am)^6}\right)^2 = \frac{1}{4(am)^4}
\end{aligned}$$

$$\begin{aligned}
\left(\frac{\partial^2 f}{\partial m^2}\right)\left(\frac{\partial^2 f}{\partial n^2}\right) - \left(\frac{\partial^2 f}{\partial m \partial n}\right)^2 &= \left(\frac{\mp a}{2(am)^2}\right)\left(\frac{\mp 1}{2a(am)^2}\right) - \frac{1}{4(am)^4}\\
&= \frac{1}{4(am)^4} - \frac{1}{4(am)^4} = 0
\end{aligned}$$

Since $\left(\frac{\partial^2 f}{\partial m^2}\right)\left(\frac{\partial^2 f}{\partial n^2}\right) - \left(\frac{\partial^2 f}{\partial m \partial n}\right)^2 = 0$, the second derivative test fails.

Conducting further test [2],

$$\forall a \in \mathbb{R}$$

$$\left.\frac{\partial^2 f}{\partial m^2}\right|_{n=am} = \frac{-4a(am)^4}{8(am)^6} = \frac{-a}{2(am)^2} < 0$$
$$\left.\frac{\partial^2 f}{\partial n^2}\right|_{n=am} = \frac{-4(am)^4}{8a(am)^6} = \frac{-1}{2a(am)^2} < 0$$
$$\left.\frac{\partial^2 f}{\partial m^2}\right|_{n=-am} = \frac{-4a(am)^4}{8(am)^6} = \frac{a}{2(am)^2} > 0$$
$$\left.\frac{\partial^2 f}{\partial n^2}\right|_{n=-am} = \frac{-4(am)^4}{8a(am)^6} = \frac{1}{2a(am)^2} > 0$$

By this further test if $\frac{\partial^2 f}{\partial m^2} > 0$, then, there is a minimum and if $\frac{\partial^2 f}{\partial m^2} < 0$, there is a maximum.

Therefore, by this further test, we can conclude that the maximum value of $f(m,n)$ is $f(m,am)$ and the minimum value of $f(m,n)$ is $f(m,-am)$.

$$f_{max}(m,n) = f(m,am) = \frac{m(am)}{(am)^2+(am)^2}$$
$$= \frac{am^2}{2(am)^2} = \frac{1}{2a}$$

$$f_{min}(m,n) = f(m,-am) = \frac{-m(am)}{(am)^2+(am)^2}$$
$$= -\frac{am^2}{2(am)^2} = -\frac{1}{2a}.$$

Since $-\frac{1}{2a} \in \mathbb{R}$, and $-\frac{1}{2a} \leq \left\{\frac{mn}{(am)^2+n^2},\ n,m \in \mathbb{Z},\ a \in \mathbb{R}\right\}$,

$$\implies \inf \mathcal{A} = -\frac{1}{2a}$$

Similarly, $\frac{1}{2a} \in \mathbb{R}$, and $\frac{1}{2a} \geq \left\{\frac{mn}{(am)^2+n^2},\ n,m \in \mathbb{Z},\ a \in \mathbb{R}\right\}$,

$$\implies \sup \mathcal{A} = \frac{1}{2a}$$

2 Applying the law of Tricotomy

Let $\mathcal{A} := \left\{ \frac{mn}{(am)^2 + n^2}, \; n,m \in \mathbb{Z}, \; a \in \mathbb{R} \setminus \{0\} \right\}$

m and n can be selected in $\mathbb{Z}$ in such a way that either one of the following three applies; $m > n, \; m < n, \; m = n$.

2.1 For Supremum [1]

If $m = n$,

$$\implies f(n,n) = \frac{n^2}{(an)^2 + n^2} = \frac{1}{a^2+1}$$

If $m > n$, m can be written as $m = an \; \forall a \in \mathbb{R}^+, \; m,n \in \mathbb{Z}^+$

$$\implies f(an,n) = \frac{an^2}{(a^2n)^2 + n^2} = \frac{a}{a^4+1}$$

If $m < n$, m can be written as $n = am \; \forall a \in \mathbb{R}^+, \; m,n \in \mathbb{Z}^+$

$$\implies f(m,am) = \frac{m(am)}{(am)^2 + (am)^2} = \frac{1}{2a}$$

It is obvious that

$$a^4 + 1 > a(a^2+1) \geq 2a^2 \; \forall a \in \mathbb{R}$$

$$\implies \frac{1}{a^4+1} < \frac{1}{a(a^2+1)} \leq \frac{1}{2a^2}$$

$$\implies \frac{a}{a^4+1} < \frac{1}{a^2+1} \leq \frac{1}{2a}$$

$\frac{1}{2a} \in \mathbb{R}$, and $\frac{1}{2a} \geq \left\{ \frac{mn}{(am)^2+n^2}, \; n,m \in \mathbb{Z}, \; a \in \mathbb{R} \right\}$,

$$\implies \sup \mathcal{A} = \frac{1}{2a}$$

2.2 For Infimum [1]

If $m < n$, m can be written as $m = -an \; \forall a \in \mathbb{R}^-, \; m,n \in \mathbb{Z}^-$

$$\implies f(-an,n) = \frac{-an^2}{(a^2n)^2 + n^2} = \frac{-a}{a^4+1}$$

If $m > n$, m can be written as $n = -am \; \forall a \in \mathbb{R}^-$, $m, n \in \mathbb{Z}^-$

$$\implies f(m,-am) = \frac{m(-am)}{(am)^2+(am)^2} = -\frac{1}{2a}$$

It can be proven that

$$-\frac{1}{2a} < -\frac{a}{a^4+1} < \frac{a}{a^4+1} < \frac{1}{a^2+1} \leq \frac{1}{2a} \quad \forall a \in \mathbb{R}$$

Proof.
It is obvious that

$$a^4+1 > 2a^2, \; \forall a \in \mathbb{R}$$

$$\implies -(a^4+1) < -2a^2$$

$$\implies -\frac{1}{a^4+1} > -\frac{1}{2a^2}$$

$$\implies -\frac{1}{2a^2} < -\frac{1}{a^4+1} < \frac{1}{a^4+1} < \frac{1}{a(a^2+1)} \leq \frac{1}{2a^2}$$

$$\implies -\frac{1}{2a} < -\frac{a}{a^4+1} < \frac{a}{a^4+1} < \frac{1}{a^2+1} \leq \frac{1}{2a}$$

Since $-\frac{1}{2a} \in \mathbb{R}$, and $-\frac{1}{2a} \leq \left\{ \frac{mn}{(am)^2+n^2}, \; n,m \in \mathbb{Z}, \; a \in \mathbb{R} \right\}$,

$$\implies \inf \mathcal{A} = -\frac{1}{2a}.$$

3 Examples

Example 3.1. If $\mathcal{A} := \left\{ \frac{mn}{4m^2+n^2}, \; n,m \in \mathbb{Z} \right\}$

Solution 3.1.1. $\implies a = 2$, so the substitution is $n = 2m$.

$$\implies f_{max}(m,n) = \frac{m(2m)}{(2m)^2+(2m)^2} = \frac{1}{2\times 2} = \frac{1}{4}$$

$$\implies f_{min}(m,n) = \frac{m(-2m)}{(2m)^2+(2m)^2} = -\frac{1}{2\times 2} = -\frac{1}{4}.$$

Since $-\frac{1}{4} \in \mathbb{R}$ and $-\frac{1}{4} \leq \left\{ \frac{mn}{4m^2+n^2}, \; n,m \in \mathbb{Z} \right\}$,

$$\implies \inf \mathcal{A} = -\frac{1}{4}$$

Similarly, $\frac{1}{4} \in \mathbb{R}$ and $\frac{1}{4} \geq \left\{\frac{mn}{4m^2+n^2},\ n,m \in \mathbb{Z}\right\}$,

$$\implies \sup \mathcal{A} = \frac{1}{4}.$$

Example 3.2. If $\mathcal{A} := \left\{\frac{mn}{3m^2+n^2},\ n,m \in \mathbb{Z}\right\}$

Solution 3.2.1. $\implies a = \sqrt{3}$, so the substitution is $n = \sqrt{3}m$.

But because $n \in \mathbb{Z} \implies n \notin \mathbb{IQ}$, we use the integral values between which $\sqrt{3}$ lies.

$$\sqrt{3} \in (1,2)$$

So, let $n = \pm 2m$ and $n = \pm m$

$$\implies f(m,2m) = \frac{m(2m)}{3m^2+(2m)^2} = \frac{2}{7}$$

$$\implies f(m,-2m) = \frac{-m(2m)}{3m^2+(2m)^2} = -\frac{2}{7}.$$

$$\implies f(m,m) = \frac{m(m)}{3m^2+(m)^2} = \frac{1}{4}$$

$$\implies f(m,-m) = \frac{-m(m)}{3m^2+(m)^2} = -\frac{1}{4}.$$

$$\frac{1}{4} < \frac{2}{7},\ -\frac{2}{7} < -\frac{1}{4}$$

$$\implies \inf \mathcal{A} = -\frac{2}{7} \wedge \sup \mathcal{A} = \frac{2}{7}$$

NB: If $a \in \mathbb{IQ} \ni m,n \in \mathbb{Z}$, choose the upper bound of a as what you will use to establish a relationship between n and m.

Example 3.3. If $\mathcal{A} := \left\{\frac{mn}{3m^2+n^2},\ n,m \in \mathbb{R}\right\}$

Solution 3.3.1. $\implies a = \sqrt{3}$, so the substitution is $n = \pm\sqrt{3}m$.

$$\implies f_{max}(m,n) = \frac{m(\sqrt{3}m)}{(\sqrt{3}m)^2+(\sqrt{3}m)^2} = \frac{1}{2\times\sqrt{3}} = \frac{1}{2\sqrt{3}}$$

$$\implies f_{min}(m,n) = \frac{-m(\sqrt{3}m)}{(\sqrt{3}m)^2+(\sqrt{3}m)^2} = -\frac{1}{2\times\sqrt{3}} = -\frac{1}{2\sqrt{3}}.$$

Since $-\frac{1}{2\sqrt{3}} \in \mathbb{R}$ and $-\frac{1}{2\sqrt{3}} \leq \left\{\frac{mn}{4m^2+n^2},\ n,m \in \mathbb{R}\right\}$,

$$\implies \inf \mathcal{A} = -\frac{1}{2\sqrt{3}}$$

Similarly, $\frac{1}{2\sqrt{3}} \in \mathbb{R}$ and $\frac{1}{2\sqrt{3}} \geq \left\{\frac{mn}{4m^2+n^2},\ n,m \in \mathbb{R}\right\}$,

$$\implies \sup \mathcal{A} = \frac{1}{2\sqrt{3}}.$$

Example 3.4. If $\mathcal{A} := \left\{\frac{mn}{m^2+4n^2},\ n,m \in \mathbb{Z}\right\}$

Solution 3.4.1. $a = 1$ but n in the denominator was replaced with $2n$, so the substitution is $2n = m, \implies m = 2n$.

$$\implies f_{max}(m,n) = \frac{n(2n)}{(2n)^2+(2n)^2} = \frac{1}{2\times 2} = \frac{1}{4}$$

$$\implies f_{min}(m,n) = \frac{n(-2n)}{(2n)^2+(2n)^2} = -\frac{1}{2\times 2} = -\frac{1}{4}.$$

Since $-\frac{1}{4} \in \mathbb{R}$ and $-\frac{1}{4} \leq \left\{\frac{mn}{m^2+4n^2},\ n,m \in \mathbb{Z}\right\}$,

$$\implies \inf \mathcal{A} = -\frac{1}{4}$$

Similarly, $\frac{1}{4} \in \mathbb{R}$ and $\frac{1}{4} \geq \left\{\frac{mn}{m^2+4n^2},\ n,m \in \mathbb{Z}\right\}$,

$$\implies \sup \mathcal{A} = \frac{1}{4}.$$

4 Extensions to other rational functions and generalizations

The following functions are going to be examined for supremum and infimum in this section

$f(m,n) := \frac{nm}{am+n}$, $f(m,n) := \frac{nm^2}{(am)^2+n^2}$, $f(m,n) := \frac{n^2m^2}{(am)^2+n^2}$, $f(m,n) := \frac{n^2m^2}{am+n}, \ldots, f(m,n) := \frac{(nm)^p}{(am)^q+n^q}$

4.1 Let $\mathcal{A}_1 := \left\{ \dfrac{nm}{am+n},\ n,m \in \mathbb{Z},\ a \in \mathbb{R} \setminus \{0\} \right\}$

m and n can be selected in $\mathbb{Z}$ in such a way that either one of the following three applies; $m > n,\ m < n,\ m = n$.

4.1.1 Supremum

If $m = n$,

$$\implies f(n,n) = \frac{n^2}{an+n} = \frac{n}{a+1}$$

$\left\{ \dfrac{n}{a+1},\ n \in \mathbb{Z} \right\}$ is neither bounded above nor below.

If $m > n$, m can be written as $m = an\ \forall a \in \mathbb{R}^+,\ m,n \in \mathbb{Z}^+$

$$\implies f(an,n) = \frac{an^2}{a^2n+n} = \frac{an}{a^2+1}$$

$\left\{ \dfrac{an}{a^2+1},\ n \in \mathbb{Z} \right\}$ is neither bounded above nor below.

If $m < n$, m can be written as $n = am\ \forall a \in \mathbb{R}^+,\ m,n \in \mathbb{Z}^+$

$$\implies f(m,am) = \frac{m(am)}{am+am} = \frac{m}{2}$$

$\left\{ \dfrac{m}{2},\ m \in \mathbb{Z} \right\}$ is neither bounded above nor below.

4.1.2 Infimum

If $m < n$, m can be written as $m = -an\ \forall a \in \mathbb{R}^-,\ m,n \in \mathbb{Z}^-$

$$\implies f(-an,n) = \frac{-an^2}{a^2n+n} = \frac{-an}{a^2+1}$$

$\left\{ \dfrac{-an}{a^2+1},\ n \in \mathbb{Z} \right\}$ is neither bounded above nor below.

If $m > n$, m can be written as $n = -am\ \forall a \in \mathbb{R}^-,\ m,n \in \mathbb{Z}^-$

$$\implies f(m,-am) = \frac{m(-am)}{am+am} = -\frac{m}{2}$$

$\left\{ -\dfrac{m}{2},\ m \in \mathbb{Z} \right\}$ is neither bounded above nor below.

Hence $\inf \mathcal{A}_1$ and $\sup \mathcal{A}_1$ do not exist.

4.2 Let $\mathcal{A}_2 := \left\{ \frac{nm^2}{(am)^2+n^2},\ n,m \in \mathbb{Z},\ a \in \mathbb{R} \right\}$

m and n can be selected in $\mathbb{Z}$ in such a way that either one of the following three applies; $m > n,\ m < n,\ m = n$.

4.2.1 Supremum

If $m = n$,

$$\implies f(n,n) = \frac{n^3}{(an)^2+n^2} = \frac{n}{a^2+1}$$

$\left\{ \frac{n}{a+1},\ n \in \mathbb{Z} \right\}$ is neither bounded above nor below.

If $m > n$, m can be written as $m = an\ \forall a \in \mathbb{R}^+,\ m,n \in \mathbb{Z}^+$

$$\implies f(an,n) = \frac{n(an)^2}{(a^2n)^2+n^2} = \frac{a^2n}{a^4+1}$$

$\left\{ \frac{a^2n}{a^4+1},\ n \in \mathbb{Z} \right\}$ is neither bounded above nor below.

If $m < n$, m can be written as $n = am\ \forall a \in \mathbb{R}^+,\ m,n \in \mathbb{Z}^+$

$$\implies f(m,am) = \frac{m^2(am)}{(am)^2+(am)^2} = \frac{m}{2a}$$

$\left\{ \frac{m}{2a},\ m \in \mathbb{Z} \right\}$ is neither bounded above nor below.

4.2.2 Infimum

If $m < n$, m can be written as $m = -an\ \forall a \in \mathbb{R}^-,\ m,n \in \mathbb{Z}^-$

$$\implies f(-an,n) = \frac{n(-an)^2}{(a^2n)^2+n^2} = \frac{a^2n}{a^4+1}$$

$\left\{ \frac{a^2n}{a^4+1},\ n \in \mathbb{Z} \right\}$ is neither bounded above nor below.

If $m > n$, m can be written as $n = -am\ \forall a \in \mathbb{R}^-,\ m,n \in \mathbb{Z}^-$

$$\implies f(m,-am) = \frac{m^2(-am)}{(am)^2+(am)^2} = -\frac{m}{2a}$$

$\left\{ -\frac{m}{2a},\ m \in \mathbb{Z} \right\}$ is neither bounded above nor below.

Hence $\inf \mathcal{A}_2$ and $\sup \mathcal{A}_2$ do not exist.

4.3 Let $\mathcal{A}_3 := \left\{ \dfrac{n^2m^2}{(am)^2+n^2},\ n,m \in \mathbb{Z},\ a \in \mathbb{R}\setminus\{0\} \right\}$

m and n can be selected in $\mathbb{Z}$ in such a way that either one of the following three applies; $m > n,\ m < n,\ m = n$.

4.3.1 Supremum

If $m = n$,

$$\implies f(n,n) = \frac{n^4}{(an)^2+n^2} = \frac{n^2}{a^2+1}$$

$\left\{ \dfrac{n^2}{a+1},\ n \in \mathbb{Z} \right\}$ is neither bounded above nor below.

If $m > n$, m can be written as $m = an\ \forall a \in \mathbb{R}^+,\ m,n \in \mathbb{Z}^+$

$$\implies f(an,n) = \frac{n^2(an)^2}{(a^2n)^2+n^2} = \frac{a^2n^2}{a^4+1}$$

$\left\{ \dfrac{a^2n^2}{a^4+1},\ n \in \mathbb{Z} \right\}$ is neither bounded above nor below.

If $m < n$, m can be written as $n = am\ \forall a \in \mathbb{R}^+,\ m,n \in \mathbb{Z}^+$

$$\implies f(m,am) = \frac{m^2(am)^2}{(am)^2+(am)^2} = \frac{m^2}{2}$$

$\left\{ \dfrac{m^2}{2},\ m \in \mathbb{Z} \right\}$ is neither bounded above nor below.

4.3.2 Infimum

If $m < n$, m can be written as $m = -an\ \forall a \in \mathbb{R}^-,\ m,n \in \mathbb{Z}^-$

$$\implies f(-an,n) = \frac{n^2(-an)^2}{(a^2n)^2+n^2} = \frac{a^2n^2}{a^4+1}$$

$\left\{ \dfrac{a^2n^2}{a^4+1},\ n \in \mathbb{Z} \right\}$ is neither bounded above nor below.

If $m > n$, m can be written as $n = -am\ \forall a \in \mathbb{R}^-,\ m,n \in \mathbb{Z}^-$

$$\implies f(m,-am) = \frac{m^2(-am)^2}{(am)^2+(am)^2} = \frac{m^2}{2}$$

$\left\{\dfrac{m^2}{2},\ m \in \mathbb{Z}\right\}$ is neither bounded above nor below.

Hence $\inf \mathcal{A}_3$ and $\sup \mathcal{A}_3$ do not exist.

4.4 Let $\mathcal{A}_4 := \left\{\dfrac{n^2m^2}{am+n},\ n,m \in \mathbb{Z},\ a \in \mathbb{R}\setminus\{0\}\right\}$

m and n can be selected in $\mathbb{Z}$ in such a way that either one of the following three applies; $m > n,\ m < n,\ m = n$.

4.4.1 Supremum

If $m = n$,

$$\implies f(n,n) = \frac{n^4}{an+n} = \frac{n^3}{a+1}$$

$\left\{\dfrac{n^3}{a+1},\ n \in \mathbb{Z}\right\}$ is neither bounded above nor below.

If $m > n$, m can be written as $m = an\ \forall a \in \mathbb{R}^+,\ m,n \in \mathbb{Z}^+$

$$\implies f(an,n) = \frac{a^2n^4}{a^2n+n} = \frac{a^2n^3}{a^2+1}$$

$\left\{\dfrac{a^2n^3}{a^2+1},\ n \in \mathbb{Z}\right\}$ is neither bounded above nor below.

If $m < n$, m can be written as $n = am\ \forall a \in \mathbb{R}^+,\ m,n \in \mathbb{Z}^+$

$$\implies f(m,am) = \frac{a^2m^4}{am+am} = \frac{am^3}{2}$$

$\left\{\dfrac{am^3}{2},\ m \in \mathbb{Z}\right\}$ is neither bounded above nor below.

4.4.2 Infimum

If $m < n$, m can be written as $m = -an\ \forall a \in \mathbb{R}^-,\ m,n \in \mathbb{Z}^-$

$$\implies f(-an,n) = \frac{a^2n^4}{a^2n+n} = \frac{a^2n^3}{a^2+1}$$

$\left\{\dfrac{a^2n^3}{a^2+1},\ n \in \mathbb{Z}\right\}$ is neither bounded above nor below.

If $m > n$, m can be written as $n = -am \, \forall a \in \mathbb{R}^-$, $m,n \in \mathbb{Z}^-$

$$\implies f(m,-am) = \frac{a^2m^4}{am+am} = \frac{am^3}{2}$$

$\left\{\frac{am^3}{2},\ m \in \mathbb{Z}\right\}$ is neither bounded above nor below.

Hence $\inf \mathcal{A}_4$ and $\sup \mathcal{A}_4$ do not exist.

4.5 For $\mathcal{A} := \left\{\frac{(nm)^p}{(am)^q + n^q},\ n,m,a \in \mathbb{Z},\ p,q \in \mathbb{Z}^+,\ a \neq 0\right\}$

4.5.1 Supremum

If $m = n$,

$$\implies f(n,n) = \frac{n^{2p}}{(a^q+1)n^q} = \frac{n^{2p-q}}{a^q+1}$$

$\mathcal{A}$ will have an infimum and supremum iff $2p - q = 0$.

If $m > n$, m can be written as $m = an \, \forall a \in \mathbb{Z}^+$, $m,n \in \mathbb{Z}$

$$\implies f(an,n) = \frac{a^p n^{2p}}{(a^{2q}+1)n^q} = \frac{a^p n^{2p-q}}{a^{2q}+1}$$

If $m < n$, m can be written as $n = am \, \forall a \in \mathbb{Z}^+$

$$\implies f(m,am) = \frac{a^p m^{2p}}{2(am)^q} = \frac{a^p m^{2p-q}}{2a^q}$$

If $2p - q = 0$,

$$f(n,n) = \frac{1}{a^q+1}$$
$$f(an,n) = \frac{a^p}{a^{2q}+1}$$
$$f(m,am) = \frac{a^p}{2a^q}$$

It is obvious that for $2p - q = 0,\ a^p(a^q+1) \geq 2a^q$.

Proof.

$$a^{p+q} + a^p \geq 2a^q$$

$$\text{But } q = 2p$$

$$a^{\frac{3q}{2}} + a^{\frac{q}{2}} \geq 2a^q$$

By $AM - GM$ inequality,

$$\frac{a^{\frac{3q}{2}} + a^{\frac{q}{2}}}{2} \geq \sqrt{a^{\frac{3q}{2}} \cdot a^{\frac{q}{2}}}$$

$$\implies a^{\frac{3q}{2}} + a^{\frac{q}{2}} \geq 2a^q$$

Hence, $a^{p+q} + a^p \geq 2a^q$ is true for all $a \in \mathbb{Z},\ p,q \in \mathbb{Z}^+$
Also,

$$a^{2q} + 1 > a^p(a^q + 1)$$

$$\implies a^{4p} + 1 > a^{3p} + a^p$$

$$\implies a^{4p} - a^{3p} - a^p + 1 > 0$$

$$\implies (a^p - 1)^2(a^{2p} + a^p + 1) > 0$$

$(a^p - 1)^2 > 0$ is always true, so to check if $a^{2p} + a^p + 1 > 0$, we add a^p to both sides.

$$\implies a^{2p} + 2a^p + 1 > a^p$$

$$\implies (a^p + 1)^2 > a^p$$

This is true, so $a^{2p} + a^p + 1 > 0$.

It implies that $(a^p - 1)^2(a^{2p} + a^p + 1) > 0$ is true.

Hence, $a^{2q} + 1 > a^p(a^q + 1)$ is true for all $a \in \mathbb{Z},\ p,q \in \mathbb{Z}^+$

Therefore,

$$a^{2q} + 1 > a^p(a^q + 1) \geq 2a^q$$

$$\implies \frac{1}{a^{2q}+1} < \frac{1}{a^p(a^q+1)} \leq \frac{1}{2a^q}$$

$$\implies \frac{a^q}{a^{2q}+1} < \frac{1}{a^{2q}+1} \leq \frac{a^p}{2a^q} = \frac{1}{2a^{q-p}}$$

$$\implies \sup \mathcal{A} = \frac{1}{2a^{q-p}}$$

4.5.2 Infimum

If $m < n$, m can be written as $m = -an\ \forall a \in \mathbb{Z}^+$ and p is odd.

$$f(n,n) = \frac{1}{a^q + 1}$$

$$f(-an,n) = \frac{(-1)^p a^p}{a^{2q}+1} = -\frac{a^p}{a^{2q}+1}$$

$$f(m,-am) = \frac{(-1)^p a^p}{2a^q} = -\frac{a^p}{2a^q}$$

$$a^{2q}+1 > a^p(a^q+1) \geq 2a^q$$

$$\implies -\left(a^{2q}+1\right) < -a^p(a^q+1) \leq -2a^q$$

$$\implies -\frac{1}{a^{2q}+1} > -\frac{1}{a^p(a^q+1)} \geq -\frac{1}{2a^q}$$

$$\implies -\frac{1}{2a^q} \leq -\frac{1}{a^p(a^q+1)} < -\frac{1}{a^{2q}+1}$$

$$\implies \inf \mathcal{A} = -\frac{1}{2a^{q-p}}$$

If p is even, we have previously that $2p - q = 0$, $q = 2p$ which implies that q is even whether p is odd or even.

So if p is even, $\mathcal{A}$ is a set of positive values, so $\mathcal{A}$ can be rewritten as

$$\mathcal{A} := \left\{ \frac{(nm)^p}{(am)^{2p} + n^{2p}},\ n,m \in \mathbb{Z}^+ \cup \{0\},\ a,p,q \in \mathbb{Z}^+ \right\}$$

Since, any negative number selected in $\mathbb{Z}$ becomes positive under an even power. Hence the minimum element in $\mathcal{A}$ is derived if $m,n = 0$.

$$\implies \inf \mathcal{A} = 0.$$

Therefore, we can generalize that

If

$$\mathcal{A} := \left\{ \frac{(nm)^p}{(am)^q + n^q},\ n,m,a \in \mathbb{Z},\ p,q \in \mathbb{Z}^+,\ a \neq 0, \wedge q = 2p \right\}$$

$$\inf \mathcal{A} = \begin{cases} -\frac{1}{2a^{q-p}}, \text{ for } p \text{ odd} \\ 0, \text{ for } p \text{ even} \end{cases}$$

and

$$\sup \mathcal{A} = \frac{1}{2a^{q-p}}$$

References

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