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Prove that

$$\sum_{n=0}^{\infty}\frac{H_{2n+1}}{\binom{n+\frac{3}{2}}{n}}=3+6G,\quad \sum_{n=0}^{\infty}\frac{H_{2n+2}}{\binom{n+\frac{3}{2}}{n}}=\frac{3\pi^2}{8}+6G$$

where H_n is the nth Harmonic number and G is Catalan's Constant.
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Since the recurrence relation of harmonic number (see [1]) is $H_{n+1}=H_n+\frac{1}{n+1}$ which implies $H_{2n+1}=H_{2n}+\frac{1}{2n+1}$ and $H_{2n+2}=H_{2n+1}+\frac{1}{2n+2}$. Further, we note that

$$\binom{n+\frac{3}{2}}{n}=\frac{\left(n+\frac{3}{2}\right)!}{\left(\frac{3}{2}\right)!n!}=\frac{\Gamma\left(n+\frac{5}{2}\right)}{\Gamma\left(\frac{5}{2}\right)\Gamma(n+1)}=\frac{4}{3\sqrt{\pi}}\left(\frac{\sqrt{\pi}}{4^{n+2}}\binom{2n+4}{n+2}\right)\tag{1}$$

Now using these tools we proceed to prove the results.

Proof of the first sum

By the recurrence relation of harmonic number and (1), we easily deduce that

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{H_{2n+1}}{\binom{n+\frac{3}{2}}{n}}&=3\sum_{n=0}^{\infty}\frac{4^nH_{2n+1}}{(2n+1)(2n+3)\binom{2n}{n}}=3\sum_{n=0}^{\infty}\frac{4^n}{(2n+1)^2(2n+3)\binom{2n}{n}}\\&+3\sum_{n=0}^{\infty}\frac{4^nH_{2n}}{(2n+1)(2n+3)\binom{2n}{n}}=3S_1+3S_2\end{aligned}$$

We solve S_1 by performing the partial fraction decomposition of the summand

$$\begin{aligned}S_1&=\frac{1}{2}\sum_{n=0}^{\infty}\frac{4^n}{(2n+1)^2\binom{2n}{n}}-\frac{1}{2}\sum_{n=0}^{\infty}\frac{4^n}{(2n+1)(2n+3)\binom{2n}{n}}\\&\overset{\mathcal{W}_n}{=}\sum_{n=0}^{\infty}\frac{1/2}{2n+1}\int_0^{\frac{\pi}{2}}\sin^{2n+1}x\,dx+\sum_{n=0}^{\infty}\frac{1/2}{2n+2}\int_0^{\frac{\pi}{2}}\sin^{2n+3}x\,dx\end{aligned}$$

Above we used the Wallis' Integral formula $\mathcal{W}_n$ [2] and on interchanging the summation and integral we get

$$S_1 = \frac{1}{2}\int_0^{\frac{\pi}{2}} \tanh^{-1}(\sin x)\,dx + \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin x \log(\cos x)dx = \frac{1}{4}\int_0^{\frac{\pi}{2}} \log\left(\frac{1+\sin x}{1-\sin x}\right)dx$$
$$+\frac{1}{2}\int_0^1 \log(y)dy = \frac{2}{4}\int_0^{\frac{\pi}{2}} \log(\sec y + \tan y)\,dy - \frac{1}{2} = G - \frac{1}{2}$$

The former integral is well know integral representation of Catalan's constant which can be easily proved by substituting $\sin x = t$ and $\frac{1-t}{1+t} = u^2$. Therefore, we have $3S_1 = 3G - \frac{3}{2}$

Alternative solution

The former sum of S_1, $\sum_{n\geq 0} \frac{4^n}{(2n+1)^2\binom{2n}{n}}$ can be observed in other way as

$$\sum_{n=0}^{\infty} \frac{4^n}{(2n+1)^2\binom{2n}{n}} = \sum_{n=0}^{\infty} \frac{4^n}{(2n+1)\binom{2n}{n}} \int_0^1 x^{2n}dx = \int_0^1 \frac{\arcsin(x)}{x\sqrt{1-x^2}}dx$$
$$= \int_0^{\frac{\pi}{2}} \frac{x}{\sin x}dx = 2\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = 2G$$

since the last integral is famous well know result which is equal to $2G$ which we obtain by using the generating function $\sum_{n\geq 0} \frac{4^n x^{2n}}{(2n+1)\binom{2n}{n}} = \frac{\text{arcsin}x}{x\sqrt{1-x^2}}$ for $|x| < 1$. Moreover, this generating function can be used to evaluate the latter sum of S_1. All we need to multiply both sides by x^2 and integrate it from 0 to 1 giving us -1 and collecting values gives us $S_1 = G - \frac{1}{2}$.

Now in order to evaluate S_2 sum, we shall deduce the following result, namely

$$G(n) = 2\sum_{n=1}^{\infty} \frac{H_{2n}}{2n+1} x^{2n+1} = -\text{arctanh}x \log\left(1-x^2\right) \tag{2}$$

Proof: We start by the series representation of arctanh$x$ and $\log(1-x^2)$, ie

$$-\text{arctanh}x \log(1-x^2) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} \sum_{n=0}^{\infty} \frac{x^{2n+2}}{n+1} \overset{\text{C.P}}{=} \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{x^{2n+3}}{(k+1)(2n-2k+1)}$$
$$\overset{\text{PFD}}{=} \sum_{n=0}^{\infty} \frac{2}{2n+3} \sum_{k=0}^{n} \left(\frac{1}{2k+2} + \frac{1}{2n-2k+1}\right) x^{2n+3} = G(n)$$

where notation C.P stands for Cauchy product and since

$$\sum_{k=0}^{n}\left(\frac{1}{2k+2}+\frac{1}{2n-2k+1}\right)=\sum_{k=0}^{\infty}\left(\frac{1}{2k+2}+\frac{1}{(2n+2)-2k-1}\right)$$
$$=\sum_{k=1}^{2n+2}\frac{1}{k}=H_{2n+2}$$

and therefore it follows $2\sum_{n=0}^{\infty}\frac{H_{2n+2}}{2n+3}x^{2n+3}=2\sum_{n=1}^{\infty}\frac{H_{2n}}{2n+1}x^{2n+1}$ and the proof is complete.
Now on differentiating (2) with respect to x gives us

$$2\sum_{n=1}^{\infty}H_{2n}x^{2n}=-\frac{d}{dx}\left(\operatorname{arctanh}x\log(1-x^2)\right)=\frac{2x\operatorname{arctanh}x}{1-x^2}-\frac{\log(1-x^2)}{1-x^2}$$

Replacing x^2 by $\sqrt{x}$ gives the generating function of $2\sum_{n=1}^{\infty}H_{2n}x^n=\frac{2\sqrt{x}\operatorname{arctanh}\left(\sqrt{x}\right)}{1-x}-\frac{\log(1-x)}{1-x}$ and on dividing both sides by 4 and on integrating from $x=0$ to $x=y^2$ for $|y|<1$, we get

$$\sum_{n=1}^{\infty}\frac{H_{2n}}{2n+2}y^{2n+3}=\frac{y}{8}\log^2(1-y^2)+\frac{y}{2}\operatorname{arctanh}^2(y)-\frac{y}{2}\log(1-y^2)-y^2\operatorname{arctanh}(y)$$

Now all we need to do is to set $y=\sin z$ and perform integration within the interval of $[0,\pi/2]$

$$\sum_{n=1}^{\infty}\frac{4^nH_{2n}}{(2n+1)(2n+3)\binom{2n}{n}}=\frac{1}{2}\underbrace{\int_0^{\frac{\pi}{2}}\left(\sin z\log^2(\cos z)-2\sin z\log(\cos z)\right)dz}_{\mathcal{J}_1}$$
$$+\frac{1}{2}\underbrace{\int_0^{\frac{\pi}{2}}\left(\sin z\operatorname{arctanh}^2(\sin z)-2\sin^2 z\operatorname{arctanh}(\sin z)\right)dz}_{\mathcal{J}_2}$$

Here in $\mathcal{J}_1$ we substitute $\cos z=t$ and following integrals are easily doable (in the same fashion of S_1) resulting $\mathcal{J}_1=4$.

To evaluate $\mathcal{J}_2$, we use identity $\operatorname{arctanh}(\sin z) = \frac{1}{2}\log\left(\frac{1+\sin z}{1-\sin z}\right)$ and perform substitution $\sin z = u$ and $\frac{1-u}{1+u} = t$ which transform the integral $\mathcal{J}_2$ to the following integrals

$$\mathcal{J}_2 = \frac{1}{2}\int_0^1 \left(\frac{(1-t)\log^2(t)}{4\sqrt{t}(1+t)^2} + \frac{(1-t)^2\log(t)}{\sqrt{t}(1+t)^3}\right)dt = A + B$$

here the former integral

$$A = \sum_{k=0}^{\infty}\binom{k+1}{k}\int_0^1 \frac{(-t)^k(1-t)\log(t)}{8\sqrt{t}}dt = \sum_{k=0}^{\infty}\left(\frac{2k+2}{(2k+1)^3} - \frac{2k+2}{(2k+3)^3}\right)(-1)^k$$
$$= \sum_{k=0}^{\infty}(-1)^k\left(\frac{1}{(2k+1)^2} - \frac{1}{(2k+1)^3} - \frac{1}{(2k+3)^2} + \frac{1}{(2k+3)^3}\right) = 2G$$

Here we have used the elementary integral result $\int_0^1 x^n \log^m(x)dx = \frac{(-1)^m m!}{(n+1)^{m+1}}$ for all $m, n > -1$. The series above are easy to see as we get $G, \frac{\pi^3}{32}, 1-G, 1-\frac{\pi^3}{32}$ respectively and on performing the operations we get $2G$.Now the latter integral (using the same elementary result) we obtain

$$B = \frac{1}{2}\sum_{k=0}^{\infty}\binom{k+2}{k}(-1)^k\int_0^1 \frac{(1-2t+t^2)\log(t)t^k}{\sqrt{t}}dt$$
$$= \frac{1}{2}\sum_{k=0}^{\infty}\binom{k+2}{k}(-1)^k\underbrace{\left(-\frac{4}{(2k+1)^2} + \frac{8}{(2k+3)^2} - \frac{4}{(2k+5)^2}\right)}_{g}$$

Noting $\binom{k+2}{k} = (k+1)(k+2)$ which shows that on calculating each sums individually we get divergent series however, it is still manageable to show convergent sum. To do so we just need to separate the summand with k^2 and $3k+2$ terms.

The series with $3k+2$ terms converges to $10G - \frac{53}{9} - \frac{3\pi}{2}$ and with k^2 converges to

$$\lim_{n\to\infty}\sum_{k=0}^{n}(-1)^k k^2 g = \frac{3\pi}{2} + \frac{97}{18} - 11G$$

Adding the obtained values we get $B = -G - \frac{1}{2}$ and hence the value of $A + B = G + \frac{3}{2}$ and $3S_2 = 3G + \frac{9}{2}$. Therefore, $3S_1 + 3S_2 = 3 + 6G$ which completes the proof.

Proof of the second sum

By recursive relation and (1), we obtain

$$\sum_{k=0}^{\infty}\frac{H_{2n+2}}{\binom{n+\frac{3}{2}}{n}}=3\sum_{n=0}^{\infty}\frac{H_{2n+2}}{(2n+1)(2n+3)\binom{2n}{n}}=3\sum_{n=0}^{\infty}\frac{4^n}{(2n+2)(2n+1)(2n+3)\binom{2n}{n}}$$
$$+3\sum_{n=0}^{\infty}\frac{4^nH_{2n+1}}{(2n+1)(2n+3)\binom{2n}{n}}=3S_3+3(S_1+S_2)=3S_3+3+6G$$

Yet again by Wallis's formula $\mathcal{W}_n$ we evaluate S_3, ie

$$S_3=\sum_{n=0}^{\infty}\frac{4^n}{(2n+2)(2n+1)(2n+3)\binom{2n}{n}}\overset{\mathcal{W}_n}{=}\sum_{n=0}^{\infty}\frac{1}{(2n+2)^2}\int_0^{\frac{\pi}{2}}\sin^{2n+3}x dx$$
$$\overset{\text{dilog.}}{=}\frac{1}{4}\int_0^{\frac{\pi}{2}}\sin x\mathrm{Li}_2\left(\sin^2 x\right)dx=\frac{1}{4}\int_0^1\mathrm{Li}_2(1-y^2)dy\overset{\text{IBP}}{=}-\int_0^1\frac{x^2\log(x)}{1-x^2}dx$$
$$=-\sum_{k=0}^{\infty}\int_0^1 x^{2k+2}\log(x)dx=\sum_{k=0}^{\infty}\frac{1}{(2k+3)^2}=\sum_{k=1}^{\infty}\frac{1}{(2k+1)^2}=\frac{\pi^2}{8}-1$$

So $3S_3=\frac{3\pi^2}{8}-3$ and $3S_3+3+6G=\frac{3\pi^2}{8}+6G$ and the solution is complete.

Alternative solution

The sum S_3 can also be evaluated by using generating function of $2\arcsin^2 x=\sum_{k=0}^{\infty}\frac{4^{n+1}x^{2n+2}}{(2n+2)(2n+1)\binom{2n}{n}}$ and hence on integrating from $x=0$ to $x=1$ it gives us the value $\frac{\pi^2}{8}-1$.

References

[1] Sondow, Jonathan and Weinstein, Eric W. "*Harmonic Number*", MathWorld, https://mathworld.wolfram.com/HarmonicNumber.html .
[2] Weisstein, Eric W. "*Wallis Cosine Formula*", MathWorld, https://mathworld.wolfram.com/WallisCosineFormula.html .