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ROMANIAN MATHEMATICAL MAGAZINE

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# ABOUT A FEW SPECIAL LIMITS AND SUMS (III)

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**Abstract:** *In this paper are presented few special limits with great integer function.*

**App.1) Find:**

$$\Omega=\lim_{p\to\infty}\frac{1}{p}\cdot\sum_{m=0}^{p-1}\sum_{n=1}^{m}\left(\sum_{k=1}^{n}\left[\sqrt{k^2+k+1}+\sqrt{k^2-k+1}\right]\right)^{-1}$$

**,where $[\cdot]$ great integer function.**

**Solution.**

Let be the notations:

$$S_1=\sum_{k=1}^{n}\left[\sqrt{k^2+k+1}+\sqrt{k^2-k+1}\right]$$

$$S_2=\sum_{n=1}^{m}\left(\sum_{k=1}^{n}\left[\sqrt{k^2+k+1}+\sqrt{k^2-k+1}\right]\right)^{-1}$$

$$S_3=\sum_{m=0}^{p-1}\sum_{n=1}^{m}\left(\sum_{k=1}^{n}\left[\sqrt{k^2+k+1}+\sqrt{k^2-k+1}\right]\right)^{-1}$$

For $n=1$ we have: $\left[\sqrt{1}+\sqrt{3}\right]=2$ and for $n=2$ we have: $\left[\sqrt{3}+\sqrt{7}\right]=4$

But, $2k\le\sqrt{k^2-k+1}+\sqrt{k^2+k+1}\le 2k+1$, then

$$\left[\sqrt{k^2-k+1}+\sqrt{k^2+k+1}\right]=2k$$

$$S_n=\sum_{k=1}^{n}\left[\sqrt{k^2+k+1}+\sqrt{k^2-k+1}\right]=\sum_{k=1}^{n}2k=n(n+1)$$

$$S_m=\sum_{n=1}^{m}\left(\sum_{k=1}^{n}\left[\sqrt{k^2+k+1}+\sqrt{k^2-k+1}\right]\right)^{-1}=$$

$$=\sum_{n=1}^{m}\left(\frac{1}{n}-\frac{1}{n+1}\right)=\frac{m}{m+1}=1-\frac{1}{m+1}$$

$$S_p=\sum_{m=0}^{p-1}\sum_{n=1}^{m}\left(\sum_{k=1}^{n}\left[\sqrt{k^2+k+1}+\sqrt{k^2-k+1}\right]\right)^{-1}=\sum_{m=0}^{p-1}\left(1-\frac{1}{m+1}\right)=$$

$$=p-\sum_{m=0}^{p}\frac{1}{m+1}=p-\left(1+\frac{1}{2}+\cdots+\frac{1}{p}\right)=p-H_p$$

Therefore,

$$\Omega=\lim_{p\to 0}\frac{p-H_p}{p}=\lim_{p\to\infty}\left(1-\frac{H_p}{p}\right)\overset{LC-S}{=}1$$

**App. 2) Find:**

$$\Omega=\lim_{n\to\infty}\frac{\left[\frac{1}{\sqrt{1-\frac{1}{2}}}+\frac{1}{\sqrt{1-\frac{1}{2^2}}}+\cdots+\frac{1}{\sqrt{1-\frac{1}{2^n}}}\right]^{\alpha}}{\left[\sqrt[3]{1}\right]+\left[\sqrt[3]{2}\right]+\left[\sqrt[3]{3}\right]+\cdots+\left[\sqrt[3]{n^3-1}\right]},[*]-\text{GIF},\alpha\in\mathbb{R}$$

**Solution.**

We have: $1<\frac{1}{\sqrt{1-\frac{1}{2^k}}}<1+\frac{1}{2^k},k=\overline{1,n}$ and summing, we get:

$$n<\sum_{k=1}^{n}\frac{1}{\sqrt{1-\frac{1}{2^k}}}<n+\sum_{k=1}^{n}\frac{1}{2^k}=n+1-\frac{1}{2^k}<n+1$$

Hence,

$$\left[\sum_{k=1}^{n}\frac{1}{\sqrt{1-\frac{1}{2^k}}}\right]=n;(1)$$

Now, we have: $\left[\sqrt[3]{k}\right]=\left[\sqrt[3]{k^3+1}\right]=\cdots=\left[\sqrt[3]{(k+1)^3-1}\right]=k$

$$\sum_{k=1}^{n-1}\left(\sum_{i=0}^{3k(k+1)}\left[\sqrt[3]{k^3+i}\right]\right)=\sum_{k=1}^{n-1}(3k^3+3k^2+k)=\frac{(n-1)n^2(3n+1)}{4};(2)$$

From (1) and (2) it follows that:

$$\Omega=\lim_{n\to\infty}\frac{\left[\frac{1}{\sqrt{1-\frac{1}{2}}}+\frac{1}{\sqrt{1-\frac{1}{2^2}}}+\cdots+\frac{1}{\sqrt{1-\frac{1}{2^n}}}\right]^{\alpha}}{\left[\sqrt[3]{1}\right]+\left[\sqrt[3]{2}\right]+\left[\sqrt[3]{3}\right]+\cdots+\left[\sqrt[3]{n^3-1}\right]}=\lim_{n\to\infty}\frac{4n^{\alpha}}{n^2(n-1)(3n+1)}=$$

$$=\begin{cases}0, \text{if } \alpha<4\\ \frac{4}{3}, \text{if } \alpha=4\\ \infty, \text{if } \alpha>4\end{cases}$$

**App. 3)**

$$\textbf{If } x_m=\lim_{n\to\infty}\sum_{k=0}^{m}\left\{\sqrt{n^2+(2k+1)n+k^2+k}\right\}, m,n\in\mathbb{N}^*, \textbf{then:}$$

$$m\sqrt[m]{m+1}<\sum_{k=1}^{m}\frac{1}{x_k}<2\sqrt{m}$$

$$\{x\}=x-[x], [\cdot]-GIF.$$

**Solution.**

$$n^2+(2k+1)n+k^2+k=n^2+2kn+k^2+k=(n+k)^2+k>(n+k)^2$$

$$n^2+(2k+1)n+k^2+k=n^2+2kn+k^2+k=(n+k)^2+k<(n+k+1)^2$$

Hence,

$$(n+k)^2<n^2+(2k+1)n+k^2+k<(n+k+1)^2$$

$$n+k<\sqrt{n^2+(2k+1)n+k^2+k}<n+k+1\Rightarrow$$

$$\left[\sqrt{n^2+(2k+1)n+k^2+k}\right]=n+k,$$

$$\because \{t\}=t-\{t\}, (\forall)t\in\mathbb{R}$$

$$\left\{\sqrt{n^2+(2k+1)n+k^2+k}\right\}=\sqrt{n^2+(2k+1)n+k^2+k}-(n+k)=$$

$$=\frac{n^2+(2k+1)n+k^2+k-(n+k)^2}{\sqrt{n^2+(2k+1)n+k^2+k}+n+k}=\frac{n^2+(2k+1)n+k^2+k-n^2-2kn-k^2}{\sqrt{n^2+(2k+1)n+k^2+k}+n+k}=$$

$$=\frac{n+k}{\sqrt{n^2+(2k+1)n+k^2+k}+n+k}$$

$$x_m=\lim_{n\to\infty}\sum_{k=0}^{m}\left\{\sqrt{n^2+(2k+1)n+k^2+k}\right\}=$$

$$=\lim_{m\to\infty}\sum_{k=0}^{m}\frac{n+k}{\sqrt{n^2+(2k+1)n+k^2+k}+n+k}=$$

$$=\frac{1}{2}+\frac{1}{2}+\overset{m-terms}{\cdots}+\frac{1}{2}=\frac{m+1}{2}\in(\sqrt{m},m),m\in\mathbb{N}^*$$

$$x_k\in(\sqrt{k},k),k\in\mathbb{N}^*\Rightarrow\frac{1}{x_k}\in\left(\frac{1}{k},\frac{1}{\sqrt{k}}\right)$$

$$\frac{1}{k}\le\frac{1}{x_k}\le\frac{1}{\sqrt{k}},k\in\mathbb{N}^*$$

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m}\le\sum_{k=1}^{m}\frac{1}{x_k}\le1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{m}};(1)$$

$$2+\frac{3}{2}+\frac{4}{3}+\cdots+\frac{m+1}{m}\ge m\sqrt[m]{m+1}\Leftrightarrow m+\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m}\right)\ge m\sqrt[m]{m+1}$$

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m}\ge m\left(\sqrt[m]{m+1}-1\right);(2)$$

$$2\left(\sqrt{k+1}-\sqrt{k}\right)<\frac{1}{\sqrt{k}}<2\left(\sqrt{k}-\sqrt{k-1}\right)\Rightarrow1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{m}}<2\sqrt{m};(3)$$

From (1), (2) and (3), we get: $m\sqrt[m]{m+1}<\sum_{k=1}^{m}\frac{1}{x_k}<2\sqrt{m},m\in\mathbb{N}^*$

**App. 4) Let $(x_n)_{n\ge1}$ and $(y_n)_{n\ge1}$ be sequences of real numbers such that**

$$x_n=\sum_{k=3}^{n}\tan\left(\frac{\pi}{k}\right)-\pi\log n,\ y_n=\sum_{k=1}^{n}2^{k-1}\cdot\left[\frac{k^2}{k+1}\right],[*]-\text{GIF}.$$

$$\text{Find: }\Omega=\lim_{n\to\infty}\frac{2^n\cdot x_n}{y_n}$$

**Solution.**

$$x_n=\sum_{k=3}^{n}\tan\left(\frac{\pi}{k}\right)-\pi\log n=\pi\left(\sum_{k=1}^{n}\frac{1}{k}-\log n\right)+\sum_{k=3}^{n}\tan\left(\frac{\pi}{k}\right)-\pi\sum_{k=1}^{n}\frac{1}{k}=$$

$$=\pi\gamma_n+\sum_{k=3}^{n}\left(\tan\left(\frac{\pi}{k}\right)-\frac{\pi}{k}\right)-\frac{3\pi}{2}=\pi\gamma_n+a_n-\frac{3\pi}{2},\text{where}$$

$$a_n=\sum_{k=3}^{n}\left(\tan\left(\frac{\pi}{k}\right)-\frac{\pi}{k}\right)<\sum_{k=3}^{n}\left(\frac{\pi}{k}\right)^3=\pi^3\sum_{k=3}^{n}\frac{1}{k^3}<\pi^3\sum_{k=3}^{n}\frac{1}{k^2}<$$

$$<\pi^3\sum_{k=3}^{n}\frac{1}{k(k-1)}=\pi^3\left(\frac{1}{2}-\frac{1}{n}\right)<\frac{\pi^3}{2};(1)$$

$$a_{n+1}-a_n=\tan\left(\frac{\pi}{n+1}\right)-\frac{\pi}{n+1}>0;(2)$$

From (1) and (2) it follows that $(a_n)_{n\geq 3}$ is convergent, so $x_n=\pi\gamma_n+a_n-\frac{3\pi}{2}$ is convergent; (3).

Now, we have: $k-1<\frac{k^2}{k+1}<k\Rightarrow\left[\frac{k^2}{k+1}\right]=k-1$, then

$$y_n=\sum_{k=1}^{n}2^{k-1}\cdot\left[\frac{k^2}{k+1}\right]=\sum_{k=1}^{n}2^{k-1}(k-1)=\frac{1}{2}\sum_{k=1}^{n}k\cdot 2^k-\sum_{k=1}^{n}2^{k-1}\overset{(*)}{=};$$

$$\left(\because\sum_{k=1}^{n}k\cdot a^k=\frac{n\cdot a^{n+2}-(n+1)\cdot a^{n+1}+a}{(a-1)^2};(*)\right)$$

$$\overset{(*)}{=}n\cdot 2^{n+1}-(n+1)\cdot 2^n+1-2^n+1=n\cdot 2^n-2^{n+1}+2;(4)$$

From (3) and (4) it follows that:

$$\Omega=\lim_{n\to\infty}\frac{2^n\cdot x_n}{y_n}=\lim_{n\to\infty}\frac{2^n\cdot x_n}{n\cdot 2^n-2^{n+1}+2}=\lim_{n\to\infty}\frac{x_n}{n-2+2^{1-n}}=0$$

**App. 5) Let $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ be sequences of real numbers such that**

$$x_n=\sum_{k=1}^{n}\sin\frac{1}{k}+\log\left(\sin\frac{1}{n}\right),y_n=\sum_{k=1}^{n^2+n}\left[\sqrt{k}+\frac{1}{2}\right],[*]-\text{GIF}.$$

$$\text{Find: }\Omega=\lim_{n\to\infty}\frac{x_n}{y_n}$$

**Solution.**

$$x_n=\sum_{k=1}^{n}\sin\frac{1}{k}+\log\left(\sin\frac{1}{n}\right)=\sum_{k=1}^{n}\left(\frac{1}{k}+\sin\frac{1}{k}\right)-\sum_{k=1}^{n}\frac{1}{k}-\log n+\log\left(\sin\frac{1}{n}\right)+\log n$$

$$=\left(\sum_{k=1}^{n}\frac{1}{k}-\log n\right)-\sum_{k=1}^{n}\left(\frac{1}{k}-\sin\frac{1}{k}\right)+\log n\sin\frac{1}{n}=$$

$$= \gamma_n + \log\left(\frac{\sin\frac{1}{n}}{\frac{1}{n}}\right) - \sum_{k=1}^{n}\left(\frac{1}{k} - \sin\frac{1}{k}\right) = \gamma_n + \log\left(\frac{\sin\frac{1}{n}}{\frac{1}{n}}\right) - a_n, \text{where}$$

$$a_n = \sum_{k=1}^{n}\left(\frac{1}{k} - \sin\frac{1}{k}\right) < \sum_{k=1}^{n}\frac{1}{k^3} < \sum_{k=1}^{n}\frac{1}{k^2} < 1 + \sum_{k=2}^{n}\frac{1}{k(k-1)} = 2 - \frac{1}{n} < 2; (1)$$

$$a_{n+1} - a_n = \frac{1}{n+1} - \sin\left(\frac{1}{n+1}\right) + \sin\left(\frac{1}{n}\right) =$$

$$= \frac{1}{n+1} - 2\sin\left(\frac{1}{2n(n+1)}\right)\cos\left(\frac{1}{n+1}\right) \Rightarrow (a_n)_{n\geq1} \text{ increasing}; \ (2)$$

From (1),(2) $(a_n)_{n\geq1}$ are convergent, then $x_n = \gamma_n + \log\left(n\sin\frac{1}{n}\right) - a_n$ are convergent; (3)

Now, we have: $\left[\sqrt{k} + \frac{1}{2}\right] = q, q \in \mathbb{Z} \Leftrightarrow q \leq \frac{1}{2} + \sqrt{k} < q + 1 \Leftrightarrow$ $(q-1)q + 1 \leq q(q+1)$, so we can write:

$$y_n = \sum_{k=1}^{n^2+n}\left[\sqrt{k} + \frac{1}{2}\right] = \sum_{q=1}^{n}\sum_{k=(q-1)q+1}^{q(q+1)}\left[\sqrt{k} + \frac{1}{2}\right] =$$

$$= \sum_{q=1}^{n}\sum_{k=(q-1)q+1}^{q(q+1)} q = \sum_{q=1}^{n}[q(q+1) - (q-1)q]q = 2\sum_{q=1}^{n} q^2$$

$$= \frac{n(n+1)(2n+1)}{3}; (4)$$

From (3) and (4) it follows that $\Omega = \lim_{n\to\infty}\frac{x_n}{y_n} = 0.$

**App. 6) Let $(a_n)_{n\geq1}, (b_n)_{n\geq1}$ be sequences of real numbers such that:**

$$\boldsymbol{a_1 = 2, na_n = b_{n+2}(a_n + b_{n+1}\cdot 2^{n-1}), b_n = \left[\sum_{k=1}^{n^2}\frac{1}{\sqrt{k}}\right], n \geq 1, [\cdot] - GIF.}$$

$$\textbf{Find: } \boldsymbol{\Omega = \lim_{n\to\infty}\sqrt[n^2]{a_1\cdot a_2\cdot \ldots \cdot a_n}}$$

**Solution.**

For all $k \geq 2$, we have: $\sqrt{k-1} + \sqrt{k} < 2\sqrt{k} < \sqrt{k} + \sqrt{k+1}$

$$\sqrt{k} < \frac{\sqrt{k+1}+\sqrt{k}}{2}$$

$$2(\sqrt{k+1}-\sqrt{k}) < \frac{1}{\sqrt{k}} < 2(\sqrt{k}-\sqrt{k-1}); (1)$$

$$1+2\sum_{k=2}^{n^2}(\sqrt{k+1}-\sqrt{k}) \le \sum_{k=1}^{n^2}\frac{1}{\sqrt{k}} \le 1+2\sum_{k=2}^{n^2}(\sqrt{k}-\sqrt{k-1})$$

$$1+2\left(\sqrt{n^2+1}-\sqrt{2}\right) \le \sum_{k=1}^{n^2}\frac{1}{\sqrt{k}} < 1+2(n-1)$$

$$2n-2 \le \sum_{k=1}^{n}\frac{1}{\sqrt{k}} < 2n-1 \Rightarrow \left[\sum_{k=1}^{n^2}\frac{1}{\sqrt{k}}\right] = 2n-2 \Rightarrow b_n = 2(n-1)$$

$$na_n = b_{n+2}(a_n + b_{n+1}\cdot 2^{n-1}) \Rightarrow na_n = 2(n+1)(a_n + n\cdot 2^n), n \ge 1 \Leftrightarrow$$

$$\frac{a_{n+1}}{n+1} = 2\left(\frac{a_n}{n}+2^n\right) \Leftrightarrow \frac{a_{n+1}}{(n+1)2^{n+1}} = \frac{a_n}{n\cdot 2^n}+1 \Leftrightarrow \frac{a_n}{n\cdot 2^n} = n \Leftrightarrow a_n = n^2\cdot 2^n$$

So, we get:

$$\sqrt[n^2]{a_1\cdot a_2\cdot\ldots\cdot a_n} = 2^{\frac{n+1}{2n}}\cdot\sqrt[n^2]{(n!)^2}$$

From $1 \le n! \le n^n$, we obtain $1 \le \sqrt[n^2]{(n!)^2} \le \left(\sqrt[n]{n}\right)^2 \to 1$

$$\Rightarrow \lim_{n\to\infty}\sqrt[n^2]{(n!)^2} = 1 \Rightarrow \Omega = \lim_{n\to\infty}\sqrt[n^2]{a_1\cdot a_2\cdot\ldots\cdot a_n} = \lim_{n\to\infty} 2^{\frac{n+1}{2n}}\cdot\sqrt[n^2]{(n!)^2} = \sqrt{2}.$$

**App. 7) Let $(x_n)_{n\ge1}$ be sequence of real numbers such that**

$$x_{n-1} = \left[\sum_{k=1}^{n^3}\frac{1}{\sqrt[3]{k^2}}\right], n \ge 1, [\cdot] - GIF.\ \text{Find:}$$

$$\Omega = \lim_{n\to\infty}\sqrt[n]{\sum_{i=0}^{n} x_i\binom{n}{i}^2}$$

**Solution.** Using the double inequality:

$$\frac{1}{3\sqrt[3]{(k+1)^2}} < \sqrt[3]{k+1}-\sqrt[3]{k} < \frac{1}{3\sqrt[3]{k^2}}$$

$$\sum_{k=1}^{n^3-1}\frac{1}{\sqrt[3]{(k+1)^2}}<3\left(\sum_{k=1}^{n^3-1}\left(\sqrt[3]{k+1}-\sqrt[3]{k}\right)\right)<\sum_{k=1}^{n^3-1}\frac{1}{\sqrt[3]{k^2}}$$

$$\sum_{k=1}^{n^3}\frac{1}{\sqrt[3]{k^2}}-1<3(n-1)<\sum_{k=1}^{n^3}\frac{1}{\sqrt[3]{k^2}}-\frac{1}{n\sqrt[3]{n}}$$

$$3n-3<3n-3+\frac{1}{n\sqrt[3]{n}}<\sum_{k=1}^{n^3}\frac{1}{\sqrt[3]{k^2}}<3n-2$$

Hence,

$$x_{n-1}=\left[\sum_{k=1}^{n^3}\frac{1}{\sqrt[3]{k^2}}\right]=3n-3\Rightarrow x_n=3n, n\geq 0$$

$(x_n)_{n\geq 0}$ –is a arithmetic progression with ratio $r=3$

$$S_n=\sum_{i=0}^{n}x_i\binom{n}{i}^2=x_0\binom{n}{0}^2+x_1\binom{n}{1}^2+\cdots+x_n\binom{n}{n}^2$$

$$2S_n=(x_0+x_n)\binom{n}{0}^2+(x_1+x_{n-1})\binom{n}{1}^2+\cdots+(x_n+x_0)\binom{n}{n}^2=$$

$$=(2x_0+nr)\left(\binom{n}{0}^2+\binom{n}{1}^2+\cdots+\binom{n}{n}^2\right)=$$

$$=3n\left(\binom{n}{0}^2+\binom{n}{1}^2+\cdots+\binom{n}{n}^2\right)=3n\binom{2n}{n}$$

Because

$$\lim_{n\to\infty}\frac{S_{n+1}}{S_n}=\lim_{n\to\infty}\frac{3(n+1)}{3n}\cdot\frac{2(2n+1)}{n+1}=4\Rightarrow\Omega=\lim_{n\to\infty}\sqrt[n]{\sum_{i=0}^{n}x_i\binom{n}{i}^2}=4$$

**REFERENCE:**

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