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PROBLEMS FOR JUNIORS

JP.466 If $a, b, c, d \in \mathbb{R}$ such that $(a^2 + b^2)(c^2 + d^2) = 25$ then:

$$3bd + 4ad + 4bc \le 3ac + 25$$

Proposed by Daniel Sitaru-Romania

JP.467 If $x, y > 0$ then in ΔABC the following relationship holds:

$$\frac{xa^n + yb^n}{xa^{n-1} + yb^{n-1}} + \frac{xb^n + yc^n}{xb^{n-1} + yc^{n-1}} + \frac{xc^n + ya^n}{xc^{n-1} + yb^{n-1}} \ge a + b + c$$

Proposed by Daniel Sitaru-Romania

JP.468 Solve for real numbers:

$$x^{12} - 15x^3 + 14 = 0$$

Proposed by Daniel Sitaru-Romania

JP.469 Let $z_1, z_2, z_3 \in \mathbb{C}^*, A(z_1), B(z_2), C(z_3)-$different in pairs such that $|z_1| = |z_2| = |z_3| = 1$. If

$$\sum_{cyc} \sqrt{\left|(2z_1 - z_2 - z_3)(2z_2 - z_1 - z_3)\right|} = 9 \text{ then } AB = BC = CA$$

Proposed by Marian Ursărescu-Romania

JP.470 Let $z_1, z_2, z_3 \in \mathbb{C}, A(z_1), B(z_2), C(z_3)-$different in pairs such that $|z_1| = |z_2| = |z_3| = 1$. If

$$\sum_{cyc} \frac{1}{(|(z_1 - z_2)|z_1 - z_3| + (z_1 - z_3)|z_1 - z_2||)^2} =$$

$$= \frac{3}{(|z_1 - z_2| + |z_2 - z_3| + |z_3 - z_1|)^2} \Leftrightarrow AB = BC = CA$$

Proposed by Marian Ursărescu-Romania

JP.471 In $\Delta ABC, AA_1, BB_1, CC_1$ internal bisectors and A_2, B_2, C_2 contact points with circumcircle of triangle ABC. Prove that:

$$A_1A_2 \cdot B_2C_2 + B_1B_2 \cdot A_2C_2 + C_1C_2 \cdot A_2B_2 \ge Rs$$

Proposed by Marian Ursărescu-Romania

JP.472 If $x, y, z \geq 0$ then:

$$\left(3x^3 - \frac{1}{x^2} + \frac{1}{x^5}\right)\left(3y^3 - \frac{1}{y^2} + \frac{1}{y^5}\right)\left(3z^3 - \frac{1}{z^2} + \frac{1}{z^5}\right) \geq (xy+yz+zx)^3$$

Proposed by Daniel Sitaru-Romania

JP.473 If $a_k > 0, k = \overline{1,5}$ then prove that exists $i, j \in \overline{1,5}$ such that:

$$0 \leq \frac{a_j - a_i}{1 + a_i a_j} \leq \sqrt{2} - 1$$

Proposed by Daniel Sitaru-Romania

JP.474 If $0 < b \leq a$ then:

$$\sqrt{a^2 + ab} + \sqrt{a^2 + \left(\frac{a+b}{2}\right)^2} \leq 2a + (\sqrt{2} - 1)\left(\sqrt{ab} + \frac{a+b}{2}\right)$$

Proposed by Daniel Sitaru-Romania

JP.475 If $x, y, z > 0$ such that $x + y + z = 3$ and $\lambda \geq 0$ then:

$$(i)\ \frac{1}{(x+\lambda)^2} + \frac{1}{(y+\lambda)^2} + \frac{1}{(z+\lambda)^2} \geq \frac{3}{(\lambda+1)^2}$$

$$(ii)\ \frac{x}{(y+\lambda)^2} + \frac{y}{(z+\lambda)^2} + \frac{z}{(x+\lambda)^2} \geq \frac{3}{(\lambda+1)^2}$$

Proposed by Marin Chirciu-Romania

JP.476 In ΔABC the following relationship holds:

$$\frac{a}{(b+\lambda c)^{n+1} s_a^n} + \frac{b}{(c+\lambda a)^{n+1} s_b^n} + \frac{c}{(a+\lambda b)^{n+1} s_c^n} \geq \frac{3}{(\lambda+1)^{n+1}}\left(\frac{1}{sR}\right)^n,$$

where $\lambda \geq 0, n \in \mathbb{N}$

Proposed by Marin Chirciu-Romania

JP.477 Let $a > 1, b > 1$ fixed. Solve for real numbers:

$$a^{\log_{2b}\left(x + \frac{b^2}{x}\right)} = \frac{(a+2b)x - b^2 - x^2}{x}$$

Proposed by Marin Chirciu-Romania

JP. 478 Let $m, n \geq 0$ and $ABC, A_1B_1C_1$ triangles with areas F, F_1 respectively, then:

$$\frac{a^{m+2} \cdot a_1^{n+1}}{h_a^m} + \frac{b^{m+2} \cdot b_1^{n+1}}{h_b^n} + \frac{c^{m+2} \cdot c_1^{n+1}}{h_c^m} \geq 2^{m+n+1} \cdot F \cdot (\sqrt{F_1})^{n+1}$$

Proposed by D.M. Bătineţu-Giurgiu, Constantin Cocea-Romania

JP.479 If $a,b,c,d>0; ab=cd; a<b, c<d; x,z\in[a,b]$ and $y,t\in[c,d]$ then:

$$ab(x+y+z+t)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}\right)\le(a+b+c+d)^2$$

Proposed by Daniel Sitaru-Romania

JP.480 In ΔABC the following relationship holds:

$$\sum_{cyc}\sqrt{\cot^2\frac{A}{2}+\cot^2\frac{B}{2}+3}\ge 9$$

Proposed by Marin Chirciu-Romania

PROBLEMS FOR SENIORS

SP.466 Let $A,B\in M_4(\mathbb{R})$. If $AB+BA=O_4$ then:

$$\det(A^4+A^2+B^2)\ge 0$$

Proposed by Marian Ursărescu-Romania

SP.467 Let $z_1,z_2,z_3\in\mathbb{C}^*, A(z_1),B(z_2),C(z_3)-$different in pairs such that $|z_1|=|z_2|=|z_3|$. If

$$\left|\frac{z_1+z_2}{z_1-z_2}\right|^2+\left|\frac{z_2+z_3}{z_2-z_3}\right|^2+\left|\frac{z_3+z_1}{z_3-z_1}\right|^2=1\Rightarrow AB=BC=CA$$

Proposed by Marian Ursărescu-Romania

SP.468 In ΔABC the following relationship holds:

$$\frac{3\sqrt{3}}{2}k\le\sum_{cyc}\frac{\sin^2 A}{\sin B+\sin C}\le\frac{\sqrt{6}}{12}\left(\frac{4}{k}+1\right)\sqrt{1-k},\ k\in\left(0,\frac{1}{2}\right]$$

Proposed by George Apostolopoulos-Greece

SP.469 In ΔABC the following relationship holds:

$$\frac{y+z}{x\cdot w_a^4}+\frac{z+x}{y\cdot w_b^4}+\frac{z+x}{z\cdot w_c^4}\ge\frac{32}{27R^4},\ x,y,z>0$$

Proposed by Marin Chirciu-Romania

SP.470 In $\Delta ABC, O_a-$circumcevian, holds:

$$\frac{6r}{R}\le\frac{r_a}{O_a}+\frac{r_b}{O_b}+\frac{r_c}{O_c}\le\frac{2R}{r}-1$$

Proposed by Marin Chirciu-Romania

SP.471 In ΔABC the following relationship holds:

$$\frac{3}{2}\cdot\sqrt[6]{\frac{4r^5}{R^2}}\le\sum_{cyc}\sqrt{m_a\cos\frac{B}{2}\cos\frac{C}{2}}\le\frac{4R+r}{\sqrt{2R}}$$

Proposed by Marin Chirciu-Romania

SP.472 If $x,y,z>0,x+y+z=1$ then:

$$\left(x+\frac{1}{y}\right)^5+\left(y+\frac{1}{z}\right)^5+\left(z+\frac{1}{x}\right)^5\ge\frac{100.000}{81}$$

Proposed by Daniel Sitaru-Romania

SP.473 If $x,y,z>0,\Delta ABC$ and $A_1\in(BC),B_1\in(CA),C_1\in(AB)$ such that $A_1B=s\cdot A_1C,B_1C=y\cdot B_1C,C_1A=z\cdot C_1B$ then holds:

$$aa_1+bb_1+cc_1\ge4\sqrt{3}\cdot\sqrt{\frac{xyz+1}{(x+1)(y+1)(z+1)}}\cdot F$$

Proposed by D.M. Bătineţu-Giurgiu, Mihàly Bencze-Romania

SP.474 If $m\ge0$ and $x,y,z>0$ then in ΔABC holds:

$$\frac{x^{m+1}\cdot a^{2m}}{(y+z)^{m+1}\cdot h_a^2}+\frac{y^{m+1}\cdot b^{2m}}{(z+x)^{m+1}\cdot h_b^2}+\frac{z^{m+1}\cdot c^{2m}}{(x+y)^{m+1}\cdot h_c^2}\ge2^{m-1}\cdot(\sqrt{3})^{1-m}\cdot F^{m-1}$$

Proposed by D.M. Bătineţu-Giurgiu, Mihàly Bencze-Romania

SP.475 In ΔABC the following relationship holds:

$$\frac{m_a^2\cdot a^3}{\sqrt{m_bm_c}}+\frac{m_b^2\cdot b^3}{\sqrt{m_cm_a}}+\frac{m_c^2\cdot c^3}{\sqrt{m_bm_c}}\ge8\sqrt{3}\cdot F^2$$

Proposed by D.M. Bătineţu-Giurgiu, Mihàly Bencze-Romania

SP.476 If $x,y,z>0$ and $0\le\lambda\le\frac{1}{25}$ then:

$$\sum_{cyc}\frac{x}{\sqrt[3]{\lambda y^3+xyz+\lambda z^3}}\ge\frac{3}{\sqrt[3]{2\lambda+1}}$$

Proposed by Marin Chirciu-Romania

SP.477 If $a,b,c,d\ge1$ then:

$$\sqrt{a-1}+\sqrt{b-1}+\sqrt{c-1}+\sqrt{d-1}\le\sqrt{2(ab+cd)}$$

Proposed by Daniel Sitaru-Romania

SP.478 Let a, b, c be the sides lengths of ΔABC, $I-$incenter, $G-$centroid. If $IG \perp BC$ and $b \neq c$ then:

$$\frac{b}{c+a} + \frac{c}{a+b} + \frac{ab+bc+ca}{a^2+b^2+c^2} < \frac{13}{6}$$

Proposed by Florică Anastase-Romania

SP.479 In ΔABC the following relationship holds:

$$\frac{R^2}{r} \geq \frac{s}{16}\left(\frac{a}{w_b} + \frac{b}{w_a}\right)\left(\frac{b}{w_c} + \frac{c}{w_b}\right)\left(\frac{a}{w_c} + \frac{c}{w_a}\right) \geq \frac{8r^2}{R}$$

Proposed by Alex Szoros-Romania

SP.480 In ΔABC, $F-$area and the points $M \in (BC)$, $N \in (CA)$, $P \in (AB)$, then:

$$(a^2 + b \cdot BM)(b^2 + c \cdot CP)(c^2 + a \cdot AM) \geq 36\sqrt{3} \cdot F^3$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru-Romania

UNDERGRATUATE PROBLEMS

UP.466 Find:

$$\Omega = \int_0^{\pi} \left(\frac{x \cos x}{1 + \sin x}\right)^2 dx$$

Proposed by Florică Anastase-Romania

UP.467 Find:

$$\Omega = \int_0^{\pi} \frac{x^2 \cos^3 x}{(1 + \sin^2 x)^2} dx$$

Proposed by Florică Anastase-Romania

UP.468 Let $m \in \mathbb{N}$ and $0 < x < m$. If $i = \left[\frac{(m+1)x}{x+1}\right]$, prove that:

$$2\binom{m}{i} \geq x^{m-i},$$

where $[x]$ is integer part of x and $\binom{m}{i}$ is a binomial coefficient.

Proposed by Ovidiu Pop-Romania

UP.469 Prove that:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{(n!)^2}} < \frac{\pi^2 e}{6}$$

Proposed by Daniel Sitaru-Romania

UP.470 **If** $(a_n)_{n\geq 1}, a_0 > 0, a_{n+1} = (2n+1)a_n, \forall n \in \mathbb{N}^*$. **Find:**

$$\Omega = \lim_{n\to\infty} \left(\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n} \right)$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

UP.471 **Let** $m \geq 0$ **and** $H_n = \sum_{k=1}^{n} \frac{1}{k}, n \in \mathbb{N}^*$. **Find:**

$$\Omega(m) = \lim_{n\to\infty} \left(\left(\sqrt[n+1]{(n+1)!} \right)^{m+1} - \left(\sqrt[n]{n!} \right) \right) \cdot e^{-mH_n}$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

UP.472 **Find:**

$$\Omega = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} e^x \cdot \sin x (x + x \cot x + 1) dx$$

Proposed by D.M. Bătineţu-Giurgiu, Florică Anastase-Romania

UP.473 **Find:**

$$\Omega = \lim_{n\to\infty} \left(\sqrt[n+1]{\frac{((n+1)!)^2}{(2n+1)!!}} - \sqrt[n]{\frac{(n!)^2}{(2n-1)!!}} \right)$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru-Romania

UP.474 **If** $(a_n)_{n\geq 1}, a_n \in \mathbb{R}_+^* = (0, \infty), n \in \mathbb{N}^*$ **and** $\lim_{n\to\infty} \frac{a_{n+1}}{a_n \cdot \sqrt[n]{n!}} = a > 0$. **Find:**

$$\Omega(a) = \lim_{n\to\infty} \left(\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n} \right)$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru-Romania

UP.475 **Let** $(a_n)_{n\geq 1}$ **be sequence of real numbers such that** $a_1 = 1$ **and** $(n+1)^2(a_{n+1} - a_n) - (a_{n+1} + n + 1) = 0$.

$$\textbf{Find: } \Omega = \lim_{n\to\infty} \sqrt[n]{1 + a_n}$$

Proposed by Florică Anastase-Romania

UP.476 **If** $(a_n)_{n\geq 1}, a_n \in \mathbb{R}_+^* = (0, \infty), n \in \mathbb{N}^*$ **and** $\lim_{n\to\infty} \frac{a_{n+1}}{a_n \cdot \sqrt[n]{n!}} = a > 0$. **Find:**

$$\Omega(a) = \lim_{n\to\infty} \left(\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n} \right)$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru-Romania

UP.477 **If** f **is nonnegative function on** $[0, 1]$ **and** $f'(x) \geq 1$ **then:**

$$\int_0^x f^n(t) dt \geq x^{n-3} \left(\int_0^x f(t) dt \right)^{n-1}; n \in \mathbb{N}, n \geq 3$$

Proposed by Florică Anastase-Romania

UP.478 If f is nonnegative function on $[0,1]$ and $f'(x) \geq 1$ then:

$$\int_0^x f^n(t)dt \geq x^{n-3}\left(\left(\int_0^x f(t)dt\right)^{n-1} + f^2(0)\left(\int_0^x f(t)dt\right)^{n-2}\right);$$

$n \in \mathbb{N}, n \geq 3$

Proposed by Florică Anastase-Romania

UP.479 Let m_a, m_b, m_c be the lengths of the medians of a triangle ABC with circumradius R and inradius r. Let r_a, r_b, r_c be the exradii of the triangle. Prove that:

$$72\frac{r^4}{R^3} \leq \frac{m_a^2}{r_a} + \frac{m_b^2}{r_b} + \frac{m_c^2}{r_c} \leq \frac{9}{8}\frac{R^{-}8r^4}{r^3}$$

Proposed by George Apostolopoulos-Greece

UP.480 If $(a_n)_{n\geq1}, (b_n)_{n\geq1}$ are the sequences of real numbers with $a_n \neq a_{n+1}, b_n \neq b_{n+1}, n \geq 1$ with $\lim\limits_{n\to\infty} a_n = a \in \mathbb{R}$, $\lim\limits_{n\to\infty} b_n = b \in \mathbb{R}$, $\lim\limits_{n\to\infty}(n(a_{n+1} - a_n)) = c \in \mathbb{R}$, $\lim\limits_{n\to\infty}(n(b_{n+1} - b_n)) = d \in \mathbb{R}$ and $f, g : \mathbb{R} \to \mathbb{R}$ are differentiable functions with continuous derivative on $\mathbb{R}$, then find in terms of a, b, c, d:

$$\Omega = \lim_{n\to\infty}(n(f(a_{n+1})g(b_{n+1}) - f(a_n)g(b_n)))$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu-Romania

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