A MAGICAL DOUBLE SERIES LEADING TO A MAGICAL CLOSED-FORM

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Show by series manipulations and no use of integrals that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{H_j}{i(i+j+1)(i+j+n+1)}=\frac{H_n^3+3H_nH_n^{(2)}+2H_n^{(3)}}{3n},$$

where $H_n^{(m)}=1+\frac{1}{2^m}+\cdots+\frac{1}{n^m},\ m\geq 1$, represents the nth generalized harmonic number of order m.

Proof. Reversing the order of summation and then reindexing, the left-hand side of the main result may be written as

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{H_j}{i(i+j+1)(i+j+n+1)}=\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\frac{H_j}{i(i+j+1)(i+j+n+1)}$$

$$=\sum_{j=1}^{\infty}\sum_{i=j+1}^{\infty}\frac{H_j}{(i-j)(i+1)(i+n+1)}=\sum_{i=1}^{\infty}\frac{1}{(i+1)(i+n+1)}\sum_{j=1}^{i-1}\frac{H_j}{i-j}$$

$$=\sum_{i=1}^{\infty}\frac{H_i^2-H_i^{(2)}}{(i+1)(i+n+1)}=\frac{H_n^3+3H_nH_n^{(2)}+2H_n^{(3)}}{3n},$$

where in the calculations I used the following results, that is $\displaystyle\sum_{k=1}^{n-1}\frac{H_k}{n-k}=H_n^2-H_n^{(2)}$, and then the fact that $\displaystyle\sum_{k=1}^{\infty}\frac{H_k^2-H_k^{(2)}}{(k+1)(k+n+1)}=\frac{H_n^3+3H_nH_n^{(2)}+2H_n^{(3)}}{3n}$, which are both presented and calculated in the book **(Almost) Impossible Integrals, Sums, and Series**, pages 287 and 291.

Q.E.D. □