A trick to solve some infinite sums.

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Let be $f(x)$ an analytic function and its reciprocal which is locally analytic where $f(x)$ is non-zero, whose expansion in series of powers is:

$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$

$$D_x^1[f(x)] = D_x^1\left[\frac{1}{\frac{1}{f(x)}}\right] = \frac{-D_x^1\left[\frac{1}{f(x)}\right]}{\left(\frac{1}{f(x)}\right)^2}$$

$$D_x^1\left[\frac{1}{f(x)}\right] = -\left(\frac{1}{f(x)}\right)^2 D_x^1[f(x)] == -\left(\frac{1}{f(x)}\right)^2 D_x^1\left[\sum_{k=0}^{\infty} a_k x^k\right]$$

For some x we can therefore evaluate the infinite sum:

$$D_x^1\left[\sum_{k=0}^{\infty} a_k x^k\right] = -D_x^1\left[\frac{1}{f(x)}\right] \cdot \left(f(x)\right)^2 \qquad (i)$$

For example let's consider the function:

$$f(x) = \frac{\ln(1-x)}{x}$$

whose expansion, in its domain, in series of powers is:

$$f(x) = -\sum_{k=0}^{\infty} \frac{1}{k} x^{k-1} = -1 - \frac{x}{2} - \frac{x^2}{3} - \frac{x^3}{4} - \cdots$$

So

$$D_x^1[f(x)] = -\sum_{k=0}^{\infty} \frac{k+1}{k+2} x^k = -\frac{1}{2} - \frac{2x}{3} - \frac{3x^2}{4} - \cdots$$

Applying (i) we get

$$-\sum_{k=0}^{\infty}\frac{k+1}{k+2}x^k = = -D_x^1\left[\frac{x}{\ln(1-x)}\right]\cdot\left(\frac{\ln(1-x)}{x}\right)^2$$

$$\sum_0^{\infty}\frac{(k+1)x^k}{(k+2)} = \frac{\frac{x}{1-x}+\ln(1-x)}{\ln^2(1-x)}\cdot\left(\frac{\ln(1-x)}{x}\right)^2$$

and then finally

$$\sum_0^{\infty}\frac{(k+1)x^k}{(k+2)} = \frac{\frac{x}{1-x}+\ln(1-x)}{\mathrm{x}^2}$$

For some x we can therefore evaluate the infinite sum.

For example at $x=\frac{1}{e}$

$$\left(\frac{1}{2}+\frac{2}{3}e^{-1}+\frac{3}{4}e^{-2}+\frac{4}{5}e^{-3}+\cdots\right) = \frac{\frac{e^{-1}}{1-e^{-1}}+\ln(1-e^{-1})}{\ln^2(1-e^{-1})}\cdot\frac{\ln^2(1-e^{-1})}{e^{-2}}$$

$$\left(\frac{1}{2}e^{-2}+\frac{2}{3}e^{-3}+\frac{3}{4}e^{-4}+\frac{4}{5}e^{-5}+\cdots\right) = \frac{e^{-1}}{1-e^{-1}}+\ln(1-e^{-1})$$

$$\left(\frac{1}{2}e^{-2}+\frac{2}{3}e^{-3}+\frac{3}{4}e^{-4}+\frac{4}{5}e^{-5}+\cdots\right) = \frac{e^{-1}}{1-e^{-1}}+\ln(1-e^{-1})$$

$$\left(\frac{1}{2}e^{-2}+\frac{2}{3}e^{-3}+\frac{3}{4}e^{-4}+\frac{4}{5}e^{-5}+\cdots\right) = \frac{1}{e-1}+\ln(1-e^{-1}) = \frac{1+\ln(1-e^{-1})^{e-1}}{e-1}$$