ABOUT A FEW INEQUALITIES IN TRIANGLE

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In [1]. Arkady M. Alt has proved the following inequality:

$$\boldsymbol{(a^2+2)(b^2+2)(c^2+2) \geq 9(ab+bc+ca);\ \forall a,b,c>0;\ (1)}$$

In ΔABC with $F-$area, holds:

$$\boldsymbol{(a^2+2)(b^2+2)(c^2+2) \geq 36\sqrt{3}\cdot F;\ (2)}$$

Proof. If in (1) $a,b,c-$are the lengths sides of a triangle ABC, then:

$$(a^2+2)(b^2+2)(c^2+2) \geq 9(ab+bc+ca) \overset{Gordon}{\geq} 9\cdot 4\sqrt{3}F = 36\sqrt{3}\cdot F$$

Theorem 1. If $x,y,z>0$ then in ΔABC with $F-$area, the following relationship holds:

$$\boldsymbol{\left(\frac{x^2a^4}{(y+z)^2}+1\right)\left(\frac{y^2b^4}{(z+x)^2}+1\right)\left(\frac{z^2c^4}{(x+y)^2}+1\right) \geq 36\cdot F^2;\ (*)}$$

Theorem 2. If $x,y,z>0$ then in ΔABC with $F-$area, the following relationship holds:

$$\boldsymbol{\left(\frac{x^2a^8}{(y+z)^2}+1\right)\left(\frac{y^2b^8}{(z+x)^2}+1\right)\left(\frac{z^2c^8}{(x+y)^2}+1\right) \geq 48\cdot F^4;\ (**)}$$

To prove the above inequality, first, we prove the next Lemma:

Lemma: If $u,v,w>0$, then holds:

$$\boldsymbol{(u^2+1)(v^2+1)(w^2+1) \geq \frac{3}{4}(u+v+w)^2;\ (3)}$$

Proof. We have:

$(i):\ (u^2+1)(v^2+1) \geq (u+v)^2 \Leftrightarrow u^2v^2+u^2+v^2+1 \geq u^2+2uv+v^2 \Leftrightarrow$

$$u^2v^2-2uv+1 \geq 0 \Leftrightarrow (uv-1)^2 \geq 0 \text{ true!}$$

and

$(ii):\ (u^2+1)(v^2+1) \geq \frac{3}{4}\left((u+v)^2+1\right) \Leftrightarrow u^2v^2+u^2+v^2+1 \geq \frac{3}{4}(u^2+2uv+v^2+1) \Leftrightarrow$

$4u^2v^2+4u^2+4v^2+4 \geq 3u^2+3v^2+6uv+3 \Leftrightarrow 4u^2v^2-4uv+1+u^2-2uv+v^2 \geq 0 \Leftrightarrow$

$$(2uv-1)^2+(u-v)^2 \geq 0 \text{ true!}$$

Hence, we have:

$$(u^2+1)(v^2+1)(w^2+1) \overset{(ii)}{\geq} \frac{3}{4}\left((u+v)^2+1\right)(w^2+1) \overset{(i)}{\geq} \frac{3}{4}\left((u+v)+w\right)^2 \Leftrightarrow$$

$$(u^2+1)(v^2+1)(w^2+1) \geq \frac{3}{4}(u+v+w)^2$$

Proof of Theorem 1.

In Lemma, we take: $u = \frac{xa^2}{y+z}, v = \frac{yb^2}{z+x}, w = \frac{zc^2}{x+y}$, we get:

$$\prod_{cyc}\left(\frac{x^2a^4}{(y+z)^2}+1\right) \geq \frac{3}{4}\left(\sum_{cyc}\frac{x}{y+z}\cdot a^2\right)^2 \overset{Tsintsifas}{\geq}$$

$$\geq \frac{3}{4}\left(4\sqrt{3}\cdot F\right)^2 = \frac{3\cdot 16\cdot 3\cdot F^2}{4} = 36\cdot F^2$$

Proof of Theorem 2.

If in Lemma, we take: $u = \frac{xa^4}{y+z}, v = \frac{yb^4}{z+x}, w = \frac{zc^4}{x+y}$, we get:

$$\prod_{cyc}\left(\frac{x^2a^8}{(y+z)^2}+1\right) \geq \frac{3}{4}\left(\sum_{cyc}\frac{xa^4}{y+z}\right)^2 =$$

$$= \frac{3}{4}\left(\sum_{cyc}\frac{x^2a^4}{xy+xz}\right)^2 \overset{Bergstrom}{\geq} \frac{3}{4}\left(\frac{xa^2+yb^2+zc^2}{2(xy+yz+zx)}\right)^2 =$$

$$= \frac{3}{4}\frac{(xa^2+yb^2+zc^2)^4}{(2(xy+yz+zx))^2} = \frac{3}{16}\frac{(xa^2+yb^2+zc^2)^4}{(xy+yz+zx)^2} \geq$$

$$\overset{Oppenheim}{\geq} \frac{3\left(16(xy+yz+zx)F^2\right)^2}{16(xy+yz+zx)^2} = 3\cdot 16\cdot F^4 = 48\cdot F^4$$

REFERENCE:

[1]. **Alt M. Arkady**, ABOUT ONE INEQUALITY FROM A.P.M.O-2004-NEW SOLUTION AND GENERALIZATIONS-Octogon Mathematical Magazine, Vol. 27, No.1, April 2019, pag.228-232.

