ABOUT ONE INEQUALITY FROM A.P.M.O. 2004

BY D.M. BĂTINEŢU-GIURGIU, MIHÁLY BENCZE, DANIEL SITARU AND FLORICĂ ANASTASE-ROMANIA

In A.P.M.O. 2004 was proposed problem A.P.M.O.-2004/5:
If $a, b, c > 0$, then:

$$\boldsymbol{(a^2+2)(b^2+2)(c^2+2) \geq 3(ab+bc+ca);\ (*)}$$

This problem has generalized by Arkady M. Alt in [1] in the form:
If $x, y, z, t > 0$ then the following relationship holds:

$$\boldsymbol{(x^2+t^2)(y^2+t^2)(z^2+t^2) \geq \frac{3}{4}t^4(x+y+z)^2;\ (**)}$$

Next, we will to prove the following problem:
If $a, b, c, x, y, z, t > 0$ the following relationship holds:

$$\boldsymbol{(a^2+x^2t^2)(b^2+y^2t^2)(c^2+z^2t^2) \geq \frac{3}{4}t^4(ayz+bzx+cxy)^2;\ (***)}$$

To prove the above inequality, first, we prove the next Lemma:
Lemma: If $u, v, w > 0$, then holds:

$$\boldsymbol{(u^2+1)(v^2+1)(w^2+1) \geq \frac{3}{4}(u+v+w)^2;\ (1)}$$

Proof. We have:

$$(i):\ (u^2+1)(v^2+1) \geq (u+v)^2 \Leftrightarrow u^2v^2+u^2+v^2+1 \geq u^2+2uv+v^2 \Leftrightarrow$$

$$u^2v^2-2uv+1 \geq 0 \Leftrightarrow (uv-1)^2 \geq 0 \text{ true!}$$

and

$$(ii):\ (u^2+1)(v^2+1) \geq \frac{3}{4}\left((u+v)^2+1\right) \Leftrightarrow u^2v^2+u^2+v^2+1 \geq \frac{3}{4}(u^2+2uv+v^2+1) \Leftrightarrow$$

$$4u^2v^2+4u^2+4v^2+4 \geq 3u^2+3v^2+6uv+3 \Leftrightarrow 4u^2v^2-4uv+1+u^2-2uv+v^2 \geq 0 \Leftrightarrow$$

$$(2uv-1)^2+(u-v)^2 \geq 0 \text{ true!}$$

Hence, we have:

$$(u^2+1)(v^2+1)(w^2+1) \overset{(ii)}{\geq} \frac{3}{4}\left((u+v)^2+1\right)(w^2+1) \overset{(i)}{\geq} \frac{3}{4}\left((u+v)+w\right)^2 \Leftrightarrow$$

$$(u^2+1)(v^2+1)(w^2+1) \geq \frac{3}{4}(u+v+w)^2$$

Now, let's prove the inequality $(**)$, we have:

$$(a^2+x^2t^2)(b^2+y^2t^2)(c^2+z^2t^2) =$$

$$= x^2y^2t^2t^6\left(\left(\frac{a}{xt}\right)^2+1\right)\left(\left(\frac{b}{yt}\right)^2+1\right)\left(\left(\frac{c}{zt}\right)^2+1\right);\ (2)$$

If we take in (1):$u=\frac{a}{xt}, v=\frac{b}{yt}, w=\frac{c}{zt}$, we get:

$$\left(\left(\frac{a}{xy}\right)^2+1\right)\left(\left(\frac{b}{yt}\right)^2+1\right)\left(\left(\frac{c}{zt}\right)^2+1\right)\geq$$

$$\geq\frac{3}{4}\left(\frac{a}{xt}+\frac{b}{yt}+\frac{c}{zt}\right)^2=$$

$$=\frac{3}{4t^2}\left(\frac{a}{x}+\frac{b}{y}+\frac{c}{z}\right)^2=\frac{3}{4t^2x^2y^2z^2}(ayz+bzx+cxy)^2;\ (3)$$

From (2) and (3), it follows:

$$(a^2+x^2t^2)(b^2+y^2t^2)(c^2+z^2t^2)\geq x^2y^2z^2t^6\cdot\frac{3}{4t^2x^2y^2z^2}(ayz+bzx+cxy)^2=$$

$$=\frac{3}{4}t^4(ayz+bzx+cxy)^2$$

If in $(***)$ we take $x=y=z$ we get:

$$(a^2+x^2t^2)(b^2+x^2t^2)(c^2+x^2t^2)\geq\frac{3}{4}t^4(ax^2+by^2+cz^2)^2=$$

$$=\frac{3}{4}t^4x^4(a+b+c)^2;\ (4)$$

If we take $x=1$ in (4), we obtain:

$$(a^2+t^2)(b^2+t^2)(c^2+t^2)\geq\frac{3}{4}t^4(a+b+c)^2;\ \text{(A.A.)}$$

i.e. inequality (4) from [1].
If we take $t=\sqrt{2}$ in A.A. we obtain:

$$(a^2+2)(b^2+2)(c^2+2)\geq\frac{3}{4}(\sqrt{2})^4(a+b+c)^2=3(a+b+c)^2\geq 9(ab+bc+ca)$$

i.e. the inequality $(*)$.

REFERENCE:

[1]. **Alt M. Arkady**, ABOUT ONE INEQUALITY FROM A.P.M.O-2004-NEW SOLUTION AND GENERALIZATIONS-Octogon Mathematical Magazine, Vol. 27, No.1, April 2019, pag.228-232.

 ©