ABOUT AN INEQUALITY FROM A.M.M.

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Let $A_1A_2...A_n$ be a convex polygon, $n \geq 3$ with area F and $a_k = A_kA_{k+1}, A_{n+1} = A_1$, $k \in \overline{1,n}$ lengths of the sides . In the Problem 1634 from A.M.M., Volume 70 (1963), E. Just and N. Shaumberger has proved that:

$$a_1^2 + a_2^2 + ... + a_n^2 \geq 4F \cdot \tan\frac{\pi}{n}; \text{ (J-S)}$$

Let $x_k \in \mathbb{R}_+^* = (0, \infty)$ such that

$$\sum_{k=1}^{n} x_k a_k = t \cdot F; \ (1)$$

where $t \in \mathbb{R}_+^*$, then:

$$\sum_{k=1}^{n} \frac{a_k^{m+2}}{x_k^m} \geq \frac{4^{m+1}}{t^m} \cdot F \cdot \tan^{m+1}\frac{\pi}{n}, \ (\forall) m \in \mathbb{R}_+; \ (*)$$

Proof.

$$\sum_{k=1}^{n} \frac{a_k^{m+2}}{x_k^m} = \sum_{k=1}^{n} \frac{a_k^{2m+2}}{(x_k a_k)^m} = \sum_{k=1}^{n} \frac{(a_k^2)^{m+1}}{(x_k a_k)^m} \overset{J.Radon}{\geq} \frac{\left(\sum\limits_{k=1}^{n} a_k^2\right)^{m+1}}{\left(\sum\limits_{k=1}^{n} x_k a_k\right)^m} \overset{(1)}{=}$$

$$= \frac{1}{t^m F^m}\left(\sum_{k=1}^{n} a_k^2\right)^{m+1} \overset{(J-S)}{\geq} \frac{1}{t^m F^m}\left(4F \cdot \tan\frac{\pi}{m}\right)^{m+1} =$$

$$= \frac{4^{m+1}F^{m+1}}{t^m F^m} \cdot \tan^{m+1}\frac{\pi}{n} = \frac{4^{m+1}}{t^m} \cdot F \cdot \tan^{m+1}\frac{\pi}{n}$$

Let M be internal point of the convex polygon $A_1A_2...A_n$, $n \geq 3$ and d_k, $k \in \overline{1,n}$ distances of M to the lines $A_kA_{k+1}, k \in \overline{1,n}$, then $\sum\limits_{k=1}^{n} a_k d_k = 2F$ and taking $x_k = d_k$ in $(*), k \in \overline{1,n}$, we get $t = 2$. So,

$$\sum_{k=1}^{n} \frac{a_k^{m+2}}{d_k^m} \geq \frac{4^{m+1}}{2^m} \cdot F \cdot \tan^{m+1}\frac{\pi}{n}; \ (2)$$

and if we take $m = 0$ in (2), we get:

$$\sum_{k=1}^{n} a_k^2 \geq 4F \cdot \tan\frac{\pi}{n}; \text{ (J-S)}$$

i.e. we find it the inequality of E. Just, N. Schaumberger.
If in (J-S) we take $m = 3$, in triangle ABC with $F-$area, the following relationship holds:

$$a^2 + b^2 + c^2 \geq 4F \cdot \tan \frac{\pi}{3} = 4\sqrt{3} \cdot F; \ (I - W)$$

i.e. Ionescu-Weitzenbock's inequality.

REFERENCE:
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