Solution to an Integral of a Hypergeometric Function from Srinivasa Raghava (AIRMC), India

Abdulhafeez A. Abdulsalam

Department of Mathematics, University of Ibadan, Ibadan, Nigeria

aabdulsalam030@stu.ui.edu.ng, hafeez147258369@gmail.com

17 Jumadal Awwal 1443AH (22 December 2021)

$$
\begin{aligned}
\int_0^\infty e^{-x}{}_2F_1\left(\frac{1}{2},\frac{1}{3};\frac{1}{4};e^{-x}\right)\mathrm{d}x &= \int_0^\infty \sum_{k=0}^\infty \frac{\left(\frac{1}{2}\right)_k\left(\frac{1}{3}\right)_k}{\left(\frac{1}{4}\right)_k}\frac{1}{k!}e^{-(k+1)x}\,\mathrm{d}x \\
&= \sum_{k=0}^\infty \frac{\left(\frac{1}{2}\right)_k\left(\frac{1}{3}\right)_k}{\left(\frac{1}{4}\right)_k k!}\int_0^\infty e^{-(k+1)x}\,\mathrm{d}x \\
&= \sum_{k=0}^\infty \frac{\left(\frac{1}{2}\right)_k\left(\frac{1}{3}\right)_k}{\left(\frac{1}{4}\right)_k (k+1)!} \\
&= \sum_{k=-1}^\infty \frac{\left(\frac{1}{2}\right)_k\left(\frac{1}{3}\right)_k}{\left(\frac{1}{4}\right)_k (k+1)!} - \frac{\left(\frac{1}{2}\right)_{-1}\left(\frac{1}{3}\right)_{-1}}{\left(\frac{1}{4}\right)_{-1}} \\
&= \frac{-\frac{3}{4}}{\left(-\frac{1}{2}\right)\cdot\left(-\frac{2}{3}\right)}\sum_{k=0}^\infty \frac{\left(-\frac{1}{2}\right)_k\left(-\frac{2}{3}\right)_k}{\left(-\frac{3}{4}\right)_k k!} - \frac{\Gamma\left(-\frac{1}{2}\right)\Gamma\left(-\frac{2}{3}\right)}{\Gamma\left(-\frac{3}{4}\right)}\cdot\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{3}\right)} \\
&= -\frac{9}{4}\sum_{k=0}^\infty \frac{\left(-\frac{1}{2}\right)_k\left(-\frac{2}{3}\right)_k}{\left(-\frac{3}{4}\right)_k k!} - \frac{\Gamma\left(-\frac{1}{2}\right)\Gamma\left(-\frac{2}{3}\right)}{\Gamma\left(-\frac{3}{4}\right)}\cdot\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{3}\right)} \\
&= \frac{-9}{4\beta\left(-\frac{1}{12},\frac{-2}{3}\right)}\int_0^1 \frac{t^{-\frac{5}{3}}(1-t)^{-\frac{13}{12}}}{(1-t)^{-\frac{1}{2}}} - \\
&\qquad \frac{\Gamma\left(-\frac{1}{2}\right)\Gamma\left(-\frac{2}{3}\right)}{\Gamma\left(-\frac{3}{4}\right)}\cdot\frac{-\frac{3}{4}\Gamma\left(-\frac{3}{4}\right)}{-\frac{1}{2}\Gamma\left(-\frac{1}{2}\right)\cdot-\frac{2}{3}\Gamma\left(-\frac{2}{3}\right)} \\
&= \frac{9}{4} - \frac{9\Gamma\left(-\frac{3}{4}\right)}{4\Gamma\left(-\frac{1}{12}\right)\Gamma\left(\frac{-2}{3}\right)}\int_0^1 t^{-\frac{5}{3}}(1-t)^{-\frac{7}{12}} \\
&= \frac{9}{4} - \frac{9\Gamma\left(-\frac{3}{4}\right)}{4\Gamma\left(-\frac{1}{12}\right)\Gamma\left(\frac{-2}{3}\right)}\int_0^1 t^{-\frac{2}{3}-1}(1-t)^{\frac{5}{12}-1} \\
&= \frac{9}{4} - \frac{9\Gamma\left(-\frac{3}{4}\right)}{4\Gamma\left(-\frac{1}{12}\right)\Gamma\left(\frac{-2}{3}\right)}\cdot\beta\left(-\frac{2}{3},\frac{5}{12}\right) \\
&= \frac{9}{4} - \frac{9\Gamma\left(-\frac{3}{4}\right)}{4\Gamma\left(-\frac{1}{12}\right)\Gamma\left(\frac{-2}{3}\right)}\cdot\frac{\Gamma\left(-\frac{2}{3}\right)\Gamma\left(\frac{5}{12}\right)}{\Gamma\left(-\frac{1}{4}\right)}
\end{aligned}
$$

$$= \frac{9}{4} - \frac{9}{4} \cdot \frac{-\frac{4}{3} \cdot \frac{4}{1}\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{5}{12}\right)}{\Gamma\left(-\frac{1}{12}\right)\Gamma\left(\frac{-1}{4}\right)}$$
$$= \frac{9}{4} + \frac{12\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{5}{12}\right)}{\Gamma\left(-\frac{1}{12}\right)\Gamma\left(\frac{-1}{4}\right)}$$

By the Multiplication theorem

$$\Gamma\left(\frac{5}{12}\right)\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{13}{12}\right) = 2\pi \cdot 3^{-\frac{3}{4}}\Gamma\left(\frac{5}{4}\right)$$

$$\Longrightarrow \frac{\Gamma\left(\frac{5}{12}\right)}{\Gamma\left(-\frac{1}{12}\right)} = \frac{2\pi \cdot 3^{-\frac{3}{4}}\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{13}{12}\right)\Gamma\left(-\frac{1}{12}\right)}$$
$$= \frac{2\pi \cdot 3^{-\frac{3}{4}}\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{3}{4}\right) \cdot \frac{\pi}{\sin\left(-\frac{\pi}{12}\right)}}$$
$$= \frac{2\pi \cdot 3^{-\frac{3}{4}}\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\pi^2\sqrt{2}} \cdot \frac{1-\sqrt{3}}{2\sqrt{2}}$$
$$= \frac{3^{-\frac{3}{4}}\Gamma\left(\frac{5}{4}\right) \cdot 4 \cdot \frac{1}{4}\Gamma\left(\frac{1}{4}\right) \cdot \left(1-\sqrt{3}\right)}{2\pi}$$
$$= \frac{2 \cdot 3^{-\frac{3}{4}}\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{5}{4}\right)\left(1-\sqrt{3}\right)}{\pi}$$
$$= \frac{2 \cdot 3^{-\frac{3}{4}}\Gamma\left(\frac{5}{4}\right)^2\left(1-\sqrt{3}\right)}{\pi}$$
$$= \frac{2 \cdot 3^{-\frac{3}{4}}\Gamma\left(\frac{5}{4}\right)^2\left(1-\sqrt{3}\right)}{\pi}$$

$$\Longrightarrow \int_0^\infty e^{-x}{}_2F_1\left(\frac{1}{2}, \frac{1}{3}; \frac{1}{4}; e^{-x}\right) \mathrm{d}x = \frac{9}{4} + \frac{12 \cdot 2 \cdot 3^{-\frac{3}{4}}\Gamma\left(\frac{5}{4}\right)^3\left(1-\sqrt{3}\right)}{\pi\Gamma\left(-\frac{1}{4}\right)}$$
$$= \frac{9}{4} - \frac{4 \cdot 2 \cdot 3 \cdot 3^{-\frac{3}{4}}\Gamma\left(\frac{5}{4}\right)^3\left(-1+\sqrt{3}\right)}{\pi\Gamma\left(-\frac{1}{4}\right)}$$
$$= \frac{9}{4} - \frac{8 \cdot 3^{\frac{1}{4}}\Gamma\left(\frac{5}{4}\right)^3\left(-1+\sqrt{3}\right)}{\pi\Gamma\left(-\frac{1}{4}\right)}$$
$$= \frac{9}{4} - \frac{8\sqrt[4]{3}\left(\sqrt{3}-1\right)\Gamma\left(\frac{5}{4}\right)^3}{\pi\Gamma\left(-\frac{1}{4}\right)}$$
$$\therefore \int_0^\infty e^{-x}{}_2F_1\left(\frac{1}{2}, \frac{1}{3}; \frac{1}{4}; e^{-x}\right) \mathrm{d}x = \frac{9}{4} - \frac{8\sqrt[4]{3}\left(\sqrt{3}-1\right)\Gamma\left(\frac{5}{4}\right)^3}{\pi\Gamma\left(-\frac{1}{4}\right)}$$