ASMAT QATEA DEC, 2021

Solution to Triple Integral $A=\int_0^1\int_0^1\int_0^1\sqrt{x^2+y^2+z^2}\,dxdydz$

$$A=\int_0^1\int_0^1\int_0^1\sqrt{x^2+y^2+z^2}\,dxdydz \;\Rightarrow\; \frac{A}{3}=\int_0^1\int_0^1\int_0^1\frac{x^2}{\sqrt{x^2+y^2+z^2}}dxdydz$$

$$A=\int_0^1\int_0^1\int_0^1\sqrt{x^2+y^2+z^2}\,dxdydz \Rightarrow \begin{cases} dx=dv \;\Rightarrow x=v \\ u=\sqrt{x^2+y^2+z^2} \;\;\Rightarrow du=\dfrac{x}{\sqrt{x^2+y^2+z^2}}\,dx \end{cases}$$

$$A=\int_0^1\int_0^1 x\sqrt{x^2+y^2+z^2}\,\Big|_0^1 dydz-\underbrace{\int_0^1\int_0^1\int_0^1\frac{x^2}{\sqrt{x^2+y^2+z^2}}dxdydz}_{\frac{A}{3}}$$

$$\frac{4A}{3}=\int_0^1\int_0^1\sqrt{1+y^2+z^2}dydz=\int_0^1\int_0^1\frac{1}{\sqrt{1+y^2+z^2}}dydz+2\int_0^1\int_0^1\frac{y^2}{\sqrt{1+y^2+z^2}}dydz \quad ...(I)$$

$$\frac{4A}{3}=\int_0^1\int_0^1\sqrt{1+y^2+z^2}dydz \Rightarrow \begin{cases} dy=dv \;\Rightarrow y=v \\ u=\sqrt{1+y^2+z^2} \;\;\Rightarrow du=\dfrac{y}{\sqrt{1+y^2+z^2}}dy \end{cases}$$

$$\frac{4A}{3}=\int_0^1 y\sqrt{1+y^2+z^2}\,\Big|_0^1 dz-\int_0^1\int_0^1\frac{y^2}{\sqrt{1+y^2+z^2}}dydz$$

$$\frac{8A}{3}=2\int_0^1\sqrt{2+z^2}\,dz-2\int_0^1\int_0^1\frac{y^2}{\sqrt{1+y^2+z^2}}dydz \quad ...(II)$$

$$4A=2\int_0^1\sqrt{2+z^2}\,dz+\underbrace{\int_0^1\int_0^1\frac{1}{\sqrt{1+y^2+z^2}}dydz}_{P} \quad ...(I+II)$$

$$\int\sqrt{a^2+x^2}\,dx=\frac{x}{2}\sqrt{a^2+x^2}-\frac{a^2}{2}\ln\left(x+\sqrt{a^2+x^2}\right)+C$$

$$4A=\sqrt{3}+2\ln\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)+P$$

$$\int_0^1\sqrt{2+z^2}\,dz=\frac{\sqrt{3}}{2}+\ln\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)$$

$$P=2\ln(1+\sqrt{3})-\ln 2-\frac{\pi}{6}$$

$$4A=\sqrt{3}+2\ln(1+\sqrt{3})-\ln 2+2\ln(1+\sqrt{3})-\ln 2-\frac{\pi}{6}$$

$$A=\int_0^1\int_0^1\int_0^1\sqrt{x^2+y^2+z^2}\,dxdydz=\frac{\sqrt{3}}{4}+\ln\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)-\frac{\pi}{24}$$

Solution to P integral

$$P = \int_0^1 \int_0^1 \frac{1}{\sqrt{1+y^2+z^2}}\,dydz$$

$$\int \frac{dx}{\sqrt{a^2+x^2}} = \ln\left(x+\sqrt{a^2+x^2}\right) + C$$

$$P = \int_0^1 \int_0^1 \frac{1}{\sqrt{1+y^2+z^2}}\,dydz = \int_0^1 \left[\ln\left(y+\sqrt{1+y^2+z^2}\right)\right] dydz\,\Big|_0^1$$

$$P = \int_0^1 \ln\left(1+\sqrt{2+z^2}\right) dz - \frac{1}{2}\int_0^1 \ln(1+z^2)\,dz$$

$$\int \ln(1+x^2)\,dx = x\ln(1+x^2) + 2\tan^{-1}x - 2x + C$$

$$P = \underbrace{\int_0^1 \ln\left(1+\sqrt{2+z^2}\right) dz}_{B} - \frac{1}{2}\left(\ln 2 + \frac{\pi}{2} - 2\right)$$

$$B = \int_0^1 \ln\left(1+\sqrt{2+z^2}\right) dz \quad \Rightarrow \begin{cases} dz = dv \ \Rightarrow z = v \\ u = \ln(1+\sqrt{2+z^2}) \ \Rightarrow du = \dfrac{z}{2+z^2+\sqrt{2+z^2}} \end{cases}$$

$$B = \ln(1+\sqrt{3}) - \underbrace{\int_0^1 \frac{z^2}{2+z^2+\sqrt{2+z^2}}\,dz}_{C}$$

$$C = \int_0^1 \frac{z^2(z^2+2) - z^2\sqrt{2+z^2}}{(z^2+2)(z^2+1)}\,dz = \int_0^1 \frac{z^2}{z^2+1}\,dz - \int_0^1 \frac{z^2\sqrt{2+z^2}}{z^2+1}\,dz + \int_0^1 \frac{z^2\sqrt{2+z^2}}{z^2+2}\,dz$$

$$C = \int_0^1 \frac{z^2}{z^2+1}\,dz - \int_0^1 \frac{z^2\sqrt{2+z^2}}{z^2+1}\,dz + \int_0^1 \frac{z^2}{\sqrt{2+z^2}}\,dz$$

$$C = \underbrace{\int_0^1 \frac{z^2}{z^2+1}\,dz}_{N} - \int_0^1 \sqrt{2+z^2}\,dz + \underbrace{\int_0^1 \frac{\sqrt{2+z^2}}{z^2+1}\,dz}_{M} + \underbrace{\int_0^1 \frac{z^2}{\sqrt{2+z^2}}\,dz}_{S}$$

$$C = 1 - \frac{\pi}{4} - \frac{\sqrt{3}}{2} - \ln\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right) + \frac{\pi}{6} + \ln\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right) + \frac{\sqrt{3}}{2} - \ln\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)$$

$$C = 1 - \frac{\pi}{12} - \ln\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)$$

$$N = \int_0^1 \frac{x^2}{1+x^2} = \int_0^1 1dx - \int_0^1 \frac{1}{1+x^2}dx = 1 - \frac{\pi}{4}$$

$$S = \int_0^1 \frac{z^2}{\sqrt{2+z^2}}dz = \int_0^1 \sqrt{2+z^2}dz - 2\int_0^1 \frac{1}{\sqrt{2+z^2}}dz = \frac{\sqrt{3}}{2} + \ln(\frac{1+\sqrt{3}}{\sqrt{2}}) - 2\ln(z+\sqrt{2+z^2})\Big|_0^1$$

$$S = \int_0^1 \frac{z^2}{\sqrt{2+z^2}}dz = \frac{\sqrt{3}}{2} + \ln(\frac{1+\sqrt{3}}{\sqrt{2}}) - 2\ln(\frac{1+\sqrt{3}}{\sqrt{2}})$$

$$S = \int_0^1 \frac{z^2}{\sqrt{2+z^2}}dz = \frac{\sqrt{3}}{2} - \ln(\frac{1+\sqrt{3}}{\sqrt{2}})$$

$$M = \int_0^1 \frac{\sqrt{2+z^2}}{z^2+1}dz = \int_0^a \frac{\sqrt{2}\sec x}{(2\tan^2 x+1)}\sqrt{2}(\sec^2 x)\mathrm{dx}$$

$$z = \sqrt{2}\mathrm{tanx}$$

$$dz = \sqrt{2}(\tan^2\mathrm{x}+1)\mathrm{dx}$$

$$a = \tan^{-1}\frac{1}{\sqrt{2}}$$

$$M = \int_0^a \frac{2\sec^3 x}{2\tan^2 x+1}\mathrm{dx} = \int_0^a \frac{\sec x\,(1+(1+2\tan^2 x))}{2\tan^2 x+1}\mathrm{dx}$$

$$M = \int_0^a \frac{\sec x}{2\tan^2 x+1}\mathrm{dx} + \int_0^a \sec x\,dx$$

$$M = \int_0^a \frac{\cos x}{2\sin^2 x+\cos^2 x}\mathrm{dx} + [\ln(\sec x+\tan x)]_0^a = \underbrace{\int_0^a \frac{\cos x}{\sin^2 x+1}\mathrm{dx}}_{\sin x=u} + \ln(\sec a+\tan a)$$

$$M = \int_0^{\sin a} \frac{du}{u^2+1} + \ln(\frac{1+\sqrt{3}}{\sqrt{2}}) = \arctan(\sin a) + \ln(\frac{1+\sqrt{3}}{\sqrt{2}})$$

$$\sin(\arctan(\frac{1}{\sqrt{2}})) = \sqrt{1-\frac{1}{1+\frac{1}{2}}} = \sqrt{1-\frac{2}{3}} = \frac{1}{\sqrt{3}}$$

$$M = \int_0^1 \frac{\sqrt{2+z^2}}{z^2+1}dz = \frac{\pi}{6} + \ln(\frac{1+\sqrt{3}}{\sqrt{2}})$$

$$P = B - \frac{1}{2}\ln(2) - \frac{\pi}{4} + 1$$

$$B = \ln(1+\sqrt{3}) - C$$

$$B = \ln(1+\sqrt{3}) - 1 + \frac{\pi}{12} + \ln(1+\sqrt{3}) - \ln\sqrt{2}$$

$$C = 1 - \frac{\pi}{12} - \ln\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)$$

$$B = 2\ln(1+\sqrt{3}) - 1 + \frac{\pi}{12} - \frac{1}{2}\ln 2$$

$$P = 2\ln(1+\sqrt{3}) - 1 + \frac{\pi}{12} - \frac{1}{2}\ln 2 - \frac{1}{2}\ln(2) - \frac{\pi}{4} + 1$$

$$P = 2\ln(1+\sqrt{3}) - \ln 2 - \frac{\pi}{6}$$

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