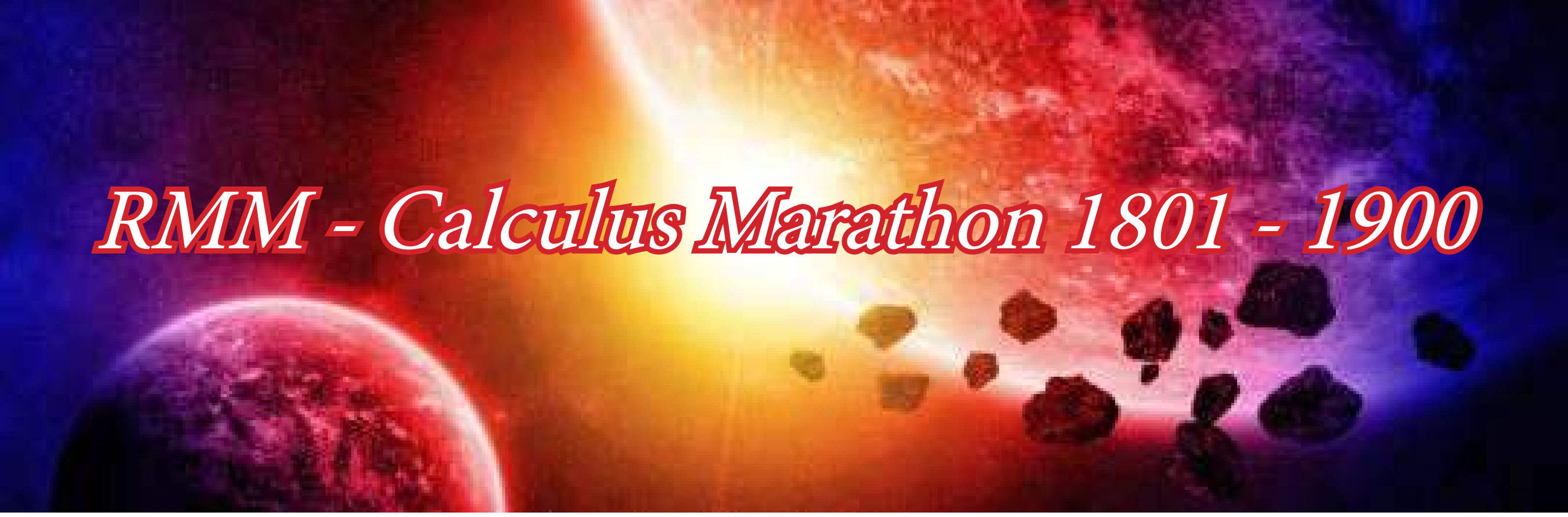

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1801. Calculate the following integral

$$\Omega = \int_0^1 \mathrm{Li}_2(x)\log(1+x)\,dx, \text{ where } \mathrm{Li}_2(x) - \text{polylogarithm function}$$

Proposed by Togrul Ehmedov-Baku-Azerbaijan

Solution 1 by proposer

$$\Omega = \int_0^1 \mathrm{Li}_2(x)\log(1+x)\,dx =$$

$$= \mathrm{Li}_2(x)\big((x+1)\log(x+1)-x\big)\Big]_{x=0}^{x=1} + \int_0^1 \frac{(x+1)\log(1+x)\log(1-x)}{x}dx$$

$$-\int_0^1 \log(1-x)\,dx =$$

$$= \zeta(2)(2\log(2)-1) + \int_0^1 \frac{\log(1+x)\log(1-x)}{x}dx + \int_0^1 \log(1+x)\log(1-x)\,dx$$

$$-\int_0^1 \log(1-x)\,dx =$$

$$= \zeta(2)(2\log(2)-1) + \left(-\frac{5}{8}\zeta(3)\right) + \left(2-2\log(2)-\zeta(2)+\log^2(2)\right) - (-1) =$$

$$= 3 + 2\zeta(2)\log(2) - 2\zeta(2) - 2\log(2) + \log^2(2) - \frac{5}{8}\zeta(3)$$

$$\text{Note: } \Omega_1 = \int_0^1 \frac{\log(1+x)\log(1-x)}{x}dx = -\frac{5}{8}\zeta(3)$$

$$\Omega_2 = \int_0^1 \log(1+x)\log(1-x)\,dx = 2 - 2\log(2) - \zeta(2) + \log^2(2)$$

Solution 2 by Syed Shahabudeen-Kerala-India

$$I = \int_0^1 Li_2(x)\log(1+x)\,dx = \int_0^1 Li_2(x)\sum_{k=1}^{\infty}\frac{(-1)^{k+1}x^k}{k}dx$$

$$= \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\int_0^1 x^k Li_2(x)\,dx =$$

$$=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\left(\frac{\zeta(2)}{k+1}-\frac{H_{k+1}}{(k+1)^2}\right)=\zeta(2)\underbrace{\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k(k+1)}}_{A}-\underbrace{\sum_{k=1}^{\infty}\frac{(-1)^{k+1}H_{k+1}}{k(k+1)^2}}_{B}$$

$$A=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k(k+1)}=\log 4-1$$

$$B=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}H_{k+1}}{k(k+1)^2}=\sum_{k=1}^{\infty}(-1)^{k+1}H_{k+1}\left(\frac{1}{k}-\frac{1}{k+1}-\frac{1}{(k+1)^2}\right)=$$

$$=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}H_k}{k}+\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k(k+1)}-\sum_{k=2}^{\infty}\frac{(-1)^{k}H_k}{k}-\sum_{k=2}^{\infty}\frac{(-1)^{k}H_k}{k^2}=$$

$$=-\frac{1}{2}\log^2 2+\frac{\zeta(2)}{2}+\log 4-1-\left(1-\frac{\zeta(2)}{2}+\frac{\log^2 2}{2}\right)-\left(1-\frac{5}{8}\zeta(3)\right)=$$

$$=\zeta(2)-\log^2 2+\log 4-3+\frac{5}{8}\zeta(3)$$

$$I=\zeta(2)\log 4-\zeta(2)-\zeta(2)+\log^2 2-\log 4+3-\frac{5}{8}\zeta(3)=$$

$$=\frac{\pi^2}{6}\log 4-\log 4+\log^2 2-\frac{\pi^2}{3}-\frac{5}{8}\zeta(3)+3$$

1802. $\mathrm{L(x)}=\frac{6}{\pi^2}\left(\mathrm{Li_2(x)}+\frac{1}{2}\log x\log(1-x)\right)$. **Find:**

$$\Omega=\int_0^1 \mathrm{L(x)Li_2(x)\,dx}$$

Proposed by Togrul Ehmedov-Azerbaijan

Solution by Artan Ajredini-Presheva-Serbie

We have:

$$\Omega=\int_0^1 L(x)Li_2(x)\,dx=$$

$$=\frac{6}{\pi^2}\left[\int_0^1 Li_2^2(x)\,dx+\frac{1}{2}\int_0^1 Li_2(x)\log x\log(1-x)\,dx\right]=\frac{6}{\pi^2}\left(\Omega_1+\frac{1}{2}\Omega_2\right)$$

$$\because Li_2^2(x)=4\sum_{n=1}^{\infty}\frac{H_n}{n^3}x^n+2\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^2}x^n-6Li_4(x)$$

Hence, we have:

$$\Omega_1\int_0^1\left(4\sum_{n=1}^{\infty}\frac{H_n}{n^3}x^n+2\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^2}x^n-6Li_4(x)\right)dx=$$

$$= 4\sum_{n=1}^{\infty}\frac{H_n}{n^3(n+1)} + 2\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^2(n+1)} - 6\int_0^1 Li_4(x)dx =$$

$$= 4\left(\sum_{n=1}^{\infty}\frac{H_n}{n^3} - \sum_{n=1}^{\infty}\frac{H_n}{n^2} + \sum_{n=1}^{\infty}\frac{H_n}{n(n+1)}\right) + 2\left(\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^2} - \sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n(n+1)}\right)$$

$$-6(\zeta(4)+\zeta(2)-\zeta(3)-1) =$$

$$= 4\left(\frac{5}{2}\zeta(4) - \frac{1}{2}\zeta^2(2) - 2\zeta(3) + \zeta(2)\right) + 2\left(\frac{7}{4}\zeta(4) - \zeta(3)\right)$$

$$-6(\zeta(4)+\zeta(2)-\zeta(3)-1) = -4\zeta(3) + 6 + \frac{5}{2}\zeta(4) - 2\zeta(2)$$

$$\Omega_2 = 5\zeta(2) + 4\zeta(3) - \frac{3}{4}\zeta(4) - 12$$

(see https://www.ssmrmh.ro/2021/08/23/integral-calculus-525/)

Hence,

$$\Omega = \frac{6}{\pi^2}\left[-4\zeta(3) + 6 + \frac{5}{2}\zeta(4) - 2\zeta(2) + \frac{1}{2}\left(5\zeta(2) + 4\zeta(3) - \frac{3}{4}\zeta(4) - 12\right)\right] =$$

$$= \frac{6}{\pi^2}\left(-2\zeta(3) + \frac{1}{2}\zeta(2) + \frac{17}{8}\zeta(4)\right)$$

1803. Calculate the following integral

$$I = \int_0^1 Li_2\left(\frac{x}{1-x}\right)\log(x)\log(1-x)\,dx$$

Where $Li_2(x)$ – polylogaritm function

Proposed by Togrul Ehmedov-Baku-Azerbaijan

Solution by proposer

$$I = \int_0^1 Li_2\left(\frac{x}{1-x}\right)\log(x)\log(1-x)\,dx =$$

$$= \int_0^1 \left(-\frac{1}{2}\log^2(1-x) - Li_2(x)\right)\log(x)\log(1-x)\,dx =$$

$$= -\frac{1}{2}\int_0^1 \log(x)\log^3(1-x)\,dx - \int_0^1 Li_2(x)\log(x)\log(1-x)\,dx = -\frac{1}{2}I_1 - I_2$$

$$I_1=\int_0^1 \log(x)\log^3(1-x)\,dx=\int_0^1 \log^3(x)\log(1-x)\,dx=-\sum_{k=1}^{\infty}\frac{1}{k}\int_0^1 x^k\log^3(x)\,dx$$

$$=6\sum_{k=1}^{\infty}\frac{1}{k(k+1)^4}=6\sum_{k=1}^{\infty}\left[\frac{1}{k}-\frac{1}{k+1}-\frac{1}{(k+1)^2}-\frac{1}{(k+1)^3}-\frac{1}{(k+1)^4}\right]$$

$$=6\big(4-\zeta(2)-\zeta(3)-\zeta(4)\big)$$

$$I_2=\int_0^1 \mathrm{Li}_2(x)\log(x)\log(1-x)\,dx=\int_0^1 \mathrm{Li}_2(1-x)\log(x)\log(1-x)\,dx$$

$$I_2=\frac{1}{2}\int_0^1 \big(\mathrm{Li}_2(x)+\mathrm{Li}_2(1-x)\big)\log(x)\log(1-x)\,dx=$$

$$=\frac{1}{2}\int_0^1 (\zeta(2)-\log(x)\log(1-x))\log(x)\log(1-x)\,dx=$$

$$=\frac{1}{2}\zeta(2)\int_0^1 \log(x)\log(1-x)\,dx-\frac{1}{2}\int_0^1 \log^2(x)\log^2(1-x)\,dx=\frac{1}{2}\zeta(2)I_3-\frac{1}{2}I_4$$

$$I_3=\int_0^1 \log(x)\log(1-x)\,dx=-\sum_{k=1}^{\infty}\frac{1}{k}\int_0^1 x^k\log(x)\,dx=\sum_{k=1}^{\infty}\frac{1}{k(k+1)^2}$$

$$=\sum_{k=1}^{\infty}\left[\frac{1}{k}-\frac{1}{k+1}-\frac{1}{(k+1)^2}\right]=2-\zeta(2)$$

$$I_4=\int_0^1 \log^2(x)\log^2(1-x)\,dx=$$

$$=-2\int_0^1 \log(x)\log^2(1-x)\,dx+2\int_0^1 \frac{x\log^2(x)\log(1-x)}{1-x}dx=$$

$$=-2\int_0^1 \log^2(x)\log(1-x)\,dx-2\int_0^1 \log^2(x)\log(1-x)\,dx+2\int_0^1 \frac{\log^2(x)\log(1-x)}{1-x}dx$$

$$=-4\int_0^1 \log^2(x)\log(1-x)\,dx+2\int_0^1 \frac{\log^2(x)\log(1-x)}{1-x}dx=-4I_5+2I_6$$

$$I_5=\int_0^1 \log^2(x)\log(1-x)\,dx=-\sum_{k=1}^{\infty}\frac{1}{k}\int_0^1 x^k\log^2(x)\,dx=-2\sum_{k=1}^{\infty}\frac{1}{k(k+1)^3}=$$

$$=-2\sum_{k=1}^{\infty}\left[\frac{1}{k}-\frac{1}{k+1}-\frac{1}{(k+1)^2}-\frac{1}{(k+1)^3}\right]=-2\big(3-\zeta(2)-\zeta(3)\big)$$

$$I_6=\int_0^1\frac{\log^2(x)\log(1-x)}{1-x}dx=-\sum_{k=1}^{\infty}H_k\int_0^1 x^k\log^2(x)\,dx=-2\sum_{k=1}^{\infty}\frac{H_k}{(k+1)^3}=$$

$$=-2\left[\sum_{k=1}^{\infty}\frac{H_k}{k^3}-\sum_{k=1}^{\infty}\frac{1}{k^4}\right]=-2\left[\frac{5}{4}\zeta(4)-\zeta(4)\right]=-\frac{1}{2}\zeta(4)$$

$$I_4=-4I_5+2I_6=8\big(3-\zeta(2)-\zeta(3)\big)-\zeta(4)$$

$$I_2=\frac{1}{2}\zeta(2)I_3-\frac{1}{2}I_4=\frac{1}{2}\zeta(2)\big(2-\zeta(2)\big)-4\big(3-\zeta(2)-\zeta(3)\big)+\frac{1}{2}\zeta(4)=$$

$$=5\zeta(2)+4\zeta(3)-\frac{3}{4}\zeta(4)-12$$

$$I=-\frac{1}{2}I_1-I_2=-3\big(4-\zeta(2)-\zeta(3)-\zeta(4)\big)-\left(5\zeta(2)+4\zeta(3)-\frac{3}{4}\zeta(4)-12\right)=$$

$$=-2\zeta(2)-\zeta(3)+\frac{15}{4}\zeta(4)$$

1804. Prove that:

$$\int_0^{\frac{\pi}{4}}\tan x\,Li_2(\tan^2 x)dx=-\frac{5}{16}\zeta(3)+\frac{\pi^2}{12}\log(2)$$

where $Li_3(x)=\sum_{n\ge1}\frac{x^n}{n^3}$ is trilogarithmic function and $\zeta(3)=Li_2(1)$.

Proposed by Naren Bhandari-Bajura- Nepal

Solution by Togrul Ehmedov-Azerbaijan

$$I=\int_0^{\frac{\pi}{4}}\tan x\,Li_2(\tan^2x)dx\Bigg]_{\tan^2x=z}=\frac{1}{2}\int_0^1\frac{Li_2(z)}{1+z}dz=-\int_0^1\int_0^1\frac{x\log(y)}{(1-zy)(1+z)}dy\,dz=$$

$$=-\frac{1}{2}\int_0^1\log(y)\int_0^1\frac{z}{(1-zy)(1+z)}dz\,dy=-\frac{1}{2}\int_0^1\frac{\log(y)}{1+y}\int_0^1\left[\frac{1}{1-zy}-\frac{1}{1+z}\right]dz\,dy=$$

$$=-\frac{1}{2}\int_0^1\frac{\log(y)}{1+y}\left[-\frac{\log(1-zy)}{y}-\log(1+z)\right]_{z=0}^{z=1}dy$$

$$=-\frac{1}{2}\int_0^1\frac{\log(y)}{1+y}\left[-\frac{\log(1-y)}{y}-\log(2)\right]dy=$$

$$=\frac{1}{2}\int_0^1\frac{\log(y)\log(1-y)}{y(1+y)}dy+\frac{1}{2}\log(2)\int_0^1\frac{\log(y)}{1+y}dy=$$

$$=\frac{1}{2}\int_0^1\frac{\log(y)\log(1-y)}{y}dy-\frac{1}{2}\int_0^1\frac{\log(y)\log(1-y)}{1+y}dy+\frac{1}{2}\log(2)\int_0^1\frac{\log(y)}{1+y}dy=$$

$$=\frac{1}{2}\zeta(3)-\frac{1}{2}\left(\frac{13}{8}\zeta(3)-\frac{\pi^2}{4}\log(2)\right)+\frac{1}{2}\log(2)\left(-\frac{1}{2}\zeta(2)\right)=$$

$$= -\frac{5}{16}\zeta(3) + \frac{\pi^2}{8}\log(2) - \frac{1}{4}\log(2)\zeta(2) = -\frac{5}{16}\zeta(3) + \frac{\pi^2}{12}\log(2)$$

$$\text{Note: } \int_0^1 \frac{\log(y)\log(1-y)}{y}dy = \zeta(3)$$

$$\int_0^1 \frac{\log(y)\log(1-y)}{1+y}dy = \frac{13}{8}\zeta(3) - \frac{\pi^2}{4}\log(2)$$

1805. $L(x) = \frac{6}{\pi^2}\left(Li_2(x) + \frac{1}{2}\log x \log(1-x)\right)$. **Find:**

$$\Omega = \int_0^1 \frac{\log x}{x} \cdot \log(1-x) \cdot L(x)\, dx$$

Proposed by Togrul Ehmedov-Azerbaijan

Solution by Artan Ajredini-Presheva-Serbie

$$\Omega = \int_0^1 \frac{\log x}{x} \cdot \log(1-x) \cdot L(x)\, dx =$$

$$= \frac{6}{\pi^2}\left[\int_0^1 Li_2(x)\frac{\log x \log(1-x)}{x}dx + \frac{1}{2}\int_0^1 \frac{\log^2 x \log^2(1-x)}{x}dx\right] = \frac{6}{\pi^2}\left(\Omega_1 + \frac{1}{2}\Omega_2\right)$$

Integrating by parts, we obtain:

$$\Omega_1 = \underbrace{-\log x\, Li_2^2(x)\Big|_0^1}_{0} + \int_0^1 \frac{Li_2^2(x)}{x}dx - \int_0^1 \frac{\log x \log(1-x)}{x}Li_2(x)dx$$

$$= \int_0^1 \frac{Li_2^2(x)}{x}dx - \Omega_1$$

Therefore,

$$2\Omega_1 = \int_0^1 \frac{Li_2^2(x)}{x}dx = \int_0^1 \frac{1}{x}\left[4\sum_{n=1}^{\infty}\frac{H_n}{n^3}x^n + 2\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^2}x^n - 6Li_4(x)\right]dx$$

$$= 4\sum_{n=1}^{\infty}\frac{H_n}{n^4} + 2\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^3} - 6\int_0^1 \frac{Li_4(x)}{x}dx =$$

$$= 4(3\zeta(5) - \zeta(2)\zeta(3)) + 2\left(3\zeta(2)\zeta(3) - \frac{9}{2}\zeta(5)\right) - 6\zeta(5) = 2\zeta(2)\zeta(3) - 3\zeta(5).$$

Also, we have:

$$\Omega_2 = \int_0^1 \frac{\log^2 x \log^2(1-x)}{x}dx = \lim_{\substack{u\to 0\\ v\to 1}} \frac{\partial^4}{\partial^2 u \partial^2 v}B(u,v) = 8\zeta(5) - 4\zeta(2)\zeta(3)$$

Finally,

$$\Omega = \frac{6}{\pi^2}\left(\Omega_1 + \frac{1}{2}\Omega_2\right) = \frac{6}{\pi^2}\left[\zeta(2)\zeta(3) - \frac{3}{2}\zeta(5) + 4\zeta(5) - 2\zeta(2)\zeta(3)\right] =$$

$$= \frac{6}{\pi^2}\left[\frac{5}{2}\zeta(5) - \zeta(2)\zeta(3)\right]$$

1806.

If we have the integral for $a > 0, \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\cos^2(4x)+1}{a\cos^2(2x)+1}dx = a$ then evaluate the expression:

$$\Omega = \sqrt{\sqrt{\frac{1}{3}(a^5 + a^4 + 8a^3 + 8a^2 - 32a)}}$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Rana Ranino-Setif-Algerie

$$a = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\cos^2(4x)+1}{a\cos^2(2x)+1}dx = \frac{8}{\pi}\int_0^{\frac{\pi}{4}} \frac{1+\cos^2(4x)}{1+a\cos^2(2x)}dx \overset{4x\to x}{=}$$

$$= \frac{2}{\pi}\int_0^{\infty} \frac{1+\cos^2 x}{1+a\cos^2\left(\frac{x}{2}\right)}dx\,; \left(\begin{cases} y = \tan\frac{x}{2} \\ a = b^2 - 1 \end{cases}\right)$$

$$b^2 - 1 = \frac{8}{\pi}\int_0^{\infty} \frac{y^4+1}{(y^2+b^2)(1+y^2)^2}dy$$

$$= \frac{8}{\pi(b^2-1)^2}\int_0^{\infty}\left(\frac{b^4+1}{b^2+y^2} - \frac{2b^2}{1+y^2} + \frac{2(b^2-1)}{(1+y^2)^2}\right)dy$$

$$b^2 - 1 = \frac{4(b^4 - b^3 - b + 1)}{(b^2-1)^2} = \frac{4(b^2+b+1)}{b(b+1)^2} \Rightarrow b^5 + 2b^4 - 6b^2 - 5b - 4 = 0$$

$$\Rightarrow 2b^4 - 2b^2 - 4 = b(5 - b^4)$$

$$b = \frac{2b^4 - 6b^2 - 4}{5 - b^4} \Rightarrow \sqrt{a+1} = \frac{2a^2 - 2a - 8}{4 - 2a - a^2}$$

$$a + 1 = \frac{4a^4 - 8a^3 - 28a^2 + 32a + 64}{a^4 + 4a^3 - 4a^2 - 16a + 16} \Rightarrow a^5 + a^4 + 8a^3 + 8a^2 - 32a = 48$$

Therefore,

$$\Omega=\sqrt{\sqrt{\frac{1}{3}(a^5+a^4+8a^3+8a^2-32a)}}=2$$

1807. Find a closed form:

$$\Omega=\int_0^{\frac{\pi}{12}}x(\tan x+\cot x)\,dx$$

Proposed by Togrul Ehmedov-Azerbaijan

Solution by Rana Ranino-Setif-Algerie

$$\Omega=\int_0^{\frac{\pi}{12}}\frac{2x}{\sin 2x}dx=x\log(\tan x)|_0^{\frac{\pi}{12}}-\int_0^{\frac{\pi}{12}}\log(\tan x)\,dx=\frac{\pi\log(2-\sqrt{3})}{12}-I$$

$$I=-2\sum_{n=0}^{\infty}\frac{1}{2n+1}\int_0^{\frac{\pi}{12}}\cos(2(2n+1)x)\,dx=-\sum_{n=0}^{\infty}\frac{\sin\left(\frac{(2n+1)\pi}{6}\right)}{(2n+1)^2}=$$

$$=-\frac{\frac{1}{2}}{1^2}-\frac{1}{3^2}-\frac{\frac{1}{2}}{5^2}+\frac{\frac{1}{2}}{7^2}+\frac{1}{9^2}+\frac{\frac{1}{2}}{11^2}-\cdots$$

$$=-\frac{1}{2}\underbrace{\left(\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\frac{1}{9^2}-\frac{1}{11^2}+\cdots\right)}_{G}-\frac{3}{2}\underbrace{\left(\frac{1}{3^2}-\frac{1}{9^2}+\frac{1}{15^2}-\cdots\right)}_{\frac{G}{9}}$$

$$I=-\left(\frac{1}{2}+\frac{1}{6}\right)G=-\frac{2G}{3}$$

$$\Omega=\int_0^{\frac{\pi}{12}}x(\tan x+\cot x)\,dx=\frac{\pi\log(2-\sqrt{3})+8G}{12}$$

1808. If $a,b,t\in\mathbb{R}, t-a^t=2$ and $\frac{\sqrt{2}}{16}(5t^2-18t+41)+a^2bi=$

$$=\sqrt{-t^4+6t^3-7t^2-6t+8}+\sqrt{2}e^{\frac{arccosh2}{2}}\left(\sqrt{2a^2b-a^3}-\sqrt{a^2b-a^3}\right)i$$

then find:

$$\Omega=\int_{a+b-t}^{\log b}\frac{\left(e^{(t+1)x}-e^{xt}\right)^{\frac{1}{a+b}}-\left(e^{\left(\frac{1}{b}+1\right)x}-e^{\frac{x}{b}}\right)^{t-1}}{\log(e^x-a)}dx$$

Proposed by Samir Cabiyev-Azerbaijan

Solution by proposer

$$\begin{cases}\frac{\sqrt{2}}{16}(5t^2-18t+41)=\sqrt{-t^4+6t^3-7t^2-6t+8};(1)\\ a^2bi=\sqrt{2}e^{\frac{arcch2}{2}}\left(\sqrt{2a^2b-a^3}-\sqrt{a^2b-a^3}\right);(2)\end{cases}$$

$$(1)\Rightarrow 5t^2-18t+41=8\sqrt{2}\sqrt{-t^4+6t^3-7t^2-6t+8}\Leftrightarrow$$

$$5t^2-18t+41=8\sqrt{(2t-2)(t+1)(4-t)(t-2)}$$

$$\because 5t^2-18t+41=(9t^2+6t+1)-4t^2+40-24t\Rightarrow$$

$$9t^2+6t+1=4t^2+24t-40+8\sqrt{(2t-2)(t+1)(4-t)(t-2)}$$

$$9t^2+6t+1=$$

$$=4\left((2t^2-2)+\left((4t-8)+(2t-t^2)\right)+2\sqrt{(2t-2)(t+1)(4-t)(t-2)}\right)$$

$$9t^2+6t+1=$$

$$4\left(\left(\sqrt{2t-2}\cdot\sqrt{t+1}\right)^2+\left(\sqrt{4-t}\cdot\sqrt{t-2}\right)^2+2\sqrt{(2t-2)(t+1)(4-t)(t-2)}\right)$$

$$9t^2+6t+1=\left(2\left(\sqrt{(2t-2)(t+1)}+\sqrt{(4-t)(t-2)}\right)\right)^2$$

$$3t+1=2\left(\sqrt{(2t-2)(t+1)}+\sqrt{(4-t)(t-2)}\right)$$

$$\left(\sqrt{2t-2}-\sqrt{t+1}\right)^2+\left(\sqrt{4-t}-\sqrt{t-2}\right)^2=0\Leftrightarrow$$

$$\begin{cases}\sqrt{2t-2}=\sqrt{t+1}\\ \sqrt{4-t}=\sqrt{t-2}\end{cases}\Rightarrow t=3$$

$$t-a^t=2\Rightarrow 3-a^3=2\Rightarrow a=1.$$

$$(2)\Rightarrow arccosh2=\log\left(2+\sqrt{2^2-1}\right)=\log(2+\sqrt{3})$$

$$\frac{\sqrt{2}e^{\log\sqrt{2+\sqrt{3}}}}{\sqrt{2a^2b-a^3}+\sqrt{a^2b-a^3}}=1\Leftrightarrow\sqrt{2a^2b-a^3}+\sqrt{a^2b-a^3}=1+\sqrt{3}$$

$$a^2b=u,a=1\Rightarrow\sqrt{2u-1}+\sqrt{u-1}=\sqrt{3}+1$$

$$u^2-(24+12\sqrt{3})u+\left(6+2\sqrt{3}\right)^2-4=0\Rightarrow u=2\Rightarrow a^2b=2\Rightarrow b=2$$

$$a=1,b=2,t=3\Rightarrow$$

$$\Omega = \int_{a+b-t}^{\log b} \frac{\left(e^{(t+1)x} - e^{xt}\right)^{\frac{1}{a+b}} - \left(e^{\left(\frac{1}{b}+1\right)x} - e^{\frac{x}{b}}\right)^{t-1}}{\log(e^x - a)} dx =$$

$$= \int_0^{\log 2} \frac{\left((e^x-1)^{\frac{1}{3}} - (e^x-1)^2\right)e^x dx}{\log(e^x-1)} \overset{e^x-1=z}{=} \int_0^1 \frac{z^{\frac{1}{3}} - z^2}{\log z} dz = 2\log\left(\frac{2}{3}\right)$$

1809. Find a closed form:

$$\Omega(a) = \int_0^\infty \frac{x\sqrt{x}}{(x^3+1)(1+a^2x^2)} dx, a \in \mathbb{R}$$

Proposed by Vasile Mircea Popa-Romania

Solution by proposer

$$\Omega(a) = \int_0^\infty \frac{x\sqrt{x}}{(x^3+1)(1+a^2x^2)} dx \overset{\sqrt{x}=t}{=} 2\int_0^\infty \frac{t^4}{(t^6+1)(1+a^2t^4)} dt$$

$$\text{Let } I(a) = 2\int_0^\infty \frac{t^4}{(t^6+1)(1+a^2t^4)} dt$$

At the beginning we consider the case $a \geq 0$. We have:

$$\frac{t^4}{(t^6+1)(1+a^2t^4)} = \frac{1}{3(a^2+1)} \cdot \frac{1}{t^2+1} + \frac{1}{3(a^4-a^2+1)} \cdot \frac{(2-a^2)t^2+2a^2-1}{t^4-t^2+1} +$$

$$+\frac{1}{a^6+1} \cdot \frac{-a^2t^2-a^4}{a^2t^4+1}$$

Let us denote:

$$P(t) = \int \frac{t^4}{(t^6+1)(1+a^2t^4)} dt = F(t) + C$$

where C is a arbitrary constant of integration.

Using the decomposition into simple fractions and performing the calculations, we have:

$$F(t) = \frac{1}{3(a^2+1)} \tan^{-1} t + \frac{\sqrt{3}(a^2-1)}{12(a^4-a^2+1)} \cdot \log\frac{t^2-\sqrt{3}t+1}{t^2-\sqrt{3}t+1} +$$

$$+\frac{a^2+1}{6(a^4-a^2+1)} \cdot \left[\tan^{-1}(\sqrt{3}+2t) - \tan^{-1}(\sqrt{3}-2t)\right] +$$

$$+\frac{\sqrt{2}\sqrt{a}(a^3-1)}{8(a^6+1)} \cdot \log\frac{at^2-\sqrt{2}\sqrt{a}t+1}{at^2+\sqrt{2}\sqrt{a}t+1} +$$

$$+\frac{\sqrt{2}\sqrt{a}(a^3+1)}{4(a^6+1)}\cdot\left[\tan^{-1}(1-\sqrt{2}\sqrt{a}t)-\tan^{-1}(1+\sqrt{2}\sqrt{a}t)\right]$$

We have:

$$I(a)=\lim_{t\to\infty}F(t)-F(0)$$

We obtain:

$$\lim_{n\to\infty}F(t)=\frac{4a^4+2a^2+4-3\sqrt{2}\sqrt{a}(1+a^3)}{12(a^6+1)}\cdot\pi;\ F(0)=0$$

$$I(a)=\frac{4a^4+2a^2+4-3\sqrt{2}\sqrt{a}(1+a^3)}{12(a^6+1)}\cdot\pi$$

$$\Omega(a)=\frac{4a^4+2a^2+4-3\sqrt{2}\sqrt{a}(1+a^3)}{6(a^6+1)}\cdot\pi$$

For the case $a<0$ we replace a with $-a$. So, the general expression of $\Omega(a)$ is:

$$\Omega(a)=\frac{4a^4+2a^2+4-3\sqrt{2}\sqrt{|a|}(1+|a|^3)}{6(a^6+1)}\cdot\pi$$

1810. Prove without any software:

$$\int_0^1\int_0^1\sqrt{1-\left(\frac{x+y}{2}\right)^2}\,dxdy>\frac{\pi}{4}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Adrian Popa-Romania

$$\frac{x+y}{2}\overset{AHQ}{\leq}\sqrt{\frac{x^2+y^2}{2}}\Leftrightarrow\left(\frac{x+y}{2}\right)^2\leq\frac{x^2+y^2}{2}$$

$$-\left(\frac{x+y}{2}\right)^2\geq-\frac{x^2+y^2}{2}\Leftrightarrow 1-\left(\frac{x+y}{2}\right)^2\geq 1-\frac{x^2+y^2}{2}$$

$$1-\left(\frac{x+y}{2}\right)^2=\frac{(1-x^2)+(1-y)^2}{2}$$

$$I\geq\int_0^1\int_0^1\sqrt{\frac{(1-x^2)+(1-y)^2}{2}}\,dxdy$$

$$\begin{cases} x = \sin t \\ y = \sin q \end{cases} \Rightarrow I \geq \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\sqrt{\frac{(1-\sin^2 t)+(1-\sin^2 q)}{2}}\cos t\cos q\, dtdq =$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\sqrt{\frac{\cos^2 t+\cos^2 q}{2}}\cos t\cos q\, dtdq$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}(\cos^2 t\cos q+\cos t\cos^2 q)dtdq =$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\cos^2 t\cos q\, dtdq = \int_0^{\frac{\pi}{2}}\cos^2 t\, dt\int_0^{\frac{\pi}{2}}\cos q\, dq = \int_0^{\frac{\pi}{2}}\frac{1+\cos 2t}{2}dt\sin q|_0^{\frac{\pi}{2}} = \frac{\pi}{4}$$

Therefore,

$$\int_0^1\int_0^1\sqrt{1-\left(\frac{x+y}{2}\right)^2}\,dxdy > \frac{\pi}{4}$$

Solution 2 by Tapas Das-India

$$\sqrt{1-\left(\frac{x+y}{2}\right)^2} > \frac{1}{\sqrt{1-x^2y^2}};\ \left(\because \frac{x+y}{2} \geq \sqrt{xy}\right)$$

$$\int_0^1\int_0^1\sqrt{1-\left(\frac{x+y}{2}\right)^2}\,dxdy > \int_0^1\int_0^1\frac{1}{\sqrt{1-x^2y^2}}dxdy$$

$$\int_0^1\int_0^1\frac{1}{\sqrt{1-x^2y^2}}dxdy = \int_0^1\left[\frac{1}{\sqrt{1-x^2y^2}}dy\right]dx = \int_0^1\left[\frac{1}{x}\sin^{-1}(xy)\right]_0^1 dx =$$

$$= \int_0^1\frac{\sin^{-1}x}{x}dx$$

$$\sin^{-1}x = x+\frac{x^3}{6}+\frac{3x^5}{40}+\cdots \Rightarrow \frac{\sin^{-1}x}{x} = 1+\frac{x^2}{6}+\frac{3x^4}{40}+\cdots$$

$$\frac{\sin^{-1}x}{x} > 1 \Rightarrow \frac{\sin^{-1}x}{x} > \frac{1}{1+x^2};\ x\in(0,1)$$

$$\int_0^1\frac{\sin^{-1}x}{x}dx > \int_0^1\frac{1}{1+x^2}dx = \tan^{-1}x\Big|_0^1 = \frac{\pi}{4}$$

Therefore,

$$\int_0^1\int_0^1\sqrt{1-\left(\frac{x+y}{2}\right)^2}\,dxdy>\frac{\pi}{4}$$

1811. Prove without any software:

$$\int_{2-\sqrt{3}}^{1}e^{-x^2}\,dx<\frac{\pi}{6}\text{ and }\int_{1}^{2+\sqrt{3}}e^{-x^2}\,dx<\frac{\pi}{6}$$

Proposed by Neculai Stanciu-Romania

Solution by Ravi Prakash-New Delhi-India

Let $f(x)=(1+x^2)e^{-x^2}-1; x\geq 0$, then

$$f'(x)=2xe^{-x^2}-2x(1+x^2)e^{-x^2}=2xe^{-x^2}(1-1-x^2)=-2x^3e^{-x^2}<0;\forall x>0$$

$\Rightarrow f$ –is strictly decreasing on $[0,\infty)\Rightarrow f(x)<f(0),\forall x>0$

$$\Rightarrow(1+x^2)e^{-x^2}<1;\forall x\geq 0\Rightarrow e^{-x^2}<\frac{1}{1+x^2};\forall x>0$$

$$\int_{2-\sqrt{3}}^{1}e^{-x^2}dx<\tan^{-1}x\Big|_{2-\sqrt{3}}^{1}=\frac{\pi}{4}-\frac{\pi}{12}=\frac{\pi}{6}$$

$$\int_{1}^{2+\sqrt{3}}e^{-x^2}\,dx=\tan^{-1}x\Big|_{1}^{2+\sqrt{3}}=\frac{5\pi}{12}-\frac{\pi}{4}<\frac{\pi}{6}$$

1812. Prove without any software:

$$\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}\frac{1}{x}\log(1+2x^2+x^4)\,dx<\sqrt{7}-\sqrt{5}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

Solution 1 by Adrian Popa-Romania

$$\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}\frac{1}{x}\log(1+2x^2+x^4)\,dx=\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}\frac{\log(1+x^2)^2}{x}dx=2\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}\frac{\log(1+x^2)}{x}dx$$

It is well-known that $\forall x\geq 0:\log(1+x)\leq\frac{x}{\sqrt{1+x}}$, then

$$2\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}\frac{\log(1+x^2)}{x}dx\leq 2\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}\frac{\frac{x^2}{\sqrt{1+x}}}{x}dx=2\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}\frac{x}{\sqrt{1+x^2}}dx=2\sqrt{1+x^2}\Big|_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}=$$

$$= 2\left(\sqrt{1+\frac{3}{4}} - \sqrt{1+\frac{1}{4}}\right) = 2\left(\frac{\sqrt{7}}{2} - \frac{\sqrt{5}}{2}\right) = \sqrt{7} - \sqrt{5}$$

Solution 2 by Amir Sofi-Kosovo

$$\Omega = \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{2x}{x^2}\log(1+2x^2+x^4)\,dx \overset{x^2\to x}{=} \frac{1}{2}\int_{\frac{1}{4}}^{\frac{3}{4}} \frac{\log(1+x)^2}{x}\,dx = \int_{\frac{1}{4}}^{\frac{3}{4}} \frac{\log(1+x)}{x}\,dx \le$$

$$\le \int_{\frac{1}{4}}^{\frac{3}{4}} \frac{dx}{\sqrt{x+1}} = 2\sqrt{x+1}\Big|_{\frac{1}{4}}^{\frac{3}{4}} = \sqrt{7} - \sqrt{5}$$

Solution 3 by Tapas Das-India

$$\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1}{x}\log(1+2x^2+x^4)\,dx = \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{\log(1+x^2)^2}{x}\,dx = 2\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{\log(1+x^2)}{x}\,dx$$

$$\frac{\log(1+x^2)}{x} = \frac{1}{x}\left(x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \cdots\right) = x - \frac{x^3}{2} + \frac{x^5}{3} - \frac{x^7}{8} + \cdots$$

$$\frac{x}{\sqrt{1+x^2}} = x\cdot(1+x^2)^{-\frac{1}{2}} = x\left(1 - \frac{1}{2}x^2 + \frac{3}{8}x^4 - \frac{5}{16}x^6 + \cdots\right)$$

$$\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{2\log(1+x^2)}{x}\,dx < \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{2x}{\sqrt{1+x^2}}\,dx = 2\sqrt{1+x^2}\Big|_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} = \sqrt{7} - \sqrt{5}$$

1813. **If $0 < a \le b < 1$ then :**

$$\int_a^b\int_a^b\int_a^b \left(\frac{1-xyz}{1+xyz}\right)^3 dxdydz \ge \left(\int_a^b \frac{1-x^3}{1+x^3}\,dx\right)^3.$$

Proposed by Daniel Sitaru-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Let's\ prove\ that:$$

$$\left(\frac{1-xyz}{1+xyz}\right)^3 \ge \left(\frac{1-x^3}{1+x^3}\right)\left(\frac{1-y^3}{1+y^3}\right)\left(\frac{1-z^3}{1+z^3}\right), \forall x,y,z \in (0,1).$$

$$Let\ x = e^u,\ \ y = e^v,\ \ z = e^w, where\ u,v,w \in (-\infty,0)\ and\ let$$

$$f(t) = \log\left(\frac{1-e^{3t}}{1+e^{3t}}\right), t \in (-\infty,0)$$

$$We\ have:\ f'(t)=-\frac{6e^{3t}}{1-e^{6t}}\ and$$

$$f''(t)=-\frac{18e^{3t}(1+e^{6t})}{(1-e^{6t})^2}\leq 0\ then\ f\ is\ concave\ on\ (-\infty,0).$$

$$By\ Jensen's\ inequality, we\ have:$$

$$\log\left(\prod_{cyc}\left(\frac{1-x^3}{1+x^3}\right)\right)=\sum_{cyc}\log\left(\frac{1-x^3}{1+x^3}\right)=\sum_{cyc}f(u)\leq 3f\left(\frac{u+v+w}{3}\right)$$

$$=\log\left(\frac{1-e^{u+v+w}}{1+e^{u+v+w}}\right)^3=\log\left(\frac{1-xyz}{1+xyz}\right)^3$$

$$Then:\ \left(\frac{1-xyz}{1+xyz}\right)^3\geq\left(\frac{1-x^3}{1+x^3}\right)\left(\frac{1-y^3}{1+y^3}\right)\left(\frac{1-z^3}{1+z^3}\right),\forall x,y,z\in(0,1).$$

$$Therefore,$$

$$\int_a^b\int_a^b\int_a^b\left(\frac{1-xyz}{1+xyz}\right)^3 dxdydz\geq\int_a^b\int_a^b\int_a^b\left(\frac{1-x^3}{1+x^3}\right)\left(\frac{1-y^3}{1+y^3}\right)\left(\frac{1-z^3}{1+z^3}\right)dxdydz$$

$$=\left(\int_a^b\frac{1-x^3}{1+x^3}dx\right)^3$$

1814. **If $0<a\leq b,\ f:[a,b]\to(0,\infty), f$ –continuous, then:**

$$(b-a)^2\int_a^b f(x)dx+\left(\int_a^b\sqrt[3]{f(x)}dx\right)^3\geq 2(b-a)\left(\int_a^b\sqrt{f(x)}dx\right)^2.$$

Proposed by Daniel Sitaru-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$By\ Schur's\ inequality, we\ have:\ u^3+v^3+w^3+3uvw$$
$$\geq uv(u+v)+vw(v+w)+wu(w+u),\forall u,v,w>0$$

$$Also, by\ AM-GM\ inequality, we\ have:\ u+v\geq 2\sqrt{uv}\ (And\ analogs)$$

$$Then:\ u^3+v^3+w^3+3uvw\geq 2\left(\sqrt{(uv)^3}+\sqrt{(vw)^3}+\sqrt{(wu)^3}\right)$$

$$Taking\ u=\sqrt[3]{f(x)},\ v=\sqrt[3]{f(y)},\ w=\sqrt[3]{f(z)},(x,y,z\in[a,b]), we\ obtain:$$

$$f(x)+f(y)+f(z)+3\sqrt[3]{f(x).f(y).f(z)}$$
$$\geq 2\left(\sqrt{f(x).f(y)}+\sqrt{f(y).f(z)}+\sqrt{f(z).f(x)}\right),\forall x,y,z\in[a,b]$$

Integrating the both sides, we obtain :

$$\int_a^b\int_a^b\int_a^b\left(f(x)+f(y)+f(z)+3\sqrt[3]{f(x).f(y).f(z)}\right)dxdydz$$

$$\geq 2\int_a^b\int_a^b\int_a^b\left(\sqrt{f(x).f(y)}+\sqrt{f(y).f(z)}+\sqrt{f(z).f(x)}\right)dxdydz$$

$$(b-a)^2\int_a^b f(x)dx+\left(\int_a^b\sqrt[3]{f(x)}dx\right)^3\geq 2(b-a)\left(\int_a^b\sqrt{f(x)}dx\right)^2$$

1815. Prove the product

$$\prod_{n=1}^{\infty}\frac{\left(1-\frac{1}{(3n)^2}\right)^{(3n)^2}}{\left(1-\frac{1}{(4n-2)^2}\right)^{(4n-2)^2}}=\sqrt{\frac{3}{2}}e^{\alpha-\frac{1}{2}},$$

where $\alpha=\frac{4G}{\pi}-\frac{47\zeta(3)}{4\pi^2}-\frac{2\pi}{3\sqrt{3}}+\frac{\psi^{(1)}\left(\frac{1}{3}\right)}{\sqrt{3}\pi}$, G **– Catalan constant,**
$\psi^{(1)}(t)$ **– Polygamma function.**

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Artan Ajredini-Presheva-Serbie

$$\prod_{n=1}^{\infty}\frac{\left(1-\frac{1}{(3n)^2}\right)^{(3n)^2}}{\left(1-\frac{1}{(4n-2)^2}\right)^{(4n-2)^2}}=$$

$$=\exp\left\{\sum_{n=0}^{\infty}\left[(3n)^2\log\left(1-\frac{1}{(3n)^2}\right)-(4n-2)^2\log\left(1-\frac{1}{(4n-2)^2}\right)\right]\right\}=$$

$$=\exp\left\{\sum_{n=1}^{\infty}\left[(3n)^2\log\left(1-\frac{1}{(3n)^2}\right)+1\right]-\sum_{n=1}^{\infty}\left[(4n-2)^2\log\left(1-\frac{1}{(4n-2)^2}\right)+1\right]\right\}=$$

$$=\exp\{S_1-S_2\}.$$

$$S_1=\sum_{n=1}^{\infty}\left[(3n)^2\log\left(1-\frac{1}{(3n)^2}\right)+1\right]=\sum_{n=1}^{\infty}\int_0^1\frac{2x^3}{x^2-9n^2}dx=18\sum_{n=1}^{\infty}\int_0^{\frac{1}{3}}\frac{x^3}{x^2-n^2}dx=$$

$$=18\int_0^{\frac{1}{3}}\sum_{n=1}^{\infty}\frac{x^3}{x^2-n^2}dx=9\int_0^{\frac{1}{3}}x^2\left(\pi\cot(\pi x)-\frac{1}{x}\right)dx=9\int_0^{\frac{1}{3}}\pi x^2\cot(\pi x)\,dx-\frac{1}{2}=$$

$$=9A-\frac{1}{2}.$$

$$A=\int_0^{\frac{1}{3}}\pi x^2\cot(\pi x)\,dx\overset{\pi x=t}{=}\frac{1}{\pi^2}\int_0^{\frac{\pi}{3}}t^2\cot t\,dt\overset{IBP}{=}\frac{1}{9}\log\left(\frac{\sqrt{3}}{2}\right)-\frac{2}{\pi^2}\int_0^{\frac{\pi}{3}}t\log(\sin t)\,dt=$$

$$=\frac{1}{9}\log\left(\frac{\sqrt{3}}{2}\right)-\frac{2}{\pi^2}B.$$

$$B=\int_0^{\frac{\pi}{3}}t\log(\sin t)\,dt=\int_0^{\frac{\pi}{3}}t\left(-\log 2-\sum_{n=1}^{\infty}\frac{\cos(2nt)}{n}\right)dt=$$

$$=-\log 2\int_0^{\frac{\pi}{3}}t\,dt-\sum_{n=1}^{\infty}\frac{1}{n}\int_0^{\frac{\pi}{3}}t\cos(2nt)\,dt=$$

$$=-\frac{\pi^2}{18}\log 2-\frac{\pi}{6}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{2n\pi}{3}\right)}{n^2}-\frac{1}{4}\sum_{n=1}^{\infty}\frac{\cos\left(\frac{2n\pi}{3}\right)}{n^3}+\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^3}=$$

$$=-\frac{\pi^2}{18}\log 2-\frac{\pi\sqrt{3}}{12}\sum_{n=1}^{\infty}\frac{1}{(3n-2)^2}+\frac{\pi\sqrt{3}}{12}\sum_{n=1}^{\infty}\frac{1}{(3n-1)^2}+\frac{1}{8}\sum_{n=1}^{\infty}\frac{1}{(3n-2)^3}+$$

$$+\frac{1}{8}\sum_{n=1}^{\infty}\frac{1}{(3n-1)^3}-\frac{1}{108}\sum_{n=1}^{\infty}\frac{1}{n^3}+\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^3}=$$

$$=-\frac{\pi^2}{18}\log 2-\frac{\pi\sqrt{3}}{108}\psi^{(1)}\left(\frac{1}{3}\right)+\frac{\pi\sqrt{3}}{108}\psi^{(1)}\left(\frac{2}{3}\right)+\frac{1}{4}\cdot\frac{117}{243}\zeta(3)-\frac{1}{108}\zeta(3)+\frac{1}{4}\zeta(3)=$$

$$=-\frac{\pi^2}{18}\log 2+\frac{\pi\sqrt{3}}{108}\left(\frac{4\pi^2}{3}-2\psi^{(1)}\left(\frac{1}{3}\right)\right)+\frac{13}{36}\zeta(3)=$$

$$=-\frac{\pi^2}{18}\log 2+\frac{\pi^3\sqrt{3}}{81}-\frac{\pi\sqrt{3}}{54}\psi^{(1)}\left(\frac{1}{3}\right)+\frac{13}{36}\zeta(3)$$

Hence,

$$A=\frac{3}{2\pi^2}\log\left(\frac{\sqrt{3}}{2}\right)-\frac{2}{\pi^2}B=$$

$$=\frac{1}{9}\log\left(\frac{\sqrt{3}}{2}\right)-\frac{2}{\pi^2}\left(-\frac{\pi^2}{18}\log 2+\frac{\pi^3\sqrt{3}}{81}-\frac{\pi\sqrt{3}}{54}\psi^{(1)}\left(\frac{1}{3}\right)+\frac{13}{36}\zeta(3)\right)=$$

$$=\frac{1}{9}\log\left(\frac{\sqrt{3}}{2}\right)+\frac{1}{9}\log 2-\frac{2\pi\sqrt{3}}{81}+\frac{\sqrt{3}}{27\pi}\psi^{(1)}\left(\frac{1}{3}\right)-\frac{13}{18\pi^2}\zeta(3)$$

Therefore,

$$S_1=9A-\frac{1}{2}=$$

$$=9\left(\frac{3}{2\pi^2}\log\left(\frac{\sqrt{3}}{2}\right)+\frac{1}{9}\log 2-\frac{2\pi\sqrt{3}}{81}+\frac{\sqrt{3}}{27\pi}\psi^{(1)}\left(\frac{1}{3}\right)-\frac{13}{18\pi^2}\zeta(3)\right)-\frac{1}{2}=$$

$$= \log\sqrt{3} - \frac{2\pi}{3\sqrt{3}} + \frac{\psi^{(1)}\left(\frac{1}{3}\right)}{\sqrt{3}\pi} - \frac{13}{2\pi^2}\zeta(3) - \frac{1}{2}.$$

Similarly, we have:

$$S_2 = \sum_{n=1}^{\infty}\left[(4n-2)^2\log\left(1-\frac{1}{(4n-2)^2}\right)+1\right] = \sum_{n=1}^{\infty}\int_0^1 \frac{2x^3}{x^2-(4n-2)^2}dx =$$

$$= \int_0^1 \sum_{n=1}^{\infty}\frac{2x^3}{x^2-(4n-2)^2}dx = \int_0^1 \sum_{n=1}^{\infty}\frac{2x^3}{x^2-4n^2}dx - \int_0^1 \sum_{n=1}^{\infty}\frac{2x^3}{x^2-16n^2}dx = C - D$$

$$C = \int_0^1 \sum_{n=1}^{\infty}\frac{2x^3}{x^2-4n^2}dx = 4\int_0^{\frac{1}{2}} \sum_{n=1}^{\infty}\frac{2x^3}{x^2-n^2}dx = 4\int_0^{\frac{1}{2}} x^2\left(\pi\cot(\pi x)-\frac{1}{x}\right)dx =$$

$$= 4\int_0^{\frac{1}{2}} \pi x^2\cot(\pi x)\,dx - \frac{1}{2} = 4E - \frac{1}{2}.$$

$$E = \int_0^{\frac{1}{2}} \pi x^2\cot(\pi x)\,dx \overset{\pi x = t}{=} \frac{1}{\pi^2}\int_0^{\frac{\pi}{2}} t^2\cot t\,dt \overset{IBP}{=} -\frac{2}{\pi^2}\int_0^{\frac{\pi}{2}} t\log(\sin t)\,dt =$$

$$= -\frac{2}{\pi^2}\int_0^{\frac{\pi}{2}} t\left(-\log 2 - \sum_{n=1}^{\infty}\frac{\cos(2nt)}{n}\right)dt =$$

$$= \frac{2}{\pi^2}\log 2\int_0^{\frac{\pi}{2}} t\,dt + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n}\int_0^{\frac{\pi}{2}} t\cos(2nt)\,dt =$$

$$= \frac{\log 2}{4} + \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{\sin(n\pi)}{n^2} + \frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{n^3} - \frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^3} =$$

$$= \frac{\log 2}{4} + \frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3} - \frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^3} = \frac{\log 2}{4} - \frac{7}{8\pi^2}\zeta(3)$$

Thus,

$$C = 4E - \frac{1}{2} = \log 2 - \frac{7}{2\pi^2}\zeta(3) - \frac{1}{2}, \qquad D = \frac{4G}{\pi} - \frac{35}{4\pi^2}\zeta(3) + \frac{1}{4}\log 4 - \frac{1}{2}$$

Hence,

$$S_2 = \log 2 - \frac{7}{2\pi^2}\zeta(3) - \frac{1}{2} - \frac{4G}{\pi} + \frac{35}{4\pi^2}\zeta(3) - \frac{1}{4}\log 4 = \frac{21}{4\pi^2}\zeta(3) - \frac{4G}{\pi} + \frac{1}{2}\log 2$$

Finally,

$$\prod_{n=1}^{\infty}\frac{\left(1-\frac{1}{(3n)^2}\right)^{(3n)^2}}{\left(1-\frac{1}{(4n-2)^2}\right)^{(4n-2)^2}} =$$

$$= \exp\left\{\log\sqrt{3} - \frac{2\pi}{3\sqrt{3}} + \frac{\psi^{(1)}\left(\frac{1}{3}\right)}{\sqrt{3}\pi} - \frac{13}{2\pi^2}\zeta(3) - \frac{1}{2} - \frac{21}{4\pi^2}\zeta(3) + \frac{4G}{\pi} - \frac{1}{2}\log 2\right\} =$$

$$= \exp\left\{\log\left(\sqrt{\frac{3}{2}}\right) - \frac{2\pi}{3\sqrt{3}} + \frac{\psi^{(1)}\left(\frac{1}{3}\right)}{\sqrt{3}\pi} - \frac{47}{4\pi^2}\zeta(3) + \frac{4G}{\pi} - \frac{1}{2}\right\} =$$

$$= \sqrt{\frac{3}{2}}\exp\left\{\frac{4G}{\pi} - \frac{47}{4\pi^2}\zeta(3)\frac{2\pi}{3\sqrt{3}} + \frac{\psi^{(1)}\left(\frac{1}{3}\right)}{\sqrt{3}\pi} - \frac{1}{2}\right\}$$

1816. Find:

$$\Omega = \prod_{n=1}^{\infty}\frac{(1+n^2)^2}{4+n^4}$$

Proposed by Neculai Stanciu-Romania

Solution 1 by Bedri Hajrizi-Mitrovica-Kosovo

$$n^4 + 4 = (n^2 - 2n + 2)(n^2 + 2n + 2) = ((n-1)^2 + 1)((n+1)^2 + 1)$$

$$\Omega(\mathrm{m}) = \prod_{n=1}^{m}\frac{(1+n^2)^2}{4+n^4} = \prod_{n=1}^{m}\frac{(1+n^2)^2}{((n-1)^2+1)((n+1)^2+1)} = \frac{2(m^2+1)}{(m+1)^2+1}$$

$$\Omega = \lim_{m\to\infty}\frac{2(m^2+1)}{(m+1)^2+1} = 2$$

Solution 2 by Tapas Das-India

$$n^4 + 4 = (n^2)^2 + 2^2 = (n^2+2)^2 - 4n^2 = (n^2+2)^2 - (2n)^2 =$$

$$= (n^2 + 2n + 2)(n^2 - 2n + 2) = [(n^2 + 2n + 1) + 1][(n^2 - 2n + 1) + 1] =$$

$$= [(n-1)^2 + 1][(n+1)^2 + 1]$$

$$\Omega(\mathrm{k}) = \prod_{n=1}^{k}\frac{(1+n^2)^2}{4+n^4} = \prod_{n=1}^{k}\frac{(1+n^2)^2}{((n-1)^2+1)((n+1)^2+1)} = \frac{2(k^2+1)}{(k+1)^2+1}$$

Therefore,

$$\Omega = \prod_{n=1}^{\infty}\frac{(1+n^2)^2}{4+n^4} = \lim_{k\to\infty}\frac{2(k^2+1)}{(k+1)^2+1} = \lim_{k\to\infty}\frac{2\left(1+\frac{1}{k^2}\right)}{\left(1+\frac{1}{k}\right)^2 + \frac{1}{k^2}} = 2$$

Solution 3 by Ravi Prakash-New Delhi-India

$$n^4 + 4 = (n^2)^2 + 2^2 = (n^2+2)^2 - 4n^2 = (n^2+2)^2 - (2n)^2 =$$

$$= (n^2 + 2n + 2)(n^2 - 2n + 2) = [(n^2 + 2n + 1) + 1][(n^2 - 2n + 1) + 1] =$$

$$=[(n-1)^2+1][(n+1)^2+1]$$

$$\text{Let } a_n=\frac{(1+n^2)^2}{((n-1)^2+1)((n+1)^2+1)}, \text{then}$$

$$\log a_n=2\log(n^2+1)-\log[(n-1)^2+1]-\log[(n+1)^2+1]$$

$$\text{Let } b_n=\prod_{k=1}^{n} a_k, \text{then } \log b_n \sum_{k=1}^{n}\log a_k=\sum_{k=1}^{n}[\log(k^2+1)-\log((k-1)^2+1)]-$$

$$-\sum_{k=1}^{n}[\log((k+1)^2+1)-\log(k^2+1)]=\log(n^2+1)-\log((n+1)^2+1)+\log 2=$$

$$=\log\left(\frac{n^2+1}{(n+1)^2+1}\right)+\log 2$$

$$\lim_{n\to\infty}\log b_n=\log 1+\log 2=\log 2\Rightarrow \lim_{n\to\infty} b_n=2.$$

Solution 4 by Hikmat Mammadov-Azerbaijan

$$\Omega=\prod_{n=1}^{\infty}\frac{(1+n^2)^2}{4+n^4}=\prod_{n=1}^{\infty}\frac{(1+n^2)^2}{(n^2+2)-(2n)^2}=\prod_{n=1}^{\infty}\frac{(1+n^2)^2}{(n^2-2n+2)(n^2+2n+2)}=$$

$$=\prod_{n=1}^{\infty}\frac{n^2+1}{(n-1)^2+1}\cdot\prod_{n=1}^{\infty}\frac{n^2+1}{(n+1)^2+1}=$$

$$\lim_{n\to\infty}\prod_{k=1}^{n}\frac{k^2+1}{(k-1)^2+1}\cdot\prod_{k=1}^{n}\frac{k^2+1}{(k+1)^2+1}=\lim_{n\to\infty}(n^2+1)\cdot\frac{2}{(n+1)^2+1}=2$$

1817. Find:

$$\Omega=\int_0^{\frac{\pi}{6}} x^2\cot x\,dx$$

Proposed by Togrul Ehmedov-Azerbaijan

Solution by Rana Ranino-Setif-Algerie

$$\Omega=\int_0^{\frac{\pi}{6}} x^2\cot x\,dx \overset{IBP}{=} x^2\log(\sin x)\Big|_0^{\frac{\pi}{6}}+\int_0^{\frac{\pi}{6}}2x\left(\log 2+\sum_{n=1}^{\infty}\frac{\cos(2nx)}{n}\right)dx=$$

$$=2\sum_{n=1}^{\infty}\frac{1}{n}\int_0^{\frac{\pi}{6}}x\cos(2nx)\,dx=2\sum_{n=1}^{\infty}\frac{1}{n}\left[\frac{x\sin(2nx)}{2n}+\frac{\cos(2nx)}{4n^2}\right]\Bigg|_0^{\frac{\pi}{6}}=$$

$$=\frac{\pi}{6}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi}{3}\right)}{n^2}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{\cos\left(\frac{nx}{3}\right)}{n^3}-\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^3}$$

$$\text{Clausen's function: } CI_2(\theta)=\sum_{n=1}^{\infty}\frac{\sin(n\theta)}{n^2}, CI_3=\sum_{n=1}^{\infty}\frac{\cos(n\theta)}{n^3}, CI_3\left(\frac{\pi}{3}\right)=\frac{\zeta(3)}{3}$$

Therefore,

$$\Omega=\int_0^{\frac{\pi}{6}}x^2\cot x\,dx=\frac{\pi}{6}CI_2\left(\frac{\pi}{3}\right)-\frac{\zeta(3)}{3}$$

1818. Prove that:

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{\cos\left(\frac{\pi+4x}{4}\right)\log\left(\log\left(\tan\left(\frac{3\pi-4x}{4}\right)\right)\right)}{\sin^3\left(\frac{4x+\pi}{4}\right)\sqrt{\log\left(\tan\left(\frac{3\pi-4x}{4}\right)\right)}}dx=\sqrt{\frac{\pi}{2}}(\gamma+\log 8)$$

Proposed by Ankush Kumar Parcha-India

Solution by Adrian Popa-Romania

$$\because\int_a^b f(x)dx=\int_b^a f(a+b-x)dx\Rightarrow$$

$$I=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{-\cos x\log(\log(\tan x))}{\sqrt{\log(\tan x)}}dx=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{-\log(\log(\tan x))}{\tan x\sin^2 x\sqrt{\log(\tan x)}}dx=$$

$$=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{-\cot x\log(-\log(\cot x))}{\sin^2 x\sqrt{-\log(\cot x)}}dx\overset{t=\cot x}{=}\int_1^0\frac{t\log(-\log t)}{\sqrt{-\log t}}dt\overset{\sqrt{-\log t}=u}{=}$$

$$=-4\int_0^{\infty}e^{-2u^2}\log u\,du\overset{u=\sqrt{\frac{y}{2}}}{=}-\frac{1}{\sqrt{2}}\int_0^{\infty}e^{-y}y^{-\frac{1}{2}}(\log y-\log 2)\,dy=$$

$$=-\frac{1}{\sqrt{2}}\left(\int_0^{\infty}e^{-y}y^{-\frac{1}{2}}\log y dy-\log 2\int_0^{\infty}e^{-y}y^{-\frac{1}{2}}dy\right)=$$

$$=-\frac{1}{\sqrt{2}\left(\psi\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)-\log 2\,\Gamma\left(\frac{1}{2}\right)\right)}=-\frac{1}{\sqrt{2}}\left((-2\log 2-\gamma)\sqrt{\pi}-\log 2\sqrt{\pi}\right)=$$

$$=\sqrt{\frac{\pi}{2}}(2\log 2+\gamma+\log 2)=\sqrt{\frac{\pi}{2}}(3\log 2+\gamma)=\sqrt{\frac{\pi}{2}}(\gamma+\log 8)$$

1819. **Calculate the following integral:**

$$\Omega = \int_0^1 \frac{Li_3(x)\log(1-x)\log\{1+x\}}{x}dx, \{x\} = x - [x], \{*\} - \text{GIF}$$

Proposed by Togrul Ehmedov-Baku-Azerbaijan

Solution by proposer

We know that $0 < x < 1 \to log\{1+x\} = \log(x)$ then

$$\Omega = \int_0^1 \frac{\mathrm{Li}_3(x)\log(1-x)\log\{1+x\}}{x}dx = \int_0^1 \frac{\mathrm{Li}_3(x)\log(1-x)\log(x)}{x}dx$$

$$= \int_0^1 \frac{\log(x)\log(1-x)}{x}\sum_{k=1}^{\infty}\frac{x^k}{k^3}dx = \sum_{k=1}^{\infty}\frac{1}{k^3}\int_0^1 x^{k-1}\log(x)\log(1-x)\,dx$$

$$= \sum_{k=1}^{\infty}\frac{1}{k^3}\frac{d^2}{dmdn}\int_0^1 x^{m-1}(1-x)^{n-1}dx\Big|_{\substack{m=k\\ n=1}} = \sum_{k=1}^{\infty}\frac{1}{k^3}\left[\frac{1}{k}\left[\frac{H_k}{k} - \zeta(2) + H_k^{(2)}\right]\right]$$

$$= \sum_{k=1}^{\infty}\frac{H_k}{k^5} - \zeta(2)\sum_{k=1}^{\infty}\frac{1}{k^4} + \sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k^4} = \frac{1}{2}\zeta^2(3) - \frac{1}{3}\zeta(6)$$

1820. **Find:**

$$\Omega = \lim_{n\to\infty}\left(\lim_{k\to\infty}\int_0^1\left(\frac{[kx]}{k}\right)^n dx\right)$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu-Romania

Solution 1 by Adrian Popa-Romania

$$I = \int_0^1\left(\frac{[kx]}{k}\right)^n dx \overset{x=\frac{t}{k}}{=} \int_0^k\left(\frac{[t]}{k}\right)^n\frac{dt}{k} = \frac{1}{k^{n+1}}\int_0^k [t]^n\,dt =$$

$$= \frac{1}{k^{n+1}}\left(\int_0^1 [t]^n\,dt + \int_1^2 [t]^n\,dt + \cdots + \int_{k-1}^k [t]^n\,dt\right) =$$

$$= \frac{1}{k^{n+1}}\left(\int_1^2 1^n dt + \int_2^3 2^n dt + \cdots + \int_{k-1}^k (k-1)^n dt\right) =$$

$$= \frac{1}{k^{n+1}}(1^n + 2^n + 3^n + \cdots + (k-1)^n)$$

$$\lim_{k\to\infty}\frac{1^n+2^n+3^n+\cdots+(k-1)^n}{k^{n+1}}\overset{C-S}{=}\lim_{k\to\infty}\frac{k^n}{(k+1)^n-k^n}=$$

$$=\lim_{k\to\infty}\frac{k^n}{\binom{n+1}{0}k^{n+1}+\binom{n+1}{1}k^n+\cdots\binom{n+1}{n+1}k^n-k^{n+1}}=\frac{1}{\binom{n+1}{1}}=\frac{1}{n+1}$$

$$\Omega=\lim_{n\to\infty}\left(\lim_{k\to\infty}\int_0^1\left(\frac{[kx]}{k}\right)^n dx\right)=0$$

Solution 2 by Ravi Prakash-New Delhi-India

$$\text{Let } a_k=\int_0^1\left(\frac{[kx]}{k}\right)^n dx=\frac{1}{k^n}\int_0^1[kx]^n dx=\frac{1}{k^n}\sum_{r=0}^{k-1}\int_{\frac{r}{k}}^{\frac{r+1}{k}}r^n dr=$$

$$=\frac{1}{k^{n+1}}\sum_{r=0}^{k-1}r^n=\frac{1}{k}\sum_{r=0}^{k-1}\left(\frac{r}{k}\right)^n$$

$$\lim_{k\to\infty}a_k=\lim_{k\to\infty}\frac{1}{k}\sum_{r=0}^{k-1}\left(\frac{r}{k}\right)^n=\int_0^1 t^n\,dt=\frac{1}{n+1}$$

Therefore,

$$\Omega=\lim_{n\to\infty}\left(\lim_{k\to\infty}\int_0^1\left(\frac{[kx]}{k}\right)^n dx\right)=0$$

1821. Find a closed form:

$$\Omega=\int_0^\infty\frac{e^{-x}\log\left(\frac{1}{x}\right)\sin(\alpha x)}{x}dx,\ (\forall)\alpha>0$$

Proposed by Ose Favour-Nigeria

Solution by Fao Ler-Iraq

$$\Omega=\int_0^\infty\frac{e^{-x}\log\left(\frac{1}{x}\right)\sin(\alpha x)}{x}dx=-\int_0^\infty e^{-x}\left(\frac{d}{dy}x^y\right)Im(e^{axi})\,dx;\,(y=-1)$$

$$=-\frac{d}{dy}Im\left(\int_0^\infty e^{-x(1-ai)x^y}\,dx\right)=-\frac{d}{dy}Im(L_{1-ai}(y))=-\frac{d}{dy}Im\left(\frac{y!}{(1-ai)^{y+1}}\right)=$$

$$=-\frac{d}{dy}Im\left((a^2+1)^{-\frac{y+1}{2}}y!\,e^{(y+1)\tan^{-1}(a)i}\right)=$$

$$= -\frac{d}{dy}\left(e^{A(y+1)}y!\sin\left((y+1)B\right)\right);\left(A = -\frac{1}{2}\log(a^2+1), B = \tan^{-1}a\right)$$

$$= -y!\,e^{A(y+1)}\left(A\sin\left((y+1)B\right) + B\cos\left((y+1)B\right) + \sin\left((y+1)B\right)\psi(y+1)\right);$$

$$(y = -1)$$

$$= -(y+1)!\left(\frac{A\sin\left((y+1)B\right)}{y+1} + \frac{B\cos\left((y+1)B\right)}{y+1} + \frac{\sin\left((y+1)B\right)}{y+1}\psi(y+1)\right) =$$

$$= -\left(AB + \frac{B\cos\left((y+1)B\right)}{y+1} + B\psi(y+1)\right) =$$

$$= -AB - B\left(\frac{B\cos\left((y+1)B\right)}{y+1} + \psi(y+2) - \frac{1}{y+1}\right) =$$

$$= -AB - B\left(\frac{\cos\left((y+1)B\right) - 1}{y+1} + \psi(1)\right) =$$

$$= -AB - B(-B\sin(y+1) - \gamma) = \gamma B - AB =$$

$$= \gamma\tan^{-1}a + \frac{1}{2}\tan^{-1}a\log(a^2+1) = \frac{1}{2}\tan^{-1}a\,(2\gamma + \log(a^2+1))$$

1822. If $0 < a \leq b$ then:

$$6\int_a^b\int_a^b(x^3+y^3)^2\,dxdy \geq (a^4+a^2b^2+b^4)(b^2-a^2)^2$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Adrian Popa-Romania

$$6\int_a^b\int_a^b(x^3+y^3)^2\,dxdy = 6\int_a^b\int_a^b(x^6+2x^3y^3+y^6))\,dxdy >$$

$$> 6\int_a^b\int_a^b(x^6+y^6)\,dxdy \overset{Murihead}{>} 6\int_a^b\int_a^b\left(x^5y+xy^5\right)dxdy =$$

$$= 12\int_a^b\int_a^b x^5y\,dxdy = 12\int_a^b x^5\,dx\int_a^b y\,dy =$$

$$= 12\left.\frac{x^6}{6}\right|_a^b\cdot\left.\frac{y^6}{6}\right|_a^b = (b^6-a^6)(b^2-a^2) =$$

$$= (b^2 - a^2)(b^4 + b^2a^2 + a^2)(b^2 - a^2) = (a^4 + a^2b^2 + b^4)(b^2 - a^2)^2$$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

$$(x^3 + y^3)^2 = x^6 + y^6 + 2x^3y^3 \geq 4x^3y^3$$

$$6\int_a^b\int_a^b (x^3 + y^3)^2\, dxdy \geq 6\int_a^b\int_a^b 4x^3y^3\, dxdy =$$

$$= 6\int_a^b 4\cdot\frac{x^4}{4}y^3\, dy = 6\int_a^b (b^4 - a^4)y^3\, dy = 6(b^4 - a^4)\cdot\frac{y^4}{4}\Big|_a^b =$$

$$= \frac{3}{2}(b^4 - a^4)^2 = \frac{3}{2}(b^8 + a^8 - 2a^4b^4) \geq$$

$$\geq (b^4 + a^2b^2 + a^4)(b^2 - a^2)^2 = (b^4 + a^2b^2 + a^4)(b^4 + a^4 - 2a^2b^2)$$

Iff $3(a^8 + b^8 - 2a^4b^4) \geq 2[a^8 + b^8 - a^2b^6 - a^6b^2]$

$a^8 + b^8 + 2a^4b^2 + 2a^2b^4 \geq 6a^6b^6$ true.

Solution 3 by Tapas Das-India

$$x^3 + y^3 \geq 2\sqrt{x^3y^3} \Rightarrow (x^3 + y^3)^2 \geq 4x^3y^3$$

$$\int_a^b\int_a^b (x^3 + y^3)^2 dxdy \geq 4\int_a^b\int_a^b x^3y^3\, dxdy =$$

$$= 4\cdot\frac{b^4 - a^4}{4}\cdot\frac{b^4 - a^4}{4} = \frac{(b^4 - a^4)^2}{4} = \frac{1}{4}(b^2 - a^2)^2(b^4 + 2a^2b^2 + a^4)$$

Now, $\frac{1}{4}(b^2 - a^2)^2(b^4 + 2a^2b^2 + a^4) - \frac{1}{6}(b^2 - a^2)^2(a^4 + a^2b^2 + b^4) =$

$$= \frac{1}{12}(b^2 - a^2)[3b^4 + 6a^2b^2 + 3a^4 - 2a^4 - 2a^2b^2 - 2b^4] =$$

$$= \frac{1}{12}(b^2 - a^2)^2(b^4 + a^4 + 4a^2b^2) > 0$$

$$\frac{1}{4}(b^2 - a^2)(b^4 + 2a^2b^2 + a^4) \geq \frac{1}{6}(b^2 - a^2)(a^4 + a^2b^2 + b^4)$$

$$\int_a^b\int_a^b (x^3 + y^3)^2 dxdy \geq \frac{1}{6}(b^2 - a^2)(a^4 + a^2b^2 + b^4)$$

$$6\int_a^b\int_a^b (x^3 + y^3)^2\, dxdy \geq (a^4 + a^2b^2 + b^4)(b^2 - a^2)^2$$

Solution 4 by Khaled Abd Imouti-Damascus-Syria

$$(x^3+y^3)^2 = x^6+y^6+2x^3y^3 \geq 4x^3y^3$$

$$6\int_a^b\int_a^b (x^3+y^3)^2\,dxdy \geq 6\int_a^b\int_a^b 4x^3y^3\,dxdy =$$

$$= 6\int_a^b 4\cdot\frac{x^4}{4}y^3\,dy = 6\int_a^b (b^4-a^4)y^3\,dy = 6(b^4-a^4)\cdot\left.\frac{y^4}{4}\right|_a^b =$$

$$= \frac{3}{2}(b^4-a^4)^2 = \frac{3}{2}(b^8+a^8-2a^4b^4) \geq$$

$$\geq \frac{3}{2}[(b^2-a^2)^2(b^2+a^2)^2] \geq \frac{3}{2}(b^4+2a^2b^2+a^4)(b^2-a^2)^2$$

$$\frac{3}{2}(b^4+2a^2b^2+a^4) \geq a^4+a^2b^2+b^4 \Leftrightarrow a^4+4a^2b^2+b^4 \geq 0(\text{true})$$

1823. If $-1 < a \leq b$ then:

$$\int_a^b \frac{e^x}{x+1}dx \leq \frac{(a-b)(a^2+ab+b^2-3)}{3} + \log\left(\frac{(a+1)^{a+1-e^a}}{(b+1)^{b+1-e^b}}\right)$$

Proposed by Pavlos Trifon-Greece

Solution by proposer

Lemma. If $x \geq -1$ then $(e^x-1)\log(x+1) \geq x^2$

Proof. Let

$$m(x) = \begin{cases} \dfrac{\log(x+1)}{x}, & x \in (-1,0)\cup(0,\infty) \\ 1, & x = 0 \end{cases}$$

$$m'(x) = \frac{\frac{x}{x+1} - \log(x+1)}{x^2} < 0 \text{ because } \log t > 1 - \frac{1}{t}; t \in (-1,0)\cup(0,\infty)$$

$$\Rightarrow m \searrow (-1,+\infty)$$

$$x > -1, e^x - 1 \geq x > -1 \Rightarrow m(e^x-1) \leq m(x)$$

$$\frac{\log(1+e^x-1)}{e^x-1} \leq \frac{\log(x+1)}{x} \Rightarrow \frac{x}{e^x-1} \leq \frac{\log(x+1)}{x}; (-e^x(x-1) > 0) \Rightarrow$$

$$(e^x-1)\log(x+1) \geq x^2$$

Equality for $x = 0$.

$$\int_a^b (e^x - 1)\log(x+1)\,dx \ge \int_a^b x^2\,dx \Rightarrow$$

$$\int_a^b e^x \log(x+1)\,dx - \int_a^b \log(x+1)\,dx \ge \frac{b^3 - a^3}{3}$$

$$[e^x \log(x+1)]_a^b - \int_a^b \frac{e^x}{x+1}dx \ge \int_a^b (x)' \log(x+1)\,dx + \frac{b^3 - a^3}{3}$$

$$e^b \log(b+1) - e^{a\log(a+1)} - \int_a^b \frac{e^x dx}{x+1} \ge [x\log(x+1)]_a^b - \int_a^b \frac{x}{x+1}dx + \frac{b^3 - a^3}{3}$$

$$\log\left(\frac{(b+1)^{e^b}}{(a+1)^{e^a}}\right) - \int_a^b \frac{e^x}{x+1}dx \ge \log\left(\frac{(b+1)^b}{(a+1)^a}\right) + \int_a^b \left(\frac{1}{x+1} - 1\right)dx + \frac{b^3 - a^3}{3}$$

$$\log\left(\frac{(a+1)^{a-e^a}}{(b+1)^{b-e^b}}\right) + \frac{a^3 - b^3}{3} \ge \int_a^b \frac{e^x}{x+1}dx + \log\left(\frac{b+1}{a+1}\right) - b + a$$

$$\log\left(\frac{(a+1)^{a+1-e^a}}{(b+1)^{b+1-e^b}}\right) + \frac{a^3 - b^3}{3} - (a-b) \ge \int_a^b \frac{e^x}{x+1}dx$$

Therefore,

$$\int_a^b \frac{e^x}{x+1}dx \le \frac{(a-b)(a^2 + ab + b^2 - 3)}{3} + \log\left(\frac{(a+1)^{a+1-e^a}}{(b+1)^{b+1-e^b}}\right)$$

1824. Find:

$$\Omega = \sum_{n=0}^{\infty}\left(\sum_{k=1}^{\infty}(-1)^{k-1}\int_0^{\infty} e^{-(n+k)x}\,dx\right)^2$$

Proposed by Hikmat Mammadov-Azerbaijan

Solution by proposer

$$\Omega = \sum_{n=0}^{\infty}\left(\sum_{k=1}^{\infty}(-1)^{k-1}\int_0^{\infty} e^{-(n+k)x}\,dx\right)^2 = \sum_{n=0}^{\infty}\left(\int_0^{\infty} e^{-nx}\cdot\frac{e^{-x}}{1+e^{-x}}dx\right)^2 =$$

$$= \sum_{n=0}^{\infty}\left[\int_0^{\infty}\int_0^{\infty}\frac{e^{-n(x+y)}\cdot e^{x+y}}{(1+e^{-x})(1+e^{-y})}dxdy\right] = \int_0^{\infty}\int_0^{\infty}\frac{dxdy}{(e^{x+y}-1)(1+e^{-x})(1+e^{-y})} =$$

$$=\int_0^\infty \frac{dx}{1+e^{-x}}\int_0^\infty \frac{dy}{(e^{x+y}-1)(e^{-y}+1)}=-\int_0^\infty \frac{\log\left(\frac{1-e^{-y}}{2}\right)}{(1+e^x)(1+e^{-x})}dx \overset{x=-\log u}{=}$$

$$=\int_0^1 \frac{\log 2-\log(1-u)}{(1+u)^2}du=\log 2\left(-\frac{1}{1+u}\right)\Big|_0^1+\left(\frac{1}{2}\log\left(\frac{1+u}{1-u}\right)+\frac{\log(1-u)}{1+u}\right)\Big|_0^1=$$

$$=\frac{1}{2}\log 2+\lim_{u\to 1}\left\{\frac{1}{2}\log(1+u)+\left(\frac{1}{1+u}-\frac{1}{2}\right)\log(1-u)\right\}=\log 2$$

Therefore,

$$\Omega=\sum_{n=0}^\infty\left(\sum_{k=1}^\infty(-1)^{k-1}\int_0^\infty e^{-(n+k)x}\,dx\right)^2=\log 2$$

1825. Find:

$$\Omega=\int_0^\infty \frac{\sqrt{x}\log(x)}{x^2+1}dx$$

Proposed by Vasile Mircea Popa-Romania

Solution 1 by Togrul Ehmedov-Azerbaijan

$$\Omega=\int_0^\infty \frac{\sqrt{x}\log(x)}{x^2+1}dx=\int_0^1 \frac{\sqrt{x}\log(x)}{x^2+1}dx+\int_1^\infty \frac{\sqrt{x}\log(x)}{x^2+1}dx\Big]_{x=\frac{1}{y}}=$$

$$=\int_0^1 \frac{\sqrt{x}\log(x)}{x^2+1}dx-\int_0^1 \frac{\log(y)}{\sqrt{y}(y^2+1)}dy=$$

$$=\sum_{k=0}^\infty(-1)^k\int_0^1 x^{2k+\frac{1}{2}}\log(x)\,dx-\sum_{k=0}^\infty(-1)^k\int_0^1 y^{2k-\frac{1}{2}}\log(y)\,dy=$$

$$=\sum_{k=0}^\infty\frac{(-1)^{k+1}}{\left(2k+\frac{3}{2}\right)^2}-\sum_{k=0}^\infty\frac{(-1)^{k+1}}{\left(2k+\frac{1}{2}\right)^2}=$$

$$=\sum_{k=0}^\infty\left[\frac{1}{\left(4k+\frac{7}{2}\right)^2}-\frac{1}{\left(4k+\frac{3}{2}\right)^2}\right]-\sum_{k=0}^\infty\left[\frac{1}{\left(4k+\frac{5}{2}\right)^2}-\frac{1}{\left(4k+\frac{1}{2}\right)^2}\right]=$$

$$=\frac{1}{16}\left[\sum_{k=0}^{\infty}\left[\frac{1}{\left(k+\frac{7}{8}\right)^2}-\frac{1}{\left(k+\frac{3}{8}\right)^2}\right]-\sum_{k=0}^{\infty}\left[\frac{1}{\left(k+\frac{5}{8}\right)^2}-\frac{1}{\left(k+\frac{1}{8}\right)^2}\right]\right]=$$

$$=\frac{1}{16}\left[\varphi^{(1)}\left(\frac{7}{8}\right)-\varphi^{(1)}\left(\frac{3}{8}\right)-\varphi^{(1)}\left(\frac{5}{8}\right)+\varphi^{(1)}\left(\frac{1}{8}\right)\right]$$

We know that

$$\varphi^{(1)}(1-z)+\varphi^{(1)}(z)=\frac{\pi^2}{\sin^2(\pi z)}\Rightarrow\begin{cases}\varphi^{(1)}\left(\frac{7}{8}\right)+\varphi^{(1)}\left(\frac{1}{8}\right)=\frac{\pi^2}{\sin^2\left(\frac{\pi}{8}\right)}\\ \varphi^{(1)}\left(\frac{5}{8}\right)+\varphi^{(1)}\left(\frac{3}{8}\right)=\frac{\pi^2}{\sin^2\left(\frac{3\pi}{8}\right)}\end{cases}$$

$$\Omega=\frac{1}{16}\left[\frac{\pi^2}{\sin^2\left(\frac{\pi}{8}\right)}-\frac{\pi^2}{\sin^2\left(\frac{3\pi}{8}\right)}\right]=\frac{1}{16}\left[\frac{2\pi^2}{1-\cos\left(\frac{\pi}{4}\right)}-\frac{2\pi^2}{1-\cos\left(\frac{3\pi}{4}\right)}\right]=$$

$$=\frac{\pi^2}{8}\left[\frac{1}{1-\cos\left(\frac{\pi}{4}\right)}-\frac{1}{1+\cos\left(\frac{\pi}{4}\right)}\right]=\frac{\sqrt{2}\pi^2}{4}$$

Solution 2 by Hikmat Mammadov-Azerbaijan

$$\Omega=\int_0^{\infty}\frac{\sqrt{x}\log x}{x^2+1}dx=\frac{\partial}{\partial a}\int_0^{\infty}\frac{x^a}{1+x^2}dx\Big|_{a=\frac{1}{2}}$$

$$\int_0^{\infty}\frac{x^a}{1+x^2}dx=\int_0^{\infty}\frac{v^{\frac{a}{2}}}{1+v}\cdot\frac{dv}{2\sqrt{v}}=\frac{1}{2}\int_0^{\infty}\frac{v^{\frac{a+1}{2}-1}}{1+v}dv=f(a)$$

$$\beta(x,y)=\int_0^{\infty}\frac{u^{x-1}}{(1+v)^{x+y}}dx$$

$$f(a)=\frac{\beta\left(\frac{a+1}{2};\frac{1-a}{2}\right)}{2}=\frac{1}{2}\left[\beta\left(\frac{1+a}{2};1-\frac{1+a}{2}\right)\right]=$$

$$=\frac{1}{2}\cdot\frac{\pi}{\sin\left(\frac{1+a}{2}\pi\right)}=\frac{\pi}{2\cos\left(\frac{\pi a}{2}\right)}$$

$$\Omega=\frac{\pi^2}{4}\cdot\frac{\sin\left(\frac{\pi a}{2}\right)}{\cos^2\left(\frac{\pi a}{2}\right)}\Big|_{a=\frac{1}{2}}=\frac{\pi^2}{4}\cdot\sqrt{2}=\frac{\pi^2}{2\sqrt{2}}$$

Therefore,

$$\Omega = \int_0^\infty \frac{\sqrt{x}\log x}{x^2+1}dx = \frac{\pi^2}{2\sqrt{2}}$$

Solution 3 by Adrian Popa-Romania

$$\Omega = \int_0^\infty \frac{\sqrt{x}\log x}{x^2+1}dx = \frac{\partial}{\partial a}\int_0^\infty \frac{x^a}{1+x^2}dx\Big|_{a=\frac{1}{2}}$$

$$= \frac{\partial}{\partial a}\left(\frac{1}{2}\int_0^\infty \frac{\left(t^{\frac{1}{2}}\right)^{a+\frac{1}{2}}}{t+1}\cdot t^{-\frac{1}{2}}dt\right)\Bigg|_{a=0} = \frac{\partial}{\partial a}\cdot\frac{1}{2}\int_0^\infty \frac{t^{\frac{a}{2}-\frac{1}{4}}}{t+1}\Big|_{a=0}dt$$

$$\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx, \qquad m-1 = \frac{a}{2}-\frac{1}{4} \Rightarrow m = \frac{a}{2}+\frac{3}{4}$$

$$m+n = 1 \Rightarrow \frac{a}{2}+\frac{3}{4}+n = 1 \Rightarrow n = -\frac{a}{2}+\frac{1}{4}$$

$$\Omega = \frac{\partial}{\partial a}\cdot\frac{1}{2}\beta\left(\frac{a}{2}+\frac{3}{4};-\frac{a}{2}+\frac{1}{4}\right)_{|a=0}$$

$$\beta\left(\frac{a}{2}+\frac{3}{4};-\frac{a}{2}+\frac{1}{4}\right) = \frac{\Gamma\left(\frac{2a+3}{4}\right)\Gamma\left(\frac{-2a+1}{4}\right)}{\Gamma(1)} = \Gamma\left(\frac{2a+3}{4}\right)\Gamma\left(1-\frac{2a+3}{4}\right)$$

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin\pi x} \Rightarrow \beta\left(\frac{2a+3}{4};\frac{-2a+1}{4}\right) = \frac{\pi}{\sin\frac{\pi(2a+3)}{4}}$$

$$\frac{\partial}{\partial a}\beta\left(\frac{2a+3}{4};\frac{-2a+1}{4}\right)\Bigg|_{a=0} = \frac{\partial}{\partial a}\frac{\pi}{\sin\left(\frac{\pi a}{2}+\frac{3\pi}{4}\right)} = \frac{-\pi\cos\left(\frac{\pi a}{2}+\frac{3\pi}{4}\right)\cdot\frac{\pi}{2}}{\sin^2\left(\frac{\pi a}{2}+\frac{3\pi}{4}\right)} = \frac{\pi^2\sqrt{2}}{2}$$

Therefore,

$$\Omega = \frac{1}{2}\cdot\frac{\pi^2\sqrt{2}}{2} = \frac{\pi^2}{2\sqrt{2}}$$

Solution 4 by Fao Ler-Iraq

$$\Omega = \int_0^\infty \frac{\sqrt{x}\log x}{x^2+1}dx = \int_0^\infty \frac{\sqrt{\sqrt{x}}\log\sqrt{x}}{x+1}d\left(\sqrt{x}\right) =$$

$$=\frac{1}{4}\int_0^{\infty}\frac{x^{\frac{1}{4}-\frac{1}{2}}\log x}{x+1}dx=\frac{1}{4}\frac{d}{dx}\int_0^{\infty}\frac{x^y}{x+1}dx;\left(y=-\frac{1}{4}\right)$$

$$=\frac{1}{4}\frac{d}{dy}\pi\csc(\pi(y+1))=\frac{\pi}{4}(-\pi\cot(\pi(y+1))\csc(\pi(y+1)))=$$

$$=-\frac{\pi^2}{4}\cot\left(\frac{3\pi}{4}\right)\csc\left(\frac{3\pi}{4}\right)=\frac{\pi^2}{4}\sqrt{2}$$

1826. Find a closed form:

$$\Omega=\int_0^1\frac{\tan^{-1}x-\tan^{-1}\left(\frac{1}{x}\right)}{1+x^2}\cdot\frac{1+x}{1-x}dx$$

Proposed by Togrul Ehmedov-Azerbaijan

Solution 1 by Rana Ranino-Setif-Algerie

$$\Omega=\int_0^1\frac{\tan^{-1}x-\tan^{-1}\left(\frac{1}{x}\right)}{1+x^2}\cdot\frac{1+x}{1-x}dx=$$

$$=\int_0^{\frac{\pi}{4}}\left(2y-\frac{\pi}{2}\right)\left(\frac{1+\tan y}{1-\tan y}\right)dy\overset{y\to\frac{\pi}{4}-y}{=}-2\int_0^{\frac{\pi}{4}}y\cot y\,dy\overset{IBP}{=}$$

$$=-2y\log(\sin y)|_0^{\frac{\pi}{4}}+2\int_0^{\frac{\pi}{4}}\log(\sin y)\,dy=\frac{\pi}{4}\log 2-\int_0^{\frac{\pi}{4}}\left(\log 2+\sum_{n=1}^{\infty}\frac{\cos(2ny)}{n}\right)dy=$$

$$=-\frac{\pi}{4}\log 2-\sum_{n=1}^{\infty}\frac{\sin\left(\frac{\pi n}{2}\right)}{n^2}$$

Therefore,

$$\Omega=\int_0^1\frac{\tan^{-1}x-\tan^{-1}\left(\frac{1}{x}\right)}{1+x^2}\cdot\frac{1+x}{1-x}dx=-\frac{\pi}{4}\log 2-G$$

Solution 2 by Hikmat Mammadov-Azerbaijan

$$\Omega=\int_0^1\frac{\tan^{-1}x-\tan^{-1}\left(\frac{1}{x}\right)}{1+x^2}\cdot\frac{1+x}{1-x}dx=$$

$$=\int_0^1\frac{\tan^{-1}\left(\frac{x^2-1}{2x}\right)}{1+x^2}\cdot\frac{1+x}{1-x}dx\overset{s=\frac{1-x}{1+x}}{=}-2\int_0^1\frac{\tan^{-1}s}{s(1+s^2)}ds=$$

$$= -2\int_0^1 \frac{\tan^{-1} s}{s} ds + 2\int_0^1 \frac{\tan^{-1} s}{1+s^2} ds = -2G + \frac{\pi}{4}\log 2 - \int_0^1 \frac{\log(1+s^2)}{1+s^2} ds =$$

$$= -2G + \frac{\pi}{4}\log 2 + 2\int_0^{\frac{\pi}{4}} \log(\cos x)\, dx =$$

$$= -2G + \frac{\pi}{4}\log 2 + 2\int_0^{\frac{\pi}{4}} \log(\cos x)\, dx - 2\int_0^{\frac{\pi}{4}} \log(\sin x)\, dx =$$

$$= -2G + \frac{\pi}{4}\log 2 - \pi\log 2 + \int_0^{\frac{\pi}{4}} \frac{x\cos x}{\sin x} dx \overset{(x=\tan^{-1} s)}{=}$$

$$= -2G - \frac{\pi}{2}\log 2 + 2\int_0^1 \frac{\tan^{-1} s}{s(1+s^2)} ds$$

Therefore,

$$\Omega = \int_0^1 \frac{\tan^{-1} x - \tan^{-1}\left(\frac{1}{x}\right)}{1+x^2} \cdot \frac{1+x}{1-x} dx = -\frac{\pi}{4}\log 2 - G$$

1827. Find:

$$\Omega = \int_0^\infty \frac{\log(1+x)}{x(x^2+x+1)} dx$$

Proposed by Vasile Mircea Popa-Romania

Solution by Togrul Ehmedov-Azerbaijan

$$\Omega = \int_0^\infty \frac{\log(1+x)}{x(x^2+x+1)} dx = \int_0^1 \int_0^\infty \frac{dxdy}{(1+xy)(x^2+x+1)} =$$

$$= \int_0^1 \frac{1}{1-y+y^2} \int_0^\infty \left[\frac{-yx+1-y}{x^2+x+1} + \frac{y^2}{1+xy}\right] dx\, dy =$$

$$= \int_0^1 \frac{1}{1-y+y^2}\left[y\log(y) - \frac{\sqrt{3}}{9}\pi y + \frac{2\sqrt{3}}{9}\pi\right] dy =$$

$$= \int_0^1 \frac{y\log(y)}{1-y+y^2} dy - \frac{\sqrt{3}}{9}\pi \int_0^1 \frac{y}{1-y+y^2} dy + \frac{2\sqrt{3}}{9}\pi \int_0^1 \frac{1}{1-y+y^2} dy =$$

$$=\frac{1}{6}\left(\frac{5\pi^2}{6}-\varphi'\left(\frac{1}{3}\right)\right)-\frac{\sqrt{3}}{9}\pi\left(\frac{\pi}{3\sqrt{3}}\right)+\frac{2\sqrt{3}}{9}\pi\left(\frac{2\pi}{3\sqrt{3}}\right)=\frac{\pi^2}{4}-\frac{1}{6}\varphi'\left(\frac{1}{3}\right)$$

$$\text{Note: }\int_0^1\frac{y\log(y)}{1-y+y^2}dy=\frac{1}{6}\left(\frac{5\pi^2}{6}-\varphi'\left(\frac{1}{3}\right)\right)$$

1828. Find:

$$\Omega=\int_0^1\frac{x^{49}}{1+x+x^2+\cdots+x^{100}}dx$$

Proposed by Hussain Reza Zadah-Afghanistan

Solution by Fao Ler-Iraq

$$\Omega=\int_0^1\frac{x^{49}}{1+x+x^2+\cdots+x^{100}}dx=\int_0^1\frac{x^{49}(1-x)}{1-x^{101}}dx=$$

$$=\sum_{k=0}^{\infty}\int_0^1(1-x)x^{101k+49}dx=\sum_{k=0}^{\infty}\left(\frac{1}{101k+50}-\frac{1}{101k+51}\right)=$$

$$=\frac{1}{101}\sum_{k=0}^{\infty}\left(\frac{1}{k+\frac{50}{101}}-\frac{1}{k+\frac{50}{101}}\right)=\frac{1}{101}\left(-\psi\left(\frac{50}{101}\right)+\psi\left(\frac{50}{101}\right)\right)=$$

$$=\frac{1}{101}\left(\psi\left(1-\frac{50}{101}\right)-\psi\left(\frac{50}{101}\right)\right)=\frac{1}{101}\left(\pi\cot\left(\frac{50\pi}{101}\right)\right)=\frac{\pi}{101}\tan\left(\frac{\pi}{202}\right)$$

1829. Prove that:

$$\int_0^{\frac{\pi}{2}}\int_1^{\infty}\frac{1}{z^2(1+2\tan^2 x)-1}dzdx=\frac{\pi^2}{8}$$

Proposed by Hikmat Mammadov-Azerbaijan

Solution by Rana Ranino-Setif-Algerie

$$\Omega=\int_0^{\frac{\pi}{2}}\int_1^{\infty}\frac{1}{z^2(1+2\tan^2 x)-1}dzdx\overset{y=\tan x}{=}\int_1^{\infty}\int_1^{\infty}\frac{1}{(2z^2y^2+z^2-1)(1+y^2)}dydz=$$

$$=\int_1^{\infty}\frac{1}{1+z^2}\int_0^{\infty}\left(\frac{2z^2}{2z^2y^2+z^2-1}-\frac{1}{1+y^2}\right)dydz=$$

$$=\int_1^{\infty}\frac{1}{1+z^2}\left[\frac{z\sqrt{2}}{\sqrt{z^2-1}}\tan^{-1}\left(\frac{yz\sqrt{2}}{\sqrt{z^2-1}}\right)-\tan^{-1}y\right]_0^{\infty}dz=$$

$$=\frac{\pi}{2}\int_1^\infty \frac{1}{1+z^2}\left(\frac{z\sqrt{2}}{\sqrt{z^2-1}}-1\right)dz=\frac{\pi}{\sqrt{2}}\int_1^\infty \frac{z}{(1+z^2)\sqrt{z^2-1}}dz-\frac{\pi}{2}\int_1^\infty \frac{dz}{1+z^2}$$

$$\int_1^\infty \frac{z}{(1+z^2)\sqrt{z^2-1}}dz \overset{z=\sec\theta}{=} \int_0^{\frac{\pi}{2}} \frac{d\theta}{1+\cos^2\theta} \overset{y=\tan\theta}{=}$$

$$=\int_0^\infty \frac{dy}{2+y^2}=\frac{\pi}{2\sqrt{2}}$$

$$\Omega=\frac{\pi}{\sqrt{2}}\cdot\frac{\pi}{2\sqrt{2}}-\frac{\pi}{2}\left(\frac{\pi}{2}-\frac{\pi}{4}\right)=\frac{\pi^2}{4}-\frac{\pi^2}{8}=\frac{\pi^2}{8}$$

Therefore,

$$\int_0^{\frac{\pi}{2}}\int_1^\infty \frac{1}{z^2(1+2\tan^2 x)-1}dzdx=\frac{\pi^2}{8}$$

1830. Find a closed form:

$$\Omega=\int_0^1 \frac{\log(1-x)}{x^2+x+1}dx$$

Proposed by Togrul Ehmedov-Azerbaijan

Solution 1 by Fao Ler-Iraq

$$\Omega=\int_0^1 \frac{\log(1-x)}{x^2+x+1}dx=\int_0^1 \frac{\log x}{(1-x)^2+1-x+1}dx=$$

$$=\int_0^1 \frac{\log x}{x^2-3x+3}dx=\int_0^1 \frac{\log x}{(x-a_1)(x-a_2)}dx\,;a_{1,2}=\frac{3\pm i\sqrt{3}}{2}$$

$$=\frac{1}{a_1-a_2}\int_0^1\left(\frac{\log x}{x-a_1}-\frac{\log x}{x-a_2}\right)dx=\frac{1}{a_1-a_2}\left(Li_2\left(-\frac{1}{-a_1}\right)-Li_2\left(-\frac{1}{-a_2}\right)\right)=$$

$$=-\frac{i}{\sqrt{3}}\left(Li_2\left(\frac{3-i\sqrt{3}}{6}\right)-Li_2\left(\frac{3+i\sqrt{3}}{6}\right)\right)=$$

$$=-\frac{i}{\sqrt{3}}(Li_2\left(\frac{1}{\sqrt{3}}e^{-\frac{\pi i}{6}}\right)-Li_2\left(\frac{1}{\sqrt{3}}e^{\frac{\pi i}{6}}\right)=$$

$$=-\frac{i}{\sqrt{3}}\left(-Li_2(e^{\frac{2\pi i}{3}}-\frac{1}{2}\log^2\left(\frac{1}{\sqrt{3}}e^{-\frac{\pi i}{6}}\right)+Li_2\left(e^{-\frac{2\pi i}{3}}\right)+\frac{1}{2}\log^2\left(\frac{1}{\sqrt{3}}e^{\frac{\pi i}{6}}\right)\right)=$$

$$=-\frac{i}{\sqrt{3}}(-2i)Im\left(Li_2\left(e^{\frac{2\pi i}{3}}\right)+\frac{1}{2}\log^2\left(\frac{1}{\sqrt{3}}e^{\frac{\pi i}{6}}\right)\right)=$$

$$= -\frac{2}{\sqrt{3}}\left(Cl_2\left(\frac{2\pi}{3}\right) - \frac{\pi}{12}\log 3\right) = \frac{\pi}{6\sqrt{3}}\log 3 - \frac{4}{3\sqrt{3}}Cl_2\left(\frac{\pi}{3}\right)$$

Solution 2 by Hamza Djahel-Algerie

$$\Omega = \int_0^1 \frac{\log(1-x)}{x^2+x+1}dx = \int_0^1 \frac{\log\left(\frac{1-x}{x}\right)}{x^2+x+1}dx + \int_0^1 \frac{\log x}{x^2+x+1}dx$$

$$A = \int_0^1 \frac{\log\left(\frac{1-x}{x}\right)}{x^2+x+1}dx \overset{y=\frac{1-x}{x}}{=} \int_0^\infty \frac{\log y}{y^2+3y+3}dy = \frac{1}{\sqrt{3}}\int_0^\infty \frac{\log(\sqrt{3})+\log y}{y^2+\sqrt{3}y+1}dy =$$

$$= \frac{1}{\sqrt{3}}\int_0^\infty \frac{\log(\sqrt{3})}{y^2+\sqrt{3}y+1}dy = \left[\frac{\log 3}{\sqrt{3}}\tan^{-1}(2y+\sqrt{3})\right]_0^\infty =$$

$$= \frac{\log 3}{\sqrt{3}}\left(\frac{\pi}{2}-\frac{\pi}{3}\right) = \frac{\pi\log 3}{6\sqrt{3}}$$

$$B = \int_0^1 \frac{\log x}{x^2+x+1}dx = \int_0^1 \frac{(1-x)\log x}{1-x^3}dx \overset{x^3=y}{=} \frac{1}{9}\int_0^1 \frac{y^{-\frac{2}{3}}-y^{-\frac{1}{3}}}{1-y}\log y\,dy =$$

$$= \frac{1}{9}\left[\psi^{(1)}\left(\frac{2}{3}\right) - \psi^{(1)}\left(\frac{1}{3}\right)\right]$$

Therefore,

$$\Omega = A + B = \frac{\pi\log 3}{6\sqrt{3}} + \frac{1}{9}\left[\psi^{(1)}\left(\frac{2}{3}\right) - \psi^{(1)}\left(\frac{1}{3}\right)\right]$$

1831. Find a closed form:

$$\Omega = \int_0^1 \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\cdot\log(1-x)\,dx$$

Proposed by Togrul Ehmedov-Azerbaijan

Solution by Rana Ranino-Setif-Algerie

$$\Omega = \int_0^1 \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\cdot\log(1-x)\,dx =$$

$$= \left[\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\cdot\log(1-x)\cdot(x-1) - x\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right]_0^1$$

$$+ 2\int_0^1 \frac{(1-x)\log(1-x)+x}{1+x^2}dx$$

$$\Omega=-\frac{\pi}{2}-2\underbrace{\int_0^1\frac{x\log(1-x)}{1+x^2}dx}_{A}+2\underbrace{\int_0^1\frac{\log(1-x)}{1+x^2}dx}_{B}+\underbrace{\int_0^1\frac{2x}{1+x^2}dx}_{\log 2}$$

$$A=-\int_0^1\int_0^1\frac{x^2}{(1-xy)(1+x^2)}dxdy=\int_0^1\frac{1}{1+y^2}\int_0^1\left(\frac{1+xy}{1+x^2}-\frac{1}{1-xy}\right)dxdy=$$

$$=\int_0^1\frac{1}{1+y^2}\left(\frac{\pi}{4}+\frac{1}{2}y\log 2+\frac{\log(1-y)}{y}\right)dy=\frac{\pi^2}{16}+\frac{1}{4}\log^2 2+\int_0^1\frac{\log(1-y)}{y(1+y^2)}dy=$$

$$=\frac{\pi^2}{16}+\frac{1}{4}\log^2 2+\underbrace{\int_0^1\frac{\log(1-y)}{y}dy}_{-\frac{\pi^2}{6}}-\underbrace{\int_0^1\frac{y\log(1-y)}{1+y^2}dy}_{A}$$

$$A=-\frac{5\pi^2}{96}+\frac{1}{8}\log^2 2$$

$$B\overset{x=\tan\theta}{=}\int_0^{\frac{\pi}{4}}\log(1-\tan\theta)\,d\theta=\underbrace{\int_0^{\frac{\pi}{4}}\log(\cos\theta-\sin\theta)\,d\theta}_{\theta=\frac{\pi}{4}-\theta}-\int_0^{\frac{\pi}{4}}\log(\cos\theta)\,d\theta=$$

$$=\int_0^{\frac{\pi}{4}}\log(\sqrt{2}\sin\theta)\,d\theta-\int_0^{\frac{\pi}{4}}\log(\cos\theta)\,d\theta=\frac{\pi}{8}\log 2+\int_0^{\frac{\pi}{4}}\log(\tan\theta)\,d\theta=\frac{\pi}{8}\log 2-G$$

Therefore,

$$\Omega=\int_0^1\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\cdot\log(1-x)\,dx=-\frac{\pi}{2}+\frac{5\pi^2}{48}-\frac{1}{4}\log^2 2+\frac{\pi}{4}\log 2-2G+\log 2$$

1832. Find a closed form:

$$\Omega(a,b)=\int_0^1\frac{\sin(a\log x)\sin(b\log x)}{x\log^2 x}dx,\qquad a,b\in\mathbb{R}$$

Proposed by Togrul Ehmedov-Azerbaijan

Solution 1 by Rana Ranino-Setif-Algerie

$$\Omega(a,b)=\int_0^1\frac{\sin(a\log x)\sin(b\log x)}{x\log^2 x}dx\overset{\log x=-t}{=}$$

$$=\int_0^\infty\frac{\sin(at)\sin(bt)}{t^2}dt=\int_0^\infty y\int_0^\infty\sin(at)\sin(bt)\,e^{-ty}dtdy$$

$$\sin(\alpha t)\sin(\beta t)=\frac{1}{2}\cos(\alpha t)-\frac{1}{2}\cos(\beta t)\,;(\alpha=a-b,\beta=a+b,b<a)$$

$$\Omega(\alpha,\beta)=\frac{1}{2}\int_0^{\infty}y\int_0^{\infty}\cos(\alpha t)\,e^{-ty}dtdy-\frac{1}{2}\int_0^{\infty}y\int_0^{\infty}\cos(\beta t)\,e^{-ty}dtdy=$$

$$=\frac{1}{2}\int_0^{\infty}\left(\frac{y^2}{\alpha^2+y^2}-\frac{y^2}{\beta^2+y^2}\right)dy=\frac{1}{2}\int_0^{\infty}\left(\frac{\beta^2}{\beta^2+y^2}-\frac{\alpha^2}{\alpha^2+y^2}\right)dy=$$

$$=\frac{\pi}{4}(\beta-\alpha)=\frac{\pi b}{2};(b<a)\Rightarrow\Omega(a,b)=\frac{\pi a}{2};(a<b)$$

Therefore,

$$\Omega(a,b)=\int_0^1\frac{\sin(a\log x)\sin(b\log x)}{x\log^2 x}dx=\frac{\pi}{2}\min(a,b)$$

Solution 2 by Muhammad Afzal-Pakistan

$$\Omega(a,b)=\int_0^1\frac{\sin(a\log x)\sin(b\log x)}{x\log^2 x}dx\overset{\log x\to -x}{=}$$

$$=\int_0^{\infty}\frac{\sin(ax)\sin(bx)}{x^2}dx=\int_0^{\infty}\mathcal{L}_x[\sin(ax)\sin(bx)]_{(p)}\mathcal{L}_x^{-1}\left[\frac{1}{x^2}\right]_{(p)}dx=$$

$$\overset{p=x}{=}\int_0^{\infty}\frac{2abp^2}{[(a-b)^2+p^2][(a+b)^2+p^2]}dp=$$

$$=2ab\int_0^{\infty}\frac{x^2}{((a-b)^2+x^2)((a+b)^2+x^2)}dx=$$

$$=2ab\int_0^{\infty}\left(\frac{(a-b)^2}{-4ab((a-b)^2+x^2)}-\frac{(a+b)^2}{-4ab((a+b)^2+x^2)}\right)dx=$$

$$=-\frac{1}{2}\int_0^{\infty}\left(\frac{(a-b)^2}{(a-b)^2+x^2}-\frac{(a+b)^2}{(a+b)^2+x^2}\right)dx=\begin{cases}\frac{b\pi}{2},\text{if } a>b\\ \frac{a\pi}{2},\text{if } a<b\end{cases}$$

Therefore,

$$\Omega(a,b)=\int_0^1\frac{\sin(a\log x)\sin(b\log x)}{x\log^2 x}dx=\frac{\pi}{2}\min(a,b)$$

1833. Find a closed form:

$$\Omega(x)=\prod_{n=1}^{\infty}\frac{n^2(n^2+x^2)}{n^4+n^2x^2+x^4},\ \ x\in\mathbb{R}$$

Proposed by Hikmat Mammadov-Azerbaijan

Solution by Fao Ler-Iraq

$$\Omega(x)=\prod_{n=1}^{\infty}\frac{n^2(n^2+x^2)}{n^4+n^2x^2+x^4}=\prod_{n=1}^{\infty}\left(1+\frac{x^2}{n^2}\right)\left(1+\frac{x^2}{n^2}+\frac{x^4}{n^4}\right)^{-1}=$$

$$=\frac{\sin(x\pi i)}{x\pi i}\prod_{n=1}^{\infty}\left(1-\frac{x^2e^{-\frac{2\pi i}{3}}}{n^2}\right)^{-1}\left(1-\frac{x^2e^{\frac{2\pi i}{3}}}{n^2}\right)^{-1}=$$

$$=\frac{\sinh(x\pi)}{x\pi}\left(\frac{\sin\left(x\pi e^{-\frac{\pi i}{3}}\right)}{x\pi e^{-\frac{\pi i}{3}}}\right)^{-1}\left(\frac{\sin\left(x\pi e^{\frac{\pi i}{3}}\right)}{x\pi e^{\frac{\pi i}{3}}}\right)^{-1}=$$

$$=x\pi\sinh(x\pi)\csc\left(x\pi e^{-\frac{\pi i}{3}}\right)\csc\left(x\pi e^{\frac{\pi i}{3}}\right)=$$

$$=x\pi\sinh(x\pi)\csc\left(\frac{x\pi}{2}-\frac{x\pi\sqrt{2}}{2}i\right)\csc\left(\frac{x\pi}{2}+\frac{x\pi\sqrt{3}}{2}i\right)=\frac{2x\pi\sinh(x\pi)}{\cosh(x\pi\sqrt{3})-\cos(x\pi)}$$

1834. Find a closed form:

$$\Omega=\int_0^{\infty}\frac{(e^{-x}+1)(e^x-1)-2x}{x^2(e^x-1)}dx$$

Proposed by Hikmat Mammadov-Azerbaijan

Solution by Hamza Djahel-Algerie

$$\Omega=\int_0^{\infty}\frac{(e^{-x}+1)(e^x-1)-2x}{x^2(e^x-1)}dx\overset{e^{-x}=y}{=}\int_0^1\frac{y^{-1}(1-y^2+2y\log y)}{\log^2 y\,(1-y)}dy=$$

$$=\int_0^1\left(\frac{y^{-1}+1}{\log^2 y}+\frac{2}{\log y(1-y)}\right)dy=\int_0^1\left(\frac{1}{\log y}+\frac{1}{1-y}\right)^2dy+\int_0^1\left(\frac{y^{-1}}{\log^2 y}-\frac{1}{1-y}\right)dy$$

$$=\log(2\pi)-\frac{3}{2}+\left(-\frac{1}{\log y}-\frac{1}{1-y}\right)_0^1=$$

$$=\log(2\pi)-\frac{3}{2}-\lim_{y\to1}\left(\frac{1}{\log y}+\frac{1}{1-y}\right)+\lim_{y\to0}\left(\frac{1}{\log y}+\frac{1}{1-y}\right)=$$

$$=\log(2\pi)-\frac{3}{2}-\frac{1}{2}+1=\log(2\pi)-1\cong 0.8378$$

$$I=\int_0^1\left(\frac{1}{\log x}-\frac{1}{1-x}\right)^2dx=\log(2\pi)-\frac{3}{2}$$

$$\int_0^1 \frac{1-x^y}{1-x}dy = \frac{1}{\log x} + \frac{1}{1-x} \Rightarrow I = \int_0^1\int_0^1\int_0^1 \frac{(x^y-1)(x^z-1)}{(1-x)^2}dxdydz$$

$$\int_0^1 \frac{(x^y-1)(x^z-1)}{(1-x)^2}dx = \frac{(x^y-1)(x^z-1)}{1-x}\Big|_0^2 - \int_0^1 \frac{yx^y(x^z-1)+zx^z(x^y-1)}{1-x}dx =$$

$$= -1 - \int_0^1 \frac{yx^{y+z-1} - yx^{y-1} + zx^{y+z-1} - zx^{z-1}}{1-x}dx =$$

$$= -1 + y\psi(y+z) - y\psi(y) + z\psi(z+y) - z\psi(z)$$

$$\int_0^1 [y\psi(y+z) - y\psi(y) + z\psi(y+z) - z\psi(z) - 1)]dy =$$

$$= y\log(\Gamma(y+z))\big|_0^1 - \int_0^1 \log(\Gamma(y+z))\,dy - y\log(\Gamma(y))\big|_0^1 + \int_0^1 \log(\Gamma(y))\,dy +$$

$$+z\log(\Gamma(y+z))\big|_0^1 - yz\psi(z) - y|_0^1 =$$

$$= \log(\Gamma(z+1)) - \int_0^1 \log(\Gamma(y+z))\,dy + \frac{\log(2\pi)}{2} + z\log\left(\frac{\Gamma(z+1)}{\Gamma(z)}\right) - z\psi(z) - 1$$

$$I = \int_0^1 \log(\Gamma(z+1))\,dz + \int_0^1 \left[\frac{\log(2\pi)}{2} - 1 + z\log z\right]dz - \int_0^1 z\psi(z)dz$$

$$-\int_0^1\int_0^1 \log(\Gamma(y+z))\,dydz =$$

$$= \int_0^1 [\log z + \log(\Gamma(z))]\,dz + \frac{\log(2\pi)}{2} - 1 - \frac{1}{4} - z\log(\Gamma(z)) + \int_0^1 \log(\Gamma(z))\,dz$$

$$-\int_0^1\int_0^1 \log(\Gamma(y+z))\,dydz =$$

$$= -1 + \frac{\log(2\pi)}{2} + \frac{\log(2\pi)}{2} - \frac{5}{4} + \frac{\log(2\pi)}{2} - A = \frac{3\log(2\pi)}{2} - \frac{9}{4} - A$$

$$A = \int_0^1\int_0^1 \log(\Gamma(y+z))\,dydz$$

$$f(z) = \int_0^1 \log(\Gamma(y+z))\,dy \Rightarrow f'(z) = \int_0^1 \psi(y+z)dy = \log(\Gamma(y+z))\big|_0^1$$

$$= \log(\left(\frac{\Gamma(z+1)}{\Gamma(z)}\right) = \log z$$

$$f(z)=z\log z-z+C;f(0)=\int_0^1\log(\Gamma(y))\,dy=\frac{\log(2\pi)}{2}=C$$

$$f(z)=z\log z-z+\frac{\log(2\pi)}{2}\Rightarrow A=\int_0^1 f(z)dz$$

$$A=\int_0^1\left[z\log z-z+\frac{\log(2\pi)}{2}\right]dz=-\frac{1}{4}-\frac{1}{2}+\frac{\log(2\pi)}{2}=-\frac{3}{4}+\frac{\log(2\pi)}{2}$$

$$I=-\frac{4}{9}+\frac{3\log(2\pi)}{2}-\left(-\frac{3}{4}+\frac{\log(2\pi)}{2}\right)=\log(2\pi)-\frac{3}{2}$$

1835. Find:

$$\Omega=\int_0^\infty\frac{\tan^{-1}x}{\sqrt{x}(x^2+1)}dx$$

Proposed by Vasile Mircea Popa-Romania

Solution 1 by Togrul Ehmedov-Azerbaijan

$$\Omega=\int_0^\infty\frac{\tan^{-1}x}{\sqrt{x}(x^2+1)}dx=2\int_0^\infty\frac{\tan^{-1}(x^2)}{x^4+1}dx$$

$$\Omega(a)=2\int_0^\infty\frac{\tan^{-1}(ax)}{x^4+1}dx\Rightarrow\Omega'(a)=2\int_0^\infty\frac{x^2}{(x^4+1)(1+a^2x^4)}dx=$$

$$=2\int_0^\infty\frac{x^4}{(x^4+1)(x^4+a^2)}dx=2\int_0^\infty\frac{dx}{a^2+x^4}-2\int_0^\infty\frac{dx}{(x^4+1)(x^4+a^2)}=$$

$$=2\int_0^\infty\frac{dx}{a^2+x^4}-\frac{2}{a^2-1}\int_0^\infty\frac{dx}{x^4+1}=\frac{2a^2}{a^2-1}\int_0^\infty\frac{dx}{a^2+x^4}-\frac{2}{a^2-1}\cdot\frac{\pi}{4\sin\frac{\pi}{4}}=$$

$$=\frac{2a^2}{a^2-1}\int_0^\infty\frac{dx}{a^2+x^4}-\frac{\pi\sqrt{2}}{2(a^2-1)}=\frac{2a^2}{a^2-1}\frac{\pi}{2\sqrt{2}a\sqrt{a}}-\frac{\pi\sqrt{2}}{2(a^2-1)}=$$

$$=\frac{\pi\sqrt{2}}{2}\left(\frac{\sqrt{a}}{a^2-1}-\frac{1}{a^2-1}\right)$$

$$\Omega(a)=\frac{\pi\sqrt{2}}{2}\left[\frac{\log(a+1)}{2}-\log(\sqrt{a}+1)+\tan^{-1}\sqrt{a}\right]+C$$

$$\Omega(0)=0\Rightarrow C=0$$

$$\Omega(1)=\frac{\pi\sqrt{2}}{2}\left[\frac{\log 2}{2}-\log 2+\tan^{-1}1\right]=\frac{\pi\sqrt{2}}{2}\left(\frac{\pi}{4}-\frac{\log 2}{2}\right)$$

Solution 2 by Ankush Kumar Parcha-India

$$\Omega(\text{a})=\int_0^\infty\frac{\tan^{-1}(ax)}{\sqrt{x}(x^2+1)}dx\Rightarrow\Omega'(a)=\int_0^\infty\frac{\sqrt{x}}{(1+a^2x^2)(1+x^2)}dx$$

$$(1-a^2)\Omega'(a)=\int_0^\infty\frac{\sqrt{x}}{1+x^2}dx-a^2\int_0^\infty\frac{\sqrt{x}}{1+a^2x^2}dx\,;(1)$$

$$\text{Let }\Omega_1=\int\frac{\sqrt{x}}{1+x^2}dx\text{ and }\Omega_2=\int\frac{\sqrt{x}}{1+a^2x^2}dx$$

$$\Omega_1\overset{\sqrt{x}=y}{=}\int\frac{2y^2}{1+y^4}dy=\int\frac{1+\frac{1}{y^2}}{\left(y-\frac{1}{y}\right)^2+2}dy+\int\frac{1-\frac{1}{y^2}}{\left(y+\frac{1}{y}\right)^2-2}dy=$$

$$=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{y^2-1}{y\sqrt{2}}\right)+\frac{1}{2\sqrt{2}}\log\left|\frac{y^2-y\sqrt{2}+1}{y^2+y\sqrt{2}+1}\right|+C=$$

$$=\frac{1}{\sqrt{2}}\tan^1\left(\frac{x-1}{\sqrt{2x}}\right)+\frac{1}{2\sqrt{2}}\log\left|\frac{x-\sqrt{2x}+1}{x+\sqrt{2x}+1}\right|+C$$

$$\Omega_2\overset{x=\frac{y^2}{a}}{=}\frac{1}{a\sqrt{a}}\int\frac{2y^2}{1+y^4}dy=\frac{1}{a\sqrt{a}}\Omega_1$$

Now put limits in $\Omega_{1,2}$ and put values in equation (1), we get:

$$(1-a^2)\Omega'(a)=\frac{\pi}{\sqrt{2}}-\frac{a^2}{a\sqrt{a}}\cdot\frac{\pi}{\sqrt{2}}\Rightarrow\Omega'(a)=\frac{\pi}{\sqrt{2}}\left[\frac{1}{1-a^2}-\frac{\sqrt{a}}{1-a^2}\right]$$

Integrating both sides w.r.t. a, we obtain:

$$\Omega(a)+C=\frac{\pi}{\sqrt{2}}\left[\int\frac{da}{1-a^2}-\int\frac{\sqrt{a}}{1-a^2}da\right]$$

$$=\frac{\pi}{\sqrt{a}}\left[\frac{1}{2}\log\left|\frac{a+1}{1-a}\right|-\int\frac{\sqrt{a}}{1+a}da-\int\frac{\sqrt{a}}{1-a}da\right]$$

$$\Omega(0)=0\Rightarrow C=0$$

$$\Omega(a)=\frac{\pi}{\sqrt{2}}\left[\log\left|\frac{(1-\sqrt{a})(1+a)}{(1-a)(1+\sqrt{a})}\right|+\tan^{-1}\sqrt{a}\right]$$

For $a=1$, it follows that:

$$\Omega=\int_0^\infty\frac{\tan^{-1}x}{(x^2+1)\sqrt{x}}dx=\frac{\pi(\pi-\log 4)}{4\sqrt{2}}$$

1836. Find a closed form:

$$\Omega=\int_0^1\int_0^1\frac{\log(1-x)\log(1-y)}{1-xy}dxdy$$

Proposed by Togrul Ehmedov-Azerbaijan

Solution by Bamidele Benjamin-Nigeria

$$\Omega=\int_0^1\int_0^1\frac{\log(1-x)\log(1-y)}{1-xy}dxdy$$

$$=\sum_{k\geq 1}\int_0^1\int_0^1(xy)^{k-1}\log(1-x)\log(1-y)\,dxdy=$$

$$=\sum_{k\geq 1}\frac{-H_k}{k}\int_0^1 y^{k-1}\log(1-y)\,dy=-\int_0^1\frac{Li_2(y)\log(1-y)}{y}dy-\int_0^1\frac{\log^3(1-y)}{2y}dy$$

$$=A+B$$

$$A=-\int_0^1\frac{Li_2(y)\log(1-y)}{y}dy\overset{IBP}{=}\left(Li_2(y)\right)^2\Big|_0^1+\int_0^1\frac{Li_2(y)\log(1-y)}{y}dy$$

$$2A=\left(Li_2(1)\right)^2\Rightarrow A=\frac{1}{2}Li_2^2(1)=\frac{5}{4}\zeta(4)$$

$$B=-\int_0^1\frac{\log^3(1-y)}{2y}dy\overset{y=1-t}{=}\int_0^1\frac{\log^3(t)}{2(1-t)}dt=\frac{1}{2}\sum_{k\geq 1}\int_0^1 t^{k-1}\log^3 t\,dt\overset{IBP}{=}$$

$$=3\sum_{k\geq 1}\frac{1}{k^4}=3\zeta(4)$$

$$\Omega=\frac{5}{4}\zeta(4)+3\zeta(4)=\frac{17}{4}\zeta(4),\qquad \sum_{k\geq 1}\frac{H_k}{k}x^{k-1}=\frac{Li_2(x)}{x}+\frac{\log^2(1-x)}{2x}$$

$$\frac{-H_k}{k}=\int_0^1 x^{k-1}\log(1-x)\,dx$$

1837. Find without any software:

$$\Omega=\int_0^1\frac{\sin^{-1}\left(\frac{2x}{1+x^2}\right)\tan^{-1}x}{1+x}dx$$

Proposed by Togrul Ehmedov-Azerbaijan

Solution by Avishek Mitra-West Bengal-India

$$\Omega=\int_0^{\frac{\pi}{4}}x\log(1+\tan x)\,dx=\int_0^{\frac{\pi}{4}}x\log(\sin x+\cos x)dx-\int_0^{\frac{\pi}{4}}x\log(\cos x)\,dx$$

$$\Omega_2=\int_0^{\frac{\pi}{4}}x\log(\cos x)\,dx=\int_0^{\frac{\pi}{4}}x\left[-\log 2-\sum_{n=1}^{\infty}\frac{(-1)^n\cos(2nx)}{n}\right]dx=$$

$$=-\log 2\cdot\frac{x^2}{2}\Big|_0^{\frac{\pi}{4}}-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^{\frac{\pi}{4}}x\cos(2nx)\,dx=$$

$$-\frac{\pi^2}{32}\log 2-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left[x\cdot\frac{\sin(2nx)}{2n}\Big|_0^{\frac{\pi}{4}}-\frac{1}{2n}\int_0^{\frac{\pi}{4}}\sin(2nx)\,dx\right]=$$

$$=-\frac{\pi^2}{32}\log 2-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left[\frac{\pi}{8n}\sin\left(\frac{n\pi}{2}\right)+\frac{1}{4n^2}\cos(2nx)\Big|_0^{\frac{\pi}{4}}\right]=$$

$$=-\frac{\pi^2}{32}\log 2-\frac{\pi}{8}\sum_{n=1}^{\infty}\frac{(-1)^n\sin\left(\frac{n\pi}{2}\right)}{n^2}-\frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^n\left(\cos\left(\frac{n\pi}{2}\right)-1\right)}{n^3}=$$

$$=-\frac{\pi^2}{32}\log 2+\frac{\pi}{8}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2}-\frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3}+\frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cos\left(\frac{n\pi}{2}\right)}{n^3}=$$

$$=-\frac{\pi^2}{32}\log 2+\frac{\pi}{8}G-\frac{1}{4}\eta(3)+\frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n)^3}=$$

$$=-\frac{\pi^2}{32}\log 2+\frac{\pi}{8}G-\frac{1}{4}\cdot\frac{3}{4}\zeta(3)+\frac{1}{32}\cdot\frac{3}{4}\zeta(3)=-\frac{\pi^2}{32}\log 2+\frac{\pi}{8}G-\frac{21}{128}\zeta(3)$$

$$\Omega_1=\int_0^{\frac{\pi}{4}}x\log(\sin x+\cos x)dx=\int_0^{\frac{\pi}{4}}x\log\left\{\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)\right\}dx=$$

$$=\frac{1}{2}\log 2\int_0^{\frac{\pi}{4}}xdx+\int_0^{\frac{\pi}{4}}x\log\left(sin\left(x+\frac{\pi}{4}\right)\right)dx=$$

$$=\frac{\pi^2}{64}\log 2+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(x-\frac{\pi}{4}\right)\log(\sin x)\,dx=$$

$$=\frac{\pi^2}{64}\log 2+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}x\log(\sin x)\,dx-\frac{\pi}{4}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\log(\sin x)\,dx$$

$$\Omega_3=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}x\log(\sin x)\,dx=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}x\left[-\log 2-\sum_{n=1}^{\infty}\frac{\cos(2nx)}{n}\right]dx=$$

$$=-\log 2\cdot\frac{x^2}{2}\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}-\sum_{n=1}^{\infty}\frac{1}{n}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}x\cos(2nx)\,dx=$$

$$=-\frac{3\pi^2}{32}\log 2-\sum_{n=1}^{\infty}\frac{1}{n}\left[x\cdot\frac{\sin(2nx)}{2n}\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}-\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{\sin(2nx)}{2n}dx\right]=$$

$$=-\frac{3\pi^2}{32}\log 2-\sum_{n=1}^{\infty}\frac{1}{n}\left[-\frac{\pi}{8n}\sin\left(\frac{n\pi}{2}\right)+\frac{1}{4n^2}\left(\cos(n\pi)-\cos\left(\frac{n\pi}{2}\right)\right)\right]=$$

$$=-\frac{3\pi^2}{32}\log 2-\frac{\pi}{8}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi}{2}\right)}{n^2}-\frac{1}{4}\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{n^3}+\frac{1}{4}\sum_{n=1}^{\infty}\frac{\cos\left(\frac{n\pi}{2}\right)}{n^3}=$$

$$=-\frac{3\pi^2}{32}\log 2+\frac{\pi}{8}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2}+\frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3}-\frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n)^3}=$$

$$=-\frac{3\pi^2}{32}\log 2+\frac{\pi}{8}G+\frac{1}{4}\cdot\frac{3}{4}\zeta(3)-\frac{1}{32}\cdot\frac{3}{4}\zeta(3)=-\frac{3\pi^2}{32}\log 2+\frac{\pi}{8}G+\frac{21}{128}\zeta(3)$$

$$\Omega_4=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\log(\sin x)dx=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left[-\log 2-\sum_{n=1}^{\infty}\frac{\cos(2nx)}{n}\right]dx=$$

$$=-\frac{\pi}{4}\log 2-\sum_{n=1}^{\infty}\frac{1}{n}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cos(2nx)\,dx=-\frac{\pi}{4}\log 2-\sum_{n=1}^{\infty}\frac{1}{n}\frac{\sin(2nx)}{2n}\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}=$$

$$=-\frac{\pi}{4}\log 2+\frac{1}{2}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi}{2}\right)}{n^2}=-\frac{\pi}{4}\log 2+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2}=-\frac{\pi}{4}\log 2+\frac{G}{2}$$

Therefore,

$$\Omega=\int_0^1\frac{\sin^{-1}\left(\frac{2x}{1+x^2}\right)\tan^{-1}x}{1+x}dx=2\int_0^1\frac{\left(\tan^{-1}x\right)^2}{1+x}dx=$$

$$=2(\tan^{-1}x)^2\log(1+x)\Big|_0^1-2\int_0^1\frac{2\tan^{-1}x}{1+x^2}\log(1+x)\,dx=$$

$$=2\cdot\frac{\pi^2}{16}\log 2-4\int_0^1\frac{\tan^{-1}x\log(1+x)}{1+x^2}dx\overset{x=\tan z}{=}$$

$$=\frac{\pi^2}{8}\log 2-4\int_0^{\frac{\pi}{4}} z\log(1+\tan z)\,dz=$$

$$=\frac{\pi^2}{8}\log 2-4\left(\frac{\pi^2}{64}\log 2-\frac{\pi}{8}G+\frac{21}{64}\zeta(3)\right)=\frac{\pi^2}{16}\log 2+\frac{\pi}{2}G-\frac{21}{16}\zeta(3)$$

1838. Find:

$$\Omega(n)=\int_1^n([x]^2\cdot\{x\}+[x]\cdot\{x\}^2)dx\,,n\in\mathbb{N},[*]-GIF,\{x\}=x-[x]$$

Proposed by Togrul Ehmedov-Baku-Azerabaijan

Solution 1 by Avishek Mitra-West Bengal-India

$$\Omega(n)=\int_1^n([x]^2\cdot\{x\}+[x]\cdot\{x\}^2)dx=\int_1^n([x]^2(x-[x])+[x](x-[x])^2)dx=$$

$$=\int_1^n(x[x]^2-[x]^3+x^2[x]-2x[x]^2+[x]^3)dx=$$

$$=\int_1^n(x^2[x]-x[x]^2)dx=\sum(n-1)\int_{n-1}^n x^2dx-\sum(n-1)^2\int_{n-1}^n xdx=$$

$$=\sum(n-1)\cdot\frac{n^3-(n-1)^3}{3}-\sum(n-1)^2\cdot\frac{n^2-(n-1)^2}{2}=$$

$$=\frac{1}{3}\sum(n-1)(n^3-n^3+3n^2-3n+1)-\frac{1}{2}\sum(n-1)^2(n^2-n^2+2n-1)=$$

$$=\sum n^3-\sum n^2+\frac{1}{3}\sum n-\sum n^2+\sum n-\frac{n}{3}-\sum n^3+$$

$$+2\sum n^2-\sum n+\frac{1}{2}\sum n^2-\sum n+\frac{n}{2}=$$

$$=\frac{1}{2}\sum n^2-\frac{2}{3}\sum n+\left(\frac{n}{2}-\frac{n}{3}\right)=\frac{n(n+1)(2n+1)}{12}-\frac{n(n+1)}{3}+\frac{n}{6}=$$

$$=\frac{n(2n^2+3n+1-4n-4+2)}{12}=\frac{n(2n^2-n-1)}{12}=\frac{n(n-1)(2n+1)}{12}$$

Solution 2 by Hikmat Mammadov-Azerbaijan

$$\Omega(n)=\int_1^n([x]^2\cdot\{x\}+[x]\cdot\{x\}^2)dx=$$

$$=\int_1^n[x]\cdot\{x\}([x]+\{x\})dx=\int_1^n x\cdot[x](x-[x])dx=$$

$$= \sum_{k=1}^{n-1} \int_k^{k+1} (x^2 \cdot k + x \cdot k^2) dx = \sum_{k=1}^{n-1} \left(k \cdot \frac{(k+1)^3 + k^3}{3} - k^2 \cdot \frac{(k+1)^2 + k^2}{2} \right) =$$

$$= \sum_{k=1}^{n-1} \left(\frac{3k^3 + 3k^2 + k}{3} - \frac{2k^3 + k^2}{2} \right) = \sum_{k=1}^{n-1} \frac{3k^2 + 2k}{6} =$$

$$= \frac{1}{6} \left(3 \cdot \frac{(n-1)n(2n-1)}{6} + 2 \cdot \frac{(n-1)n}{2} \right) = \frac{n(n-1)(2n+1)}{12}$$

Solution 3 by Felix Marin-USA

$$\Omega(n+1) - \Omega(n) = \int_n^{n+1} ([x]^2\{x\} + [x]\{x\}^2) dx =$$

$$= n^2 \int_n^{n+1} \{x\} dx + n \int_n^{n+1} \{x\}^2 dx = n^2 \int_n^{n+1} (x-n) dx + n \int_n^{n+1} (x-n)^2 dx =$$

$$= n^2 \int_0^1 x dx + n \int_0^1 x^2 dx = \frac{1}{2} n^2 + \frac{1}{3} n$$

$$\Omega(n) = \sum_{k=1}^{n-1} [\Omega(k+1) - \Omega(k)] = \frac{1}{2} \sum_{k=1}^{n-1} k^2 + \frac{1}{3} \sum_{k=1}^{n-1} k =$$

$$= \frac{1}{2} \cdot \frac{(n-1)n(2n-1)}{6} + \frac{1}{3} \cdot \frac{(n-1)n}{2}$$

Therefore,

$$\Omega(n) = \int_1^n ([x]^2 \cdot \{x\} + [x] \cdot \{x\}^2) dx = \frac{2n^3 - n^2 - n}{12}$$

Solution 4 by proposer

$$\Omega(n) = \int_1^n ([x]^2 \cdot \{x\} + [x] \cdot \{x\}^2) dx = \int_1^n [x]\{x\}([x] + \{x\}) dx =$$

$$= \int_1^n x^2 \{x\} dx - \int_1^n x\{x\}^2 dx = \sum_{k=1}^{n-1} \int_k^{k+1} x^2 (x-k) dx - \sum_{k=1}^{n-1} \int_k^{k-1} x(x-k)^2 dx =$$

$$= \sum_{k=1}^{n-1} \int_k^{k+1} (kx^2 - k^2 x) dx = \sum_{k=1}^{n-1} \left[k \cdot \frac{x^3}{3} - k^2 \cdot \frac{x^2}{2} \right]_k^{k+1} =$$

$$=\sum_{k=1}^{n-1}\left[\frac{k(k+1)^3}{3}-\frac{k^4}{3}-\frac{k^2(k+1)^2}{2}+\frac{k^4}{2}\right]=$$

$$=\sum_{k=1}^{n-1}\left(\frac{k^2}{2}+\frac{k}{3}\right)=\frac{1}{12}n(n-1)(2n-1)+\frac{1}{6}n(n-1)=\frac{n(n-1)(2n+1)}{12}$$

1839. Let $n\in\mathbb{Z}_+$. Prove that:

$$\int_0^{\infty}\frac{\sin(x^{-n})\log x}{x}dx=\frac{\pi\gamma}{2n^2}$$

where γ is the Euler-Mascheroni

Proposed by Max Wong-Hong Kong

Solution 1 Muhammad Afzal-Pakistan

$$\Omega=\int_0^{\infty}\frac{\sin(x^{-n})\log x}{x}dx=-\frac{1}{n^2}\int_0^{\infty}\frac{\sin y\cdot\log y}{y}dy=$$

$$=-\frac{1}{n^2}\int_0^{\infty}\frac{\sin x\cdot\log x}{x}dx=-\frac{1}{n^2}\int_0^{\infty}\mathcal{L}\{\sin x\}\mathcal{L}^{-1}\left\{\frac{\log x}{x}\right\}dx=$$

$$=-\frac{1}{n^2}\int_0^{\infty}\frac{1}{1+t^2}(-\gamma-\log t)dt=\frac{\gamma}{n^2}\underbrace{\int_0^{\infty}\frac{1}{1+t^2}dt}_{\frac{\pi}{2}}+\frac{1}{n^2}\underbrace{\int_0^{\infty}\frac{\log t}{1+t^2}dt}_{t=\frac{1}{t}}=\frac{\gamma\pi}{2n^2}$$

$$\mathcal{L}\{\log x\}=-\frac{\gamma}{t}-\frac{\log t}{t},\qquad \frac{\log t}{t}=-\frac{\gamma}{t}-\mathcal{L}\{\log x\}$$

$$\mathcal{L}^{-1}\left\{\frac{\log x}{x}\right\}=\mathcal{L}^{-1}\left\{\frac{-\gamma}{t}\right\}-\mathcal{L}^{-1}\{\mathcal{L}\{\log x\}\}=-\gamma-\log x$$

Solution 2 by Rana Ranino-Setif-Algerie

$$\Omega=\int_0^{\infty}\frac{\sin(x^{-n})\log x}{x}dx\overset{x^{-n}\to x}{=}-\frac{1}{n^2}\int_0^{\infty}\frac{\sin x\log x}{x}dx=$$

$$=-\frac{1}{n^2}\frac{\partial}{\partial s}\left[\int_0^{\infty}x^{s-1}\sin x\,dx\right]\Bigg|_{s=0}$$

Mellin transform:

$$M\{\sin x\}=\int_0^{\infty}x^{s-1}\sin x\,dx=\sin\left(\frac{\pi s}{2}\right)\Gamma(s)$$

$$\Omega=-\frac{1}{n^2}\frac{\partial}{\partial s}\left[\sin\left(\frac{\pi s}{2}\right)\Gamma(s)\right]\Bigg|_{s=0}=-\frac{1}{n^2}\lim_{s\to 0}\Gamma(s)\left(\frac{\pi}{2}\cos\left(\frac{\pi s}{2}\right)+\sin\left(\frac{\pi s}{2}\right)\psi(s)\right)=$$

$$= -\frac{1}{n^2}\lim_{s\to 0}\Gamma(s+1)\frac{\frac{\pi}{2}\cos\left(\frac{\pi s}{2}\right)+\sin\left(\frac{\pi s}{2}\right)\psi(s)}{s}$$

$$\text{when } s\to 0: \sin\left(\frac{\pi s}{2}\right)\sim\frac{\pi s}{2}; \psi(s)\sim -\frac{1}{s}-\gamma$$

$$\sin\left(\frac{\pi s}{2}\right)\psi(s)\sim -\frac{\pi}{2}-\frac{\pi s\gamma}{2}, \qquad \Omega = -\frac{1}{n^2}\lim_{s\to 0}\frac{\frac{\pi}{2}-\frac{\pi}{2}-\frac{\pi s\gamma}{2}}{s} = \frac{\pi\gamma}{2n^2}$$

Therefore,

$$\int_0^\infty \frac{\sin(x^{-n})\log x}{x}dx = \frac{\pi\gamma}{2n^2}$$

Solution 3 by Hikmat Mammadov-Azerbajan

$$\Omega = \int_0^\infty \frac{\sin(x^{-n})\log x}{x}dx = \int_0^\infty \frac{\sin(x^n)\log(x^n)}{nx^n}\cdot x^{n-1}dx =$$

$$= \frac{1}{n^2}\int_0^\infty \frac{\sin(x^n)\log(x^n)}{x^n}dx^n = \frac{1}{n^2}\int_0^\infty \frac{\sin s}{s}\log s\, ds$$

$$f(a) = \int_0^\infty \sin s\cdot s^a\, ds = Im\int_0^\infty e^{is}\cdot s^a\, ds = Im\int_0^{i\infty} e^{-s}(is)^a\, ids =$$

$$= Im\, i^{a+1}\int_0^\infty s^a\cdot e^{-s}ds = \sin\left(\frac{a+1}{2}\pi\right)\Gamma(1+a)$$

$$\Omega = \frac{1}{n^2}f'(-1) = \frac{\pi}{2a^2}\lim_{a\to 0}\left[\cos\left(\frac{a}{2}\pi\right)\Gamma(a)+\Gamma'(a)a\right]$$

$$\because \Gamma'(a) = \psi(a)\Gamma(a) = \lim_{a\to 0}\frac{\pi}{2n^2}[\Gamma(a)+\psi(a)\Gamma(a)a]$$

$$\because \psi(1+a) = \psi(a)+\frac{1}{a} \Rightarrow a\psi(a) = a\psi(1+a)-1$$

$$= \lim_{a\to 0}\frac{\pi}{2n^2}[\Gamma(a)+a\psi(1+a)\Gamma(a)-\Gamma(a)] =$$

$$= \lim_{a\to 0}\frac{\pi}{2n^2}[\Gamma(1+a)\psi(1+a)] = \frac{\pi}{2n^2}\psi(1) = \frac{\pi\gamma}{2n^2}$$

1840. Find without any software:

$$\Omega = \int_0^{\frac{\pi}{4}} x\log(1+\tan x)\, dx$$

Proposed by Togrul Ehmedov-Azerbaijan

Solution 1 by Avishek Mitra-West Bengal-India

$$\Omega=\int_0^{\frac{\pi}{4}}x\log(1+\tan x)\,dx=\int_0^{\frac{\pi}{4}}x\log(\sin x+\cos x)dx-\int_0^{\frac{\pi}{4}}x\log(\cos x)\,dx$$

$$\Omega_2=\int_0^{\frac{\pi}{4}}x\log(\cos x)\,dx=\int_0^{\frac{\pi}{4}}x\left[-\log 2-\sum_{n=1}^{\infty}\frac{(-1)^n\cos(2nx)}{n}\right]dx=$$

$$=-\log 2\cdot\frac{x^2}{2}\Big|_0^{\frac{\pi}{4}}-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^{\frac{\pi}{4}}x\cos(2nx)\,dx=$$

$$-\frac{\pi^2}{32}\log 2-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left[x\cdot\frac{\sin(2nx)}{2n}\Big|_0^{\frac{\pi}{4}}-\frac{1}{2n}\int_0^{\frac{\pi}{4}}\sin(2nx)\,dx\right]=$$

$$=-\frac{\pi^2}{32}\log 2-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left[\frac{\pi}{8n}\sin\left(\frac{n\pi}{2}\right)+\frac{1}{4n^2}\cos(2nx)|_0^{\frac{\pi}{4}}\right]=$$

$$=-\frac{\pi^2}{32}\log 2-\frac{\pi}{8}\sum_{n=1}^{\infty}\frac{(-1)^n\sin\left(\frac{n\pi}{2}\right)}{n^2}-\frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^n\left(\cos\left(\frac{n\pi}{2}\right)-1\right)}{n^3}=$$

$$=-\frac{\pi^2}{32}\log 2+\frac{\pi}{8}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2}-\frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3}+\frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cos\left(\frac{n\pi}{2}\right)}{n^3}=$$

$$=-\frac{\pi^2}{32}\log 2+\frac{\pi}{8}G-\frac{1}{4}\eta(3)+\frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n)^3}=$$

$$=-\frac{\pi^2}{32}\log 2+\frac{\pi}{8}G-\frac{1}{4}\cdot\frac{3}{4}\zeta(3)+\frac{1}{32}\cdot\frac{3}{4}\zeta(3)=-\frac{\pi^2}{32}\log 2+\frac{\pi}{8}G-\frac{21}{128}\zeta(3)$$

$$\Omega_1=\int_0^{\frac{\pi}{4}}x\log(\sin x+\cos x)dx=\int_0^{\frac{\pi}{4}}x\log\left\{\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)\right\}dx=$$

$$=\frac{1}{2}\log 2\int_0^{\frac{\pi}{4}}xdx+\int_0^{\frac{\pi}{4}}x\log\left(sin\left(x+\frac{\pi}{4}\right)\right)dx=$$

$$=\frac{\pi^2}{64}\log 2+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(x-\frac{\pi}{4}\right)\log(\sin x)\,dx=$$

$$=\frac{\pi^2}{64}\log 2+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}x\log(\sin x)\,dx-\frac{\pi}{4}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\log(\sin x)\,dx$$

$$\Omega_3=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}x\log(\sin x)\,dx=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}x\left[-\log 2-\sum_{n=1}^{\infty}\frac{\cos(2nx)}{n}\right]dx=$$

$$=-\log 2\cdot\frac{x^2}{2}\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}-\sum_{n=1}^{\infty}\frac{1}{n}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}x\cos(2nx)\,dx=$$

$$=-\frac{3\pi^2}{32}\log 2-\sum_{n=1}^{\infty}\frac{1}{n}\left[x\cdot\frac{\sin(2nx)}{2n}\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}-\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{\sin(2nx)}{2n}dx\right]=$$

$$=-\frac{3\pi^2}{32}\log 2-\sum_{n=1}^{\infty}\frac{1}{n}\left[-\frac{\pi}{8n}\sin\left(\frac{n\pi}{2}\right)+\frac{1}{4n^2}\left(\cos(n\pi)-\cos\left(\frac{n\pi}{2}\right)\right)\right]=$$

$$=-\frac{3\pi^2}{32}\log 2-\frac{\pi}{8}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi}{2}\right)}{n^2}-\frac{1}{4}\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{n^3}+\frac{1}{4}\sum_{n=1}^{\infty}\frac{\cos\left(\frac{n\pi}{2}\right)}{n^3}=$$

$$=-\frac{3\pi^2}{32}\log 2+\frac{\pi}{8}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2}+\frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3}-\frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n)^3}=$$

$$=-\frac{3\pi^2}{32}\log 2+\frac{\pi}{8}G+\frac{1}{4}\cdot\frac{3}{4}\zeta(3)-\frac{1}{32}\cdot\frac{3}{4}\zeta(3)=$$

$$=-\frac{3\pi^2}{32}\log 2+\frac{\pi}{8}G+\frac{21}{128}\zeta(3)$$

$$\Omega_4=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\log(\sin x)dx=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left[-\log 2-\sum_{n=1}^{\infty}\frac{\cos(2nx)}{n}\right]dx=$$

$$=-\frac{\pi}{4}\log 2-\sum_{n=1}^{\infty}\frac{1}{n}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cos(2nx)\,dx=-\frac{\pi}{4}\log 2-\sum_{n=1}^{\infty}\frac{1}{n}\frac{\sin(2nx)}{2n}\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}=$$

$$=-\frac{\pi}{4}\log 2+\frac{1}{2}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi}{2}\right)}{n^2}=-\frac{\pi}{4}\log 2+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2}=-\frac{\pi}{4}\log 2+\frac{G}{2}$$

Solution 2 by Hikmat Mammadov-Azerbaijan

$$\Omega=\int_0^{\frac{\pi}{4}}x\log(1+\tan x)\,dx=\int_0^{\frac{\pi}{4}}x\log(\cos x)\,dx\overset{(1)}{=}$$

$$\because\ \log(\cos x)=-\sum_{k\geq 1}\frac{(-1)^k\cos(2kx)}{k}-\log 2$$

$$\overset{(1)}{=} 0\sum_{k\ge1}\frac{(-1)^k}{k}\int_0^{\frac{\pi}{4}} x\cos(2kx)\,dx - \log 2\int_0^{\frac{\pi}{4}} xdx =$$

$$= -\sum_{k\ge1}\frac{(-1)^k}{k}\cdot\frac{x\,sn(2kx)}{2k}\Big|_0^{\frac{\pi}{4}} + \frac{1}{4k^2}\cos(2kx)|_0^{\frac{\pi}{4}} - \frac{\pi^2}{32}\log 2 =$$

$$= -\sum_{k\ge1}\left[\frac{(-1)^k\sin\left(\frac{k\pi}{2}\right)}{2k^2} + \frac{(-1)^k\cos\left(\frac{k\pi}{2}\right)}{4k^3}\right] \overset{(2)}{=}$$

$$k = 2m; k = 2m+1 \Rightarrow \sin\left(\pi m + \frac{\pi}{2}\right) = (-1)^m; \cos(\pi m) = (-1)^m$$

$$\sin(\pi m) = 0; \cos\left(\pi m + \frac{\pi}{2}\right) = (-1)^m$$

$$\overset{(2)}{=} \sum_{m\ge1}\frac{(-1)^m}{2(2m+1)^2} - \sum_{m\ge1}\frac{(-1)^m}{32m^3} =$$

$$= \frac{1}{2}\beta(-1) - \frac{1}{32}Li_3(-1) = \frac{G}{2} + \frac{3}{4\cdot 32}\zeta(3) - \frac{\pi^2\log 2}{32}$$

$$\int_0^{\frac{\pi}{4}}\log(\cos x)\,dx = \int_0^{\frac{\pi}{4}}\log(\tan x)\,dx = G$$

$$\int_0^{\frac{\pi}{2}}\log(\sin x)\,dx = -\frac{\pi}{2}\log 2$$

$$\cos x = \frac{1}{2}\log\left[\frac{\sin x\cos x}{\sin x}\cos x\right] = \frac{1}{2}\log(sin(2x)) = -\frac{1}{2}\log(\tan x - x)\,; x\in\left[0,\frac{\pi}{2}\right]$$

$$\int_0^{\frac{\pi}{4}}\log(\cos x)\,dx = \frac{\pi}{8}\log 2 - \frac{G}{2}$$

$$\Omega = \int_0^{\frac{\pi}{4}} x\log\left[\frac{\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)}{\cos x}\right]dx =$$

$$= \frac{1}{2}\log 2\int_0^{\frac{\pi}{4}} xdx + \int_0^{\frac{\pi}{4}} x\log\left[\sin\left(x+\frac{\pi}{4}\right)\right]dx - \int_0^{\frac{\pi}{4}} x\log(\cos x)\,dx =$$

$$= \frac{\log 2}{32}\pi^2 + \int_0^{\frac{\pi}{4}}\log(\cos x)\,dx - 2\int_0^{\frac{\pi}{4}} x\log(\cos x)\,dx =$$

$$=\frac{\log 2}{32}\pi^2+\frac{\pi}{4}\left(\frac{\pi\log 2}{8}-\frac{G}{2}\right)-2\left(\frac{G}{2}+\frac{2\zeta(1)}{128}-\frac{\pi^2\log 2}{32}\right)=$$

$$=-\left(\frac{\pi}{8}+1\right)G+\frac{\pi^2\log 2}{8}-\frac{3}{64}\zeta(3)=\frac{\pi^2}{64}\log 2-\frac{\pi}{8}G+\frac{21}{64}\zeta(3)$$

1841. Find a closed form:

$$\Omega=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1}{\pi^n}\cdot\left(\frac{\pi}{e}\right)^k$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Yen Tung Chung-Taichung-Taiwan

$$\Omega=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1}{\pi^n}\cdot\left(\frac{\pi}{e}\right)^k=\sum_{n=0}^{\infty}\frac{1}{\pi^n}\left(\sum_{k=0}^{n}\left(\frac{\pi}{e}\right)^k\right)=\sum_{n=0}^{\infty}\frac{1}{\pi^n}\cdot\frac{1-\left(\frac{\pi}{e}\right)^{n+1}}{1-\frac{\pi}{e}}=$$

$$=\frac{e}{e-\pi}\sum_{n=0}^{\infty}\left(\frac{1}{\pi^n}-\frac{\pi}{e^{n+1}}\right)=\frac{e}{e-\pi}\left(\frac{1}{1-\frac{1}{\pi}}-\pi\cdot\frac{\frac{1}{e}}{1-\frac{1}{e}}\right)=\frac{e}{e-\pi}\left(\frac{\pi}{\pi-1}-\frac{\pi}{e-1}\right)$$

$$=\frac{\pi e}{(\pi-1)(e-1)}$$

Solution 2 by Hussain Reza Zadah-Afghanistan

$$\Omega=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1}{\pi^n}\cdot\left(\frac{\pi}{e}\right)^k=\lim_{j\to\infty}\sum_{n=0}^{j}\sum_{k=0}^{n}\frac{1}{\pi^n}\cdot\left(\frac{\pi}{e}\right)^k=\lim_{j\to\infty}\sum_{n=0}^{j}\frac{1}{\pi^n}\left(1+\frac{\pi}{e}+\cdots+\left(\frac{\pi}{e}\right)^n\right)=$$

$$=\lim_{j\to\infty}\sum_{n=0}^{j}\frac{1}{\pi^n}\left(\frac{1-\left(\frac{\pi}{e}\right)^{n+1}}{1-\frac{\pi}{e}}\right)=\frac{e}{e-\pi}\lim_{j\to\infty}\sum_{n=0}^{j}\left(\frac{1}{\pi^n}-\frac{\pi}{e^{n+1}}\right)=$$

$$=\frac{e}{e-\pi}\lim_{j\to\infty}\left(\frac{1-\left(\frac{1}{\pi}\right)^{j+1}}{1-\frac{1}{\pi}}-\frac{\pi\frac{1}{e}-\pi\left(\frac{1}{e}\right)^{j+2}}{1-\frac{1}{e}}\right)=$$

$$=\frac{e}{e-\pi}\cdot\frac{\pi}{\pi-1}-\lim_{j\to\infty}\left(\frac{1}{\pi}\right)^{j+1}-\frac{\pi}{e-1}\cdot\frac{e}{e-1}-\lim_{j\to\infty}\left(\frac{1}{e}\right)^{j+2}=$$

$$\frac{e}{e-\pi}\left(\frac{\pi}{\pi-1}-\frac{\pi}{e-1}\right)=\frac{\pi e}{(\pi-1)(e-1)}$$

1842. **If we have the integral**

$$\phi(n)=\int_{-\infty}^{\infty}e^{-\pi x(n\,sgn(x)+1)}\sinh\left(\frac{\pi x}{n}\right)dx$$

then prove the sum:

$$\sum_{n=-\infty}^{\infty}\phi(n)=\frac{\tanh\left(\frac{\sqrt{3}}{2}\pi\right)}{\sqrt{3}}-\frac{\tan\left(\frac{\sqrt{5}}{2}\pi\right)}{\sqrt{5}}$$

Where $sgn(x)$ is Signum function.

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Rana Ranino-Setif-Algerie

$$\phi(n)=\int_{-\infty}^{\infty}e^{-\pi x(n\,sgn(x)+1)}\sinh\left(\frac{\pi x}{n}\right)dx=$$

$$=\int_{0}^{\infty}e^{-\pi x(n+1)}\sinh\left(\frac{\pi x}{n}\right)dx+\int_{-\infty}^{0}e^{-\pi x(1-n)}\sinh\left(\frac{\pi x}{n}\right)dx=$$

$$=\int_{0}^{\infty}e^{-\pi x(n+1)}\sinh\left(\frac{\pi x}{n}\right)dx-\int_{0}^{\infty}e^{-\pi x(n-1)}\sinh\left(\frac{\pi x}{n}\right)dx=$$

$$\overset{t=\frac{\pi x}{n}}{=}\frac{n}{\pi}\int_{0}^{\infty}\sinh t\,e^{-n(n+1)t}dt-\frac{n}{pi}\int_{0}^{\infty}\sinh t\,t^{-n(n-1)t}\,dt=$$

$$=\frac{n}{\pi}\mathcal{L}\{\sinh t\}\big(n(n+1)\big)-\frac{n}{\pi}\mathcal{L}\{\sinh t\}\big(n(n-1)\big)$$

$$\because\ \mathcal{L}\{\sinh t\}_{(s)}=\frac{1}{s^2-1}$$

$$\phi(n)=\frac{1}{\pi}\left(\frac{n}{n^2(n+1)^2-1}-\frac{n}{n^2(n-1)^2-1}\right)=$$

$$=\frac{1}{2\pi}\left(\frac{n}{n^2-n+1}-\frac{n}{n^2+n+1}+\frac{n}{n^2+n-1}-\frac{n}{n^2-n-1}\right)$$

$$S=\sum_{-\infty}^{\infty}\phi(n)=\frac{1}{2\pi}\sum_{-\infty}^{\infty}\left(\frac{n}{n^2-n+1}-\frac{n}{n^2+n+1}+\frac{n}{n^2+n-1}-\frac{n}{n^2-n-1}\right)=$$

$$=\frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{n}{n^2-n+1}-\frac{n}{n^2+n+1}\right)-\frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{n}{n^2+n-1}-\frac{n}{n^2-n-1}\right)$$

$$\sum_{n=1}^{\infty}\left(\frac{n}{n^2-n+1}-\frac{n}{n^2+n+1}\right)=\frac{1}{1}-\frac{1}{3}+\frac{2}{3}-\frac{2}{7}+\frac{3}{7}-\frac{3}{13}+\frac{4}{13}-\cdots=$$

$$=\frac{1}{1}+\frac{1}{3}+\frac{1}{7}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n^2-n+1}$$

$$\sum_{n=1}^{\infty}\left(\frac{n}{n^2+n-1}-\frac{n}{n^2-n-1}\right)-\frac{1}{1}-\frac{1}{1}+\frac{2}{1}-\frac{2}{5}+\frac{3}{5}-\frac{3}{11}+\frac{4}{11}+\cdots=$$

$$=-\frac{1}{1}+\frac{1}{1}+\frac{1}{5}+\frac{1}{11}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n^2-n-1}$$

$$\tanh(\pi x)=\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{x}{(2n-1)^2+4x^2}$$

$$x=\frac{\sqrt{3}}{2}:\ \sum_{n=1}^{\infty}\frac{1}{n^2-n+1}=\frac{\pi\tanh\left(\frac{\pi\sqrt{3}}{2}\right)}{\sqrt{3}}$$

$$x=\frac{i\sqrt{5}}{2}:\ \sum_{n=1}^{\infty}\frac{1}{n^2-n-1}=\frac{\pi\tan\left(\frac{\pi\sqrt{5}}{2}\right)}{\sqrt{5}}$$

Therefore,

$$\sum_{n=-\infty}^{\infty}\phi(n)=\frac{\tanh\left(\frac{\sqrt{3}}{2}\pi\right)}{\sqrt{3}}-\frac{\tan\left(\frac{\sqrt{5}}{2}\pi\right)}{\sqrt{5}}$$

1843.

$$\Omega_1=1-\frac{\pi}{2}+\sum_{n=2}^{\infty}\left(-\frac{1}{\pi}\right)^n\cdot\frac{1}{n+1};\ \Omega_2=1-\frac{\pi}{2}+\sum_{n=2}^{\infty}\left(-\frac{1}{e}\right)^n\cdot\frac{1}{n+1}$$

$A.\ \Omega_1<\Omega_2,\quad B.\ \Omega_1=\Omega_2,\quad C.\ \Omega_1>\Omega_2$

Proposed by Daniel Sitaru-Romania

Solution by Adrian Popa-Romania

Let $x\in(0,1)$

$$1-x+x^2+\cdots+(-1)^n x^n=\frac{(-x)^{n+1}-1}{-x-1},(n\to\infty)\Rightarrow$$

$$1-x+x^2+\cdots=\frac{1}{1+x},\qquad x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots=\log(1+x)$$

$$1-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\cdots=\frac{\log(1+x)}{x}.\ \text{Let } x=\frac{1}{\pi}$$

$$1-\frac{1}{\pi}\cdot\frac{1}{2}+\frac{1}{\pi^2}\cdot\frac{1}{3}-\frac{1}{\pi^3}\cdot\frac{1}{4}+\cdots=\frac{\log\left(\frac{1}{\pi}+1\right)}{\frac{1}{\pi}}$$

$$\sum_{n=0}^{\infty}\left(-\frac{1}{\pi}\right)^n\cdot\frac{1}{n+1}=\pi\log\left(\frac{1}{\pi}+1\right)$$

$$\Omega_1=1+\frac{\pi}{2}+1-\frac{1}{2\pi}=\pi\log\left(\frac{1}{\pi}+1\right)=\pi\log\left(\frac{1}{\pi}+1\right)+\frac{1}{2\pi}-\frac{\pi}{2}$$

Let $x=\frac{1}{e}$

$$1-\frac{1}{e}\cdot\frac{1}{2}+\frac{1}{e^2}\cdot\frac{1}{3}-\frac{1}{e^3}\cdot\frac{1}{4}+\cdots=\frac{\log\left(1+\frac{1}{e}\right)}{\frac{1}{e}}$$

$$\sum_{n=0}^{\infty}\left(-\frac{1}{e}\right)^n\cdot\frac{1}{n+1}=e\log\left(1+\frac{1}{e}\right)$$

$$\Omega_2=-1+\frac{\pi}{2}+1-\frac{1}{2e}=e\log\left(1+\frac{1}{e}\right)=e\log\left(1+\frac{1}{e}\right)+\frac{1}{2e}-\frac{\pi}{2}$$

Let $f(x)=x\log\left(1+\frac{1}{x}\right)+\frac{1}{2x}, x>1$

$$f'(x)=\log\left(1+\frac{1}{x}\right)+\frac{x\cdot\left(-\frac{1}{x^2}\right)}{1+\frac{1}{x}}-\frac{1}{2x^2}=\log\left(1+\frac{1}{x}\right)-\frac{1}{x+1}-\frac{1}{2x^2}$$

$$f''(x)=\frac{-\frac{1}{x^2}}{1+\frac{1}{x}}+\frac{1}{(x+1)^2}+\frac{1}{x^3}=-\frac{1}{x(x+1)}+\frac{1}{(x+1)^2}+\frac{1}{x^3}=\frac{2x+1}{x^3(x+1)^2}>0$$

x	1 $\qquad$ e $\qquad$ π $\qquad$ ∞
$f''(x)$	+ + + + + + + + + + + + + + + + + +
$f'(x)$	$\log 2-1$ − − − − − − − − − − − − ↗ 0
$f(x)$	$\log 2+\frac{1}{2}$ ↘ 1

$$f(e) > f(\pi) \to \Omega_1 < \Omega_2$$

1844. Find a closed form:

$$\Omega = \sum_{m=0}^{\infty} \frac{(-1)\cdot\binom{2m}{m}}{2^{2m}\cdot(4m+1)^3}$$

Proposed by Hikmat Mammadov-Azerbaijan

Solution by Fao Ler-Iraq

$$\Omega = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k}(4k+1)^3}\binom{2k}{k} = \frac{1}{4^3}\sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k}\left(k+\frac{1}{4}\right)^3}\binom{2k}{k} =$$

$$= \frac{1}{128}\sum_{k=0}^{\infty}\left(-\frac{1}{4}\right)^k\binom{2k}{k}\int_0^1 x^{k-\frac{3}{4}}\log^2 x\,dx = \frac{1}{128}\int_0^1 x^{-\frac{3}{4}}\log^2 x\sum_{k=0}^{\infty}\left(-\frac{x}{4}\right)^k\binom{2k}{k}dx =$$

$$= \frac{1}{128}\int_0^1 \frac{x^{-\frac{3}{4}}\log^2 x}{\sqrt{1+x}}dx = \frac{1}{128}\frac{d^2}{dy^2}\int_0^1 \frac{x^y}{\sqrt{1+x}}dx;\left(y=-\frac{3}{4}\right)$$

$$= \frac{1}{128}\frac{d^2}{dy^2}\beta\left(y+1,\frac{1}{2}-y\right) = \frac{1}{64\sqrt{\pi}}\frac{d^2}{dy^2}\Gamma(y+1)\Gamma\left(\frac{1}{2}-y\right) =$$

$$= \frac{1}{64\sqrt{\pi}}\left(\Gamma''(y+1)\Gamma\left(\frac{1}{2}-y\right) - 2\Gamma'(y+1)\Gamma'\left(\frac{1}{2}-y\right) + \Gamma(y+1)\Gamma''\left(\frac{1}{2}-y\right)\right) =$$

$$= \frac{1}{64\sqrt{\pi}}\left(\Gamma''\left(\frac{1}{4}\right)\Gamma\left(\frac{5}{4}\right) - 2\Gamma'\left(\frac{1}{4}\right)\Gamma'\left(\frac{5}{4}\right) + \Gamma\left(\frac{1}{4}\right)\Gamma''\left(\frac{1}{4}\right)\right) =$$

$$= \frac{1}{64\sqrt{\pi}}8\Gamma^2\left(\frac{5}{4}\right)(8C+\pi^2) = \frac{8C+\pi^2}{8\sqrt{\pi}}\Gamma^2\left(\frac{5}{4}\right)$$

1845. Let $f(y,n) = F_x\left[\frac{\sin(nx)}{x}\right](y)$ then prove the sum

$$\sum_{n=1}^{m}\cos(2\pi m)\,f\left(\frac{\pi}{n},n\right)^k = \left(\frac{\pi}{2}\right)^{\frac{k}{2}}(m-1)$$

where $m \geq 2, k \geq 1$ and $F_x[f](y)$ is Fourier transform.

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Rana Ranino-Setif-Algerie

$$f(y,n)=F_x\left[\frac{\sin(nx)}{x}\right](y)$$

Since $x\to\frac{\sin(nx)}{xx}$ is an even function

$$f(y,n)=F_x\left[\frac{\sin(nx)}{x}\right](y)=F_x^c\left[\frac{\sin(nx)}{x}\right](y)=\sqrt{\frac{2}{\pi}}\int_0^\infty\frac{\sin(nx)\cos(xy)}{x}dx$$

$$f(y,n)=\frac{1}{\sqrt{2\pi}}\int_0^\infty\frac{\sin\big((n+y)x\big)+\sin\big((n-y)x\big)}{x}dx$$

$$\because\int_0^\infty\frac{\sin(ax)}{x}dx=\frac{\pi}{2}sgn(a)$$

$$f(y,n)=\frac{1}{2}\sqrt{\frac{\pi}{2}}\big(sgn(n+y)+sgn(n-y)\big)$$

$$f\left(\frac{\pi}{n},n\right)=\frac{1}{2}\sqrt{\frac{\pi}{2}}\left(sgn\left(n+\frac{\pi}{n}\right)+sgn\left(n-\frac{\pi}{n}\right)\right),\qquad f\left(\frac{\pi}{n},n\right)=\sqrt{\frac{\pi}{2}},\forall n>1$$

$$\sum_{n=1}^{m}\cos(2\pi m)f\left(\frac{\pi}{n},n\right)^k=\left(\frac{\pi}{2}\right)^{\frac{k}{2}}\sum_{n=2}^{m}\cos(2\pi n)=\left(\frac{\pi}{2}\right)^{\frac{k}{2}}(m-1)$$

1846. Prove that:

$$\psi_1\left(\frac{1}{8}\right)+\psi_1\left(\frac{5}{8}\right)=32G+4\pi^2$$

where $\psi_1(x)$ is the trigamma function and G is Catalan constant.

Proposed by Vasile Mircea Popa-Romania

Solution 1 by Togrul Ehmedov-Azerbaijan

We know that:

$$\frac{\Gamma(x)}{\Gamma(2x)}=\frac{\sqrt{\pi}2^{1-2x}}{\Gamma\left(x+\frac{1}{2}\right)}\Rightarrow\log\big(\Gamma(x)\big)-\log\big(\Gamma(1-x)\big)=$$

$$=\log(\sqrt{\pi})-(1-2x)\log(2)-\log\left(\Gamma\left(x+\frac{1}{2}\right)\right)$$

$$\psi_1'(x)-4\psi_1'(2x)=-\psi_1'\left(x+\frac{1}{2}\right)$$

$$x=\frac{1}{8}\Rightarrow \psi_1'\left(\frac{1}{8}\right)-4\psi_1'\left(\frac{1}{4}\right)=-\psi_1'\left(\frac{5}{8}\right)$$

We know that:

$$\psi_1'\left(\frac{1}{4}\right)=\pi^2+8G\Rightarrow \psi_1'\left(\frac{1}{8}\right)-4(\pi^2+8G)=-\psi_1'\left(\frac{5}{8}\right)$$

Therefore,

$$\psi_1\left(\frac{1}{8}\right)+\psi_1\left(\frac{5}{8}\right)=32G+4\pi^2$$

Solution 2 by Rana Ranino-Setif-Algerie

$$\because\ \psi^{(1)}(z)=-\int_0^1\frac{t^{z-1}\log(t)}{1-t}dt$$

$$\psi_1\left(\frac{1}{8}\right)+\psi_1\left(\frac{5}{8}\right)=-\int_0^1\frac{\left(t^{-\frac{7}{8}}+t^{-\frac{3}{8}}\right)\log(t)}{1-t}dt\overset{t=x^8}{=}-64\int_0^1\frac{(1+x^4)\log(x)}{1-x^8}dx=$$

$$=-64\int_0^1\frac{\log(x)}{1-x^4}dx=-32\int_0^1\frac{\log(x)}{1+x^2}dx-32\int_0^1\frac{\log(x)}{1-x^2}dx$$

$$\int_0^1\frac{\log(x)}{1+x^2}dx\overset{x=\tan\theta}{=}\int_0^{\frac{\pi}{4}}\log(\tan\theta)\,d\theta=-G$$

$$\int_0^1\frac{\log(x)}{1-x^2}dx=\sum_{n=0}^{\infty}\int_0^1 x^{2n}\log(x)\,dx=-\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}=-\frac{\pi^2}{8}$$

Therefore,

$$\psi_1\left(\frac{1}{8}\right)+\psi_1\left(\frac{5}{8}\right)=32G+4\pi^2$$

Solution 3 by Hikmat Mammadov-Azerbaijan

$$\psi\left(\frac{p}{q}\right)=\frac{\pi^2}{2\sin^2\left(\frac{p\pi}{q}\right)}+2q\sum_{k=1}^{\left[\frac{q-1}{2}\right]}\sin\left(\frac{2k\pi p}{q}\right)Cl_2\left(\frac{2\pi k}{q}\right)$$

Cl_2 −Claus function.

$$\psi_1\left(\frac{1}{8}\right)=\frac{\pi^2}{2\sin^2\left(\frac{\pi}{8}\right)}+16\sum_{k=1}^{3}\sin\left(\frac{\pi k}{4}\right)Cl_2\left(\frac{\pi k}{4}\right)=$$

$$=\frac{2\pi^2}{2-\sqrt{2}}+16\left(\sin\left(\frac{\pi}{4}\right)Cl_2\left(\frac{\pi}{4}\right)+Cl_2\left(\frac{\pi}{2}\right)+\sin\left(\frac{\pi}{4}\right)Cl_2\left(\frac{3\pi}{4}\right)\right)$$

$$\psi_1\left(\frac{5}{8}\right)=\frac{2\pi^2}{2+\sqrt{2}}+16\left(\sin\left(\frac{10\pi}{4}\right)Cl_2\left(\frac{\pi}{4}\right)+\sin\left(\frac{20\pi}{8}\right)Cl_2\left(\frac{\pi}{2}\right)+\sin\left(\frac{30\pi}{4}\right)Cl_2\left(\frac{3\pi}{4}\right)\right)$$

$$=\frac{2\pi^2}{2+\sqrt{2}}+16\left(-\sin\left(\frac{\pi}{4}\right)Cl_2\left(\frac{\pi}{4}\right)+Cl_2\left(\frac{\pi}{2}\right)-\sin\left(\frac{\pi}{4}\right)Cl_2\left(\frac{3\pi}{4}\right)\right)$$

$$\psi_1\left(\frac{1}{8}\right)+\psi_1\left(\frac{5}{8}\right)=\frac{2\pi^2}{2+\sqrt{2}}+\frac{2\pi^2}{2-\sqrt{2}}+32Cl_2\left(\frac{\pi}{2}\right)=4\pi^2+32Cl_2\left(\frac{\pi}{2}\right)$$

$$Cl_2(x)=\sum_{n\geq 1}\frac{\sin(nx)}{n^2}\Rightarrow Cl_2(x)=\sum_{n\geq 1}\frac{\sin\left(\frac{n\pi}{2}\right)}{n^2}$$

$$\sin(nx)=0;\forall n\in\mathbb{Z}$$

$$Cl_2\left(\frac{\pi}{2}\right)=\sum_{n\geq 0}\frac{\sin\left(n\pi+\frac{\pi}{2}\right)}{(2n+1)^2}=\sum_{n\geq 1}\frac{(-1)^n}{2n+1}=\beta(2)=G$$

Therefore,

$$\psi_1\left(\frac{1}{8}\right)+\psi_1\left(\frac{5}{8}\right)=32G+4\pi^2$$

Solution 4 by Hikmat Mammadov-Azerbaijan

$$\psi_1\left(\frac{1}{8}\right)+\psi_1\left(\frac{5}{8}\right)=\sum_{n=0}^{\infty}\left(\frac{1}{\left(n+\frac{1}{8}\right)^2}+\frac{1}{\left(n+\frac{5}{8}\right)^2}\right)=64\sum_{n=0}^{\infty}\left(\frac{1}{(8n+1)^2}+\frac{1}{(8n+5)^2}\right)=$$

$$=-64\sum_{n=0}^{\infty}\int_0^1(x^{8n}+x^{8n+4})\log(x)\,dx=-64\int_0^1(1+x^4)\log(x)\sum_{n=0}^{\infty}x^{8n}dx=$$

$$=-64\int_0^1\frac{(1+x^4)\log(x)}{1-x^8}dx=-32\int_0^1\left(\frac{1}{1-x^2}+\frac{1}{1+x^2}\right)\log(x)\,dx=$$

$$=-32\sum_{n=0}^{\infty}\int_0^1(x^{2n}+(-1)^nx^{2n})\log(x)\,dx=32\sum_{n=0}^{\infty}\left(\frac{1}{(2n+1)^2}+\frac{(-1)^n}{(2n+1)^2}\right)=$$

$$=32(1-2^{-2})\zeta(2)+32G=4\pi^2+32G$$

1847. Find a closed form:

$$\Omega=\int_0^1\frac{\sin^{-1}(\sqrt{x})\cdot\log^2(x)\cdot\log^2(1-x)}{x(1-x)}dx$$

Proposed by Togrul Ehmedov-Azerbaijan

Solution by Bamidele Benjamin-Nigeria

$$\Omega=\int_0^1\frac{\sin^{-1}(\sqrt{x})\cdot\log^2(x)\cdot\log^2(1-x)}{x(1-x)}dx\overset{x=\sin^2 y}{=}$$

$$=\int_0^{\frac{\pi}{2}}\frac{2y\cdot\log^2(\sin^2 y)\cdot\log^2(\cos^2 y)}{\sin y\cos y}dy=$$

$$=\frac{\pi}{2}\int_0^{\frac{\pi}{2}}\frac{\log^2(\sin^2 y)\cdot\log^2(\cos^2 y)}{\sin y\cos y}dy\overset{\sin^2 y=x}{=}\frac{\pi}{4}\int_0^1\frac{\log^2(x)\cdot\log^2(1-x)}{x(1-x)}dx=$$

$$=\frac{\pi}{4}\left(\int_0^1\frac{\log^2(x)\cdot\log^2(1-x)}{x}dx+\int_0^1\frac{\log^2(x)\log^2(1-x)}{1-x}dx\right)=$$

$$=\frac{\pi}{2}\int_0^1\frac{\log^2(x)\log^2(1-x)}{x}dx$$

$$\sum_{n=1}^{\infty}x^{n+1}\frac{H_n}{n+1}=\frac{1}{2}\log^2(1-x)$$

$$\Omega=\pi\sum_{n=1}^{\infty}\frac{H_n}{n+1}\int_0^1 x^n\log^2(x)\,dx=2\pi\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^4}=2\pi\sum_{n=1}^{\infty}\frac{H_{n+1}-\frac{1}{n+1}}{(n+1)^4}=$$

$$=2\pi\left(\sum_{n=1}^{\infty}\frac{H_n}{n^4}-\sum_{n=1}^{\infty}\frac{1}{n^5}\right)=2\pi\sum_{n=1}^{\infty}\frac{H_n}{n^4}-2\pi\zeta(5)$$

$$2\sum_{n=1}^{\infty}\frac{H_n}{n^k}=(k+2)\zeta(k+1)-\sum_{n=1}^{k-2}\zeta(k-n)\zeta(n+1),k\in\mathbb{N},k\geq 2$$

$$\Omega=6\pi\zeta(5)-\left(\pi\sum_{n=1}^{2}\zeta(4-n)\zeta(n+1)\right)-2\pi\zeta(5)=$$

$$=4\pi\zeta(5)-2\pi\zeta(3)\zeta(2)=2\pi\big(2\zeta(5)-\zeta(2)\zeta(3)\big)$$

1848. Prove that:

$$\int_0^1 \frac{(\tan^{-1} x)^3}{x+1} dx = \frac{\pi(24\pi C - 63\zeta(3) + 2\pi^2 \log 2)}{256}$$

Proposed by Fao Ler-Iraq

Solution 1 by Togrul Ehmedov-Azerbaijan

Let $x = \frac{1-y}{1+y} \Rightarrow dx = -\frac{2}{(1+y)^2} dy$, then we have:

$$\Omega = \int_0^1 \frac{\left(\tan^{-1}\left(\frac{1-y}{1+y}\right)\right)^3}{1+y} dy = \int_0^1 \frac{\left(\frac{\pi}{4} - \tan^{-1}(y)\right)^3}{1+y} dy =$$

$$= \frac{\pi^3}{64}\int_0^1 \frac{dy}{1+y} - \frac{3\pi^2}{16}\int_0^1 \frac{\tan^{-1}(y)}{1+y} dy + \frac{3\pi}{4}\int_0^1 \frac{(\tan^{-1}(y))^2}{1+y} dy - \Omega$$

$$2\Omega = \frac{\pi^3}{64}\int_0^1 \frac{dy}{1+y} - \frac{3\pi^2}{16}\int_0^1 \frac{\tan^{-1}(y)}{1+y} dy + \frac{3\pi}{4}\int_0^1 \frac{(\tan^{-1}(y))^2}{1+y} dy$$

$$\because \int_0^1 \frac{dy}{1+y} = \log(2);\ \int_0^1 \frac{\tan^{-1}(y)}{1+y} dy = \frac{\pi}{8}\log(2);$$

$$\because \int_0^1 \frac{(\tan^{-1}(y))^3}{1+y} dy = \frac{\pi^2}{32}\log(2) + \frac{\pi}{4}G - \frac{21}{32}\zeta(3)$$

Hence,

$$2\Omega = \frac{\pi^3}{64}\log(2) + \frac{3\pi^2}{32}G - \frac{63\pi}{128}\zeta(3)$$

Therefore,

$$\int_0^1 \frac{(\tan^{-1} x)^3}{x+1} dx = \frac{\pi(24\pi C - 63\zeta(3) + 2\pi^2 \log 2)}{256}$$

Solution 2 by Ankush Kumar Parcha-India

$$\text{Let } I_1 = \int_0^{\frac{\pi}{4}} y^2 \log(1 + \tan y)\, dy = \int_0^{\frac{\pi}{4}} \left(\frac{\pi}{4} - y\right)^2 \log\left(\frac{2}{1+\tan y}\right) dy =$$

$$= \frac{\pi^3}{192}\log(2) - \frac{\pi^2}{16}\int_0^{\frac{\pi}{4}} \log(1+\tan y)\, dy - I_1 + \frac{\pi}{2}\int_0^{\frac{\pi}{4}} y \log(1+\tan y)\, dy$$

$$2I_1=\frac{\pi^3}{192}\log(2)-\frac{\pi^2}{16}\left(\frac{\pi}{8}\log(2)\right)+\frac{\pi}{2}\int_0^{\frac{\pi}{4}}y\log(1+\tan y)\,dy$$

$$\because\int_0^{\frac{\pi}{4}}y\log(1+\tan y)\,dy=\frac{\pi}{8}\log(2)$$

$$\text{Let } I_3=\int_0^{\frac{\pi}{4}}y\log(1+\tan y)\,dy=\int_0^{\frac{\pi}{4}}y\log\left[\sqrt{2}\,sin\left(\frac{\pi}{4}-y\right)\right]dy-\int_0^{\frac{\pi}{4}}y\log(\cos y)\,dy$$

$$=\frac{\pi^2}{64}\log(2)+\int_0^{\frac{\pi}{4}}\left(\frac{\pi}{4}-y\right)\log\left(\frac{\pi}{2}-y\right)dy-\int_0^{\frac{\pi}{4}}y\log(\cos y)\,dy$$

$$I_3=\frac{\pi^2}{64}\log(2)+\frac{\pi}{4}\int_0^{\frac{\pi}{4}}\log(\cos y)\,dy-2\int_0^{\frac{\pi}{4}}y\log(\cos y)\,dy=$$

$$=\frac{\pi^2}{64}\log(2)+\frac{\pi}{4}\int_0^{\frac{\pi}{4}}\log(2)\,dy-2\int_0^{\frac{\pi}{4}}y\log(1+e^{2yi})\,dy+\log(2)\int_0^{\frac{\pi}{4}}ydy=$$

$$=\frac{\pi^2}{64}\log(2)+\frac{\pi}{4}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\int_0^{\frac{\pi}{4}}(e^{2yi})^n dy-2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\int_0^{\frac{\pi}{4}}y(e^{2iy})^n dy=$$

$$=\frac{\pi^2}{64}\log(2)+\frac{\pi}{4}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\frac{e^{2iny}}{2ni}\Bigg|_0^{\frac{\pi}{4}}-2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left[\frac{ye^{2niy}}{2ni}\Bigg|_0^{\frac{\pi}{4}}-\frac{e^{2niy}}{(2ni)^2}\Bigg|_0^{\frac{\pi}{4}}\right]=$$

$$=\frac{\pi^2}{64}\log(2)+\frac{\pi}{8}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\sin\left(\frac{n\pi}{2}\right)+\frac{\pi}{4}\sum_{n=1}^{\infty}\frac{-1}{n^2}\sin\left(\frac{n\pi}{2}\right)$$

$$+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\left[\cos\left(\frac{n\pi}{2}\right)-1\right]$$

$$=\frac{\pi^2}{64}\log(2)-\frac{\pi}{8}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}-\frac{\pi}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\cos\left(\frac{n\pi}{2}\right)$$

$$+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^3}$$

$$\because\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}=C-Catalan's\ constant$$

$$\Omega=\frac{\pi^2}{64}\log(2)-\frac{\pi}{8}C-\frac{1}{16}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^3}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^3}=$$

$$=\frac{\pi^2}{16}\log(2)-\frac{\pi}{8}C-\frac{1}{16}\eta(3)+\frac{\eta(3)}{2}$$

$$\because \eta(s)=(1-2^{1-s})\zeta(s)$$

$$I_3=\frac{\pi^2}{64}\log(2)-\frac{\pi}{8}C+\frac{21}{64}\zeta(3)$$

Therefore,

$$\int_0^1\frac{(\tan^{-1}x)^3}{x+1}dx=\frac{\pi(24\pi C-63\zeta(3)+2\pi^2\log 2)}{256}$$

Solution 3 by Hikmat Mammadov-Azerbaijan

$$\int_0^1\frac{(\tan^{-1}x)^3}{x+1}dx=\frac{\pi^3}{64}\log(2)-3\int_0^1\frac{\log(1+x)}{1+x^2}(\tan^{-1}x)^2dx=$$

$$=-3\int_0^{\frac{\pi}{4}}\log(1+\tan t)\,t^2\,dt$$

$$\int_0^{\frac{\pi}{4}}\log(1+\tan t)\,t^2\,dt=\int_0^{\frac{\pi}{4}}\log\left(1+tan\left(\frac{\pi}{4}-t\right)\right)\left(\frac{\pi}{4}-t\right)^2dt=$$

$$=2\int_0^{\frac{\pi}{4}}\log(1+\tan t)\,t^2\,dt=\int_0^{\frac{\pi}{4}}\log(2)\,t^2dt-\frac{\pi^2}{16}\int_0^{\frac{\pi}{4}}\log(1+\tan t)\,dt+$$

$$+\frac{\pi}{2}\int_0^{\frac{\pi}{4}}t\log(1+\tan t)\,dt=A-\frac{\pi^2}{16}B+\frac{\pi}{2}C$$

$$A=\frac{\log(2)}{3}\cdot\frac{\pi^3}{64}$$

$$C=\int_0^{\frac{\pi}{4}}\log\left(\frac{\sqrt{2}\,sin\left(\frac{\pi}{4}+t\right)}{\cos t}\right)t\,dt=-2\int_0^{\frac{\pi}{4}}\log(\cos t)\,tdt+\frac{\log(\sqrt{2})}{2}\cdot\frac{\pi}{4}$$

$$Y=2\int_0^{\frac{\pi}{4}}\log(\cos t)\,tdt=2\int_0^{\frac{\pi}{4}}\sum_{k\geq 1}(-1)^k\frac{\cos(2kx)}{k}+2\int_0^{\frac{\pi}{4}}\log 2\,t\,dt=$$

$$= 2\sum_{k\ge1}\frac{(-1)^k}{k}\left[\frac{\pi}{8k}\sin\left(\frac{k\pi}{2}\right)+\frac{\cos\left(\frac{k\pi}{2}\right)-1}{4k^2}\right]+\frac{\log(2)\,\pi^2}{16}=$$

$$= -2\sum_{k\ge0}\frac{(-1)^k\pi}{8(2k+1)^2}-\frac{1}{2}\sum_{k\ge1}\frac{(-1)^k}{k^3}+\frac{1}{2}\sum_{k=1}\frac{(-1)^k}{8k^3}+\frac{\log(2)\,\pi^2}{16}-$$

$$-2\left(\frac{\pi}{8}C-\frac{21}{128}\zeta(3)-\frac{\pi^2}{32}\log(2)\right)=-2\,\frac{16\pi C-21\zeta(3)-4\pi^2\log(2)}{128}+\frac{\pi\log(2)}{32}$$

$$B=\int_0^{\frac{\pi}{4}}\log(1+\tan t)\,dt=\int_0^{\frac{\pi}{4}}\log\left(\frac{2}{1+tan\,t}\right)dt=\frac{\pi}{8}\log 2$$

Therefore,

$$\int_0^{\frac{\pi}{4}}t^2\log(1+\tan t)\,dt=$$

$$=\frac{1}{2}\left(\frac{\log(2)}{3}\cdot\frac{\pi^3}{64}-\frac{\pi^2}{16}\cdot\frac{\pi\log(2)}{8}-2\pi\cdot\frac{16\pi C-21\zeta(3)-4\pi^2\log(2)}{128}+\frac{\pi^2}{2}\cdot\frac{\log(2)}{32}\right)$$

$$=\Omega$$

$$\int_0^1\frac{(\tan^{-1}x)^3}{x+1}dx=\frac{\pi^3\log(2)}{64}-3\Omega$$

1849. Find:

$$\Omega(n)=\int_0^n\log\left(\sqrt{n+x}+\sqrt{n-x}\right)dx,n\in\mathbb{N}-\{0\}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Tapas Das-India

$$(n+x)-(n-x)=2x$$

$$\left(\sqrt{n+x}+\sqrt{n-x}\right)\left(\sqrt{n+x}-\sqrt{n-x}\right)=2x$$

$$\Omega(n)=\int_0^n\log\left(\sqrt{n+x}+\sqrt{n-x}\right)dx\overset{IBP}{=}\left[x\log\left(\sqrt{n+x}+\sqrt{n-x}\right)\right]_0^n-$$

$$-\int_0^n\frac{1}{\sqrt{n+x}+\sqrt{n-x}}\cdot\frac{1}{2}\left(\frac{1}{\sqrt{n+x}}-\frac{1}{\sqrt{n-x}}\right)xdx=$$

$$=n\log\sqrt{2n}-\frac{1}{2}\int_0^n\frac{1}{\sqrt{n+x}+\sqrt{n-x}}\cdot\frac{\sqrt{n+x}-\sqrt{n-x}}{\sqrt{n^2-x^2}}dx=$$

$$= n\log\sqrt{2n} - \frac{1}{4}\int_0^n \frac{2x}{\sqrt{n+x}+\sqrt{n-x}} \cdot \frac{\sqrt{n+x}-\sqrt{n-x}}{\sqrt{n^2-x^2}}dx =$$

$$= \frac{n}{2}\log(2n) + \frac{1}{4}\int_0^n \frac{\left(\sqrt{n+x}-\sqrt{n-x}\right)^2}{\sqrt{n^2-x^2}}dx = \frac{n}{2}\log(2n) + \frac{1}{4}\int_0^n \frac{2n-2\sqrt{n^2-x^2}}{\sqrt{n^2-x^2}}dx =$$

$$= \frac{n}{2}\log(2n) + \frac{n}{2}\int_0^n \frac{dx}{\sqrt{n^2-x^2}} - \frac{1}{2}\int_0^n dx = \frac{n}{2}\log(2n) + \frac{n}{2}\sin^{-1}\left(\frac{x}{n}\right)\Big|_0^n - \frac{x}{2}\Big|_0^n =$$

$$= \frac{n}{2}\log(2n) + \frac{n}{2}\cdot\frac{\pi}{2} - \frac{n}{2} = \frac{n}{2}\left[\log(2n) + \frac{\pi}{2} - 1\right]$$

Solution 2 by Ankush Kumar Parcha-India

$$\Omega(n) = \int_0^n \log\left(\sqrt{n+x}+\sqrt{n-x}\right)dx \overset{IBP}{=} \left[x\log\left(\sqrt{n+x}+\sqrt{n-x}\right)\right]_0^n -$$

$$-\int_0^n \frac{1}{\sqrt{n+x}+\sqrt{n-x}} \cdot \frac{1}{2}\left(\frac{1}{\sqrt{n+x}} - \frac{1}{\sqrt{n-x}}\right)xdx =$$

$$= n\log\sqrt{2n} - \frac{1}{2}\int_0^n \frac{1}{\sqrt{n+x}+\sqrt{n-x}} \cdot \frac{\sqrt{n+x}-\sqrt{n-x}}{\sqrt{n^2-x^2}}dx =$$

$$= n\log\sqrt{2n} - \frac{1}{4}\int_0^n \frac{2x}{\sqrt{n+x}+\sqrt{n-x}} \cdot \frac{\sqrt{n+x}-\sqrt{n-x}}{\sqrt{n^2-x^2}}dx =$$

$$= n\log\sqrt{2n} + \frac{1}{2}\int_0^n \frac{x\left(\sqrt{n+x}-\sqrt{n-x}\right)^2}{(n+x-n+x)\sqrt{n^2-x^2}}dx =$$

$$= n\log\sqrt{2n} + \frac{1}{2}\int_0^n \frac{2\left(n-\sqrt{n^2-x^2}\right)}{2\sqrt{n^2-x^2}}dx =$$

$$= n\log\sqrt{2n} + \frac{1}{2}\int_0^n \frac{n}{\sqrt{n^2-x^2}}dx - \frac{1}{2}\int_0^n \frac{\sqrt{n^2-x^2}}{\sqrt{n^2-x^2}}dx =$$

$$= \frac{n}{2}\log(2n) + \frac{n}{2}\sin^{-1}\left(\frac{x}{n}\right)\Big|_0^n - \frac{x}{2}\Big|_0^n =$$

$$= \frac{n}{2}\log(2n) + \frac{n}{2}\cdot\frac{\pi}{2} - \frac{n}{2} = \frac{n}{2}\left[\log(2n) + \frac{\pi}{2} - 1\right]$$

Solution 3 by Ravi Prakash-New Delhi-India

$$\Omega(n) = \int_0^n \log\left(\sqrt{n+x}+\sqrt{n-x}\right)dx \overset{(x=n\cos(2\theta))}{=}$$

$$= n\int_{\frac{\pi}{4}}^{0}\log\left(\sqrt{2n}(\cos\theta+\sin\theta)\right)(-2\sin 2\theta)d\theta = n\left[\frac{1}{2}\log(2n)-I\right]$$

Where,

$$I=\int_{\frac{\pi}{4}}^{0}\log(\cos\theta+\sin\theta)\frac{d}{d\theta}(\cos 2\theta)d\theta =$$

$$=\log(\cos\theta+\sin\theta)\cos(2\theta)\Big|_{\frac{\pi}{4}}^{0}+\int_0^{\frac{\pi}{4}}\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}(\cos^2\theta-\sin^2\theta)d\theta =$$

$$=\int_0^{\frac{\pi}{4}}(\cos\theta-\sin\theta)^2 d\theta=\int_0^{\frac{\pi}{4}}(1-\sin(2\theta))d\theta=\left(\theta+\frac{1}{2}\cos(2\theta)\right)\Big|_0^{\frac{\pi}{4}}=\frac{\pi-2}{4}$$

Thus,

$$\Omega(n)=n\left[\frac{1}{2}\log(2n)+\frac{\pi-2}{4}\right]=\frac{n}{4}[2\log(2n)+\pi-2]$$

Solution 4 by Hikmat Mammadov-Azerbaijan

$$\sqrt{n+x}+\sqrt{n-x}=\sqrt{2\left(n+\sqrt{n^2-x^2}\right)}$$

$$\Omega(n)=\int_0^n\log\left(\sqrt{2\left(n+\sqrt{n^2-x^2}\right)}\right)dx=$$

$$=\frac{\log(2)}{2}\int_0^n dx+\frac{1}{2}\int_0^n\log\left(n+\sqrt{n^2-x^2}\right)dx \overset{x=n\sin\alpha}{=}$$

$$=n\log(\sqrt{2})+\frac{n}{2}\log(n)\int_0^{\frac{\pi}{2}}d(\sin\alpha)+\frac{n}{2}\int_0^{\frac{\pi}{2}}\log(1+\cos\alpha)\,d(\sin\alpha)=$$

$$=n\log(\sqrt{2})+n\log(\sqrt{n})\sin\alpha|_0^{\frac{\pi}{2}}+\frac{n}{2}\sin\alpha\log(1+\cos\alpha)|_0^{\frac{\pi}{2}}-\frac{n}{2}\int_0^{\frac{\pi}{2}}\frac{\sin\alpha\,(-\sin\alpha)}{1+\cos\alpha}d\alpha$$

$$=n\log(\sqrt{2n})+\frac{n}{2}\int_0^{\frac{\pi}{2}}\frac{(1-\cos\alpha)(1+\cos\alpha)}{1+\cos\alpha}d\alpha=$$

$$=n\log(\sqrt{2n})+\frac{n}{2}\int_0^{\frac{\pi}{2}}d\alpha-\frac{n}{2}\int_0^{\frac{\pi}{2}}d(\sin\alpha)=$$

$$=n\log(\sqrt{2n})+\frac{n\pi}{4}-\frac{n}{2}=\frac{n}{4}(2\log(2n)+n\pi-2n)$$

Solution 5 by Timson Folorunsho-Nigeria

$$\Omega(n)=\int_0^n \log\left(\sqrt{n+x}+\sqrt{n-x}\right)dx \overset{IBP}{=} \left[x\log\left(\sqrt{n+x}+\sqrt{n-x}\right)\right]_0^n-$$

$$-\int_0^n \frac{1}{\sqrt{n+x}+\sqrt{n-x}}\cdot\frac{1}{2}\left(\frac{1}{\sqrt{n+x}}-\frac{1}{\sqrt{n-x}}\right)xdx=$$

$$=n\log\sqrt{2n}-\frac{1}{2}\int_0^n \frac{1}{\sqrt{n+x}+\sqrt{n-x}}\cdot\frac{\sqrt{n+x}-\sqrt{n-x}}{\sqrt{n^2-x^2}}dx=$$

$$=n\log\sqrt{2n}-\frac{1}{4}\int_0^n \frac{2x}{\sqrt{n+x}+\sqrt{n-x}}\cdot\frac{\sqrt{n+x}-\sqrt{n-x}}{\sqrt{n^2-x^2}}dx=$$

$$=n\log\sqrt{2n}+\frac{1}{2}\int_0^n \frac{x\left(\sqrt{n+x}-\sqrt{n-x}\right)^2}{(n+x-n+x)\sqrt{n^2-x^2}}dx=$$

$$=n\log\sqrt{2n}+\frac{1}{2}\int_0^n \frac{2n-2\sqrt{n^2-x^2}}{2\sqrt{n^2-x^2}}dx=n\log\sqrt{2n}+\frac{1}{2}\int_0^n \frac{n-\sqrt{n^2-x^2}}{\sqrt{n^2-x^2}}dx$$

$$I=\int_0^n \frac{n-\sqrt{n^2-x^2}}{\sqrt{n^2-x^2}}dx \overset{x=n\sin y}{=} \int_0^{\frac{\pi}{2}} \frac{n-\sqrt{n^2-n^2\sin^2 y}}{\sqrt{n^2-n^2\sin^2 y}}\cdot n\cos y\,dy$$

$$=\int_0^{\frac{\pi}{2}} \frac{n-n\cos y}{n\cos y}\cdot n\cos y\,dy=n\int_0^{\frac{\pi}{2}}(1-\cos y)dy=n(y-\sin y)|_0^{\frac{\pi}{2}}=n\left(\frac{\pi}{2}-1\right)$$

Therefore,

$$\Omega(n)=\frac{n}{4}(2\log(2n)+n\pi-2n)$$

1850. Find:

$$\Omega=\int_0^1 \frac{\tan^{-1}x\cdot Li_2\left(\frac{1-x}{1+x}\right)}{(1+x)^2}dx$$

Proposed by Togrul Ehmedov-Azerbaijan

Solution by Fao Ler-Iraq

$$\Omega=\int_0^1 \frac{\tan^{-1}x\cdot Li_2\left(\frac{1-x}{1+x}\right)}{(1+x)^2}dx \overset{x\to\frac{1-x}{1+x}}{=} \int_0^1 \frac{\tan^{-1}\left(\frac{1-x}{1+x}\right)\cdot Li_2(x)}{\left(1+\frac{1-x}{1+x}\right)^2}d\left(\frac{1-x}{1+x}\right)=$$

$$= 2\int_0^1 \frac{(\tan^{-1} 1 - \tan^{-1} x)Li_2(x)}{4} dx = \frac{\pi}{8}\int_0^1 Li_2(x)\,dx - \frac{1}{2}\int_0^1 \tan^{-1} x\, Li_2(x)\,dx =$$

$$= \frac{\pi}{8}\left(\frac{\pi^2}{6} - 1\right) - \frac{1}{2}\int_0^1 \tan^{-1} x\, Li_2(x)\,dx =$$

$$= \frac{24C(4-\pi) - 96Re\big(Li_3(1-i)\big) + 87\zeta(3) - 5\pi^2 - 48\pi + 12\log^2 2 - 48\log 2 + 11\pi^2\log 2 - 12\pi\log 2}{192}$$

$$\int_0^1 Li_2(x)\tan^{-1} x\,dx = (xLi_2(x)\tan^{-1} x)\Big|_0^1 - \int_0^1 x\left(\frac{Li_2(x)}{x^2+1} - \frac{\tan^{-1} x\log(1-x)}{x}\right)dx =$$

$$= \frac{\pi^3}{24} - \int_0^1 x\left(\frac{Li_2(x)}{x^2+1} - \frac{\tan^{-1} x\log(1-x)}{x}\right)dx =$$

$$= \frac{\pi^3}{24} - \int_0^1 x\frac{Li_2(x)}{x^2+1}dx + \int_0^1 \tan^{-1} x\log(1-x)\,dx =$$

$$= \frac{\pi^3}{24} - \frac{1}{2}\log(\mathrm{x}^2+1)\,Li_2(x)\Big|_0^1 + \frac{1}{2}\int_0^1 \frac{\log(x^2+1)\log(1-x)}{x}dx$$

$$+ \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\int_0^1 x^{2k+1}\log(1-x)\,dx =$$

$$= \frac{\pi^3}{24} - \frac{\pi^2}{12}\log 2 - \frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\int_0^1 x^{2k-1}\log(1-x)\,dx + \sum_{k=0}^{\infty}\frac{(-1)^k H_{2k+2}}{(2k+1)(2k+2)} =$$

$$= \frac{\pi^3}{24} - \frac{\pi^2}{12}\log 2 - \frac{1}{4}\sum_{k=1}^{\infty}(-1)^k H_{2k}\left(\frac{1}{k^2} - \frac{4}{2k-1} + \frac{2}{k}\right) =$$

$$= \frac{\pi^3}{24} - \frac{\pi^2}{12}\log 2 -$$

$$-\frac{1}{4}\Bigg(C(4-\pi) - 4Re\big(Li_3(1-i)\big) + \frac{29}{8}\zeta(3) - \frac{5}{24}\pi^2 + \pi + \frac{\log^2 2}{2}$$

$$+ \log 2\left(\frac{\pi^2}{8} - \frac{\pi}{2} - 2\right)\Bigg)$$

$$= \frac{24C(\pi-4) + 96Re\big(Li_3(1-i)\big) - 87\zeta(3) + 4\pi^3 + 5\pi^2 - 24\pi - 12\log^2 2 + 48\log 2 - 11\pi^2\log 2 + 12\pi\log 2}{192}$$

$$H_1(x) = \sum_{k=1}^{\infty}\frac{H_k}{k}x^k = Li_2(x) + \frac{1}{2}\log^2(1-x)$$

$$Re\big((1-i)H_1(i)\big) = C - \frac{5\pi^2}{96} + \frac{\log^2 2}{8} - \frac{\pi}{8}\log 2$$

$$H_2(x) = \sum_{k=1}^{\infty}\frac{H_k}{k^2}x^k =$$

$$= \zeta(3) + Li_3(x) - Li_3(1-x) + Li_2(1-x)\log(1-x) + \frac{\log x}{2}\log^2(1-x)$$

$$Re(H_2(i)) = \frac{29}{32}\zeta(3) + \frac{\pi^2}{16}\log 2 - \frac{\pi}{32}(8C + \pi\log 2) - Re(Li_3(1-i))$$

$$\sum_{k=1}^{\infty}(-1)^k H_{2k}\left(\frac{1}{k^2} - \frac{4}{2k-1} + \frac{2}{k}\right) = \frac{1}{2}\sum_{k=2}^{\infty}(-1)^{\frac{k}{2}}H_k\left(\frac{4}{k^2} - \frac{4}{k-1} + \frac{4}{k}\right)((-1)^k + 1) =$$

$$= 2\sum_{k=2}^{\infty}H_k((-i)^k + i^k)\left(\frac{1}{k^2} - \frac{1}{k-1} + \frac{1}{k}\right) = 4Re\left(\sum_{k=2}^{\infty}H_k i^k\left(\frac{1}{k^2} - \frac{1}{k-1} + \frac{1}{k}\right)\right) =$$

$$= 4Re\left(\sum_{k=1}^{\infty}H_k i^k\left(\frac{1}{k^2} + \frac{1}{k}\right) - \sum_{k=1}^{\infty}\frac{H_{k+1}i^{k+1}}{k}\right) =$$

$$= 4Re\left(\sum_{k=1}^{\infty}H_k i^k\left(\frac{1}{k^2} + \frac{1-i}{k}\right) - \sum_{k=1}^{\infty}\frac{i^{k+1}}{k(k+1)}\right) =$$

$$= -2\log 2 + \pi + 4Re(H_2(i) + (1-i)H_1(i)) =$$

$$= C(4-\pi) - 4Re(Li_3(1-i)) + \frac{29}{8}\zeta(3) - \frac{5}{24}\pi^2 + \pi + \frac{\log^2 2}{2} + \log 2\left(\frac{\pi^2}{8} - \frac{\pi}{2} - 2\right)$$

1851. Find the closed form:

$$\Omega = \int_0^1 \frac{\sin^{-1}x\log(1+x)}{x^2}dx$$

Proposed by Naren Bhandari-Bajura-Nepal

Solution 1 by Rana Ranino-Setif-Algerie

$$\Omega = \int_0^1 \frac{\sin^{-1}x\log(1+x)}{x^2}dx \overset{x\to\sin x}{=} \int_0^{\frac{\pi}{2}} \frac{x\log(1+\sin x)\cos x}{\sin^2 x}dx \overset{IBP}{=}$$

$$= -\frac{x\log(1+\sin x)}{\sin x}\Bigg|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}}\frac{\log(1+\sin x)}{\sin x}dx + \int_0^{\frac{\pi}{2}}\frac{x\cos x}{\sin x(1+\sin x)}dx =$$

$$= -\frac{\pi}{2}\log 2 + \int_0^{\frac{\pi}{2}}\frac{\log(1+\sin x)}{\sin x}dx + \int_0^{\frac{\pi}{2}}x\cot x\,dx - \int_0^{\frac{\pi}{2}}\frac{x\cos x}{1+\sin x}dx$$

$$A = \int_0^{\frac{\pi}{2}}\frac{\log(1+\sin x)}{\sin x}dx = \int_0^1\int_0^{\frac{\pi}{2}}\frac{1}{1+y\sin x}dxdy \overset{t=\tan\frac{x}{2}}{=}$$

$$= 2\int_0^1\int_0^1 \frac{1}{(t+y)^2+(1-y^2)}dtdy = 2\int_0^1\left[\frac{\tan^{-1}\left(\frac{t+y}{\sqrt{1-y^2}}\right)}{\sqrt{1-y^2}}\right]_0^1 dy =$$

$$= 2\int_0^1 \frac{\tan^{-1}\left(\frac{1+y}{\sqrt{1-y^2}}\right)-\tan^{-1}\left(\frac{y}{\sqrt{1-y^2}}\right)}{\sqrt{1-y^2}}dy \overset{y=\cos\varphi}{=}$$

$$= 2\int_0^{\frac{\pi}{2}}\left[\tan^{-1}\left(\cos\frac{\varphi}{2}\right)-\tan^{-1}(\cot\varphi)\right]d\varphi = \int_0^{\frac{\pi}{2}}\varphi\, d\varphi = \frac{\pi^2}{8}$$

$$B = \int_0^{\frac{\pi}{2}} x\cot x\, dx \overset{IBP}{=} x\log(\sin x)|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}}\log(\sin x)\, dx = \frac{\pi}{2}\log 2$$

$$C = \int_0^{\frac{\pi}{2}}\frac{x\cos x}{1+\sin x}dx \overset{IBP}{=} x\log(1+\sin x)|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}}\log(1+\sin x)\, dx \overset{x\to\frac{\pi}{2}-x}{=}$$

$$= \frac{\pi}{2}\log 2 - \int_0^{\frac{\pi}{2}}\log(1+\cos x)\, dx = \frac{\pi}{2}\log 2 - \int_0^{\frac{\pi}{2}}\log\left(2\, cos^2\frac{x}{2}\right)dx =$$

$$= -4\int_0^{\frac{\pi}{4}}\log(\cos x)\, dx = \pi\log 2 - 2G$$

Therefore,

$$\Omega = \int_0^1 \frac{\sin^{-1}x\log(1+x)}{x^2}dx = 2G + \frac{\pi^2}{8} - \pi\log 2$$

Solution 2 by Hamza Djahel-Msila-Algerie

$$\Omega = \int_0^1 \frac{\sin^{-1}x\log(1+x)}{x^2}dx =$$

$$= -\frac{\sin^{-1}x\log(1+x)}{x}\Bigg|_0^1 + \int_0^1\frac{\log(1+x)}{x\sqrt{1-x^2}}dx + \int_0^1\frac{\sin^{-1}x}{x(1+x)}dx =$$

$$= -\frac{\pi}{2}\log 2 + A + B$$

$$A = \int_0^1\frac{\log(1+x)}{x\sqrt{1-x^2}}dx \overset{\frac{1-x}{1+x}=y}{=} \int_0^1\frac{\log 2 - \log(1+y)}{(1-y)\sqrt{y}}dy =$$

$$= 2\int_0^1 \frac{-\log(1+x^2)+2\log(1+x)}{x}dx = \frac{1}{2}Li_2(-x^2)-2Li(-x)\Big|_0^1 =$$

$$= \frac{1}{2}\left(\frac{-\pi^2}{12}\right)+2\frac{\pi^2}{12}=\frac{\pi^2}{8}$$

$$B = \int_0^1 \frac{\sin^{-1}x}{x(1+x)}dx = \sin^{-1}x\log\left(\frac{x}{1+x}\right)\Big|_0^1 - \int_0^1 \frac{\log x-\log(1+x)}{\sqrt{1-x^2}}dx =$$

$$= \frac{\pi\log\left(\frac{1}{2}\right)}{2}+\frac{\pi\log 2}{2}+\int_0^1 \frac{\log(1+x)}{\sqrt{1-x^2}}dx = \int_0^1 \frac{\log(1+x)}{\sqrt{1-x^2}}dx \overset{\frac{1-x}{1+x}=y}{=}$$

$$= \int_0^1 \frac{\log 2-\log(1+x)}{(1+x)\sqrt{x}}dx = 2\int_0^1 \frac{\log 2-\log(1+x^2)}{1+x^2}dx =$$

$$= \frac{\pi\log 2}{2}-2\left(\frac{\pi\log 2}{2}-G\right)=2G-\frac{\pi}{2}\log 2$$

Therefore,

$$\Omega = \int_0^1 \frac{\sin^{-1}x\log(1+x)}{x^2}dx = 2G+\frac{\pi^2}{8}-\pi\log 2$$

1852. Find:

$$\Omega = \int_0^1 \frac{x\cdot Li_2(1-x)}{1+x^2}dx$$

Proposed by Togrul Ehmedov-Azerbaijan

Solution 1 by Fao Ler-Iraq

$$\Omega = \int_0^1 \frac{x\cdot Li_2(1-x)}{1+x^2}dx = \frac{1}{2}\log(x^2+1)\,Li_2(1-x)\Big|_0^1 - \frac{1}{2}\int_0^1 \log(x^2+1)\cdot\frac{\log x}{1-x}dx =$$

$$= -\frac{1}{2}\int_0^1 \log(x^2+1)\cdot\frac{\log x}{1-x}dx = \frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\int_0^1 \frac{x^{2k}\log x}{1-x}dx =$$

$$= -\frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\psi_1(2k+1) = -\frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^{\frac{k}{2}}}{k}\psi_1(k+1)\left((-1)^k+1\right)=$$

$$= -\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{\pi^2}{6}-H_{(k)}^{(2)}\right)\left((-i)^k+i^k\right) = -Re\left(\frac{\pi^2}{6}\sum_{k=1}^{\infty}\frac{i^k}{k}-\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k}i^k\right)=$$

$$=\frac{\pi^2}{12}\log 2+Re\left(H_1^{(2)}(i)\right)=$$

$$=\frac{\pi^2}{12}\log 2+\frac{24\pi C+192Re\big(Li_3(1-i)\big)-201\zeta(3)-5\pi^2\log 2}{96}=$$

$$=\frac{24\pi C+192Re\big(Li_3(1-i)\big)-201\zeta(3)+3\pi^2\log 2}{96}$$

$$H_1^{(2)}(x)=\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k}x^k=\int_0^1\frac{1}{y}\sum_{k=1}^{\infty}H_k^{(2)}(xy)^k\,dy=$$

$$=\int_0^1\frac{Li_2(xy)}{y(1-xy)}dy=\int_0^1\frac{Li_2(xy)}{y}dy+x\int_0^1\frac{Li_2(xy)}{1-xy}dy=$$

$$=Li_3(x)+\int_0^x\frac{Li_2(y)}{1-y}dy=$$

$$Li_3(x)+2Li_3(1-x)-2Li_2(1-x)\log(1-x)-Li_2(x)\log(1-x)$$
$$-\log x\log^2(1-x)-2\zeta(3)$$

$$Re\left(H_1^{(2)}(i)\right)=\frac{24\pi C+192Re\big(Li_3(1-i)\big)-201\zeta(3)-5\pi^2\log 2}{96}$$

Solution 2 by proposer

$$\because\ Li_2(x)+Li_2(1-x)=\zeta(2)-\log x\log(1-x)$$

$$\Omega=\int_0^1\frac{x\cdot Li_2(1-x)}{1+x^2}dx=$$

$$=\zeta(2)\int_0^1\frac{xdx}{1+x^2}-\int_0^1\frac{xLi_2(x)}{1+x^2}dx-\int_0^1\frac{x\log x\log(1-x)}{1+x^2}dx=$$

$$=\frac{\zeta(2)}{2}\log 2-I_1-I_2$$

$$I_1=\int_0^1\frac{xLi_2(x)}{1+x^2}dx=\frac{23}{64}\zeta(2)+\frac{\pi^2}{12}\log 2-\frac{\pi}{4}G$$

$$I_2=\int_0^1\frac{x\log x\log(1-x)}{1+x^2}dx=\frac{41}{64}\zeta(3)-\frac{3\pi^2}{32}\log 2$$

Therefore,

$$\Omega=\frac{\zeta(2)}{2}\log 2-\left(\frac{23}{64}\zeta(2)+\frac{\pi^2}{12}\log 2-\frac{\pi}{4}G\right)-\left(\frac{41}{64}\zeta(3)-\frac{3\pi^2}{32}\log 2\right)=$$

$$= -\zeta(3) + \frac{\pi}{4}G + \frac{9}{16}\zeta(2)\log 2$$

1853. Find:

$$\Omega = \int_0^\infty \frac{\sqrt{x}\log x}{x^2 - x + 1}dx$$

Proposed by Vasile Mircea Popa-Romania

Solution 1 by Fao Ler-Iraq

$$\Omega = \int_0^\infty \frac{\sqrt{x}\log x}{x^2 - x + 1}dx = \int_0^\infty \frac{(1+x)\sqrt{x}\log x}{1 + x^3}dx = \frac{1}{9}\int_0^\infty \frac{x^{-\frac{1}{2}} + x^{-\frac{1}{6}}}{1+x}\log x\, dx$$

$$\text{Let: } I(y) = \int_0^\infty \frac{x^{y-\frac{1}{2}} + x^{y-\frac{1}{6}}}{1+x}dx \Rightarrow I'(y) = \int_0^\infty \frac{x^{y-\frac{1}{2}} + x^{y-\frac{1}{6}}}{1+x}\log x\, dx$$

$$\Omega = \frac{1}{9}I'(0)$$

$$\text{We use } \int_0^\infty \frac{x^{p-1}}{1+x}dx = \pi\csc(p\pi), 0 < p < 1 \text{ and we obtain:}$$

$$I(y) = \int_0^\infty \frac{x^{y-\frac{1}{2}} + x^{y-\frac{1}{6}}}{1+x}dx = \pi\csc\left(\pi\left(y + \frac{1}{2}\right)\right) + \pi\csc\left(\pi\left(y + \frac{5}{6}\right)\right)$$

$$I'(y) =$$

$$= -\pi^2\left[\cot\left(\pi\left(y+\frac{1}{2}\right)\right)\csc\left(\pi\left(y+\frac{1}{2}\right)\right) + \cot\left(\pi\left(y+\frac{5}{6}\right)\right)\csc\left(\pi\left(y+\frac{5}{6}\right)\right)\csc\left(\pi\left(y+\frac{5}{6}\right)\right)\right]$$

$$I'(0) = -\pi^2\left(\cot\left(\frac{\pi}{2}\right)\csc\left(\frac{\pi}{2}\right) + \cot\left(\frac{5\pi}{6}\right)\csc\left(\frac{5\pi}{6}\right)\right) = 2\pi^2\sqrt{3}$$

Therefore,

$$\Omega = \frac{1}{9}2\pi^2\sqrt{3} = \frac{2\pi^2}{3\sqrt{3}}$$

Solution 2 by Togrul Ehmedov-Azerbaijan

$$\Omega = \int_0^\infty \frac{\sqrt{x}\log x}{x^2 - x + 1}dx = \int_0^1 \frac{\sqrt{x}\log x}{x^2 - x + 1}dx + \int_1^\infty \frac{\sqrt{x}\log x}{x^2 - x + 1}dx$$

$$= \int_0^1 \frac{\sqrt{x}\log x}{x^2 - x + 1}dx - \int_0^1 \frac{\log x}{\sqrt{x}(x^2 - x + 1)}dx =$$

$$=\int_0^1\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)\frac{\log x}{x^2-x+1}dx=4\int_0^1\frac{(x^2-1)\log x}{x^4-x^2+1}dx=$$

$$=4\int_0^1\frac{(x^4-1)\log x}{x^6+1}dx=4\sum_{k=0}^{\infty}(-1)^k\int_0^1 x^{6k}(x^4-1)\log x\,dx=$$

$$=4\sum_{k=0}^{\infty}(-1)^k\left[\frac{1}{(6k+1)^2}-\frac{1}{(6k+5)^2}\right]=$$

$$=4\sum_{k=0}^{\infty}\left[\frac{1}{(12k+1)^2}-\frac{1}{(12k+7)^2}-\frac{1}{(12k+5)^2}+\frac{1}{(12k+11)^2}\right]=$$

$$=\frac{1}{36}\left[\psi_1\left(\frac{1}{12}\right)-\psi_1\left(\frac{7}{12}\right)-\psi_1\left(\frac{5}{12}\right)+\psi_1\left(\frac{11}{12}\right)\right]=$$

$$=\frac{1}{36}\left[\frac{\pi^2}{\sin^2\left(\frac{\pi}{12}\right)}-\frac{\pi^2}{\sin^2\left(\frac{5\pi}{12}\right)}\right]$$

Therefore,

$$\Omega=\frac{1}{9}2\pi^2\sqrt{3}=\frac{2\pi^2}{3\sqrt{3}}$$

1854. Find:

$$\Omega=\int_0^1\int_0^1\log x\cdot\log(1-x)\cdot\log(1-xy)\,dxdy$$

Proposed by Togrul Ehmedov-Azerbaijan

Solution by Dawid Bialek-Poland

$$\Omega=\int_0^1\left(\log x\log(1-x)\underbrace{\int_0^1\log(1-xy)\,dy}_{t=1-xy}\right)dx=$$

$$=\int_0^1\left(\frac{\log x\log(1-x)}{x}\int_{1-x}^1\log t\,dt\right)dx=$$

$$=\int_0^1\left(\frac{\log x\log(1-x)}{x}[t\cdot\log t-t]_{1-x}^1\right)dx=$$

$$=-\int_0^1\frac{\log x\log(1-x)}{x}[(1-x)\log(1-x)+x]dx=$$

$$=\int_0^1 \log x\log^2(1-x)\,dx-\int_0^1 \frac{\log x\log^2(1-x)}{x}dx-\int_0^1 \log x\log(1-x)\,dx=$$

$$=I_1-I_2-I_3$$

$$I_1=\int_0^1 \log x\log^2(1-x)\,dx \overset{x\to 1-x}{=} \int_0^1 \log(1-x)\log^2 x\,dx=-\sum_{n=1}^{\infty}\frac{1}{n}\int_0^1 x^n\log^2 x\,dx=$$

$$\overset{IBP}{=} 2\sum_{n=1}^{\infty}\frac{1}{n(n+1)}\int_0^1 x^n\log x\,dx \overset{IBP}{=} -2\sum_{n=1}^{\infty}\frac{1}{n(n+1)^2}\int_0^1 x^n dx=-2\sum_{n=1}^{\infty}\frac{1}{n(n+1)^3}=$$

$$=-2\underbrace{\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)}_{telescoping\ series=1}+2\sum_{n=1}^{\infty}\frac{1}{(n+1)^2}+2\sum_{n=1}^{\infty}\frac{1}{(n+1)^2}=$$

$$=-2+2\sum_{n=1}^{\infty}\frac{1}{n^2}-2+2\sum_{n=1}^{\infty}\frac{1}{n^2}-2$$

$$I_1=2\zeta(2)+2\zeta(3)-6$$

$$I_2=\int_0^1 \frac{\log x\log^2(1-x)}{x}dx \overset{1-x\to x}{=} -\int_0^1 \frac{-\log(1-x)}{1-x}\log^2 x\,dx=$$

$$=-\sum_{n=1}^{\infty}H_n\int_0^1 x^n\log^2 x\,dx \overset{IBP}{=} 2\sum_{n=1}^{\infty}\frac{H_n}{n+1}\int_0^1 x^n\log x \overset{IBP}{=}$$

$$=-2\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^2}\int_0^1 x^n\,dx=-2\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^3}=$$

$$=-2\sum_{n=1}^{\infty}\frac{1}{(n+1)^3}\left(H_{n+1}-\frac{1}{n+1}\right)=-2\sum_{n=1}^{\infty}\frac{H_{n+1}}{(n+1)^3}+2\sum_{n=1}^{\infty}\frac{1}{(n+1)^4}=$$

$$=-2\sum_{n=1}^{\infty}\frac{H_n}{n^3}+2\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$I_2=-2\cdot\frac{5}{4}\zeta(4)+2\zeta(4)=-\frac{1}{2}\zeta(4)$$

$$I_3=\int_0^1 \log x\log(1-x)\,dx=-\sum_{n=1}^{\infty}\frac{1}{n}\int_0^1 x^n\log x\,dx \overset{IBP}{=}$$

$$=\sum_{n=1}^{\infty}\frac{1}{n(n+1)}\int_0^1 x^n dx=\sum_{n=1}^{\infty}\frac{1}{n(n+1)^2}=$$

$$=\underbrace{\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)}_{telescoping\ series=1}-\sum_{n=1}^{\infty}\frac{1}{(n+1)^2}=1-\sum_{n=1}^{\infty}\frac{1}{n^2}+1$$

$$I_3=2-\zeta(2)$$

Finally, we have:

$$\Omega=I_1-I_2-I_3=2\zeta(2)+2\zeta(3)-6+\frac{1}{2}\zeta(4)-2+\zeta(2)=$$

$$=3\zeta(2)+2\zeta(3)+\frac{1}{2}\zeta(4)-8$$

1855. If $0<a\le b<\frac{\pi}{2}$ then:

$$2\pi\int_a^b\frac{x}{\sin x}dx\le(b-a)\big(2\pi(\pi-2)(b+a)\big)$$

Proposed by Daniel Sitaru-Romania

Solution by Hikmat Mammadov-Azerbaijan

$$\text{Let: } f(x)=\frac{1}{\sin x}=\csc x=\sum_{n=0}^{\infty}\frac{(-1)^n(2-2^{2n})B_{2n}}{(2n)!},x\in[-\pi,\pi]$$

$$\frac{(-1)^n(2-2^{2n})B_{2n}}{(2n)!}>0;\forall x\in\mathbb{Z}_+$$

$$f''(x)\ge 0,\forall x\in[-\pi,\pi]\Rightarrow f\text{ −convex function}$$

$$f\left(t\cdot 0+(1-t)\cdot\frac{\pi}{2}\right)\le t\cdot f(0)+(1-t)f\left(\frac{\pi}{2}\right)=t+(1-t)f\left(\frac{\pi}{2}\right)=$$

$$=t+(1-t)\cdot\frac{\pi}{2},t\in[0,1]$$

$$f(x)\le\left(1-\frac{2}{\pi x}\right)+x,x\in\left[0,\frac{\pi}{2}\right]$$

$$x=t\cdot 0+(1-t)\cdot\frac{\pi}{2}$$

$$2\pi\int_a^b\frac{x}{\sin x}dx=2\pi\int_a^b f(x)dx\le 2\pi\int_a^b\left[\left(1-\frac{2}{\pi}x\right)+x\right]dx=$$

$$= 2\pi \int_a^b \left[1 + \left(1 - \frac{2}{\pi}\right)x\right] dx = 2\pi \left[x + \left(1 - \frac{2}{\pi}\right) \cdot \frac{x^2}{2}\right]_a^b =$$

$$= 2\pi \left[(b-a) + \left(1 - \frac{2}{\pi}\right) \cdot \frac{b^2 - a^2}{2}\right] = (b-a)[2\pi + (\pi - 2)(b+a)$$

Therefore,

$$2\pi \int_a^b \frac{x}{\sin x} dx \le (b-a)\big(2\pi(\pi-2)(b+a)\big)$$

1856. If $1 < a \le b$ then:

$$\int_a^b \int_a^b \frac{dxdy}{1+xy} \le (b-a)\tan^{-1}\left(\frac{b-a}{1+ab}\right)$$

Proposed by Daniel Sitaru-Romania

Solution by Hikmat Mammadov-Azerbaijan

$$\int_a^b \int_a^b \left(\frac{1}{1+x^2} - \frac{1}{1+xy}\right) dxdy = \int_a^b \int_a^b \frac{x(y-x)}{(1+x^2)(1+xy)} dxdy =$$

$$= \frac{1}{2}\left(\int_a^b \int_a^b \frac{x(y-x)}{1+x^2(1+xy)} dxdy + \int_a^b \int_a^b \frac{y(x-y)}{(1+y^2)(1+yx)} dydx\right) =$$

$$= \frac{1}{2}\int_a^b \int_a^b \frac{(y-x)[x(1+y^2) - y(1+x^2)]}{(1+x^2)(1+y^2)(1+xy)} dxdy =$$

$$= \frac{1}{2}\int_a^b \int_a^b \frac{(y-x)(x+xy^2-y-yx^2)}{(1+x^2)(1+y^2)(1+xy)} dxdy =$$

$$= \frac{1}{2}\int_a^b \int_a^b \frac{(y-x)(-1+xy)}{(1+x^2)(1+y^2)(1+xy)} dxdy \ge$$

$$\ge \frac{1}{2}\int_a^b \frac{(y-x^2)(-1+a^2)}{(1+x^2)(1+y^2)(1+xy)} dxdy \ge 0$$

Therefore,

$$\int_a^b \int_a^b \frac{dxdy}{1+xy} \le \int_a^b \int_a^b \frac{1}{1+x^2} dxdy = y|_a^b \tan^{-1} x\Big|_a^b = (b-a)(\tan^{-1} b - \tan^{-1} a) =$$

$$= (b-a)\tan^{-1}\left(\frac{b-a}{1+ab}\right)$$

1857.

$$\Omega(n)=\sum_{k=1}^{n}e^{4\left(2-\frac{k}{n}\right)}\cdot\sum_{k=1}^{n}e^{6\left(2-\frac{k}{n}\right)}-\left(\sum_{k=1}^{n}e^{5\left(2-\frac{k}{n}\right)}\right)^{2}$$

Find:

$$\Omega=\lim_{n\to\infty}n^{7}\cdot\Omega(n)\cdot\sin\left(\frac{1}{n^{4}}\right)\cdot\tan\left(\frac{1}{n^{5}}\right)$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Ankush Kumar Parcha-India

$$\Omega(n)=\sum_{k=1}^{n}e^{4\left(2-\frac{k}{n}\right)}\cdot\sum_{k=1}^{n}e^{6\left(2-\frac{k}{n}\right)}-\left(\sum_{k=1}^{n}e^{5\left(2-\frac{k}{n}\right)}\right)^{2}=$$

$$=e^{20}\left[\sum_{k=1}^{n}e^{-\frac{4k}{n}}\cdot\sum_{k=1}^{n}e^{-\frac{6k}{n}}-\left(\sum_{k=1}^{n}e^{-\frac{5k}{n}}\right)^{2}\right]=$$

$$=e^{20}\left[\frac{e^{-\frac{4}{n}}(1-e^{-4})}{1-e^{-\frac{4}{n}}}\cdot\frac{e^{-\frac{6}{n}}(1-e^{-6})}{1-e^{-\frac{6}{n}}}-\left(\frac{e^{-\frac{5}{n}}(1-e^{-5})}{1-e^{-\frac{5}{n}}}\right)^{2}\right]=$$

$$=\frac{e^{10}(e^{4}-1)(e^{6}-1)}{\left(e^{\frac{4}{n}}-1\right)\left(e^{\frac{6}{n}}-1\right)}-\frac{e^{10}\left(e^{5}-1\right)^{2}}{\left(e^{\frac{5}{n}}-1\right)^{2}}$$

Hence,

$$\Omega=\lim_{n\to\infty}n^{7}\frac{e^{10}(e^{4}-1)(e^{6}-1)}{\left(e^{\frac{4}{n}}-1\right)\left(e^{\frac{6}{n}}-1\right)}\cdot\sin\left(\frac{1}{n^{4}}\right)\cdot\tan\left(\frac{1}{n^{5}}\right)-$$

$$-\lim_{n\to\infty}\frac{e^{10}\left(e^{5}-1\right)^{2}}{\left(e^{\frac{5}{n}}-1\right)^{2}}\cdot\sin\left(\frac{1}{n^{4}}\right)\cdot\tan\left(\frac{1}{n^{5}}\right)=$$

$$=\lim_{n\to\infty}\frac{e^{10}(e^{4}-1)(e^{6}-1)}{24\frac{\left(e^{\frac{4}{n}}-1\right)\left(e^{\frac{6}{n}}-1\right)}{\frac{24}{n^{2}}}}\cdot\frac{\sin\left(\frac{1}{n^{4}}\right)}{\frac{1}{n^{4}}}\cdot\frac{\tan\left(\frac{1}{n^{5}}\right)}{\frac{1}{n^{5}}}-$$

$$-\lim_{n\to\infty}\frac{e^{10}(e^5-1)^2}{25\frac{\left(e^{\frac{5}{n}}-1\right)^2}{\frac{25}{n^2}}}\cdot\frac{\sin\left(\frac{1}{n^4}\right)}{\frac{1}{n^4}}\cdot\frac{\tan\left(\frac{1}{n^5}\right)}{\frac{1}{n^5}}=e^{10}\left[\frac{(e^4-1)(e^6-1)}{24}-\frac{(e^5-1)^2}{25}\right]$$

Solution 2 by Ravi Prakash-New Delhi-India

For $m\in\mathbb{N}$, let

$$S(m)=\sum_{k=1}^{\infty}e^{m\left(2-\frac{k}{n}\right)}=e^{2m}\sum_{k=1}^{\infty}\left(e^{-\frac{m}{n}}\right)^k=e^{2m}\frac{e^{-\frac{m}{n}}\left[1-\left(e^{-\frac{m}{n}}\right)^n\right]}{1-e^{-\frac{m}{n}}}=$$

$$=\frac{e^{2m}(1-e^{-m})}{e^{\frac{m}{n}}-1}=\frac{e^{2m}-e^{n}}{e^{\frac{m}{n}}-1}$$

For $m_1,m_2\in\mathbb{N}$ we have:

$$\lim_{n\to\infty}n^7\cdot S(m_1)\cdot S(m_2)\cdot\sin\left(\frac{1}{n^4}\right)\cdot\tan\left(\frac{1}{n^5}\right)=$$

$$=\lim_{n\to\infty}\frac{(e^{2m_1}-e^{m_1})(e^{2m_2}-e^{m_2})}{m_1\left(\frac{e^{\frac{m_1}{n}}-1}{\frac{m_1}{n}}\right)m_2\left(\frac{e^{\frac{m_2}{n}}-1}{\frac{m_2}{n}}\right)}\cdot\frac{\sin\left(\frac{1}{n^4}\right)}{\frac{1}{n^4}}\cdot\frac{\tan\left(\frac{1}{n^5}\right)}{\frac{1}{n^5}}=$$

$$=\frac{e^{2m_1}-e^{m_1}}{m_1}\cdot\frac{e^{2m_2}-e^{m_2}}{m_2}$$

Therefore,

$$\Omega=\lim_{n\to\infty}n^7\cdot\Omega(n)\cdot\sin\left(\frac{1}{n^4}\right)\cdot\tan\left(\frac{1}{n^5}\right)=$$

$$=e^{10}\left[\frac{(e^4-1)(e^6-1)}{24}-\frac{(e^5-1)^2}{25}\right]$$

Solution 3 by Adrian Popa-Romania

$$\Omega=\lim_{n\to\infty}n^7\cdot\Omega(n)\cdot\sin\left(\frac{1}{n^4}\right)\cdot\tan\left(\frac{1}{n^5}\right)=$$

$$=\lim_{n\to\infty}n^9\left(\frac{1}{n}\cdot\sum_{k=1}^{n}e^{4\left(2-\frac{k}{n}\right)}\cdot\frac{1}{n}\cdot\sum_{k=1}^{n}e^{6\left(2-\frac{k}{n}\right)}-\left(\frac{1}{n}\sum_{k=1}^{n}e^{5\left(2-\frac{k}{n}\right)}\right)^2\right)\cdot\sin\left(\frac{1}{n^4}\right)\cdot\tan\left(\frac{1}{n^5}\right)=$$

$$= \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n^4}\right)}{\frac{1}{n^4}} \cdot \frac{\tan\left(\frac{1}{n^5}\right)}{\frac{1}{n^5}} \left(\int_0^1 e^{4(2-x)}dx \cdot \int_0^1 e^{6(2-x)}dx - \left(\int_0^1 e^{5(2-x)}dx\right)^2\right) =$$

$$= \left(-e^8 \frac{e^{-4x}}{4}\Big|_0^1\right) \cdot \left(-e^{12} \frac{e^{-6x}}{6}\Big|_0^1\right) - \left(-e^{10} \frac{e^{-5x}}{5}\Big|_0^1\right)^2 =$$

$$= e^{10}\left[\frac{(e^4-1)(e^6-1)}{24} - \frac{(e^5-1)^2}{25}\right]$$

1858. Find:

$$\Omega = \int_0^1 \int_0^1 \log(1-x) \cdot Li_2(1-x)\, dxdy$$

Proposed by Togrul Ehmedov-Azerbaijan

Solution by Benjamin Bamidele-Nigeria

$$\Omega = \int_0^1 \int_0^1 \log(1-x) \cdot Li_2(1-x)\, dxdy = \int_0^1 \log(1-x) \int_0^1 Li_2(1-xy)\, dxdy$$

$$\int_0^1 Li_2(1-xy)\, dy \overset{1-xy=z}{=} \frac{1}{x}\int_1^{1-x} Li_2(z)\, dz = \frac{1}{x}(zLi_2(z) + (z-1)\log(1-z))\Big|_1^{1-x} =$$

$$= (x-1)\frac{Li_2(1-x)}{x} + \frac{\zeta(2)}{x} + \log x - 1$$

$$\Omega = \int_0^1 \log(1-x)\left[(x-1)\frac{Li_2(1-x)}{x} + \frac{\zeta(2)}{x} + \log x - 1\right] dx$$

$$Let\ A = \int_0^1 \frac{(x-1)\log(1-x)\, Li_2(1-x)}{x} dx =$$

$$= \int_0^1 \left(\log(1-x)\, Li_2(1-x) - \frac{\log(1-x)\, Li_2(1-x)}{x}\right) dx$$

$$\int_0^1 Li_2(1-x)\log(1-x)\, dx = \int_0^1 Li_2(x)\log x\, dx \overset{IBP}{=}$$

$$= x\log x|_0^1 - \int_0^1 Li_2(x)\, dx + \int_0^1 \log x \log(1-x)\, dx \overset{IBP}{=}$$

$$= -Li_2(1) + 1 - \left(\int_0^1 \log(1-x)\, dx - \int_0^x \frac{x\log x}{1-x} dx\right) =$$

$$= -\zeta(2) + 1 + \left(1 + \int_0^1 \frac{\log(1-x)}{x} dx - \int_0^1 \log(1-x)\, dx\right) =$$

$$= -\zeta(2) + 1 + \left(2 - Li_2(1)\right)$$

$$\int_0^1 Li_2(1-x)\log(1-x)\, dx = \int_0^1 \frac{Li_2(x)\log x}{1-x} dx \overset{IBP}{=}$$

$$= -\int_0^1 \frac{\log^2(1-x)\log x}{x} dx - \int_0^1 \frac{Li_2(x)\log x}{1-x} dx \overset{IBP}{=}$$

$$= \frac{1}{3}\int_0^1 \frac{\log^3 x}{1-x} dx + \left(Li_2(x)\right)^2\Big|_0^1 - \int_0^1 \frac{\log(1-x)\, Li_2(x)}{x} dx =$$

$$= \frac{1}{3}\sum_{k=1}^{\infty}\int_0^1 x^k \log^3 x\, dx + \frac{1}{2}\left(Li_2(1)\right)^2 = \frac{1}{3}\sum_{k=1}^{\infty}\frac{-6}{k^4} + \zeta^2(2) = \zeta^2(2) - 2\zeta(4)$$

$$\Rightarrow \int_0^1 \frac{Li_2(1-x)\log(1-x)}{x} dx = \zeta^2(2) - 2\zeta(4)$$

Hence, $A = 3 - 2\zeta(2) - \frac{1}{2}\zeta^2(2) + 2\zeta(4)$

$$B = \zeta(2)\int_0^1 \frac{\log(1-x)}{x} dx = \zeta(2)\int_0^1 \frac{\log x}{1-x} dx = -\zeta^2(2)$$

Let $C = \int_0^1 \log x \log(1-x)\, dx$

Integral C solution is above bracketed for easy recognition.

$$C = 2 - Li_2(1) = 2 - \zeta(2)$$

Let $D = \int_0^1 \log(1-x)\, dx = -1$

$$\Omega = A + B + C - D = 6 - 3\zeta(2) + 2\zeta(4) - \frac{3}{2}\zeta^2(2)$$

1859.

$$\omega = \sum_{n=1}^{\infty} \cos^{-1}\left(\frac{n^2 - n + 1}{\sqrt{1 + (n^2 - n + 1)^2}}\right)$$

Find:

$$\Omega = \left(\int_0^{4\omega} \frac{dx}{3+\sin x}\right)\left(\int_0^{\frac{2\omega}{\pi}} \frac{x^3}{\sqrt{1+x+x^2}}dx\right)$$

Proposed by Daniel Sitaru-Romania

Solution by Ravi Prakash-New Delhi-India

Put $n^2 - n + 1 = \cot\theta$, then

$$\cos^{-1}\left(\frac{n^2-n+1}{\sqrt{1+(n^2-n+1)^2}}\right) = \cos^{-1}(\cos\theta) = \theta =$$

$$= \cot^{-1}(n^2-n+1) = \tan^{-1}\left(\frac{n-(n-1)}{1+n(n-1)}\right) = \tan^{-1} n - \tan^{-1}(n-1)$$

$$\omega = \sum_{n=1}^{\infty} \cos^{-1}\left(\frac{n^2-n+1}{\sqrt{1+(n^2-n+1)^2}}\right) = \sum_{n=1}^{\infty}(\tan^{-1} n - \tan^{-1}(n-1)) = \frac{\pi}{2}$$

$$I_1 = \int_0^{4\omega} \frac{dx}{3+\sin x} = \int_0^{2\pi} \frac{dx}{3+\sin x} = \int_0^{\pi} \frac{dx}{3+\sin x} + \int_{\pi}^{2\pi} \frac{dx}{3+\sin x} \overset{x=\pi+\theta}{=}$$

$$= \int_0^{\pi} \frac{dx}{3+\sin x} + \int_0^{\pi} \frac{d\theta}{3-\sin\theta} = \int_0^{\pi} \frac{6}{9-\sin^2 x}dx = 6\int_0^{\pi} \frac{dx}{8+\cos^2 x} =$$

$$= 6\int_0^{\frac{\pi}{2}} \frac{dx}{8+\cos^2 x} + 6\int_{\frac{\pi}{2}}^{\pi} \frac{dx}{8+\cos^2 x} \overset{x=\frac{\pi}{2}+\theta}{=} 6\int_0^{\frac{\pi}{2}} \frac{dx}{8+\cos^2 x} + 6\int_0^{\frac{\pi}{2}} \frac{d\theta}{8+\cos^2\theta} =$$

$$= 6\int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{8\sec^2 x + 1}dx + 6\int_0^{\frac{\pi}{2}} \frac{\csc^2\theta}{8\csc^2\theta + 1}d\theta =$$

$$= 6\int_0^{\frac{\pi}{2}} \frac{\sec^2 x\,dx}{8\tan^2 x + 9} + 6\int_0^{\frac{\pi}{2}} \frac{\csc^2 x\,dx}{8\cot^2 x + 9} =$$

$$= \frac{6}{6\sqrt{2}}\tan^{-1}\left(\frac{2\sqrt{2}\tan x}{3}\right)\Bigg|_0^{\frac{\pi}{2}} - \frac{6}{6\sqrt{2}}\tan^{-1}\left(\frac{2\sqrt{2}\cot\theta}{3}\right)\Bigg|_0^{\frac{\pi}{2}} = \frac{\pi}{\sqrt{2}}$$

$$I_2 = \int_0^{\frac{2\omega}{\pi}} \frac{x^3}{\sqrt{1+x+x^2}}dx = \int_0^1 \frac{x^3}{\sqrt{1+x+x^2}}dx =$$

$$= \int_0^1 \frac{x(x^2+x+1) - (x^2+x+1) + 1}{\sqrt{1+x+x^2}}dx =$$

$$= \int_0^1 (x-1)\sqrt{x^2+x+1}\,dx + \int_0^1 \frac{dx}{\sqrt{x^2+x+1}} =$$

$$= \frac{1}{2}\int_0^1 (2x+1-3)\sqrt{x^2+x+1}\,dx + \int_0^1 \frac{dx}{\sqrt{x^2+x+1}} =$$

$$= \frac{1}{2}\int_0^1 (2x+1)\sqrt{x^2+x+1}\,dx - \frac{3}{2}\int_0^1 \sqrt{x^2+x+1}\,dx + \int_0^1 \frac{dx}{\sqrt{x^2+x+1}} =$$

$$= \frac{1}{3}(x^2+x+1)^{\frac{3}{2}}\Big|_0^1 - \frac{3}{2}\int_0^1 \sqrt{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\,dx + \int_0^1 \frac{1}{\sqrt{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}}\,dx =$$

$$= \frac{1}{3}(3\sqrt{3}-1) - \frac{3}{2}\left[\frac{x+\frac{1}{2}}{2}\sqrt{x^2+x+1} + \frac{3}{4}\log\left(x+\frac{1}{2}+\sqrt{x^2+x+1}\right)\right]\Bigg|_0^1 +$$

$$+\log\left(x+\frac{1}{2}+\sqrt{x^2+x+1}\right)\Big|_0^1 == \frac{1}{24} + \frac{5}{8}\sqrt{3} + \frac{1}{8}\log\left(\frac{3}{2}\right) - \frac{1}{8}\log\left(\frac{3}{2}+\sqrt{3}\right)$$

Therefore,

$$\Omega = \frac{\pi}{\sqrt{2}}\left[\frac{1}{24} + \frac{5}{8}\sqrt{3} + \frac{1}{8}\log\left(\frac{3}{2}\right) - \frac{1}{8}\log\left(\frac{3}{2}+\sqrt{3}\right)\right]$$

Note by editor:

$$\int_0^{4\omega} \frac{dx}{3+\sin x} = \int_0^{2\pi} \frac{dx}{3+\sin x} \overset{y=x-\pi}{\cong} \int_{-\pi}^{\pi} \frac{dy}{3-\sin y} =$$

$$= \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \int_{-\pi+\varepsilon}^{\pi-\varepsilon} \frac{dy}{3-\sin y} \overset{z=\tan\frac{y}{2}}{\cong} \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \int_{\tan\left(\frac{-\pi+\varepsilon}{2}\right)}^{\tan\left(\frac{\pi-\varepsilon}{2}\right)} \frac{\frac{2}{1+z^2}dz}{3-\frac{2z}{1+z^2}} =$$

$$= 2\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \int_{\tan\left(\frac{-\pi+\varepsilon}{2}\right)}^{\tan\left(\frac{\pi-\varepsilon}{2}\right)} \frac{dz}{3+3z^2-2z} = \frac{2}{3}\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \int_{\tan\left(\frac{-\pi+\varepsilon}{2}\right)}^{\tan\left(\frac{\pi-\varepsilon}{2}\right)} \frac{dz}{\left(z-\frac{1}{3}\right)^2+\frac{8}{9}} =$$

$$=\frac{2}{3}\cdot\frac{1}{\frac{2\sqrt{2}}{3}}\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}\left(arctan\frac{tan\left(\frac{\pi-\varepsilon}{2}\right)-\frac{1}{3}}{\frac{2\sqrt{2}}{\sqrt{3}}}-arctan\frac{tan\left(\frac{-\pi+\varepsilon}{2}\right)-\frac{1}{3}}{\frac{2\sqrt{2}}{\sqrt{3}}}\right)=$$

$$=\frac{1}{\sqrt{2}}\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right)=\frac{\pi\sqrt{2}}{2}$$

Dan S.

1860. Find:

$$\Omega=\int_0^1\log^2 x\log^2(1-x)\,dx$$

Proposed by Togrul Ehmedov-Azerbaijan

Solution by Avishek Mitra-West Bengal-India

$$\Omega=\int_0^1\log^2 x\log^2(1-x)\,dx=(x\log x-x)\log x\log^2(1-x)\Big|_0^1+$$

$$+\int_0^1\left(\frac{2\log x\log(1-x)}{1-x}-\frac{\log^2(1-x)}{x}\right)(x\log x-x)dx=$$

$$=2\int_0^1\frac{x\log^2 x\log(1-x)}{1-x}dx-2\int_0^1\frac{x\log(1-x)\log x}{1-x}dx-\int_0^1\log x\log^2(1-x)\,dx+$$

$$+\int_0^1\log^2(1-x)\,dx$$

$$I_1=\int_0^1\log^2(1-x)\,dx=\int_0^1\log^2 x\,dx=x\log^2 x\Big|_0^1-2\int_0^1\frac{\log x}{x}xdx=-2\int_0^1\log x\,dx$$

$$=$$

$$=-2(x\log x-x)|_0^1=2$$

$$I_2=\int_0^1\log x\log^2(1-x)\,dx=\int_0^1\log(1-x)\log^2 x\,dx=$$

$$=-\sum_{n=1}^{\infty}\frac{1}{n}\int_0^1 x^n\log^2 x\,dx=-\sum_{n=1}^{\infty}\frac{2}{n(n+1)^3}=$$

$$\left(\because \int_0^1 x^n \log^2 x\,dx = \frac{(-1)^2(2!)}{(n+1)^{2+1}}; IBP\ and\ reduction\ formula\right)$$

$$= -2\left[\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)-\sum_{n=1}^{\infty}\frac{1}{(n+1)^2}-\sum_{n=1}^{\infty}\frac{1}{(n+1)^3}\right] =$$

$$= -2[1-(\zeta(2)-1)-(\zeta(3)-1)] = -2[3-\zeta(2)-\zeta(3)] = 2\zeta(2)+2\zeta(3)-6$$

$$I_3 = \int_0^1 \frac{x\log(1-x)\log x}{1-x}dx = \int_0^1 \frac{(1-x)\log x\log(1-x)}{x}dx =$$

$$= \int_0^1 \frac{\log x\log(1-x)}{x}dx - \int_0^1 \log x\log(1-x)\,dx =$$

$$= -\sum_{n=1}^{\infty}\frac{1}{n}\int_0^1 x^{n-1}\log x\,dx + \sum_{n=1}^{\infty}\frac{1}{n}\cdot\frac{(-1)(1!)}{(n+1)^2} =$$

$$= \sum_{n=1}^{\infty}\frac{1}{n^3} - \sum_{n=1}^{\infty}\frac{1}{n(n+1)^2} = \zeta(3) - \left[\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)-\sum_{n=1}^{\infty}\frac{1}{(n+1)^2}\right] =$$

$$= \zeta(3) - [1-(\zeta(2)-1)] = \zeta(3)+\zeta(2)-2$$

$$I_4 = \int_0^1 \frac{x\log^2 x\log(1-x)}{(1-x)}dx = \int_0^1 \frac{(1-x)\log x\log^2(1-x)}{x}dx =$$

$$= \int_0^1 \frac{\log x\log^2(1-x)}{x}dx - \int_0^1 \log x\log^2(1-x)\,dx =$$

$$= \int_0^1 \frac{\log(1-x)\log^2 x}{1-x}dx - I_2 = -\int_0^1 \frac{Li_2(x)\log^2 x}{(1-x)}dx - I_2 =$$

$$= -\sum_{n=1}^{\infty} H_n \int_0^1 x^n \log^2 x\,dx - I_2 = -2\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^3} - I_2 =$$

$$= -2\sum_{n=1}^{\infty}\frac{H_{n+1}-\frac{1}{n+1}}{(n+1)^3} - I_2 = -2\sum_{n=1}^{\infty}\frac{H_{n+1}}{(n+1)^3} + 2\sum_{n=1}^{\infty}\frac{1}{(n+1)^4} - I_2 =$$

$$= -2\left(\sum_{n=1}^{\infty}\frac{H_n}{n^3}-1\right) + 2(\zeta(4)-1) - I_2 = -2\cdot\frac{5}{4}\zeta(4) + 2\zeta(4) - I_2 =$$

$$= -\frac{1}{2}\zeta(4) - I_2$$

$$\Omega = 2I_4 - 2I_3 - I_2 + I_1 = 24 - \zeta(4) - 8\zeta(3) - 8\zeta(2)$$

1861. Find a closed form:

$$\Omega = \int_0^{\frac{\pi}{4}} x\tan x\log(\cos x)\,dx$$

Proposed by Naren Bhandari-Bajura-Nepal

Solution by Rana Ranino-Setif-Algerie

$$\Omega = \int_0^{\frac{\pi}{4}} x\tan x\log(\cos x)\,dx \overset{IBP}{=} -\frac{1}{2}x\log^2(\cos x)\Big|_0^{\frac{\pi}{4}} + \frac{1}{2}\int_0^{\frac{\pi}{4}}\log^2(\cos x)\,dx =$$

$$= -\frac{\pi}{32}\log^2 2 + \frac{1}{2}I$$

Note: $\log(1+e^{2ix}) = \log(e^{ix}+e^{-ix}) + \log(e^{ix}) = \log(2\cos x) + ix$

Squaring both sides and integrating, we get:

$$\int_0^{\frac{\pi}{4}}\log^2(1+e^{2ix})\,dx = \int_0^{\frac{\pi}{4}}(\log(2\cos x)+ix)^2dx =$$

$$= \int_0^{\frac{\pi}{4}}(\log(2\cos x) - x^2 + 2ix\log(2\cos x))dx$$

$$\Re\left\{\int_0^{\frac{\pi}{4}}\log^2(1+e^{2ix})\,dx\right\} =$$

$$= \int_0^{\frac{\pi}{4}}\log^2(\cos x)\,dx + \underbrace{\int_0^{\frac{\pi}{4}}(\log^2 2 - x^2)dx}_{\frac{\pi}{4}\log^2 2 - \frac{\pi^3}{192}} + 2\log 2\underbrace{\int_0^{\frac{\pi}{4}}\log(\cos x)\,dx}_{\frac{G}{2}-\frac{\pi}{4}\log 2}$$

$$\int_0^{\frac{\pi}{4}}\log^2(\cos x)\,dx = \frac{\pi^3}{192} + \frac{\pi}{4}\log^2 2 - G\log 2 + \underbrace{\Re\left\{\int_0^{\frac{\pi}{4}}\log^2(1+e^{2ix})\,dx\right\}}_{J}$$

$$J \overset{t=e^{2ix}}{=} \frac{1}{2}\Im\left\{\int_1^i \frac{\log^2(1+t)}{t}dt\right\} =$$

$$= \frac{1}{2}\Im[\log(-t)\log^2(1+t) + 2Li_2(1+t)\log(1+t) - 2Li_2(1+t)]_1^i =$$

$$= \frac{1}{2}\Im(\log(-i)\log^2(1+i) + 2Li_2(1+i)\log(1+i) - 2Li_3(1+i))$$

$$\log(-i)=-\frac{i\pi}{2},\qquad \log(1+i)=\frac{1}{2}\log 2+\frac{i\pi}{4},\qquad Li_2(1+i)=\frac{\pi^2}{16}+i\left(G+\frac{\pi}{4}\log 2\right)$$

$$J=\frac{\pi^2}{32}+\frac{\pi}{16}\log^2 2+\frac{G}{2}\log 2-\Im\big(Li_3(1+i)\big)=$$

$$=\frac{7\pi^3}{192}-\frac{G}{2}\log 2+\frac{5\pi}{16}\log^2 2-\Im\big(Li_3(1+i)\big)$$

Therefore,

$$\Omega=\int_0^{\frac{\pi}{4}} x\tan x\log(\cos x)\,dx=\frac{\pi}{8}\log^2 2+\frac{7\pi^2}{384}-\frac{G}{4}\log 2-\frac{1}{2}\Im\big(Li_3(1+i)\big)$$

1862. Prove that:

$$\frac{8}{\pi}\int_0^{\frac{\pi}{2}} x\cot x\log(1-\sin^4 x)\,dx=\frac{5\pi^2}{12}-\frac{25}{4}\log^2 2-4Li_2\left(\frac{1}{\sqrt{2}}\right)+$$

$$+2Li_2\left(\frac{2-\sqrt{2}}{4}\right)-\log^2(1+\sqrt{2})+3\log 2\log(1+\sqrt{2})$$

where $Li_2(x)=\sum_{k=1}^{\infty}\frac{x^k}{k^2}$ is dilogarithm function.

Proposed by Naren Bhandari-Bajura-Nepal

Solution by Rana Ranino-Setif-Algerie

$$\Omega=\frac{8}{\pi}\int_0^{\frac{\pi}{2}} x\cot x\log(1-\sin^4 x)\,dx$$

$$\int\cot x\log(1-\sin^4 x)\,dx \overset{t=\sin^4 x}{=} \frac{1}{4}\int\frac{\log(1-t)}{t}dt=-\frac{1}{4}Li_2(t)=-\frac{1}{4}Li_2(\sin^4 x)$$

$$\int\frac{\log(1-y)}{1+y}dy \overset{x=\frac{1+y}{2}}{=} \int\frac{\log 2+\log(1-x)}{x}dx=\log 2\log x-Li_2(x)=$$

$$=\log 2\log\left(\frac{1+y}{2}\right)-Li_2\left(\frac{1+y}{2}\right)=\log 2\log x+Li_2(1-x)+\log(x-1)\log x=$$

$$=Li_2\left(\frac{1-y}{2}\right)+\log(y-1)\log\left(\frac{1+y}{2}\right)$$

$$\because Li_2(z^2)=2Li_2(z)+2Li_2(-z)$$

$$\Omega=\left[-\frac{2}{\pi}xLi_2(\sin^4 x)\right]_0^{\frac{\pi}{2}}+\frac{2}{\pi}\int_0^{\frac{\pi}{2}}Li_2(\sin^4 x)\,dx=-\frac{\pi^2}{6}+\frac{2}{\pi}\int_0^{\frac{\pi}{2}}Li_2(\sin^4 x)dx=$$

$$= -\frac{\pi^2}{6} + \frac{4}{\pi}\underbrace{\int_0^{\frac{\pi}{2}} Li_2(\sin^2 x)\,dx}_{A} + \frac{4}{\pi}\underbrace{\int_0^{\frac{\pi}{2}} Li_2(-\sin^2 x)dx}_{B}$$

$$A = -\int_0^1\int_0^{\frac{\pi}{2}} \frac{\sin^2 x\log x}{1 - y\sin^2 x}dxdy \overset{x=\tan t}{=} -\int_0^1 \log y\int_0^{\infty}\frac{t^2}{(1+(1-y)t^2)(1+t^2)}dtdy =$$

$$= \int_0^1 \frac{\log y}{y}\int_0^{\infty}\left(\frac{1}{1+t^2} - \frac{1}{1+(1-y)t^2}\right)dtdy = \frac{\pi}{2}\int_0^1\frac{\log y}{y}\left(\frac{\sqrt{1-y}-1}{\sqrt{1-y}}\right)dy =$$

$$= -\frac{\pi}{2}\int_0^1 \frac{\log y}{1-y+\sqrt{1-y}}dy \overset{y\to 1-y}{=} -\frac{\pi}{2}\int_0^1\frac{\log(1-y)}{y+\sqrt{y}}dy \overset{y\to y^2}{=}$$

$$= -\pi\int_0^1\frac{\log(1-y^2)}{1+y}dy = -\pi\int_0^1\frac{\log(1-y)}{1+y}dy - \pi\int_0^1\frac{\log(1+y)}{1+y}dy =$$

$$= -\pi\left[\log 2\log\left(\frac{1+y}{2}\right) - Li_2\left(\frac{1+y}{2}\right)\right]_0^1 - \frac{\pi}{2}\log^2 2 =$$

$$= \pi Li_2(1) - \frac{3\pi}{2}\log^2 2 - \pi Li_2\left(\frac{1}{2}\right)$$

$$A = \frac{\pi^3}{12} - \pi\log^2 2$$

$$B = \int_0^1\int_0^{\frac{\pi}{2}}\frac{\sin^2 x\log y}{1+y\sin^2 x}dxdy \overset{x=\tan t}{=} \int_0^1\log y\int_0^{\infty}\frac{t^2}{(1+(1+y)t^2)(1+t^2)}dtdy =$$

$$= \frac{\pi}{2}\int_0^1\frac{\log y}{1+y+\sqrt{1+y}}dy = \pi\int_1^{\sqrt{2}}\frac{\log(y^2-1)}{y+1}dy =$$

$$= \pi\int_1^{\sqrt{2}}\frac{\log(y-1)}{y+1}dy + \pi\int_1^{\sqrt{2}}\frac{\log(y+1)}{y+1}dy =$$

$$= \frac{\pi}{2}\log^2(1+\sqrt{2}) - \frac{\pi}{2}\log^2 2 + \pi\left[Li_2\left(\frac{1-y}{2}\right) + \log(y-1)\log\left(\frac{1+y}{2}\right)\right]_1^{\sqrt{2}} =$$

$$= \frac{\pi}{2}\log^2(1+\sqrt{2}) - \frac{\pi}{2}\log^2 2 + \pi Li_2\left(\frac{1-\sqrt{2}}{2}\right) + \pi\log(\sqrt{2}-1)\log\left(\frac{1+\sqrt{2}}{2}\right) =$$

$$= -\frac{\pi}{2}\log^2(1+\sqrt{2}) - \frac{\pi}{2}\log^2 2 + \pi Li_2\left(\frac{1-\sqrt{2}}{2}\right) + \pi\log(1+\sqrt{2})\log 2$$

$$\frac{8}{\pi}\int_0^{\frac{\pi}{2}} x\cot x\log(1-\sin^4 x)\,dx =$$

$$= \frac{\pi^2}{6} - 6\log^2 2 - 2\log^2(1+\sqrt{2}) + 4\log 2\log(1+\sqrt{2}) + 4Li_2\left(\frac{1-\sqrt{2}}{2}\right)$$

1863. Prove that:

$$\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{4^n(4n+5)(2n+2)} = 1 - \frac{\sqrt{\pi}\Gamma\left(\frac{1}{4}\right)}{8\Gamma\left(\frac{7}{4}\right)} = 1 - \frac{\Gamma^2\left(\frac{1}{4}\right)}{6\sqrt{2\pi}}$$

where $\Gamma(z) = \int_0^{\infty} x^{z-1}e^{-x}\,dx$ is gamma function for $\Re(z) > 0$.

Proposed by Naren Bhandari-Bajura-Nepal

Solution 1 by Syed Shahabudeen-Kerala-India

$$\Omega = \sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{4^n(4n+5)(2n+2)} = \frac{1}{2}\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{4^n(n+1)} - 2\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{4^n(4n+5)}$$

$$A = \int_0^1\sum_{n=0}^{\infty}\frac{\binom{2n}{n}x^n}{4^n}dx = \int_0^1\frac{1}{\sqrt{1-x}}dx = 2$$

$$B = \frac{1}{4}\int_0^1\sum_{n=0}^{\infty}\frac{\binom{2n}{n}x^{n+\frac{1}{4}}}{4^n}dx = \frac{1}{4}\int_0^1\frac{x^{\frac{1}{4}}}{\sqrt{1-x}}dx = \frac{1}{4}\cdot\frac{\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{7}{4}\right)}$$

$$\Omega = \frac{1}{2}A - 2B = 1 - \frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{7}{4}\right)} = 1 - \frac{\sqrt{\pi}}{8}\cdot\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{7}{4}\right)};\ (\because \Gamma(s+1) = s\Gamma(s))$$

$$\Gamma\left(\frac{7}{4}\right) = \frac{3}{4}\Gamma\left(\frac{3}{4}\right) = \frac{3}{4}\cdot\frac{\pi\sqrt{2}}{\Gamma\left(\frac{1}{4}\right)}$$

$$\Omega = 1 - \frac{\sqrt{\pi}}{8}\Gamma\left(\frac{1}{4}\right)\cdot\frac{4}{3}\cdot\frac{\Gamma\left(\frac{1}{4}\right)}{\pi\sqrt{2}} = 1 - \frac{\Gamma^2\left(\frac{1}{4}\right)}{6\sqrt{2\pi}}$$

Solution 2 by Tapas Das-India

$$(1+x)^a = \sum_{k=0}^{\infty}\binom{a}{k}x^k,\ \text{for } |x| < 1$$

where $a \in \mathbb{R}$ ord the coefficients are

$$\binom{a}{k} = \frac{a(a-1)\cdot \ldots \cdot (a-k+1)}{k!}$$

For $a = -\frac{1}{2}$ we have:

$$(1+x)^{-\frac{1}{2}} = \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} x^k = \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdot \ldots \cdot \left(-\frac{2k-1}{2}\right)}{k!} =$$

$$= \frac{1\cdot 3\cdot 5\cdot \ldots \cdot (2k-1)(-1)^k}{2^k \cdot k!}; (1)$$

$$2^k \cdot k! = 2^k \cdot 1\cdot 2\cdot 3\cdot \ldots \cdot k = 2\cdot 4\cdot 6\cdot \ldots \cdot 2k; (2)$$

So, multiplying (1) by $\frac{2^k k!}{2^k k!}$ we have:

$$\binom{-\frac{1}{2}}{k} = (-1)^k \cdot \frac{2^k \cdot k!}{2^k \cdot k!} \cdot \frac{1\cdot 3\cdot 5\cdot \ldots \cdot (2k-1)}{2^k \cdot k!} = (-1)^k \cdot \frac{1}{2^{2k}} \cdot \frac{(2k)!}{k!\,k!} =$$

$$= (-1)^k \cdot \frac{1}{2^{2k}} \binom{2k}{k} \cdot \binom{2k}{k}$$

$$(1+x)^{-\frac{1}{2}} = \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k} \binom{2k}{k} x^k \Rightarrow (1-x)^{-\frac{1}{2}} = \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k} \binom{2k}{k} (-x)^k$$

$$\frac{\binom{2n}{n}}{4^n(4n+5)(2n+2)} = \frac{1}{2} \cdot \frac{\binom{2n}{n}}{4^n(4n+5)(n+1)} =$$

$$= \frac{1}{2} \frac{[(4n+5)-4(n+1)]\binom{2n}{n}}{4^n(4n+5)(n+1)} =$$

$$= \frac{1}{2} \cdot \frac{\binom{2n}{n}}{4^n(n+1)} - 2\frac{\binom{2n}{n}}{4^n(4n+5)}$$

$$\Omega = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n(4n+5)(2n+2)} = \frac{1}{2}\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n(n+1)} - 2\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n(4n+5)}$$

Now,

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n(n+1)} = \int_0^1 \sum_{n=0}^{\infty} \frac{\binom{2n}{n} x^n}{4^n} dx = \int_0^1 \frac{dx}{\sqrt{1-x}} = 2$$

$$\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{4^n(4n+5)}=\frac{1}{4}\int_0^1\sum_{n=0}^{\infty}\frac{\binom{2n}{n}x^{n+\frac{1}{4}}}{4^n}dx=\frac{1}{4}\int_0^1\frac{x^{\frac{1}{4}}}{\sqrt{1-x}}dx=$$

$$=\frac{1}{4}\int_0^1 x^{\frac{5}{4}-1}(1-x)^{\frac{1}{2}-1}dx=\frac{1}{4}\beta\left(\frac{5}{4},\frac{1}{2}\right)=\frac{1}{4}\cdot\frac{\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{7}{4}\right)}$$

$$\Omega=\frac{1}{2}\cdot 2-2\cdot\frac{1}{4}\frac{\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{7}{4}\right)}=1-\frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{7}{4}\right)}=1-\frac{\sqrt{\pi}}{8}\cdot\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{7}{4}\right)}$$

$$\Gamma\left(\frac{7}{4}\right)=\Gamma\left(1+\frac{3}{4}\right)=\frac{3}{4}\Gamma\left(\frac{3}{4}\right)=\frac{3}{4}\cdot\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}=\frac{3}{4}\cdot\frac{\pi\csc\left(\frac{\pi}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}=\frac{3}{4}\cdot\frac{\pi\sqrt{2}}{\Gamma\left(\frac{1}{4}\right)}$$

$$\Omega=1-\frac{\sqrt{\pi}}{8}\Gamma\left(\frac{1}{4}\right)\cdot\frac{4}{3}\cdot\frac{\Gamma\left(\frac{1}{4}\right)}{\pi\sqrt{2}}=1-\frac{\Gamma^2\left(\frac{1}{4}\right)}{6\sqrt{2\pi}}$$

Solution 3 by Ankush Kumar Parcha-India

$$S=\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{4^n(4n+5)(2n+2)}\Rightarrow$$

$$2S=\sum_{n=0}^{\infty}\frac{(2n)!}{4^n(n!)^2(n+1)}-4\sum_{n=0}^{\infty}\frac{(2n)!}{4^n(n!)^2(4n+5)};(1)$$

$$S_1=\sum_{n=0}^{\infty}\frac{(2n)!}{4^n(n!)^2(n+1)};\ S_2=\sum_{n=0}^{\infty}\frac{(2n)!}{4^n(n!)^2(4n+5)}$$

$$S_1=\sum_{n=0}^{\infty}\frac{(2n-1)!!}{2^n(n+1)!}\Rightarrow\sqrt{\pi}\Omega=\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{(n+1)!};(2)$$

Using Maclaurin series for $\sin^{-1}(y)$:

$$\sqrt{\pi}\sin^{-1}(y)=\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{(2n+1)n!}y^{2n+1}$$

Differentiating w.r.t. y, we get:

$$\sqrt{\frac{\pi}{1-y^2}}=\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{n!}y^{2n}$$

Put $y^2=x$ and integrate it w.r.t. x, we get:

$$\sqrt{\pi}\int\frac{dx}{\sqrt{1-x}}=\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{(n+1)!}x^{n+1}+C$$

$$\left(2\sqrt{\pi}-2\sqrt{\pi(1-x)}\right)_{x=1}=\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{(n+1)!}\left(x^{n+1}\right)_{x=1}$$

Analogous, using Maclaurin series for $\sin^{-1}(y)$:

$$\sqrt{\pi}\sin^{-1}(y)=\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{(2n+1)n!}y^{2n+1}$$

$$S_2=\sum_{n=0}^{\infty}\frac{(2n)!}{4^n(n!)^2(4n+5)}=\sum_{n=0}^{\infty}\frac{2^n(2n-1)!!}{4^n n!\,(4n+5)}\Rightarrow$$

$$5\sqrt{\pi}S_2=\sum_{n=1}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{n!\left(\frac{4n}{5}+1\right)};(3)$$

Differentiating w.r.t. y, we get:

$$\sqrt{\frac{\pi}{1-y^2}}=\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{n!}y^{2n}$$

Put $y=x^{\frac{2}{5}}$ in the above equation and integrate w.r.t x from 0 to 1, we get:

$$\sqrt{\pi}\int_0^1\frac{dx}{\sqrt{1-x^{\frac{4}{5}}}}=\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{n!\left(\frac{4n}{5}+1\right)}\overset{x=u^{\frac{5}{4}}}{=}\frac{5\sqrt{\pi}}{4}\int_0^1u^{\frac{5}{4}-1}(1-u)^{\frac{1}{2}-1}du$$

$$\because\ \beta(m,n)=\int_0^1x^{m-1}(1-x)^{n-1}dx=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$$\frac{5\sqrt{\pi}}{4}\cdot\frac{\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{7}{4}\right)}=\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{n!\left(\frac{4n}{5}+1\right)}$$

Put the value in (3), we get:

$$5\sqrt{\pi}S_2=\frac{5}{4}\pi\frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{7}{4}\right)}=\frac{\sqrt{\pi}\Gamma\left(\frac{5}{4}\right)}{4\Gamma\left(\frac{7}{4}\right)};\left(\because\Gamma(n+1)=n\Gamma(n)\right)$$

Put the value of S_1 and S_2 in equation (2), we get:

$$2S=2-4\cdot\frac{\sqrt{\pi}}{4}\frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{7}{4}\right)}=1-\frac{\sqrt{\pi}}{8}\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{7}{4}\right)}$$

Using Euler reflection formula at $x=\frac{3}{4}$ we get

$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)=\frac{\pi}{\sin\left(\frac{3\pi}{4}\right)},\qquad\Gamma\left(\frac{3}{4}\right)=\frac{\pi\sqrt{2}}{\Gamma\left(\frac{1}{4}\right)}$$

By putting the value in S we get:

$$S=\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{4^n(4n+5)(2n+2)}=1-\frac{\sqrt{\pi}\Gamma\left(\frac{1}{4}\right)}{8\Gamma\left(\frac{7}{4}\right)}=1-\frac{\Gamma^2\left(\frac{1}{4}\right)}{6\sqrt{2\pi}}$$

Solution 4 by Hikmat Mammadov-Azerbaijan

$$\Omega=\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{4^n(4n+5)(2n+2)}=\frac{1}{2}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}(-1)^n\left(\frac{1}{n+1}-\frac{4}{4n+5}\right)=$$

$$=\frac{1}{2}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}(-1)^n\int_0^1(x^n-4x^{4n+4})dx=\int_0^1\left(\frac{1}{2}(1-x)^{-\frac{1}{2}}-2x^4(1-x^4)^{-\frac{1}{2}}\right)dx=$$

$$=\frac{1}{2}\int_0^1\left((1-x)^{-\frac{1}{2}}-x^{-\frac{1}{4}}(1-x)^{-\frac{1}{2}}\right)dx=\frac{1}{2}\left[\beta\left(1;\frac{1}{2}\right)-\beta\left(\frac{5}{4};\frac{1}{2}\right)\right]=1-\frac{\Gamma^2\left(\frac{1}{4}\right)}{6\sqrt{2\pi}}$$

1864. Prove that:

$$\sum_{n=1}^{\infty}\frac{\overline{H}_{2n}}{n4^n}\binom{2n}{n}=\frac{\pi^2}{12}$$

where $\overline{H}_n - n^{th}$ skew-harmonic number $\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}$.

Proposed by Naren Bhandari-Bajura-Nepal

Solution 1 by Shivam Sharma-New Delhi-India

$$\text{Let: }\Omega=\sum_{n=1}^{\infty}\frac{\overline{H}_{2n}}{n4^n}\binom{2n}{n}$$

As we know, $\overline{H}_{2n}=H_{2n}-H_n;(1)$

Using (1), we get,

$$\Omega=\sum_{n=1}^{\infty}\frac{H_{2n}}{n4^n}\binom{2n}{n}-\sum_{n=1}^{\infty}\frac{H_n}{n4^n}\binom{2n}{n};(2)$$

$$\text{Let }A=\sum_{n=1}^{\infty}\frac{H_{2n}}{n4^n}\binom{2n}{n},\text{ then:}$$

$$A=-2\sum_{n=1}^{\infty}\frac{\binom{2n}{n}}{4^n}\int_0^1 x^{2n-1}\log(1-x)\,dx=$$

$$=-2\int_0^1\frac{\log(1-x)}{x\sqrt{1-x^2}}dx+2\int_0^1\frac{\log(1-x)}{x}dx\overset{\substack{x\to\frac{1-x}{1+x}\\ \text{in the first integral}}}{=}$$

$$=-2\left[2\int_0^1\frac{\log(1-x)}{x}dx-\int_0^1\frac{\log(1+x^2)}{x}dx\right]+2\int_0^1\frac{\log(1-x)}{x}dx=$$

$$=-2\left(-\frac{\pi^2}{3}-\frac{\pi^2}{24}\right)-\frac{\pi^2}{3}$$

We get, $A=\frac{5\pi^2}{12}$

$$\text{Let: }B=\sum_{n=1}^{\infty}\frac{H_n}{n4^n}\binom{2n}{n},\text{ then:}$$

$$B=-\sum_{n=1}^{\infty}\frac{\binom{2n}{n}}{4^n}\int_0^1 x^{n-1}\log(1-x)\,dx=$$

$$=-\int_0^1\frac{\log(1-x)}{x\sqrt{1-x}}dx+\int_0^1\frac{\log(1-x)}{x}dx\overset{\substack{\sqrt{1-x}=u\\ \text{in the first integral}}}{=}$$

$$=-4\int_0^1\frac{\log u}{1-u^2}du-\frac{\pi^2}{6}=\frac{\pi^2}{3}$$

Put the values of A and B in (2), we finally get

$$\Omega=\sum_{n=1}^{\infty}\frac{\overline{H}_{2n}}{n4^n}\binom{2n}{n}=\frac{\pi^2}{12}$$

Solution 2 by Rana Ranino-Setif-Algerie

$$\Omega=\sum_{n=1}^{\infty}\frac{\overline{H}_{2n}}{n4^n}\binom{2n}{n}$$

$$\because\ \overline{H}_{2n}=\log 2-\int_0^1\frac{x^{2n}}{1+x}dx\,;\ \frac{\binom{2n}{n}}{n4^n}=\frac{B\left(n+\frac{1}{2},\frac{1}{2}\right)}{\pi n}=\frac{2}{\pi n}\int_0^{\frac{\pi}{2}}\sin^{2n}\theta\, d\theta$$

$$\Omega=\frac{2\log 2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\int_0^{\frac{\pi}{2}}\sin^{2n}\theta\, d\theta+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\int_0^1\frac{1}{1+x}\int_0^{\frac{\pi}{2}}x^{2n}\sin^{2n}\theta\, d\theta\, dx=$$

$$=-\frac{2}{\pi}\log 2\int_0^{\frac{\pi}{2}}\log(1-\sin^2\theta)\, d\theta+\frac{2}{\pi}\int_0^1\frac{1}{1+x}\int_0^{\frac{\pi}{2}}\log(1-x^2\sin^2\theta)\, d\theta\, dx=$$

$$=-\frac{4}{\pi}\log 2\int_0^{\frac{\pi}{2}}\log(\cos\theta)\, d\theta+\frac{2}{\pi}\int_0^1\frac{1}{1+x}\int_0^{\frac{\pi}{2}}\log(\cos^2\theta+(1-x^2)\sin^2\theta)\, d\theta\, dx$$

$$\int_0^{\frac{\pi}{2}}\log(\cos\theta)\, d\theta=-\frac{\pi}{2}\log 2$$

$$\int_0^{\frac{\pi}{2}}\log(\cos^2\theta+(1-x^2)\sin^2\theta)\, d\theta=\pi\log\left(\frac{a+b}{2}\right)$$

$$\Omega=2\log^2 2+2\int_0^1\frac{\log\left(\frac{1+\sqrt{1-x^2}}{2}\right)}{1+x}dx=2\int_0^1\frac{\log\left(1+\sqrt{1-x^2}\right)}{1+x}dx\overset{x=\cos t}{=}$$

$$=2\int_0^{\frac{\pi}{2}}\frac{\log(1+\sin t)}{1+\cos t}\sin t\, dt=2\int_0^{\frac{\pi}{2}}\log(1+\sin t)\tan\left(\frac{t}{2}\right)dt=$$

$$=2\int_0^1\int_0^{\frac{\pi}{2}}\frac{\sin t\sin\left(\frac{t}{2}\right)}{(1+y\sin t)\cos\left(\frac{t}{2}\right)}dtdy=4\int_0^1\int_0^{\frac{\pi}{2}}\frac{\sin^2\left(\frac{t}{2}\right)}{1+y\sin t}dtdy\overset{x=\tan\left(\frac{t}{2}\right)}{=}$$

$$=8\int_0^1\int_0^1\frac{x^2}{(1+2xy+x^2)(1+x^2)}dxdy=$$

$$= 2\int_0^1 \frac{1}{y}\int_0^1 \left(\frac{2x}{x^2+1} - \frac{2x+2y}{1+2xy+x^2} + \frac{2y}{(x+y)^2+(1-y^2)}\right) dxdy =$$

$$= 2\int_0^1 \frac{1}{y}\left[\log\left(\frac{1+x^2}{1+2xy+x^2}\right) + \frac{2y\tan^{-1}\left(\frac{x+y}{\sqrt{1-y^2}}\right)}{\sqrt{1-y^2}}\right]_0^1 dy =$$

$$= -2\int_0^1 \frac{\log(1+y)}{y}dy + 4\int_0^1 \frac{\tan^{-1}\left(\frac{1+y}{\sqrt{1-y^2}}\right) - \tan^{-1}\left(\frac{y}{\sqrt{1-y^2}}\right)}{\sqrt{1-y^2}}dy =$$

$$= 2Li_2(-1) + 4\int_0^1 \frac{\tan^{-1}\sqrt{\frac{1-y}{1+y}}}{\sqrt{1-y^2}}dy \overset{y=\cos\theta}{=} -\frac{\pi^2}{6} + 4\int_0^{\frac{\pi}{2}} \tan^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}d\theta =$$

$$= -\frac{\pi^2}{6} + \int_0^{\frac{\pi}{2}} 2\theta\, d\theta = -\frac{\pi^2}{6} + \frac{\pi^2}{4} = \frac{\pi^2}{12}$$

Therefore,

$$\Omega = \sum_{n=1}^{\infty} \frac{\overline{H}_{2n}}{n4^n}\binom{2n}{n} = \frac{\pi^2}{12}$$

Solution 3 by Felix Marin-USA

$$\Omega = \sum_{n=1}^{\infty} \frac{\overline{H}_{2n}}{n4^n}\binom{2n}{n} = \sum_{n=1}^{\infty} \frac{\overline{H}_{2n}}{n4^n}\left[\binom{-\frac{1}{2}}{n}(-4)^n\right] = 2\sum_{n=1}^{\infty}\binom{-\frac{1}{2}}{n}(-1)^n\frac{\overline{H}_{2n}}{2n}; (1)$$

$$\frac{\overline{H}_{2n}}{2n} = \frac{1}{2n}\sum_{k=1}^{2n}(-1)^{k-1}\int_0^1 t^{k-1}dt = \frac{1}{2n}\int_0^1 \frac{(-t)^{2n}-1}{(-t)-1}dt =$$

$$= \frac{1}{2n}\int_0^1 \frac{1-t^{2n}}{1+t}dt \overset{IBP}{=} -\frac{1}{2n}\int_0^1 \log(1+t)\,(-2nt^{2n-1})dt = \int_0^1 \log(1+t)\,t^{2n-1}dt; (2)$$

From (1) and (2) it follows:

$$\Omega = \sum_{n=1}^{\infty} \frac{\overline{H}_{2n}}{n4^n}\binom{2n}{n} = 2\sum_{n=1}^{\infty}\binom{-\frac{1}{2}}{n}(-1)^n\int_0^1 \log(1+t)\,t^{2n-1}dt =$$

$$= 2\int_0^1 \log(1+t)\left[\sum_{n=1}^{\infty}\binom{-\frac{1}{2}}{n}(-t^2)^n\right]\frac{dt}{t} = 2\int_0^1 \log(1+t)\left[(1-t^2)^{-\frac{1}{2}}-1\right]\frac{dt}{t} =$$

$$= 2\underbrace{\int_0^1 \frac{\log(1+t)}{t\sqrt{1-t^2}}dt}_{t=\sin\theta} - 2\underbrace{\int_0^{-1}\frac{\log(1-t)}{t}dt}_{-Li_2(-1)=\frac{\pi^2}{12}} = 2\underbrace{\int_0^{\frac{\pi}{2}}\frac{\log(1+\sin\theta)}{\sin\theta}d\theta}_{\frac{\pi^2}{12}} - \frac{\pi^2}{6} =$$

$$= 2\frac{\pi^2}{8} - \frac{\pi^2}{6} = \frac{\pi^2}{12}$$

1865.

$$\text{For } S_n = \sum_{k=1}^{2^{n+1}-1}\log_2\left(1+\frac{1}{\sqrt{k^2+k+1}}\right) \text{ find:}$$

$$\Omega = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k\sqrt{k+1}+(k+1)\sqrt{k}}\right)^{\frac{S_n}{\sqrt{n}}}$$

Proposed by Florică Anastase-Romania

Solution 1 by Ruxandra Daniela Tonilă-Romania

$$\sum_{k=1}^{n}\frac{1}{k\sqrt{k+1}+(k+1)\sqrt{k}} = \sum_{k=1}^{n}\frac{1}{\sqrt{k+1}\cdot\sqrt{k}}\cdot\frac{1}{\sqrt{k+1}+\sqrt{k}} =$$

$$= \sum_{k=1}^{n}\frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k(k+1)}} = \sum_{k=1}^{n}\left(\frac{1}{\sqrt{k}}-\frac{1}{\sqrt{k+1}}\right) = 1-\frac{1}{\sqrt{n+1}}; (1)$$

We have:

$$\frac{1}{\sqrt{(k+1)^2}} \le \frac{1}{\sqrt{k^2+k+1}} \le \frac{1}{\sqrt{\left(k+\frac{1}{2}\right)^2}}; \forall k \ge 1$$

$$\sum_{k=1}^{2^{n+1}-1}\log_2\left(1+\frac{1}{k+1}\right) < \sum_{k=1}^{2^{n+1}-1}\log_2\left(1-\frac{1}{\sqrt{k^2+k+1}}\right) < \sum_{k=1}^{2^{n+1}-1}\log_2\left(1+\frac{1}{k+\frac{1}{2}}\right)$$

$$\sum_{k=1}^{2^{n+1}-1}\log_2\left(\frac{k+2}{k+1}\right) < S_n < \sum_{k=1}^{2^{n+1}-1}\log_2\left(\frac{2k+3}{2k+1}\right)$$

$$\log_2\left(\prod_{k=1}^{2^{n+1}-1}\frac{k+2}{k+1}\right) < S_n < \log_2\left(\prod_{k=1}^{2^{n+1}-1}\frac{2k+3}{2k+1}\right)$$

$$\log_2\left(\frac{2^{n+1}+1}{2}\right) < S_n < \log_2\left(\frac{2^{n+2}+1}{3}\right)$$

$$\frac{\log_2\left(\frac{2^{n+1}+1}{2}\right)}{\sqrt{n(n+1)}} < \frac{S_n}{\sqrt{n(n+1)}} < \frac{\log_2\left(\frac{2^{n+2}+1}{3}\right)}{\sqrt{n(n+1)}}$$

$$\lim_{n\to\infty}\frac{\log_2\left(\frac{2^{n+1}+1}{2}\right)}{\sqrt{n(n+1)}} \le \lim_{n\to\infty}\frac{S_n}{\sqrt{n(n+1)}} \le \lim_{n\to\infty}\frac{\log_2\left(\frac{2^{n+2}+1}{3}\right)}{\sqrt{n(n+1)}}$$

$$\lim_{n\to\infty}\frac{S_n}{\sqrt{n(n+1)}} = 1;\ (2)$$

From (1) and (2) it follows that:

$$\Omega = \lim_{n\to\infty}\left[\left(1-\frac{1}{\sqrt{n+1}}\right)^{-\sqrt{n+1}}\right]^{-\frac{S_n}{\sqrt{n(n+1)}}} = \frac{1}{e}$$

Solution 2 by proposer

$$S_n = \sum_{k=1}^{2^{n+1}-1}\log_2\left(1+\frac{1}{\sqrt{k^2+k+1}}\right)$$

We have: $k < \sqrt{k^2+k+1} < k+1; \forall k \in \mathbb{N}^*$**, hence,**

$$\log_2\left(1+\frac{1}{k+1}\right) < \log_2\left(1+\frac{1}{\sqrt{k^2+k+1}}\right) < \log_2\left(1+\frac{1}{k}\right)$$

$$\sum_{k=1}^{2^{n+1}-1}\log_2\left(\frac{k+2}{k+1}\right) < S_n < \sum_{k=1}^{2^{n+1}-1}\log_2\left(\frac{k+1}{k}\right)$$

$$\log_2\left(\prod_{k=1}^{2^{n+1}-1}\frac{k+2}{k+1}\right) < S_n < \log_2\left(\prod_{k=1}^{2^{n+1}-1}\frac{k+1}{k}\right)$$

$$\log_2\left(\frac{2^{n+1}-1}{2}\right) < S_n < \log_2 2^{n+1}$$

$$n < S_n < n+1$$

$$x_n=\sum_{k=1}^{n}\frac{1}{k\sqrt{k+1}+(k+1)\sqrt{k}}=\sum_{k=1}^{n}\frac{k\sqrt{k+1}-(k+1)\sqrt{k}}{k^2(k+1)-(k+1)^2k}=$$

$$=\sum_{k=1}^{n}\left(\frac{1}{\sqrt{k}}-\frac{1}{\sqrt{k+1}}\right)=1-\frac{1}{\sqrt{n+1}}$$

Therefore,

$$\left(1-\frac{1}{\sqrt{n+1}}\right)^{\frac{n+1}{\sqrt{n}}}<\left(\sum_{k=1}^{n}\frac{1}{k\sqrt{k+1}+(k+1)\sqrt{k}}\right)^{\frac{S_n}{\sqrt{n}}}<\left(1-\frac{1}{\sqrt{n+1}}\right)^{\frac{n}{\sqrt{n}}}$$

$$\lim_{n\to\infty}\left(1-\frac{1}{\sqrt{n+1}}\right)^{\frac{n}{\sqrt{n}}}=\lim_{n\to\infty}\left(1-\frac{1}{\sqrt{n+1}}\right)^{\sqrt{n}}=\lim_{n\to\infty}\left(1-\frac{1}{\sqrt{n+1}}\right)^{\frac{\sqrt{n+1}}{1}\cdot\frac{\sqrt{n}}{\sqrt{n+1}}}=\frac{1}{e}$$

$$\lim_{n\to\infty}\left(1-\frac{1}{\sqrt{n+1}}\right)^{\frac{n+1}{\sqrt{n}}}=\lim_{n\to\infty}\left(1-\frac{1}{\sqrt{n+1}}\right)^{\frac{n+1}{\sqrt{n}}\cdot\frac{\sqrt{n+1}}{\sqrt{n+1}}}=\lim_{n\to\infty}\left(1-\frac{1}{\sqrt{n+1}}\right)^{\frac{\sqrt{n+1}}{1}\cdot\frac{(n+1)}{\sqrt{n(n+1)}}}=\frac{1}{e}$$

$$\Omega=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k\sqrt{k+1}+(k+1)\sqrt{k}}\right)^{\frac{S_n}{\sqrt{n}}}=\frac{1}{e}$$

1866.

$$\Omega_1=\int_0^1\log x\sqrt{\frac{x+1}{x-x^3}}dx,\ \Omega_2=\int_1^\infty\frac{\log x}{x\sqrt{x-1}}dx$$

$$\Omega_3=\int_0^1\log\left(\frac{1}{1-x^2}\right)\sqrt{\frac{x+1}{x-x^3}}dx,\ \Omega_4=\int_1^\infty\frac{\log(1+x)}{x\sqrt{x-1}}dx$$

Prove that:

$$\Omega_1+\Omega_2-(\Omega_3+\Omega_4)=-\pi\log 16$$

Proposed by Ankush KumarParcha-India

Solution 1 by Togrul Ehmedov-Azerbaijan

$$\Omega_1-\Omega_3=\int_0^1\log(x-x^3)\sqrt{\frac{x+1}{x-x^3}}dx\overset{\frac{1}{x}=y}{=}\int_1^\infty\log\left(\frac{y^2-1}{y^3}\right)\frac{dy}{y\sqrt{y-1}}=$$

$$=\int_1^\infty\frac{\log(y-1)+\log(y+1)-3\log y}{y\sqrt{y-1}}dy$$

$$\Omega_1-\Omega_3+\Omega_2-\Omega_4=\int_1^{\infty}\frac{\log(x-1)-2\log x}{x\sqrt{x-1}}dx=$$

$$=\int_1^{\infty}\frac{\log(x-1)}{x\sqrt{x-1}}dx-2\int_1^{\infty}\frac{\log x}{x\sqrt{x-1}}dx\overset{x=\frac{1}{z}}{=}$$

$$=\int_0^1\frac{\log(1-z)-\log z}{\sqrt{z(1-z)}}dz+2\int_0^1\frac{\log z}{\sqrt{z(1-z)}}dz=$$

$$=\int_0^1\frac{\log(1-z)}{\sqrt{z(1-z)}}dz+\int_0^1\frac{\log z}{\sqrt{z(1-z)}}dz=2\int_0^1\frac{\log z}{\sqrt{z(1-z)}}\overset{z=\sin^2 m}{=}$$

$$=8\int_0^{\frac{\pi}{2}}\log(\sin m)\,dm=8\left(-\frac{\pi}{2}\log 2\right)=-\pi\log 16$$

Solution 2 by Rana Ranino-Setif-Algerie

$$\Omega_1=\int_0^1\log x\sqrt{\frac{x+1}{x-x^3}}dx=\int_0^1\frac{\log x}{\sqrt{x(1-x)}}dx\overset{x=\sin^2\theta}{=}$$

$$=4\int_0^{\frac{\pi}{2}}\log(\sin\theta)\,d\theta=-2\pi\log 2$$

$$\Omega_2=\int_1^{\infty}\frac{\log x}{x\sqrt{x-1}}dx\overset{x\to\frac{1}{x}}{=}-\int_0^1\frac{\log x}{\sqrt{x(1-x)}}dx=-\Omega_1=2\pi\log 2$$

$$\Omega_1+\Omega_2=0$$

$$\Omega_3=\int_0^1\log\left(\frac{1}{1-x^2}\right)\sqrt{\frac{x+1}{x-x^3}}dx=\int_0^1\frac{-\log(1-x)-\log(1+x)}{\sqrt{x(1-x)}}dx$$

$$\Omega_4=\int_1^{\infty}\frac{\log(1+x)}{x\sqrt{x-1}}dx\overset{x\to\frac{1}{x}}{=}\int_0^1\frac{\log(1+x)-\log x}{\sqrt{x(1-x)}}dx$$

$$\Omega_3+\Omega_4=-\int_0^1\frac{\log(1+x)+\log x}{\sqrt{x(1-x)}}dx=-\underbrace{\int_0^1\frac{\log(1-x)}{\sqrt{x(1-x)}}dx}_{x\to 1-x}-\int_0^1\frac{\log x}{\sqrt{x(1-x)}}dx$$

Therefore,

$$\Omega_3+\Omega_4=-2\int_0^1\frac{\log x}{\sqrt{x(1-x)}}dx=-2\Omega_1=4\pi\log 2=\pi\log 16$$

$$\Omega_1+\Omega_2-(\Omega_3+\Omega_4)=-\pi\log 16$$

Solution 3 by Syed Shahabudeen-India

$$\Omega_1=\int_0^1 \log x\sqrt{\frac{x+1}{x-x^3}}dx=\int_0^1\frac{\log x}{\sqrt{x(1-x)}}dx$$

$$\frac{d}{da}\int_0^1 x^{a-1}(1-x)^{b-1}\,dx=\frac{d}{da}\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

$$\int_0^1 x^{a-1}(1-x)^{b-1}\log x\,dx=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\big(\psi(a)-\psi(a+b)\big)$$

Let $a=b=\frac{1}{2}$, we get:

$$\Omega_1=\pi\left(\pi\left(\frac{1}{2}\right)-\psi(1)\right)=-2\pi\log 2$$

$$\Omega_2=\int_1^\infty\frac{\log x}{\sqrt{x(1-x)}}dx=-\int_0^1\frac{\log x}{\sqrt{x(1-x)}}dx=2\pi\log 2$$

Similarly,

$$\Omega_3+\Omega_4=-\int_0^1\frac{\log x}{\sqrt{x(1-x)}}dx-\int_0^1\frac{\log(1-x)}{\sqrt{x(1-x)}}dx$$

It is known:

$$\int_0^1 x^{a-1}(1-x)^{b-1}\log x\,dx=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\big(\psi(a)-\psi(a+b)\big)$$

For $a=b=\frac{1}{2}$, we get:

$$\int_0^1\frac{\log x}{\sqrt{x(1-x)}}dx=-2\pi\log 2\Rightarrow\Omega_3+\Omega_4=\pi\log 16$$

Therefore,

$$\Omega_1+\Omega_2-(\Omega_3+\Omega_4)=-\pi\log 16$$

1867. Find:

$$\Omega=\int_0^1\frac{Li_2(x)\cdot\log(1-x)}{x}dx+\int_0^1\frac{Li_2(x)\cdot\log x}{1-x}dx$$

Proposed by Togrul Ehmedov-Baku-Azerbaijan

Solution 1 by Avishek Mitra-West Bengal-India

$$\mathbf{I}=\int_0^1\frac{Li_2(x)\cdot\log(1-x)}{1-x}dx=\int_0^1\frac{Li_2(1-x)\cdot\log(1-x0}{x}dx=$$

$$=\int_0^1\frac{\log(1-x)\left(\zeta(2)-Li_2(x)-\log x\log(1-x)\right)}{x}dx=$$

$$=\zeta(2)\int_0^1\frac{\log(1-x)}{x}dx-\int_0^1\frac{Li_2(x)\log(1-x)\,dx}{x}-\int_0^1\frac{\log x\log^2(1-x)\,dx}{x}$$

$$\Omega=\int_0^1\frac{Li_2(x)\cdot\log(1-x)}{x}dx+\int_0^1\frac{Li_2(x)\cdot\log x}{1-x}dx=$$

$$=\zeta(2)\left(-\sum_{n=1}^{\infty}\frac{1}{n}\int_0^1x^{n-1}\,dx\right)-\int_0^1\frac{\log(1-x)\log^2x}{1-x}dx$$

$$I_1+I_2=-\zeta(2)\sum_{n=1}^{\infty}\frac{1}{n^2}+\int_0^1\frac{Li_1(x)\log^2x\,dx}{1-x}=$$

$$=-\left(\zeta(2)\right)^2+\sum_{n=1}^{\infty}H_n\int_0^1x^{n-1}\log^2x\,dx=$$

$$=-\frac{\pi^4}{36}+2\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^3}=-\frac{\pi^4}{36}+2\sum_{n=1}^{\infty}\frac{H_{n+1}-\frac{1}{n+1}}{(n+1)^3}=$$

$$=-\frac{\pi^4}{36}+2\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^3}=-\frac{\pi^4}{36}+2\sum_{n=1}^{\infty}\frac{1}{(n+1)^4}=$$

$$=-\frac{\pi^4}{36}+2\sum_{n=1}^{\infty}\frac{H_n}{n^3}-2\sum_{n=1}^{\infty}\frac{1}{n^4}=-\frac{\pi^4}{36}+2\cdot\frac{5}{4}\zeta(4)=-\frac{\pi^4}{36}+\frac{5}{2}\cdot\frac{\pi^4}{90}-2\cdot\frac{\pi^4}{90}=-\frac{\pi^4}{45}$$

Solution 2 by Benjamin Bamidele-Nigeria

$$\Omega=\int_0^1\frac{Li_2(x)\cdot\log(1-x)}{x}dx+\int_0^1\frac{Li_2(x)\cdot\log x}{1-x}dx=A+B$$

$$A=\int_0^1\frac{Li_2(x)\cdot\log(1-x)}{x}dx\overset{IBP}{=}-\left[Li_2(x)\right]^2\Big|_0^1-\int_0^1\frac{Li_2(x)\log(1-x)}{x}dx$$

$$2A=-Li_2^2(1)\Rightarrow A=-\frac{1}{2}Li_2^2(1)$$

$$B=\int_0^1\frac{Li_2(x)\cdot\log x}{1-x}dx=-\int_0^1\int_0^1\frac{x\log x\log y}{(1-x)(1-xy)}dxdy=$$

$$=-\int_0^1\int_0^1\left(\frac{\log x\log y}{(1-x)(1-y)}-\frac{\log x\log y}{(1-y)(1-xy)}\right)dxdy=$$

$$=-Li_2^2(1)+\int_0^1\int_0^1\frac{\log x\log y}{(1-y)(1-xy)}dxdy=;\left(\because\int_0^1\frac{\log x}{x-a}dx=Li_2\left(\frac{1}{a}\right)\right)$$

$$=-Li_2^2(1)-\int_0^1\frac{\log x\log y}{y(1-y)}dy=$$

$$=-Li_2^2(1)-\int_0^1\frac{Li_2(y)\log y}{y}dy-\int_0^1\frac{Li_2(y)\log y}{1-y}dy=$$

$$=-Li_2^2(1)-\int_0^1\frac{Li_2(y)\log y}{y}dy-A$$

$$A=-\frac{1}{2}Li_2^2(1)-\frac{1}{2}\int_0^1\frac{Li_2(y)\log y}{y}dy$$

$$\int_0^1\frac{Li2\,(y)\log y}{y}dy\overset{IBP}{=}\frac{1}{2}\int_0^1\frac{\log^2 y\log(1-y)}{y}dy\overset{IBP}{=}$$

$$=-\frac{1}{6}\int_0^1\frac{\log^3 y}{1-y}dy=-\frac{1}{6}\sum_{k=1}^{\infty}\int_0^1 y^{k-1}\log^3 y\,dy=-\sum_{k=1}^{\infty}\frac{1}{k^4}$$

$$\int_0^1\frac{Li_2(y)\log y}{y}dy=-\zeta(4)$$

$$B=-\frac{1}{2}Li_2^2(1)+\frac{1}{2}\zeta(4)$$

$$\Omega=A+B=-Li_2^2(1)+\frac{1}{2}\zeta(4)=-\frac{\pi^4}{45}$$

Solution 3 by Syed Shahabudeen-Kerala-India

$$\Omega=\int_0^1\frac{Li_2(x)\cdot\log(1-x)}{x}dx+\int_0^1\frac{Li_2(x)\cdot\log x}{1-x}dx=\Omega_1+\Omega_2$$

$$\Omega_1=-\int_0^1\frac{Li_2(x)}{x}\sum_{k=1}^{\infty}\frac{x^k}{k}dx=-\sum_{k=1}^{\infty}\frac{1}{k}\int_0^1 x^{k-1}Li_2(x)\,dx$$

$$\because\int_0^1 x^{k-1}Li_2(x)\,dx=\frac{\zeta(2)}{k}-\frac{H_k}{k^2}$$

$$\Omega_1=-\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{\zeta(2)}{k}-\frac{H_k}{k^2}\right)=\sum_{k=1}^{\infty}\frac{-\zeta(2)}{k^2}+\sum_{k=1}^{\infty}\frac{H_k}{k^3}$$

$$\because\sum_{k=1}^{\infty}\frac{1}{k^2}=\zeta(2)\text{ and }\sum_{k=1}^{\infty}\frac{H_k}{k^3}=\frac{5\zeta(4)}{2}-\frac{\zeta^2(2)}{2}$$

Therefore,

$$\Omega_1=\frac{-3\zeta^2(2)}{2}+\frac{5\zeta(4)}{2}$$

$$\Omega_2=\sum_{k=1}^{\infty}H_k^2\int_0^1 x^k\log x\,dx=-\sum_{k=1}^{\infty}\frac{H_k^2}{(k+1)^2}=;\left(\because\frac{Li_2(x)}{1-x}=\sum_{k=1}^{\infty}H_k^2x^k\right)$$

$$=-\sum_{k=1}^{\infty}\frac{H_{k+1}^2}{(k+1)^2}+\sum_{k=1}^{\infty}\frac{1}{(k+1)^4}$$

To evaluate the harmonic sum we'll make use of the equation

$$\sum_{k=1}^{\infty}\frac{H_k^p}{k^p}=\frac{\zeta^2(p)+\zeta(2p)}{2}$$

$$\sum_{k=1}^{\infty}\frac{H_{k+1}^2}{(k+1)^2}=\frac{7}{4}\zeta(4)-1$$

On substituting, we get:

$$\Omega_2=-\frac{3}{4}\zeta(4)$$

Therefore,

$$\Omega_1+\Omega_2=-\frac{3\zeta^2(2)}{2}+\frac{5\zeta(4)}{2}-\frac{3}{4}\zeta(4)$$

$$\Omega_1+\Omega_2=-\frac{\pi^4}{45}$$

1868. Find:

$$\Omega=\int_0^{\infty}\frac{x(\tan^{-1}x)^2}{(x+1)(x^2+1)}dx$$

Proposed by Togrul Ehmedov-Azerbaijan

Solution by Benjamin Bamidele-Nigeria

$$\Omega=\int_0^\infty \frac{x(\tan^{-1}x)^2}{(x+1)(x^2+1)}dx=\int_0^\infty \frac{x(\tan^{-1}x)^2+(\tan^{-1}x)^2-(\tan^{-1}x)^2}{(1+x)(1+x^2)}dx=$$

$$=\int_0^\infty \frac{(\tan^{-1}x)^2}{x^2+1}dx-\int_0^\infty \frac{(\tan^{-1}x)^2}{(1+x)(1+x^2)}dx\overset{x=\tan z}{=}$$

$$=\int_0^{\frac{\pi}{2}} z^2dz-\int_0^{\frac{\pi}{2}}\frac{z^2}{\tan z+1}dz=\frac{\pi^3}{24}-\int_0^{\frac{\pi}{2}}\frac{z^2\cos z}{\sin z+\cos z}dz=$$

$$=\frac{\pi^3}{24}-\frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{2}} z^2\cos z\cdot\csc\left(z+\frac{\pi}{4}\right)dz$$

$$\because a\sin\theta+b\cos\theta=\sqrt{a^2+b^2}\sin\left(\theta+tan^{-1}\left(\frac{a}{b}\right)\right)$$

$$z=t-\frac{\pi}{4}$$

$$\Omega=\frac{\pi^3}{24}-\frac{1}{\sqrt{2}}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\left(t-\frac{\pi}{4}\right)^2\cdot\frac{\cos\left(t-\frac{\pi}{4}\right)}{\sin t}dt=$$

$$=\frac{\pi^3}{24}-\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\left(t-\frac{\pi}{4}\right)^2dt-\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\left(t-\frac{\pi}{4}\right)^2\cot t\,dt=$$

$$=\frac{\pi^3}{48}-\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\left(t-\frac{\pi}{4}\right)^2\cot t\,dt$$

Consider $\Omega=\frac{\pi^3}{48}-\Phi$,

$$\Phi=\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\left(t-\frac{\pi}{4}\right)^2\cot t\,dt\overset{IBP}{=}\frac{1}{2}\left(t-\frac{\pi}{4}\right)^2\log(\sin t)\Big|_{\frac{\pi}{4}}^{\frac{3\pi}{4}}-\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\left(t-\frac{\pi}{4}\right)\log(\sin t)\,dt=$$

$$=-\frac{\pi^2\log 2}{16}-\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\left(t-\frac{\pi}{4}\right)\left(-\log 2-\sum_{k=1}^{\infty}\frac{\cos(2kt)}{k}\right)dt=$$

$$=\frac{\pi^2\log 2}{16}+\sum_{k=1}^{\infty}\frac{1}{k}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\left(t-\frac{\pi}{4}\right)\cos(2kt)\,dt=$$

$$=\frac{\pi^2\log 2}{16}+\sum_{k=1}^{\infty}\left(\frac{\pi\sin\left(\frac{k\pi}{2}\right)}{4k^2}-\underbrace{\frac{\sin\frac{k\pi}{2}\sin k\pi}{2k^3}}_{0}+\frac{\pi\sin\frac{k\pi}{2}\cos k\pi}{2k^2}\right)=$$

$$=\frac{\pi^2\log 2}{16}+\sum_{k=1}^{\infty}\left(\frac{\pi\sin\left(\frac{k\pi}{2}\right)}{4k^2}+\frac{\pi\sin\frac{k\pi}{2}\cos k\pi}{2k^2}\right)=$$

$$=\frac{\pi^2\log 2}{16}+\frac{\pi}{4}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)^2}+\frac{\pi}{2}\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k-1)^2}=$$

$$=\frac{\pi^2\log 2}{16}+\frac{\pi}{4}G-\frac{\pi}{2}G\Rightarrow\Phi=\frac{\pi^2\log 2}{16}-\frac{\pi}{4}G$$

$$\Omega=\frac{\pi^3}{48}-\Phi=\frac{\pi^3}{48}-\frac{\pi^2}{16}\log 2+\frac{\pi}{4}G$$

1869. Find:

$$\Omega=\int_0^1\int_0^1\frac{x\cdot\tan^{-1}(xy)}{1+xy}dxdy$$

Proposed by Togrul Ehmedov-Azerbaijan

Solution by Rana Ranino-Setif-Algerie

$$\Omega=\int_0^1\int_0^1\frac{x\cdot\tan^{-1}(xy)}{1+xy}dxdy\overset{t=xy}{=}\int_0^1\int_0^x\frac{(\tan^{-1}x)^2}{1+t}dt\,dx\overset{IBP}{=}$$

$$=\left[x\int_0^x\frac{(\tan^{-1}x)^2}{1+t}dt\right]_0^1-\int_0^1\frac{x\cdot(\tan^{-1}x)^2}{1+x}dx=$$

$$=2\underbrace{\int_0^1\frac{(\tan^{-1}x)^2}{1+x}dx}_{A}-\underbrace{\int_0^1(\tan^{-1}x)^2\,dx}_{B}$$

$$A\overset{x=\tan t}{=}\int_0^{\frac{\pi}{4}}\frac{t^2}{(\sin t+\cos t)\cos t}dt\overset{IBP}{=}[t^2\log(1+\tan t)]_0^{\frac{\pi}{4}}-$$

$$-2\underbrace{\int_0^{\frac{\pi}{4}}t\log(\sin t+\cos t)\,dt}_{t\to\frac{\pi}{4}-t}+2\int_0^{\frac{\pi}{4}}t\log(\cos t)\,dt=$$

$$=\frac{\pi^2}{16}\log 2-2\int_0^{\frac{\pi}{4}}\left(\frac{\pi}{4}-t\right)\log(\sqrt{2}\,cos\,t)\,dt+2\int_0^{\frac{\pi}{4}}t\log(\cos t)\,dt=$$

$$=\frac{\pi^2}{16}-\log 2\int_0^{\frac{\pi}{4}}\left(\frac{\pi}{4}-t\right)dt-\frac{\pi}{2}\int_0^{\frac{\pi}{4}}\log(\cos t)\,dt+4\int_0^{\frac{\pi}{4}}t\log(\cos t)\,dt=$$

$$=\frac{\pi^2}{16}\log 2-\frac{\pi^2}{32}\log 2-\frac{\pi}{2}\left(\frac{G}{2}-\frac{\pi}{4}\log 2\right)+4\int_0^{\frac{\pi}{4}} t\log(\cos t)\,dt$$

$$\Omega=\frac{5\pi^2}{32}\log 2-\frac{\pi G}{4}+4I$$

$$I=-\int_0^{\frac{\pi}{4}} t\left(\log 2+\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\cos(2nt)\right)dt=$$

$$=-\frac{\pi^2}{32}\log 2-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^{\frac{\pi}{4}} t\cos(2nt)\,dt=$$

$$=-\frac{\pi^2}{32}\log 2-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left[\frac{t\sin(2nt)}{2n}+\frac{\cos(2nt)}{4n^2}\right]_0^{\frac{\pi}{4}}=$$

$$=-\frac{\pi^2}{32}\log 2-\frac{\pi}{8}\sum_{n=1}^{\infty}\frac{(-1)^n\sin\left(\frac{n\pi}{2}\right)}{n^2}-\frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^n\cos\left(\frac{n\pi}{2}\right)}{n^3}+\frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}=$$

$$=-\frac{\pi^2}{32}\log 2+\frac{\pi}{8}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2}-\frac{1}{32}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}+\frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}$$

$$I=-\frac{\pi^2}{32}\log 2+\frac{\pi G}{8}-\frac{21}{128}\zeta(3),\ \ A=\frac{\pi^2}{32}\log 2+\frac{\pi G}{4}-\frac{21}{32}\zeta(3)$$

$$B\overset{IBP}{=}[x\cdot(\tan^{-1}x)^2]_0^1-2\int_0^1\frac{x\cdot\tan^{-1}x}{1+x^2}dx\overset{x=\tan t}{=}$$

$$=\frac{\pi^2}{16}-2\int_0^{\frac{\pi}{4}} t\tan t\,dt=\frac{\pi^2}{16}+2[t\log(\cos t)]_0^{\frac{\pi}{4}}-2\int_0^{\frac{\pi}{4}}\log(\cos t)\,dt=$$

$$=\frac{\pi^2}{16}+\frac{\pi}{4}\log 2-G$$

$$\Omega=\int_0^1\int_0^1\frac{x\cdot\tan^{-1}(xy)}{1+xy}dxdy=\frac{\pi^2}{16}(\log 2-1)-\frac{\pi}{4}\log 2+\left(\frac{\pi}{2}+1\right)G-\frac{21}{16}\zeta(3)$$

1870. Find:

$$\Omega=\int_0^1(\tan^{-1}x)^2\,dx$$

Proposed by Togrul Ehmedov-Azerbaijan

Solution 1 by Rana Ranino-Setif-Algerie, Solution 2 by Avishek Mitra-West Bengal-India, Solution 3 by Ahman Isnawahyudi-Indonesia

Find:

$$\Omega=\int_0^1(\tan^{-1}x)^3\,dx$$

Proposed by Togrul Ehmedov-Azerbaijan

Solution 1 by Avishek Mitra-West Bengal-India, Solution 2 by Ankush Kumar Parcha-India

If

$$\mathbf{J} = \int_0^1 \arctan^n(x)\,dx \text{ where } n > 1$$

Show that

$$\mathbf{J} = \left(\frac{\pi}{4}\right)^n + \frac{n}{2}\log(2)\left(\frac{\pi}{4}\right)^{n-1} - n(n-1)\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\mathbf{I}(k,n)$$

$$\text{Where } \mathbf{I}(k,n) = \Re\int_0^{\frac{\pi}{4}} x^{n-2}e^{2ikx}\,dx$$

Proposed by Akerele Olofin-Nigeria

Solution by Togrul Ehmedov-Baku-Azerbaijan

Solution 1 by Rana Ranino-Setif-Algerie

$$\Omega = \int_0^1 (\tan^{-1}x)^2\,dx \overset{IBP}{=} x(\tan^{-1}x)^2\Big|_0^1 - 2\int_0^1 \frac{x\cdot\tan^{-1}x}{1+x^2}\,dx \overset{x=\tan t}{=}$$

$$= \frac{\pi^2}{16} - 2\int_0^{\frac{\pi}{4}} t\cdot\tan t\,dt = \frac{\pi^2}{16} + 2t\log(\cos t)\Big|_0^{\frac{\pi}{4}} - 2\int_0^{\frac{\pi}{4}}\log(\cos t)\,dt =$$

$$= \frac{\pi^2}{16} + \frac{\pi}{4}\log 2 - G$$

Therefore,

$$\Omega = \int_0^1 (\tan^{-1}x)^2\,dx = \frac{\pi^2}{16} + \frac{\pi}{4}\log 2 - G$$

Solution 2 by Avishek Mitra-West Bengal-India

$$I_1 = \int_0^1 \frac{\tan^{-1}x}{x^2+1}\,dx = \int_0^1 \tan^{-1}x\,d(\tan^{-1}x)\,dx = \frac{(\tan^{-1}x)^2}{2}\Bigg|_0^1 = \frac{\pi^2}{32}$$

$$I_2 = \int_0^1 \frac{x\cdot\tan^{-1}x}{x^2+1}\,dx \overset{x=\tan z}{=} \int_0^{\frac{\pi}{4}} z\cdot\tan z\,dz = z\log(\sec z)\Big|_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}}\log(\cos z)\,dz =$$

$$= \frac{\pi}{8}\log 2 + \int_0^{\frac{\pi}{4}}\left[-\log 2 - \sum_{n=1}^{\infty}\frac{(-1)^n\cos(2nz)}{n}\right]dz =$$

$$=\frac{\pi}{8}\log 2-\frac{\pi}{4}\log 2-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^{\frac{\pi}{4}}\cos(2nz)\,dz=$$

$$=-\frac{\pi}{8}\log 2-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{\sin(2nz)}{2n}\Bigg|_0^{\frac{\pi}{4}}=-\frac{\pi}{8}\log 2+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\sin\left(\frac{n\pi}{2}\right)}{n^2}=$$

$$=-\frac{\pi}{8}\log 2+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2}=-\frac{\pi}{8}\log 2+\frac{9}{2}$$

$$I_1=\int_0^1(\tan^{-1}x)^2dx=x\cdot\tan^{-1}x\Big|_0^1-\int_0^1\frac{2\tan^{-1}x}{1+x^2}xdx=$$

$$=\frac{\pi^2}{16}-2I_2=\frac{\pi^2}{16}-2\left(-\frac{\pi}{8}\log 2+\frac{G}{2}\right)=\pi^2+\frac{\pi}{4}\log 2-G$$

Solution 3 by Ahman Isnawahyudi-Indonesia

$$\Omega=\int_0^1(\tan^{-1}x)^2\,dx\overset{IBP}{=}x(\tan^{-1}x)^2\Big|_0^1-2\int_0^1\frac{x\cdot\tan^{-1}x}{1+x^2}dx\overset{IBP}{=}$$

$$=\frac{\pi^2}{16}-2\cdot\frac{1}{2}\tan^{-1}x\cdot\log(1+x^2)\Big|_0^1+2\int_0^1\frac{\log(1+x^2)}{2(1+x^2)}dx=$$

$$=\frac{\pi^2}{16}-\frac{\pi}{4}\log 2+2\int_0^1\frac{\log(1+x^2)}{1+x^2}dx=\frac{\pi^2}{16}-\frac{\pi}{4}\log 2-2\int_0^{\frac{\pi}{4}}\log|\cos t|\,dt=$$

$$=\frac{\pi^2}{16}-\frac{\pi}{4}\log 2-2\left(\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\int_0^{\frac{\pi}{4}}\cos(2kx)\,dx\right)+\frac{\pi}{2}\log 2=$$

$$=\frac{\pi^1}{16}+\frac{\pi}{4}\log 2-\sum_{k=1}^{\infty}\frac{\sin\left(\frac{\pi}{k}\right)}{k^2}=\frac{\pi^2}{16}+\frac{\pi}{4}\log 2-\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k+1)^2}=\frac{\pi^2}{16}+\frac{\pi}{4}\log 2-G$$

Solution 1 by Avishek Mitra-West Bengal-India

$$\Omega=\int_0^1(\tan^{-1}x)^3\,dx=x\tan^{-1}x\Big|_0^{\frac{\pi}{4}}-3\int_0^1\frac{x(\tan^{-1}x)^2}{1+x^2}dx\overset{x=\tan z}{=}$$

$$=\frac{\pi^3}{64}-3\int_0^{\frac{\pi}{4}}z^2\tan z\,dz=\frac{\pi^3}{64}-\Omega_1$$

$$\Omega_1=\int_0^{\frac{\pi}{4}}z^2\tan z\,dz=z^2\log(\sec z)\Big|_0^{\frac{\pi}{4}}+2\int_0^{\frac{\pi}{4}}z\log(\cos z)\,dz=$$

$$=\frac{\pi^2}{32}\log 2+2\int_0^{\frac{\pi}{4}}z\left[-\log 2-\sum_{n=1}^{\infty}\frac{(-1)^n\cos(2nz)}{n}\right]dz=$$

$$=\frac{\pi^2}{32}\log 2-2\log 2\frac{z^2}{2}\Bigg|_0^{\frac{\pi}{4}}-2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^{\frac{\pi}{4}}z\cos(2nz)\,dz=$$

$$= -\frac{\pi^2}{32}\log 2 + 2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left[z\cdot\frac{\sin(2nz)}{2n}\Big|_0^{\frac{\pi}{4}} - \frac{1}{2n}\int_0^{\frac{\pi}{4}}\sin(2nz)\,dz\right] =$$

$$= -\frac{\pi^2}{32}\log 2 + 2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}[\frac{\pi}{8n}\sin\left(\frac{n\pi}{2}\right) + \frac{1}{4n^2}\cos(2nz)\Big|_0^{\frac{\pi}{4}} =$$

$$= -\frac{\pi^2}{32}\log 2 + 2\cdot\frac{\pi}{8}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\sin\left(\frac{n\pi}{2}\right)}{n^2} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cos\left(\frac{n\pi}{2}\right)}{n^3} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3} =$$

$$= -\frac{\pi^2}{32}\log 2 + \frac{\pi}{4}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n)^3} - \frac{1}{2}\cdot\frac{3}{4}\zeta(3) =$$

$$= -\frac{\pi^2}{32}\log 2 + \frac{\pi}{4}G - \frac{21}{64}\zeta(3)$$

$$\Omega = \int_0^1 (\tan^{-1} x)^3\,dx = \frac{\pi^3}{64} + \frac{3\pi^2}{32}\log 2 - \frac{3\pi}{4}G + \frac{63}{64}\zeta(3)$$

Solution 2 by Ankush Kumar Parcha-India

$$\Omega = \int_0^1 (\tan^{-1} x)^3\,dx \overset{y=\tan^{-1}x}{=} \int_0^{\frac{\pi}{4}} y^3\sec^2 y\,dy = y^3\tan y\Big|_0^{\frac{\pi}{4}} - 3\int_0^{\frac{\pi}{4}} y^2\tan y\,dy =$$

$$= \frac{\pi^3}{64} - 3[-y^2\log|\cos y|\Big|_0^{\frac{\pi}{4}} + 2\int_0^{\frac{\pi}{4}} y\log|\cos y|\,dy =$$

$$= \frac{\pi^3}{64} - \frac{3\pi^2}{32}\log 2 - 6\int_0^{\frac{\pi}{4}} y\log|\cos y|\,dy =$$

$$= \frac{\pi^3}{64} - \frac{3\pi^2}{32}\log 2 - 6\left[\int_0^{\frac{\pi}{4}} y\log(1-e^{-2iy})\,dy - \int_0^{\frac{\pi}{4}} y\log 2\,dy\right] =$$

$$= \frac{\pi^3}{64} - \frac{3\pi^2}{32}\log 2 + 3\log 2\,y^2\Big|_0^{\frac{\pi}{4}} - 6\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\int_0^{\frac{\pi}{4}} ye^{-2iny}dy =$$

$$= \frac{\pi^3}{64} - \frac{3\pi^2}{32}\log 2 - 6\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left[\frac{ye^{-2iny}}{-2iy}\Big|_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}}\frac{e^{-2iny}}{2in}dy\right] =$$

$$= \frac{\pi^3 + 6\pi^2\log 2}{64} - 6\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left[\frac{\pi}{8n}\sin\left(\frac{n\pi}{2}\right) + \frac{1}{4n^2}\left(e^{-\frac{i\pi}{2}} - 1\right)\right] =$$

$$= \frac{\pi^3 + 6\pi^2\log 2}{64} - \frac{3\pi}{4}G + \frac{9}{8}\zeta(3) - \frac{9}{64}\zeta(3)$$

$$\Omega = \int_0^1 (\tan^{-1} x)^3\,dx = \frac{\pi^3}{64} + \frac{3\pi^2}{32}\log 2 - \frac{3\pi}{4}G + \frac{63}{64}\zeta(3)$$

Solution by Togrul Ehmedov

$$J=\int_0^1 \arctan^n(x)\,dx = x\arctan^n(x)\Big]_{x=0}^{x=1} - n\int_0^1 \frac{x}{1+x^2}\arctan^{n-1}(x)\,dx$$

$$=\left(\frac{\pi}{4}\right)^n - n\int_0^1 \frac{x}{1+x^2}\arctan^{n-1}(x)\,dx$$

$$=\left(\frac{\pi}{4}\right)^n - n\left[\left[\frac{1}{2}\log(1+x^2)\arctan^{n-1}(x)\right]_{x=0}^{x=1} - \frac{n-1}{2}\int_0^1 \frac{\log(1+x^2)}{1+x^2}\arctan^{n-2}(x)\,dx\right]$$

$$=\left(\frac{\pi}{4}\right)^n - n\left[\frac{1}{2}\log(2)\left(\frac{\pi}{4}\right)^{n-1} - \frac{n-1}{2}\int_0^1 \frac{\log(1+x^2)}{1+x^2}\arctan^{n-2}(x)\,dx\right]$$

$$=\left(\frac{\pi}{4}\right)^n - n\left[\frac{1}{2}\log(2)\left(\frac{\pi}{4}\right)^{n-1} + (n-1)\int_0^{\frac{\pi}{4}} x^{n-2}\log(\cos x)\,dx\right]$$

$$=\left(\frac{\pi}{4}\right)^n - n\left[\frac{1}{2}\log(2)\left(\frac{\pi}{4}\right)^{n-1} + (n-1)\int_0^{\frac{\pi}{4}} x^{n-2}\left[-\log(2) + \Re\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}e^{2ikx}\right]dx\right]$$

$$=\left(\frac{\pi}{4}\right)^n - \frac{n}{2}\log(2)\left(\frac{\pi}{4}\right)^{n-1} - n(n-1)\int_0^{\frac{\pi}{4}} x^{n-2}\left[-\log(2) + \Re\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}e^{2ikx}\right]dx$$

$$=\left(\frac{\pi}{4}\right)^n - \frac{n}{2}\log(2)\left(\frac{\pi}{4}\right)^{n-1} + \log(2)\,n(n-1)\int_0^{\frac{\pi}{4}} x^{n-2}\,dx$$

$$-n(n-1)\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\Re\int_0^{\frac{\pi}{4}} x^{n-2}e^{2ikx}\,dx$$

$$=\left(\frac{\pi}{4}\right)^n - \frac{n}{2}\log(2)\left(\frac{\pi}{4}\right)^{n-1} + \log(2)\,n\left(\frac{\pi}{4}\right)^{n-1} - n(n-1)\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}I(k,n)$$

$$=\left(\frac{\pi}{4}\right)^n + \frac{n}{2}\log(2)\left(\frac{\pi}{4}\right)^{n-1} - n(n-1)\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}I(k,n)$$

1871. Prove that:

$$\int_0^{\infty}\frac{(5x^2+2x+2)\tan^{-1}x}{(1+x)^2(1+x^2)}dx = \frac{12G+\pi^2+\pi(5-\log 8)}{8}$$

Proposed by Ankush Kumar Parcha-India

Solution by Togrul Ehmedov-Azerbaijan

$$\Omega=\int_0^\infty\frac{(5x^2+2x+2)\tan^{-1}x}{(1+x)^2(1+x^2)}dx=5\int_0^\infty\frac{x\cdot\tan^{-1}x}{(1+x)(1+x^2)}dx-$$

$$-3\int_0^\infty\frac{\tan^{-1}x}{(1+x)(1+x^2)}dx+5\int_0^\infty\frac{\tan^{-1}x}{(1+x)^2(1+x^2)}dx=5I_1-3I_2+5I_3$$

$$I_1=\int_0^\infty\frac{x\cdot\tan^{-1}x}{(1+x)(1+x^2)}dx=\frac{1}{2}G-\frac{\pi}{8}\log 2+\frac{\pi^2}{16}$$

$$I_2=\int_0^\infty\frac{\tan^{-1}x}{(1+x)(1+x^2)}dx=\int_0^1\frac{\tan^{-1}x}{(1+x)(1+x^2)}dx+\int_1^\infty\frac{\tan^{-1}x}{(1+x)^2(1+x^2)}dx$$

$$=\int_0^1\frac{\tan^{-1}x}{(1+x)(1+x^2)}dx+\int_0^1\frac{x\tan^{-1}x}{(1+x)(1+x^2)}dx=$$

$$=\frac{\pi}{2}\int_0^1\frac{xdx}{(1+x)(1+x^2)}+\int_0^1\frac{(1-x)\tan^{-1}x}{(1+x)(1+x^2)}dx=I_{2a}+I_{2b}$$

$$I_{2a}=\int_0^1\frac{xdx}{(1+x)(1+x^2)}=\frac{\pi}{8}-\frac{1}{4}\log 2$$

$$I_{2b}=\int_0^1\frac{(1-x)\tan^{-1}x}{(1+x)(1+x^2)}dx=\frac{\pi^2}{16}+\frac{\pi}{8}\log 2-\frac{1}{2}G$$

$$I_3=\int_1^\infty\frac{\tan^{-1}x}{(1+x)^2(1+x^2)}dx=$$

$$=\frac{\pi}{2}\int_0^1\frac{x^2}{(1+x)^2(1+x^2)}dx+\int_0^1\frac{(1-x^2)\tan^{-1}x}{(1+x)^2(1+x^2)}dx=$$

$$=\frac{\pi}{2}\int_0^1\frac{x^2dx}{(1+x)^2(1+x^2)}+\int_0^1\frac{(1-x)\tan^{-1}x}{(1+x)(1+x^2)}dx=\frac{\pi}{2}I_{3a}+I_{2b}$$

$$I_{3a}=\int_0^1\frac{x^2dx}{(1+x)^2(1+x^2)}=\frac{1}{4}-\frac{1}{4}\log 2$$

$$I_3=\frac{\pi}{2}I_{3a}+I_{2b}=\frac{\pi}{8}+\frac{\pi}{8}\log 2-\frac{1}{2}G$$

$$\Omega=5I_1-3I_2+5I_3=\frac{12G+\pi^2+\pi(5-\log 8)}{8}$$

1872. Find:

$$\Omega=\int_0^{\frac{\pi}{6}}\frac{\sin x\cdot\sin\left(x+\frac{\pi}{3}\right)\cdot\sin\left(x+\frac{2\pi}{3}\right)}{\sin 3x+\cos 3x}dx$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Togrul Ehmedov-Azerbaijan

$$\because\ \sin\left(x+\frac{\pi}{3}\right)\sin\left(\frac{\pi}{3}-x\right)=\frac{1}{4}\sin 3x$$

$$\Omega=\int_0^{\frac{\pi}{6}}\frac{\sin x\cdot\sin\left(x+\frac{\pi}{3}\right)\cdot\sin\left(x+\frac{2\pi}{3}\right)}{\sin 3x+\cos 3x}dx=$$

$$=\frac{1}{4}\int_0^{\frac{\pi}{6}}\frac{\sin 3x}{\sin 3x+\cos 3x}dx=\frac{1}{4}\int_0^{\frac{\pi}{6}}\frac{\sin 3\left(\frac{\pi}{6}-x\right)}{\sin 3\left(\frac{\pi}{6}-x\right)+\cos 3\left(\frac{\pi}{6}-x\right)}dx=$$

$$=\frac{1}{4}\int_0^{\frac{\pi}{6}}\frac{\cos 3x}{\sin 3x+\cos 3x}dx$$

$$2\Omega=\frac{1}{4}\int_0^{\frac{\pi}{6}}\frac{\sin 3x+\cos 3x}{\sin 3x+\cos 3x}dx=\frac{1}{4}\int_0^{\frac{\pi}{6}}dx=\frac{\pi}{24}$$

Therefore,

$$\Omega=\int_0^{\frac{\pi}{6}}\frac{\sin x\cdot\sin\left(x+\frac{\pi}{3}\right)\cdot\sin\left(x+\frac{2\pi}{3}\right)}{\sin 3x+\cos 3x}dx=\frac{\pi}{48}$$

Solution 2 by Avishek Mitra-West Bengal-India

$$\sin x\sin\left(x+\frac{\pi}{3}\right)\sin\left(x+\frac{2\pi}{3}\right)=\sin x\left(\frac{1}{2}\sin x+\frac{\sqrt{3}}{2}\cos x\right)\left(-\frac{1}{2}\sin x+\frac{\sqrt{3}}{2}\cos x\right)=$$

$$=\frac{1}{4}\sin x\left(3\cos^2 x-\sin^2 x\right)=\frac{1}{4}\sin x\left(3-4\sin^2 x\right)=$$

$$=\frac{1}{4}\left(3\sin x-4\sin^3 x\right)=\frac{1}{4}\sin 3x$$

$$\Omega=\int_0^{\frac{\pi}{6}}\frac{\sin x\cdot\sin\left(x+\frac{\pi}{3}\right)\cdot\sin\left(x+\frac{2\pi}{3}\right)}{\sin 3x+\cos 3x}dx\overset{3x=z}{=}$$

$$=\frac{1}{12}\int_0^{\frac{\pi}{2}}\frac{\sin z}{\sin z+\cos z}dz=\frac{1}{12}\int_0^{\frac{\pi}{2}}\frac{(\sin z+\cos z)+(\sin z-\cos z)}{2(\sin z+\cos z)}dz=$$

$$=\frac{1}{24}\int_0^{\frac{\pi}{2}}dz-\frac{1}{24}\int_0^{\frac{\pi}{2}}\frac{d(\sin z+\cos z)}{\sin z+\cos z}=\frac{1}{24}\left(\frac{\pi}{2}-\log(\sin z+\cos z)|_0^{\frac{\pi}{2}}\right)=\frac{\pi}{48}$$

Solution 3 by Hikmat Mammadov-Azerbaijan

$$\sin x\sin\left(\frac{\pi}{3}-x\right)\sin\left(\frac{\pi}{3}+x\right)=\frac{1}{4}\sin(3x)$$

$$\sin\left(\frac{2\pi}{3}+x\right)=\sin\left(\pi-\frac{2\pi}{3}-x\right)=\sin\left(\frac{\pi}{3}-x\right)$$

$$\Omega=\int_0^{\frac{\pi}{6}}\frac{\sin x\cdot\sin\left(x+\frac{\pi}{3}\right)\cdot\sin\left(x+\frac{2\pi}{3}\right)}{\sin 3x+\cos 3x}dx=$$

$$=\frac{1}{12}\int_0^{\frac{\pi}{2}}\frac{\sin x}{\sin x+\cos x}dx=\frac{1}{12}\cdot\frac{\pi}{4}=\frac{\pi}{48}$$

Solution 4 by Tapas Das-India

$$\sin x\sin\left(\frac{\pi}{3}-x\right)\sin\left(\frac{\pi}{3}+x\right)=\frac{1}{4}\sin(3x)$$

$$\sin\left(\frac{2\pi}{3}+x\right)=\sin\left(\pi-\frac{2\pi}{3}-x\right)=\sin\left(\frac{\pi}{3}-x\right)$$

$$\Omega=\int_0^{\frac{\pi}{6}}\frac{\sin x\cdot\sin\left(x+\frac{\pi}{3}\right)\cdot\sin\left(x+\frac{2\pi}{3}\right)}{\sin 3x+\cos 3x}dx=$$

$$=\frac{1}{4}\int_0^{\frac{\pi}{6}}\frac{\sin 3x}{\sin 3x+\cos 3x}dx=\frac{1}{12}\int_0^{\frac{\pi}{2}}\frac{\sin p}{\cos p+\sin p}dp$$

$$2\Omega=\frac{1}{12}\int_0^{\frac{\pi}{2}}\frac{\sin p+\cos p}{\sin p+\cos p}dp=\frac{1}{12}\cdot\frac{\pi}{4}=\frac{\pi}{48}$$

Solution 5 by Ravi Prakash-New Delhi-India

$$\sin x\sin\left(x+\frac{\pi}{3}\right)\sin\left(x+\frac{2\pi}{3}\right)=\sin x\left(\frac{1}{2}\sin x+\frac{\sqrt{3}}{2}\cos x\right)\left(-\frac{1}{2}\sin x+\frac{\sqrt{3}}{2}\cos x\right)=$$

$$=\frac{1}{4}\sin x\,(3\cos^2 x-\sin^2 x)=\frac{1}{4}\sin x\,(3-4\sin^2 x)=$$

$$=\frac{1}{4}(3\sin x-4\sin^3 x)=\frac{1}{4}\sin 3x$$

$$\Omega=\int_0^{\frac{\pi}{6}}\frac{\sin x\cdot\sin\left(x+\frac{\pi}{3}\right)\cdot\sin\left(x+\frac{2\pi}{3}\right)}{\sin 3x+\cos 3x}dx=$$

$$=\int_0^{\frac{\pi}{2}}\frac{\sin 3x}{\sin 3x+\cos 3x}dx\overset{3x=t}{=}$$

$$=\frac{1}{12}\int_0^{\frac{\pi}{2}}\frac{\sin t}{\sin t+\cos t}dt=\frac{1}{12}\int_0^{\frac{\pi}{2}}\frac{\sin\left(\frac{\pi}{2}-t\right)dt}{\sin\left(\frac{\pi}{2}-t\right)+\cos\left(\frac{\pi}{2}-t\right)}=$$

$$=\frac{1}{12}\int_0^{\frac{\pi}{2}}\frac{\cos t}{\cos t+\sin t}dt$$

$$2\Omega=\frac{1}{12}\int_0^{\frac{\pi}{2}}\frac{\sin t+\cos t}{\sin t+\cos t}dt=\frac{1}{12}\cdot\frac{\pi}{2}=\frac{\pi}{48}$$

1873. Find all value $\alpha\geq 0$ such that

$$\int_0^{\infty}\log(1+\alpha x^2)\,dx<+\infty$$

Find all value $\beta\geq 0$ such that:

$$\int_{-\infty}^{\infty}2021^{\beta x}x\,dx<\infty$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution by Florentin Vișescu-Romania

$$\int\log(1+\alpha x^2)\,dx=\int x'\log(1+\alpha x^2)dx=$$

$$=x\log(1+\alpha x^2)-\int\frac{2\alpha x^2}{1+\alpha x^2}dx=x\log(1+\alpha x^2)-2\int\left(1-\frac{1}{1+\alpha x^2}\right)dx=$$

$$=x\log(1+\alpha x^2)-2\left(x-\frac{1}{\sqrt{\alpha}}\tan^{-1}(x\sqrt{\alpha})\right)+C$$

$$\int_0^{\infty}\log(1+\alpha x^2)\,dx=\lim_{x\to\infty}\left(x\log(1+\alpha x^2)-2\left(x-\frac{1}{\sqrt{\alpha}}\tan^{-1}(x\sqrt{\alpha})\right)\right)=$$

$$=\lim_{n\to\infty}x(\log(1+\alpha x^2)-2)+\frac{\pi}{\sqrt{\alpha}}=+\infty$$

For $\alpha = 0$ we have:

$$\int_0^{\infty} \log 1\, dx = 0(\text{not verify})$$

So, does not exist so that

$$\int_0^{\infty} \log(1+\alpha x^2)\, dx < +\infty$$

$$\int 2021^{\beta x} x\, dx = \int \left(\frac{2021^{\beta x}}{\log 2021^{\beta}}\right)' x dx = \frac{2021^{\beta x}}{\beta \log 2021} x - \int \frac{2021^{\beta x}}{\beta \log 2021} dx =$$

$$= \frac{2021^{\beta x}}{\beta \log 2021} - \frac{2021^{\beta x}}{\beta^2 \log^2 2021} + C = \frac{2021^{\beta x}}{\beta \log 2021}\left(1 - \frac{1}{\beta \log 2021}\right) + C$$

$$\int_{-\infty}^{\infty} 2021^{\beta x} x\, dx = \lim_{x\to\infty} \frac{2021^{\beta x}}{\beta \log 2021}\left(1 - \frac{1}{\beta \log 2021}\right) = \begin{cases} +\infty\, , \beta > \dfrac{1}{\log 2021} \\ -\infty, \beta < \dfrac{1}{\log 2021} \\ 0, \beta = \dfrac{1}{\log 2021} \end{cases}$$

$$\text{For } \beta = 0: \int_{-\infty}^{\infty} x\, dx = \frac{x^2}{2}\bigg|_{-\infty}^{\infty} \Rightarrow \beta \in \left(0, \frac{1}{\log 2021}\right]$$

1874. If $0 < a \le b$ then:

$$\int_a^b \frac{x^{19}}{\sqrt{1+x^{30}}} dx \ge \log \sqrt[10]{\frac{2+b^{20}}{2+a^{20}}}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$\text{For } x \ge 0 \Rightarrow x^{20}(x^{10}-2)^2 \ge 0 \Rightarrow x^{20}(x^{20} - 4x^{10} + 4) \ge 0$$

$$x^{40} - 4x^{30} + 4x^{20} \ge 0 \Rightarrow x^{40} + 4x^{20} + 4 \ge 4(1+x^{30})$$

$$(x^{20}+2)^2 \ge 4(1+x^{30}) \Rightarrow \frac{1}{\sqrt{1+x^{30}}} \ge \frac{2}{x^{20}+2}$$

$$\frac{x^{19}}{\sqrt{1+x^{30}}} \ge \frac{2x^{19}}{x^{20}+2}$$

$$\int_a^b \frac{x^{19}}{\sqrt{1+x^{30}}}dx \geq 2\int_a^b \frac{x^{19}}{x^{20}+2}dx = \frac{1}{10}\log(x^{20}+2)\Big|_a^b = \log\sqrt[10]{\frac{2+b^{20}}{2+a^{20}}}$$

Solution 2 by Tapas Das-India

$$\log\sqrt[10]{\frac{2+b^{20}}{2+a^{20}}} = \frac{1}{10}\log\left(\frac{2+b^{20}}{2+a^{20}}\right) = \int_a^b \frac{2x^{19}}{x^{20}+2}dx$$

Now, we need to prove:

$$\frac{x^{19}}{\sqrt{1+x^{30}}} \geq \frac{2x^{19}}{x^{20}+2} \Leftrightarrow \frac{1}{\sqrt{1+x^{30}}} \geq \frac{2}{x^{20}+2} \Leftrightarrow$$

$$(x^{20}+2)^2 \geq 4(1+x^{30}) \Leftrightarrow x^{40}+4x^{20}+4 \geq 4(1+x^{30}) \Leftrightarrow$$

$$x^{40}-4x^{30}+4x^{20} \geq 0 \Leftrightarrow x^{20}(x^{10}-2)^2 \geq 0 \text{ true for all } x \geq 0.$$

1875. Prove that:

$$\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n+1}\left(H_{\frac{2n+1}{4}}-H_{\frac{2n-1}{4}}\right)H_n = \frac{7\pi^3}{24}+\frac{5\pi}{2}\log^2 2 - 4G\log 2 - 8\Im\big(Li_3(1+i)\big)$$

where $H_n = \int_0^1 \frac{1-x^n}{1-x}dx$, G – is Catalan's constant, $\sum_{k=0}^{n}\frac{(-1)^{k+1}}{(2k+1)^2}$

$Li_3(z) = \sum_{k=1}^{\infty}\frac{z^k}{k^3}$ is trilogarithm function, $\Im(z)$ is imaginary part of z and i is imaginary part.

Proposed by Naren Bhandari-Bajura-Nepal

Solution by Rana Ranino-Setif-Algerie

$$\Omega = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n+1}\left(H_{\frac{2n+1}{4}}-H_{\frac{2n-1}{4}}\right)H_n = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n}\left(H_{\frac{2n+1}{4}}-H_{\frac{2n-1}{4}}\right)\left(H_n-\frac{1}{n}\right) =$$

$$= \underbrace{\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left(H_{\frac{2n-1}{4}}-H_{\frac{2n-3}{4}}\right)H_n}_{A} - \underbrace{\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\left(H_{\frac{2n-1}{4}}-H_{\frac{2n-3}{4}}\right)}_{B}$$

$$H_{\frac{2n-1}{4}}-H_{\frac{2n-3}{4}} = \int_0^1 \frac{t^{\frac{2n-3}{4}}-t^{\frac{2n-1}{4}}}{1-t}dt \overset{t=x^4}{=} 4\int_0^1 \frac{x^{2n}-x^{2n+2}}{1-x^4}dx = 4\int_0^1 \frac{x^{2n}}{1+x^2}dx$$

$$A = 4\int_0^1 \frac{1}{1+x^2}\left(\sum_{n=1}^{\infty}\frac{H_n}{n}(-x^2)^n\right)dx = 4\int_0^1 \frac{Li_2(-x^2)+\frac{1}{2}\log^2(1+x^2)}{1+x^2}dx =$$

$$= 4\int_0^1 \frac{Li_2(-x^2)}{1+x^2}dx + 2\int_0^1 \frac{\log^2(1+x^2)}{1+x^2}dx$$

$$B = 4\int_0^1 \frac{1}{1+x^2}\left(\sum_{n=1}^{\infty}\frac{(-x^2)^n}{n^2}\right)dx = 4\int_0^1 \frac{Li_2(1+x^2)}{1+x^2}dx$$

$$\Omega = 2\int_0^1 \frac{\log^2(1+x^2)}{1+x^2}dx \overset{x=\tan\theta}{=} 8\int_0^{\frac{\pi}{2}} \log^2(\cos\theta)\,d\theta$$

Therefore,

$$\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n+1}\left(H_{\frac{2n+1}{4}} - H_{\frac{2n-1}{4}}\right)H_n = \frac{7\pi^3}{24} + \frac{5\pi}{2}\log^2 2 - 4G\log 2 - 8\Im\left(Li_3(1+i)\right)$$

1876. Let $S(x) = \sum_{n=0}^{\infty}(3x)^{n+2}$. Using the above sum, find:

$$\Omega = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{3^{n+1}(n+3)}$$

Proposed by Tobi Joshua-Nigeria

Solution 1 by Akerele Olofin-Nigeria

$$S(x) = \sum_{n=0}^{\infty}(3x)^{n+2} \Rightarrow S(x) = \frac{9x^2}{1-3x}, |3x| < 1$$

$$S(x) = \sum_{n=0}^{\infty}(3n)^{n+2} = \frac{9x^2}{1-3x}, |3x| < 1$$

$$\int S(x)\,dx = \sum_{n=0}^{\infty}3^{n+1}\int x^{n+2}\,dx = \int \frac{9x^2}{1-3x}dx \Rightarrow$$

$$\Omega = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{3^{n+1}(n+3)} = \frac{-2\log|3x-1| - 9x^2 - 6x}{6}$$

$$x = -\frac{1}{9} \Rightarrow \sum_{n=0}^{\infty}\frac{3^{n+2}\left(-\frac{1}{9}\right)^{n+2}}{n+3} = \sum_{n=0}^{\infty}\frac{3^{n+2}\cdot 3^{-2n-6}\cdot(-1)^{n+3}}{n+3} =$$

$$= 3^{-3}\sum_{n=0}^{\infty}\frac{3^{-n-1}(-1)^{n+1}}{n+3} = 3^{-3}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{3^{n+1}(n+3)} =$$

$$=\frac{-2\log\left|-\frac{1}{3}-1\right|-\frac{9}{81}-6\left(-\frac{1}{9}\right)}{6}=\frac{-2\log\left(\frac{4}{3}\right)+\frac{5}{9}}{6}=\frac{-18\log\left(\frac{4}{3}\right)+5}{9\cdot 6}$$

$$\Rightarrow \Omega=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{3^{n+1}(n+3)}=3^3\cdot\frac{5-18\log\left(\frac{4}{3}\right)}{9\cdot 6}=\frac{5}{2}-9\log\left(\frac{4}{3}\right)$$

Solution 2 by Tapas Das-India

$$\Omega=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{3^{n+1}(n+3)}=\frac{-1}{3\cdot 3}+\frac{1}{3^2\cdot 4}+\frac{-1}{3^3\cdot 5}+\frac{1}{3^4\cdot 6}-\cdots$$

$$=-\left[\frac{\frac{1}{3}}{3}-\frac{\frac{1}{3^2}}{4}+\frac{\frac{1}{3^3}}{5}-\frac{\frac{1}{3^4}}{6}-\cdots\right]=-3^2\left[\frac{\frac{1}{3^3}}{3}-\frac{\frac{1}{3^4}}{4}+\frac{\frac{1}{4^5}}{5}-\frac{\frac{1}{3^6}}{6}+\cdots\right]=$$

$$=-3^2\left[\frac{1}{3}-\frac{\frac{1}{3^2}}{2}+\frac{\frac{1}{3^3}}{3}-\frac{\frac{1}{3^4}}{4}+\frac{\frac{1}{3^5}}{5}+\cdots\right]+3^2\left(\frac{1}{3}-\frac{\frac{1}{3^2}}{2}\right)=$$

$$=-3^2\log\left(1+\frac{1}{3}\right)+3^2\left(\frac{1}{3}-\frac{1}{18}\right)=9\log\left(\frac{3}{4}\right)+\frac{5}{2}$$

$$\because \log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots$$

Solution 3 by Hikmat Mammadov-Azerbaijan

$$S(x)=\sum_{n=0}^{\infty}(3x)^{n+2}=\frac{9x^2}{1-3x}=\frac{(1-3x)^2-2(1-3x)+1}{1-3x}$$

$$\int S(x)dx=\frac{1}{3}\sum_{n=0}^{\infty}\frac{(3x)^{n+3}}{n+3}+C=x-\frac{3}{2}x^2-2x-\frac{1}{3}\log(1-3x)+C$$

$$\int S(x)dx=0=C\Rightarrow\int S(x)dx=\frac{1}{3}\sum_{n=0}^{\infty}\frac{3x^{n+3}}{n+3}=-x-\frac{3}{2}x^2-\frac{\log(1-3x)}{3}$$

$$\int S\left(-\frac{1}{9}\right)dx=\frac{1}{3}\sum_{n=0}^{\infty}\frac{(-1)^{n+3}}{3^{n+3}(n+3)}=\frac{1}{9}-\frac{1}{54}-\frac{1}{3}\log\left(\frac{4}{3}\right)$$

$$\Omega=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{3^{n+1}(n+3)}=3-\frac{1}{2}-9\log\left(\frac{4}{3}\right)=\frac{5}{2}-9\log\left(\frac{4}{3}\right)$$

1877. Find $f,g:(1,\infty)\to R$ such that:

$$g(x)=ax-xf'(x)\cdot\log x \text{ and } f(x)=ax-xg'(x)\cdot\log x,\ x>1,a\in R.$$

Proposed by Florică Anastase-Romania

Solution 1 by Debjit Mullick-India

$$\begin{cases}g(x)=ax-xf'(x)\log x;(1)\\f(x)=ax-xg'(x)\log x;(2)\end{cases}\Rightarrow f(x)-g(x)=\big(f'(x)-g'(x)\big)x\log x$$

$$\frac{dx}{x\log x}=\frac{d\Big(\big(f(x)-g(x)\big)\Big)}{f(x)-g(x)}$$

$$\log(\log x)=\log(c\big(f(x)-g(x)\big),c\in\mathbb{R},\qquad f(x)=f(x)+k\log x$$

Now, equation (2) becomes:

$$g(x)+k\log x=ax-x\log x\,g'(x)\Rightarrow g'(x)=\frac{g(x)}{x\log x}=ax+k\log x$$

By multiplying both sides with $\log x$, we get:

$$\frac{d}{dx}(g(x)\log x)=ax\log x+k\log^2 x$$

$$g(x)\log x=\int ax\log x\,dx+k\int\log^2 x\,dx$$

$$g(x)\log x=\frac{x^2}{2}\left(\log x-\frac{1}{2}\right)+k\left(x(\log x-1)^2-\frac{x^2}{2}\right)$$

$$g(x)=\frac{1}{2}\cdot\frac{x^2}{\log x}\left(\log x-\frac{1}{2}\right)+\frac{k}{\log x}\left(x(\log x-1)^2-\frac{x^2}{2}\right)$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Let\ x>1.\ We\ have:\ \frac{g(x)}{x}+f'(x).\log x=a\ \ and\ \ \frac{f(x)}{x}+g'(x).\log x=a.$$

$$Then:\ \left(\frac{g(x)}{x}+g'(x).\log x\right)+\left(\frac{f(x)}{x}+f'(x).\log x\right)=2a$$

$$\Leftrightarrow (g(x).\log x)'+(f(x).\log x)'=2a\ \Rightarrow\ g(x).\log x+f(x).\log x=2ax+c_1,\ c_1\in R.$$

$$Then:\ f(x)+g(x)=\frac{2ax+c_1}{\log x},\ \ \forall x>1,\ c_1\in R.\ \ (1)$$

$$\textit{Also we have}: \left(\frac{g(x)}{x}+f'(x).\log x\right)-\left(\frac{f(x)}{x}+g'(x).\log x\right)=0$$

$$\Leftrightarrow \frac{f'(x).\log x-\frac{f(x)}{x}}{(\log x)^2}-\frac{g'(x).\log x-\frac{g(x)}{x}}{(\log x)^2}=0 \Leftrightarrow \left(\frac{f(x)}{\log x}-\frac{g(x)}{\log x}\right)'=0$$

$$\Rightarrow \frac{f(x)}{\log x}-\frac{g(x)}{\log x}=c_2,\ \ c_2\in R.$$

$$\textit{Then}: \ f(x)-g(x)=c_2.\log x,\ \ \forall x>1,\ \ c_2\in R.\ \ (2)$$

$$\textit{From } (1) \textit{ and } (2) \textit{ we get}:$$

$$f(x)=\frac{2ax+c_1}{2\log x}+\frac{c_2}{2}.\log x \ \ \textit{and} \ \ g(x)=\frac{2ax+c_1}{2\log x}-\frac{c_2}{2}.\log x,\ \ \forall x>1,\ \ c_1,c_2\in R.$$

$$\textit{Therefore},\ \ f(x)=\frac{ax+d_1}{\log x}+d_2.\log x \ \ \textit{and}$$

$$g(x)=\frac{ax+d_1}{\log x}-d_2.\log x,\ \ \forall x>1,\ \ d_1,d_2\in R.$$

Solution 3 by proposer

$$\begin{cases} f(x)=ax-xf'(x)\cdot\log x \\ g(x)=ax-xg'(x)\cdot\log x \end{cases}\Rightarrow$$

$$\begin{cases} f(x)+g(x)=2ax-\big(f'(x)+g'(x)\big)x\cdot\log x \\ \quad f(x)-g(x)=\big(xf'(x)-xg'(x)\big)x\cdot\log x \end{cases}$$

$$f(x)+g(x)+\big(f'(x)+g'(x)\big)\cdot x\log x=2ax, x>1\Leftrightarrow$$

$$\frac{f(x)+g(x)}{x}+\big(f'(x)+g'(x)\big)\cdot\log x=2a$$

$$\big[\big(f(x)+g(x)\big)\log x\big]'=2a\Leftrightarrow \big(f(x)+g(x)\big)\log x=2ax+b, b\in\mathbb{R}$$

$$x>1\Rightarrow f(x)+g(x)=\frac{1}{\log x}(2ax+b);(1)$$

$$\left(\frac{f(x)-g(x)}{\log x}\right)'=\frac{\big(f'(x)-g'(x)\big)\log x-\frac{1}{x}\big(f(x)-g(x)\big)}{\log^2 x}=$$

$$=\frac{\big(f'(x)-g'(x)\big)x\log x-\big(f(x)-g(x)\big)}{x\log^2 x}=$$

$$=\frac{\big(f'(x)-g'(x)\big)x\log x-\big(xf'(x)-xg'(x)\big)x\cdot\log x}{x\log^2 x}=0$$

$\Rightarrow (\exists) c \in \mathbb{R}$ such that $\frac{f(x)-g(x)}{\log x} = c \Rightarrow f(x) - g(x) = c \log x$; (2)

From (1) and (2), we get:

$$f(x) = \frac{1}{2}\left(\frac{2ax+b}{\log x} + c\log x\right) \text{ and } g(x) = \frac{1}{\log x}(2ax+b) - \frac{1}{2}\left(\frac{2ax+b}{\log x} + c\log x\right)$$

$$g(x) = \frac{1}{2}\left(\frac{2ax+b}{\log x} - c\log x\right)$$

1878. Find:

$$\Omega = \int_0^\infty \frac{x\tan^{-1}x}{(x+1)(x^2+1)}dx$$

Proposed by Vasile Mircea Popa-Romania

Solution 1 by Hikmat Mammadov-Azerbaijan

$$\text{A} = \int_0^\infty \frac{x\tan^{-1}x}{(x+1)(x^2+1)}dx = \int_0^\infty \left(\frac{\tan^{-1}x}{1+x^2} - \frac{\tan^{-1}x}{(1+x)(1+x^2)}\right)dx =$$

$$= \frac{\pi^2}{8} - \int_0^{\frac{\pi}{2}} \frac{x\cos x}{\cos x + \sin x}dx, \qquad B = \int_0^{\frac{\pi}{2}} \frac{x\cos x}{\cos x + \sin x}dx \Rightarrow A + B = \frac{\pi^2}{8}$$

$$A - B = \int_0^{\frac{\pi}{2}} \frac{x(\cos x - \sin x)}{\cos x + \sin x}dx = x\log(\cos x + \sin x)\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \log(\cos x + \sin x)\,dx =$$

$$= -\int_0^{\frac{\pi}{2}} \log\left(\sqrt{2}\,sin\left(x + \frac{\pi}{4}\right)\right)dx = -\frac{\pi}{2}\log\sqrt{2} - \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \log(\sin x)\,dx =$$

$$= -\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \log(\sin x)\,dx - \int_0^{\frac{\pi}{4}} \log(\cos x)\,dx = -2\int_0^{\frac{\pi}{4}} \log(\cos x)\,dx$$

$$= -2 \cdot \frac{1}{4}(2G - \pi\log 2)$$

$$A = \frac{1}{2}(A - B + A + B) = \frac{1}{2}\left(\frac{\pi^2}{8} - \frac{\pi}{2}\log\sqrt{2} - G + \frac{\pi}{2}\log 2\right) =$$

$$= \frac{\pi^2}{16} + \frac{\pi}{8}\log 2 - \frac{G}{2} = \frac{\pi^2}{16} + \frac{G}{2} - \frac{\pi}{8}\log 2$$

Solution 2 by Ankush KumarParcha-India

$$\mathrm{I}=\int_0^{\infty}\frac{x\tan^{-1}x}{(x+1)(x^2+1)}dx=\int_0^1\frac{x\tan^{-1}x}{(x+1)(x^2+1)}dx+\int_1^{\infty}\frac{x\tan^{-1}x}{(x+1)(x^2+1)}dx$$

$$=I_1+I_2$$

$$2I_1=\int_0^1\left(\frac{1+x}{1+x^2}\right)\tan^{-1}x\,dx-\int_0^1\frac{\tan^{-1}x}{1+x}dx=\int_0^1\left(\frac{1+x}{1+x^2}\right)\tan^{-1}x\,dx+$$

$$+\int_0^1\frac{\log(1+x)}{1+x^2}dx-[\tan^{-1}x\log(1+x)]_0^1$$

$$2I\overset{\tan^{-1}x=y}{=}\frac{\pi^2}{32}-\frac{\pi}{4}\log 2+\int_0^{\frac{\pi}{4}}y\tan y\,dy+\int_0^{\frac{\pi}{4}}\log(1+\tan y)\,dy=$$

$$=\frac{\pi^2}{32}-\frac{\pi}{4}\log 2-[-y\log(\cos y)]_0^{\frac{\pi}{4}}+\int_0^{\frac{\pi}{4}}\log(\cos y)\,dy+\int_0^{\frac{\pi}{4}}\log\left(1+tan\left(\frac{\pi}{4}-y\right)\right)dy$$

$$=\frac{\pi^2}{32}-\frac{\pi}{4}\log 2-\frac{\pi}{8}\log 2+\int_0^{\frac{\pi}{4}}\log(1+e^{-2iy})\,dy$$

$$+\int_0^{\frac{\pi}{4}}(\log 2-\log(1+\tan y))dy=$$

$$=\frac{\pi^2}{32}-\frac{\pi}{4}\log 2+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\int_0^{\frac{\pi}{4}}e^{-2niy}\,dy=\frac{\pi^2}{32}-\frac{\pi}{4}\log 2+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\sin\left(\frac{n\pi}{2}\right)$$

$$I=\int_0^1\frac{x\tan^{-1}x}{(1+x)(1+x^2)}dx=\frac{\pi^2}{64}-\frac{\pi}{8}\log 2+\frac{G}{4}$$

$$I_2=\int_0^1\frac{x\tan^{-1}x}{(1+x)(1+x^2)}dx\overset{y=\frac{1}{x}}{=}\int_0^1\frac{\tan^{-1}\left(\frac{1}{y}\right)}{(1+y)(1+y^2)}dy$$

$$2I_2=\int_0^1\frac{\tan^{-1}\left(\frac{1}{y}\right)}{1+y}dy+\int_0^1\frac{(1-y)\tan^{-1}\left(\frac{1}{y}\right)}{1+y^2}dy=$$

$$=\left[\tan^{-1}\left(\frac{1}{y}\right)\log(1+y)\right]_0^1+\int_0^1\frac{\log(1+y)}{1+y^2}dy+\int_0^1\left(\frac{1-y}{1+y^2}\right)\tan^{-1}\left(\frac{1}{y}\right)dy\overset{\tan^{-1}\left(\frac{1}{y}\right)=x}{=}$$

$$=\frac{\pi}{4}\log 2+\int_0^{\frac{\pi}{4}}\log(1+\tan x)\,dx+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(x-\frac{x}{\tan x}\right)dx=$$

$$=\frac{\pi}{4}\log 2+\frac{\pi}{8}\log 2+\left[\frac{x^2}{2}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}-\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{x}{\tan x}dx=$$

$$=\frac{3\pi^2}{32}+\frac{3\pi}{8}\log 2-[x\log(\sin x)]_{\frac{\pi}{4}}^{\frac{\pi}{2}}+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\log(\sin x)\,dx=$$

$$=\frac{3\pi^2}{32}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\Rightarrow I_2=\frac{3\pi^2+16G}{64}$$

By adding I_1, I_2, we get:

$$\Omega=\int_0^{\infty}\frac{x\tan^{-1}x}{(x+1)(x^2+1)}dx=\frac{8G+\pi^2-\pi\log 4}{16}$$

Solution 3 by Daniel Immarube-Nigeria

$$\Omega=\int_0^{\infty}\frac{x\tan^{-1}x}{(x+1)(x^2+1)}dx=\int_0^{\infty}\frac{\tan^{-1}x}{1+x^2}dx-\int_0^{\infty}\frac{\tan^{-1}x}{(1+x)(1+x^2)}dx=\frac{\pi^2}{8}-I$$

$$I=\int_0^{\infty}\frac{\tan^{-1}x}{(1+x)(1+x^2)}dx=\int_0^{\frac{\pi}{2}}\frac{u}{1+\tan u}du\overset{IBP}{=}$$

$$=u\log(1+\tan u)|_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}\log(1+\tan u)\,du$$

$$I_0=\int_0^{\frac{\pi}{2}}\log(1+\tan u)\,du=\int_0^{\frac{\pi}{2}}\log(\cos u+\sin u)\cos u\,du=$$

$$=\int_0^{\frac{\pi}{2}}\log(\cos u)\,du+\int_0^{\frac{\pi}{2}}\log(\cos u+\sin u)du=$$

$$=\int_0^{\frac{\pi}{2}}\log\left(\sqrt{2}\,cos\left(\frac{\pi}{4}-x\right)\right)dx-\frac{\pi}{2}\log 2=$$

$$=\int_0^{\frac{\pi}{2}}\log\sqrt{2}\,dx+\int_0^{\frac{\pi}{2}}\log\cos\left(\frac{\pi}{4}-x\right)dx-\frac{\pi}{2}\log 2=$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}\log 2\,dx-2\int_0^{\frac{\pi}{4}}\log\cos x\,dx-\frac{\pi}{2}\log 2=$$

$$=\frac{\pi}{4}\log 2-2\left(\frac{C}{2}-\frac{\pi}{4}\log 2\right)-\frac{\pi}{2}\log 2=\frac{\pi}{4}\log 2-C$$

$$\Omega=\frac{\pi^2}{16}-\frac{\pi}{8}\log 2+\frac{C}{2}$$

Solution 4 by Togrul Ehmedov-Azerbaijan

$$\Omega=\int_0^{\infty}\frac{x\tan^{-1}x}{(x+1)(x^2+1)}dx=\int_0^1\frac{x\tan^{-1}x}{(x+1)(x^2+1)}dx+\int_1^{\infty}\frac{x\tan^{-1}x}{(x+1)(x^2+1)}dx=$$

$$=\int_0^1\frac{x\tan^{-1}x}{(x+1)(x^2+1)}dx+\int_0^1\frac{\tan^{-1}\left(\frac{1}{x}\right)}{(x+1)(x^2+1)}dx=$$

$$=\int_0^1\frac{x\tan^{-1}x}{(x+1)(x^2+1)}dx+\frac{\pi}{2}\int_0^1\frac{dx}{(x+1)(x^2+1)}-\int_0^1\frac{\tan^{-1}x}{(x+1)(x^2+1)}dx=$$

$$=\int_0^1\frac{(x-1)\tan^{-1}x}{(x+1)(x^2+1)}dx+\frac{\pi}{2}\int_0^1\frac{dx}{(x+1)(x^2+1)}=$$

$$=\int_0^1\frac{x\tan^{-1}x}{x^2+1}dx-\int_0^1\frac{\tan^{-1}x}{x+1}dx+\frac{\pi}{2}\int_0^1\frac{dx}{(x+1)(x^2+1)}=I_1-I_2+\frac{\pi}{2}I_3$$

$$I_1=\int_0^1\frac{x\tan^{-1}x}{x^2+1}dx=\frac{1}{2}G-\frac{\pi}{8}\log 2$$

$$I_2=\int_0^1\frac{\tan^{-1}x}{x+1}dx=\frac{\pi}{8}\log 2$$

$$I_3=\int_0^1\frac{dx}{(x+1)(x^2+1)}=\frac{1}{4}\log 2+\frac{\pi}{8}$$

$$\Omega=I_1-I_2+\frac{\pi}{2}I_3=\frac{1}{2}G-\frac{\pi}{8}\log 2+\frac{\pi^2}{16}$$

1879. Find:

$$\Omega=\int_0^1\int_0^1\frac{\log(1-x)\cdot\log(xy)}{1-xy}dxdy$$

Proposed by Togrul Ehmedov-Azerbaijan

Solution 1 by Rana Ranino-Setif-Algerie

$$\Omega=\int_0^1\int_0^1\frac{\log(1-x)\cdot\log(xy)}{1-xy}dxdy\overset{1-xy=t}{=}$$

$$= -\underbrace{\int_0^1 \frac{\log(1-x)\,Li_2(1-x)}{x}dx}_{x=1-x} - Li_2(1)\underbrace{\int_0^1 \frac{\log(1-x)}{x}dx}_{-Li_2(1)} =$$

$$= \frac{\pi^4}{36} + \int_0^1 \frac{\log x\,Li_2(x)}{1-x}dx\,;\text{Since}\,\frac{Li_2(x)}{1-x} = \sum_{n=1}^{\infty} H_n^{(2)} x^n$$

$$\int_0^1 \frac{\log x\,Li_2(x)}{1-x}dx = \sum_{n=1}^{\infty} H_n^{(2)} \int_0^1 x^n \log x\,dx = -\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{(n+1)^2} = -\sum_{n=2}^{\infty} \frac{H_{n-1}^2}{n^2} =$$

$$= \sum_{n=2}^{\infty} \frac{1}{n^4} - \sum_{n=2}^{\infty} \frac{H_n^{(2)}}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^4} - \sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^2} = \zeta(4) - \frac{7}{4}\zeta(4) = -\frac{3}{4}\zeta(4)$$

Therefore,

$$\Omega = \int_0^1 \int_0^1 \frac{\log(1-x)\cdot\log(xy)}{1-xy}dxdy = \frac{7}{4}\zeta(4)$$

Solution 2 by proposer

$$\Omega = \int_0^1 \int_0^1 \frac{\log(1-x)\cdot\log(xy)}{1-xy}dxdy \overset{xy=z}{=} \int_0^1 \frac{\log(1-x)}{x}\int_0^x \frac{\log z}{1-z}dz\,dx =$$

$$= -Li_2(x)\int_0^x \frac{\log z}{1-z}dz\Big|_0^1 + \int_0^1 \frac{Li_2(x)\log x}{1-x}dx =$$

$$= -\zeta(2)\int_0^1 \frac{\log z}{1-z}dz + \int_0^1 \frac{Li_2(x)\log x}{1-x}dx = -\zeta(2)I_1 + I_2$$

$$I_1 = \int_0^1 \frac{\log z}{1-z}dz = -\zeta(2)$$

$$I_2 = \int_0^1 \frac{Li_2(x)\log x}{1-x}dx = -\frac{3}{4}\zeta(4)$$

Therefore,

$$\Omega = \int_0^1 \int_0^1 \frac{\log(1-x)\cdot\log(xy)}{1-xy}dxdy = \frac{7}{4}\zeta(4)$$

Solution 3 by Syed Shahabudeen-Kerala-India

$$\Omega = \int_0^1 \int_0^1 \frac{\log(1-x)\cdot\log(xy)}{1-xy}dxdy =$$

$$=\int_0^1\int_0^1\frac{\log(1-x)\cdot\log(x)}{1-xy}dxdy+\int_0^1\int_0^1\frac{\log(1-x)\cdot\log(y)}{1-xy}dxdy=\Omega_1+\Omega_2$$

$$\Omega_1=\sum_{k=0}^{\infty}\int_0^1 y^k\,dy\int_0^1\log(1-x)\log x\,x^k\,dx=$$

$$=-\sum_{k=0}^{\infty}\frac{1}{k+1}\frac{\partial}{\partial a}\left(\frac{H_{a+k+1}}{a+k+1}\right)_{a=0}=-\sum_{k=0}^{\infty}\frac{1}{k+1}\left(\frac{\zeta(2)}{k+1}-\frac{H_{k+1}^{(2)}}{k+1}-\frac{H_{k+1}}{(k+1)^2}\right)=$$

$$=-\zeta^2(2)+\frac{7}{4}\zeta(4)+\frac{5}{2}\zeta(4)-\frac{1}{2}\zeta^2(2)$$

$$\Omega_2=\sum_{k=0}^{\infty}\int_0^1\log y\,y^k dy\int_0^1\log(1-x)\,x^k dx=\sum_{k=0}^{\infty}\frac{H_{k+1}}{(k+1)^3}=\frac{5}{2}\zeta(4)-\frac{1}{2}\zeta^2(2)$$

Therefore,

$$\Omega=\Omega_1+\Omega_2=\frac{7}{4}\zeta(4)$$

Solution 4 by Benjamin Bamidele-Nigeria

$$\Omega=\int_0^1\int_0^1\frac{\log(1-x)\cdot\log(xy)}{1-xy}dxdy=$$

$$=\sum_{k=1}^{\infty}\int_0^1\int_0^1(xy)^{k-1}\log(1-x)\log(xy)\,dydx\overset{IBP}{=}$$

$$=\sum_{k=1}^{\infty}\frac{1}{k}\left(\int_0^1 x^{k-1}\log(1-x)\left(\log x-\frac{1}{k}\right)\right)dx=$$

$$=\sum_{k=1}^{\infty}\frac{1}{k}\int_0^1 x^{k-1}\log(1-x)\log x\,dx+\sum_{k=1}^{\infty}\frac{H_k}{k^3}$$

$$\because\int_0^1 x^{k-1}\log(1-x)\,dx=-\frac{H_k}{k}$$

$$\Omega\overset{IBP}{=}2\sum_{k=1}^{\infty}\frac{H_k}{k^3}+\sum_{k=1}^{\infty}\frac{1}{k^2}\int_0^1\frac{x^k\log x}{1-x}dx$$

$$\because\sum_{k=1}^{\infty}\frac{H_k}{k^n}=(n+2)\zeta(k+1)-\sum_{k=1}^{n-2}\zeta(n-k)\zeta(k+1),n\geq 2$$

$$\Omega=\frac{5}{2}\zeta(4)+\int_0^1\frac{Li_2(x)\log x}{1-x}dx\overset{let}{=}\frac{5}{2}\zeta(4)+\Upsilon$$

$$\Upsilon=\int_0^1\frac{Li_2(x)\log x}{1-x}dx\overset{IBP}{=}$$

$$=\int_0^1\frac{\log(1-x)\,Li_2(x)}{x}dx-\int_0^1\frac{\log^2(1-x)\log x}{x}dx$$

$$\text{Let } A=\int_0^1\frac{\log(1-x)\,Li_2(x)}{x}dx=-Li_2^2(1)-A\Rightarrow 2A=Li_2^2(1)$$

$$A=-\frac{1}{2}Li_2^2(1)=-\frac{5}{4}\zeta(4)$$

$$\because\sum_{k=1}^{\infty}\frac{H_k}{k+1}x^{k+1}=\frac{1}{2}\log^2(1-x)$$

$$\text{Let } B=\int_0^1\frac{\log^2(1-x)\log x}{x}dx=2\sum_{k=1}^{\infty}\frac{H_k}{k+1}\int_0^1 x^k\log x\,dx=-2\sum_{k=1}^{\infty}\frac{H_k}{(k+1)^3}$$

$$B=-2\sum_{k=1}^{\infty}\frac{H_{k+1}-(k+1)^{-1}}{(k+1)^3}=-2\sum_{k=1}^{\infty}\frac{H_k}{k^3}+2\sum_{k=1}^{\infty}\frac{1}{k^4}=$$

$$=-\frac{5}{2}\zeta(4)+2\zeta(4)=-\frac{1}{2}\zeta(4)$$

$$\Upsilon=A-B=-\frac{5}{4}\zeta(4)+\frac{1}{2}\zeta(4)=-\frac{3}{4}\zeta(4)$$

$$\Omega=\frac{5}{2}\zeta(4)+\Upsilon=\frac{5}{2}\zeta(4)-\frac{3}{4}\zeta(4)=\frac{7}{4}\zeta(4)$$

1880. Find:

$$\Omega=\int_1^{\infty}\frac{\log x}{x^4+x^2+1}dx$$

Proposed by Vasile Mircea Popa-Romania

Solution 1 by Rana Ranino-Setif-Algerie

$$\Omega=\int_1^{\infty}\frac{\log x}{x^4+x^2+1}dx\overset{x=\frac{1}{x}}{=}-\int_0^1\frac{x^2\log x}{1+x^2+x^4}dx=\int_0^1\frac{(x^4-x^2)\log x}{1-x^6}dx=$$

$$\overset{t=x^6}{=}\frac{1}{36}\int_0^1\frac{\left(t^{\frac{5}{6}-1}-t^{\frac{1}{2}-1}\right)\log t}{1-t}dt=\frac{1}{36}\left\{\psi^{(1)}\left(\frac{1}{2}\right)-\psi^{(1)}\left(\frac{5}{6}\right)\right\}=\frac{\pi^2}{72}-\frac{1}{36}\psi^{(1)}\left(\frac{5}{6}\right)$$

Therefore,

$$\Omega=\int_1^{\infty}\frac{\log x}{x^4+x^2+1}dx=\frac{\pi^2}{72}-\frac{1}{36}\psi^{(1)}\left(\frac{5}{6}\right)$$

Solution 2 by Ankush Kumar Parcha-India

$$\Omega=\int_1^{\infty}\frac{\log x}{x^4+x^2+1}dx\overset{x=\frac{1}{y}}{=}\int_0^1\frac{y^2(y^2-1)\log y}{(1-y^2)(y^4+y^2+1)}dy=$$

$$=\int_0^1\frac{y^4\log y}{1-y^6}dy-\int_0^1\frac{y^2\log y}{1-y^6}dy=\sum_{n=0}^{\infty}\int_0^1 y^{6n+4}\log y\,dy-\sum_{n=0}^{\infty}\int_0^1 y^{6n+2}\log y\,dy=$$

$$=\frac{1}{36}\sum_{n=0}^{\infty}\frac{1}{\left(n+\frac{1}{2}\right)^2}-\frac{1}{36}\sum_{n=0}^{\infty}\frac{1}{\left(n+\frac{5}{6}\right)^2}$$

$$\because\int_0^1 x^m\log^n x\,dx=\frac{(-1)^n n!}{(m+1)^{n+1}},m\neq-1,n>-1\text{ and }\sum_{n=0}^{\infty}(n+a)^{-2}=\psi^{(1)}(a)$$

Therefore,

$$\Omega=\int_1^{\infty}\frac{\log x}{x^4+x^2+1}dx=\frac{1}{36}\left\{\psi^{(1)}\left(\frac{1}{2}\right)-\psi^{(1)}\left(\frac{5}{6}\right)\right\}=\frac{\pi^2}{72}-\frac{1}{36}\psi^{(1)}\left(\frac{5}{6}\right)$$

Solution 3 by Daniel Immarube-Nigeria

$$\Omega=\int_1^{\infty}\frac{\log x}{x^4+x^2+1}dx=\int_0^{\infty}\frac{\log x}{x^4+x^2+1}dx-\int_0^1\frac{\log x}{x^4+x^2+1}dx=$$

$$=\int_0^{\infty}\frac{(1-x^2)\log x}{1-x^6}dx-\int_0^1\frac{(1-x^2)\log x}{1-x^6}dx-\Omega_1=$$

$$=\int_1^{\infty}\frac{\log x}{1-x^6}dx-\int_0^{\infty}\frac{x^2\log x}{1-x^6}dx-\Omega_1=$$

$$=\frac{d}{da}\Gamma\left(\frac{a}{6}+\frac{1}{6}\right)\Gamma\left(\frac{5}{6}-\frac{a}{6}\right)-\frac{1}{18}\psi^{(1)}\left(\frac{1}{2}\right)-\Omega_1=-\frac{\pi^2}{9}+\frac{\pi^2}{36}+\Omega_1=-\frac{\pi^2}{12}$$

$$\Omega_1=\int_0^1\frac{\log x}{1-x^6}dx-\int_0^1\frac{x^2\log x}{1-x^6}dx=$$

$$=-\frac{1}{6}\int_0^1\log\left(u^{\frac{1}{6}}\right)\left(u^{-\frac{5}{6}}\right)(1-u)^{-1}du+\frac{1}{6}\int\log\left(v^{\frac{1}{6}}\right)\left(v^{-\frac{1}{2}}\right)(1-v)^{-1}dv=$$

$$=-\frac{1}{36}\psi^{(1)}\left(\frac{1}{6}\right)+\frac{1}{36}\psi^{(1)}\left(\frac{1}{2}\right)=-\frac{\pi^2}{12}-\frac{\pi^2}{72}+\frac{\psi^{(1)}\left(\frac{1}{6}\right)}{36}=-\frac{7\pi^2}{72}+\frac{\psi^{(1)}\left(\frac{1}{6}\right)}{36}$$

1881. Let $f:[0,\infty)\to[0,\infty)$ continuous function, strictly increasing with

$$f(0)>0.$$

If $m\ge 0, u,v>0, a\ge 0, b\in Imf$ then prove:

$$\frac{1}{u^m}\left(\int_0^a f(x)dx\right)^{m+1}+\frac{1}{v^m}\left(\int_0^b f^{-1}(y)dy\right)^{m+1}\ge\frac{(ab)^{m+1}}{(u+v)^m}$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

Solution by Adrian Popa-Romania

Lemma (Young's identity): If $f:[a,b]\to[c,d]$ is bijective and increasing function, then:

$$\int_a^b f(x)dx+\int_c^d f^{-1}(y)\,dy=bd-ac$$

We have:

$$\frac{1}{u^m}\left(\int_0^a f(x)dx\right)^{m+1}+\frac{1}{v^m}\left(\int_0^b f^{-1}(y)dy\right)^{m+1}\overset{Radon}{\ge}$$

$$\ge\frac{1}{(u+v)^m}\left(\int_0^a f(x)dx+\int_0^b f^{-1}(y)dy\right)^{m+1}\overset{Young}{=}\frac{(a\cdot b-0\cdot 0)^{m+1}}{(u+v)^m}=\frac{(ab)^{m+1}}{(u+v)^m}$$

1882. If $z\ge 0$ and $f''(z)\ge 0$ then:

$$\int_0^z\big(f(2t)-f(t)\big)dt\le\frac{z\big(f(2z)-f(z)\big)}{2}$$

Proposed by Hikmat Mammadov-Azerbaijan

Solution by Aggeliki Papaspyropoulou-Greece

$$I(z)=\int_0^z f(2t)dt\overset{2z\to u}{=}\frac{1}{2}\int_0^{2z}f(t)dt;(1)$$

$$I'(z)=\frac{1}{2}(2z)'f(2z)=f(2z);(2)$$

$$\text{Let: }H(z)=\frac{z}{2}\big(f(2z)-f(z)\big)-\int_0^z f(2t)dt+\int_0^z f(t)dt\overset{(?)}{\ge}0;(*)$$

$H(0)=0$. So, we have to prove: $H'(z)\ge 0;(3)$

$$H'(z)=\frac{1}{2}\big(f(2z)-f(z)\big)+\frac{z}{2}\cdot 2f'(2z)-\frac{z}{2}f'(z)-f(2z)+f(z)$$

$$2H'(z)=f(2z)-f(z)+2zf'(z)-zf'(z)-2f(2z)+zf(z)$$

$$2H'(z)=[2zf'(2z)-f(2z)]-[zf'(z)-f(z)]\overset{(?)}{\geq}0\Leftrightarrow$$

$$2zf'(z)-f(2z)\geq zf'(z)-f(z);(**)$$

$$\text{Let } G(z)=zf'(z)-f(z)\Rightarrow G'(z)=zf''(z)+f'(z)-f'(z)=f''(z)>0\Rightarrow$$

$$G-\nearrow\Rightarrow G(2z)\geq G(z);(z>0)\Rightarrow 2zf'(2z)-f(2z)\geq zf'(z)-f(z)\Rightarrow(**)\text{ true}$$

$$\Rightarrow H'(z)\geq 0\Rightarrow(*)\text{ true}\Rightarrow H(z)\geq 0\text{ true.}$$

1883. Prove that:

$$\lim_{n\to\infty}\left(n\zeta(q+1)-\sum_{k=1}^{n-1}\left(\frac{n-k}{k^{q+1}}\right)\right)=\zeta(q)$$

where $\zeta(q)$ is zeta function.

Proposed by Syed Shahabudeen-Kerala-India

Solution by Togrul Ehmedov-Baku-Azerbaijan

$$\Omega=\lim_{n\to\infty}\left(n\zeta(q+1)-\sum_{k=1}^{n-1}\left(\frac{n-k}{k^{q+1}}\right)\right)=\lim_{n\to\infty}\left(n\zeta(q+1)-\left(\sum_{k=1}^{n}\frac{n-k}{k^{q+1}}-\left[\frac{n-k}{k^{q+1}}\right]\right)\right)=$$

$$=\lim_{n\to\infty}\left(n\zeta(q+1)-n\sum_{k=1}^{n}\frac{1}{k^{q+1}}+\sum_{k=1}^{n}\frac{1}{k^{q}}\right)=$$

$$=\lim_{n\to\infty}(n\zeta(q+1)-n\zeta(q+1)+\zeta(q))=\zeta(q)$$

1884. Prove the summation

$$\sum_{n=1}^{\infty}\frac{\left(\sqrt{1}\sin(\pi n)+\sqrt{3}\sin\left(\frac{\pi n}{3}\right)+\sqrt{5}\sin\left(\frac{\pi n}{5}\right)\right)^2}{n^4}=$$

$$=\frac{4\pi^4(29\sqrt{15}+4)}{10125}+\frac{2\pi^4}{45}$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Rana Ranino-Setif-Algerie

$$\Omega=\sum_{n=1}^{\infty}\frac{\left(\sqrt{1}\sin(\pi n)+\sqrt{3}\sin\left(\frac{\pi n}{3}\right)+\sqrt{5}\sin\left(\frac{\pi n}{5}\right)\right)^2}{n^4}=$$

$$=\sum_{n=1}^{\infty}\frac{\left(\sqrt{3}\sin\left(\frac{n\pi}{3}\right)+\sqrt{5}\sin\left(\frac{n\pi}{5}\right)\right)^2}{n^4}=$$

$$=\sum_{n=1}^{\infty}\frac{3\sin^2\left(\frac{n\pi}{3}\right)+5\sin^2\left(\frac{n\pi}{5}\right)+2\sqrt{15}\sin\left(\frac{n\pi}{3}\right)\sin\left(\frac{n\pi}{5}\right)}{n^4}=$$

$$=4\sum_{n=1}^{\infty}\frac{1}{n^4}+\sum_{n=1}^{\infty}\frac{-\frac{3}{2}\cos\left(\frac{2n\pi}{3}\right)-\frac{5}{2}\cos\left(\frac{2n\pi}{5}\right)+\sqrt{15}\cos\left(\frac{2n\pi}{15}\right)-\sqrt{15}\cos\left(\frac{8n\pi}{15}\right)}{n^4}=$$

$$=\frac{2\pi^4}{45}-\frac{3}{2}\sum_{n=1}^{\infty}\frac{\cos\left(\frac{2n\pi}{3}\right)}{n^4}-\frac{5}{2}\sum_{n=1}^{\infty}\frac{\cos\left(\frac{2n\pi}{5}\right)}{n^4}+\sqrt{15}\sum_{n=1}^{\infty}\frac{\cos\left(\frac{2n\pi}{15}\right)}{n^4}-\sqrt{15}\sum_{n=1}^{\infty}\frac{\cos\left(\frac{8n\pi}{15}\right)}{n^4}$$

Using Fourier series:

$$\sum_{n=1}^{\infty}\frac{\sin(2nx)}{n}=\frac{\pi}{2}-x,x\in(0,\pi)$$

Integrating both sides gives:

$$-\frac{1}{2}\sum_{n=1}^{\infty}\frac{\cos(2nx)}{n^2}=\frac{\pi x}{2}-\frac{x^2}{2}+C_1$$

$$\text{when } x=0: C_1=-\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}=-\frac{\pi^2}{12},\because\sum_{n=1}^{\infty}\frac{\cos(2nx)}{n^2}=x^2-\pi x+\frac{\pi^2}{6}$$

Integrating both sides gives:

$$\frac{1}{2}\sum_{n=1}^{\infty}\frac{\sin(2nx)}{n^3}=\frac{x^3}{3}-\frac{\pi x^2}{2}+\frac{\pi^2 x}{6}+C_2,(C_2=0)$$

$$\sum_{n=1}^{\infty}\frac{\sin(2nx)}{n^3}=\frac{2x^3}{3}-\pi x^2+\frac{\pi^2 x}{3}$$

Integrating both sides gives:

$$-\frac{1}{2}\sum_{n=1}^{\infty}\frac{\cos(2nx)}{n^4}=\frac{x^4}{6}-\frac{\pi x^3}{3}+\frac{\pi^2 x^2}{6}+C_3$$

$$\text{when } x=0: C_3=-\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^4}=-\frac{\pi^4}{180},$$

$$\because f(x)=\sum_{n=1}^{\infty}\frac{\cos(2nx)}{n^4}=-\frac{x^4}{3}+\frac{2\pi x^3}{3}-\frac{\pi^2 x^2}{3}+\frac{\pi^4}{90}$$

$$f\left(\frac{\pi}{3}\right)=-\frac{13\pi^4}{2430},f\left(\frac{\pi}{5}\right)=\frac{29\pi^4}{11250},f\left(\frac{\pi}{15}\right)=\frac{2983\pi^4}{303750},f\left(\frac{4\pi}{15}\right)=-\frac{497\pi^4}{303750}$$

$$\Omega=\frac{2\pi^4}{45}-\frac{3}{2}f\left(\frac{\pi}{3}\right)-\frac{5}{2}f\left(\frac{\pi}{5}\right)+\sqrt{15}f\left(\frac{\pi}{15}\right)-\sqrt{15}f\left(\frac{4\pi}{15}\right)$$

Therefore,

$$\sum_{n=1}^{\infty}\frac{\left(\sqrt{1}\sin(\pi n)+\sqrt{3}\sin\left(\frac{\pi n}{3}\right)+\sqrt{5}\sin\left(\frac{\pi n}{5}\right)\right)^2}{n^4}=\frac{4\pi^4(29\sqrt{15}+4)}{10125}+\frac{2\pi^4}{45}$$

1885. Find:

$$\Omega(m)=\lim_{n\to\infty}\left(\left(\sqrt[n+1]{(n+1)!}\right)^{m+1}-\left(\sqrt[n]{n!}\right)^{m+1}\right)\cdot\tan^m\frac{\pi}{n}$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

Solution by Marian Ursărescu-Romania

$$\Omega(m)=\lim_{n\to\infty}\left(\left(\sqrt[n+1]{(n+1)!}\right)^{m+1}-\left(\sqrt[n]{n!}\right)^{m+1}\right)\cdot\tan^m\frac{\pi}{n}=$$

$$=\lim_{n\to\infty}\left(\sqrt[n]{n!}\right)^{m+1}\left(\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^{m+1}-1\right)\cdot\tan^m\frac{\pi}{n};(1)$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{n!}{n^n}}\overset{C-D}{=}\lim_{n\to\infty}\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!}=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n=\frac{1}{e};(2)$$

$$\lim_{n\to\infty}\left(\sqrt[n]{n!}\cdot\tan\frac{\pi}{n}\right)^m=\lim_{n\to\infty}\left(\frac{\sqrt[n]{n!}}{n}\cdot\frac{\tan\frac{\pi}{n}}{\frac{\pi}{n}}\cdot\pi\right)^m=\left(\frac{1}{e}\cdot1\cdot\pi\right)^m=\frac{\pi^m}{e^m};(3)$$

$$\lim_{n\to\infty}\left(\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^{m+1}-1\right)=\lim_{n\to\infty}\frac{e^{\log\left(\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^{m+1}\right)}-1}{\log\left(\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^{m+1}\right)}\cdot\log\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^{m+1}$$

$$=\lim_{n\to\infty}n(m+1)\cdot\log\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)$$

$$=(m+1)\cdot\lim_{n\to\infty}\log\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^n=$$

$$=(m+1)\cdot\lim_{n\to\infty}\log\left(\frac{\sqrt[n+1]{(n+1)!}^{\,n}}{n!}\right)=(m+1)\cdot\lim_{n\to\infty}\log\left(\frac{(n+1)!}{n!}\cdot\frac{1}{\sqrt[n+1]{(n+1)!}}\right)=$$

$$= (m+1)\cdot\lim_{n\to\infty}\log\left(\frac{n+1}{\sqrt[n+1]{(n+1)!}}\right) = (m+1)\cdot\log e = m+1; (4)$$

From (1),(2),(3) and (4), we get:

$$\Omega(m) = \lim_{n\to\infty}\left(\left(\sqrt[n+1]{(n+1)!}\right)^{m+1} - \left(\sqrt[n]{n!}\right)^{m+1}\right)\cdot\tan^m\frac{\pi}{n} = \frac{(m+1)\pi^m}{e^{m+1}}$$

1886. Prove that:

$$\lim_{k\to\infty}\sum_{q=0}^{k}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{(m+n+2)^{q+3}} = \frac{\pi^2}{6} - 1$$

Proposed by Syed Shahabudeen-Kerala-India

Solution by proposer

$$\Omega = \lim_{n\to\infty}\sum_{q=0}^{k}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{(m+n+2)^{q+3}} = \lim_{k\to\infty}\underbrace{\sum_{q=2}^{k+2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{(m+n+2)^{q+3}}}_{S}$$

$$S = \sum_{m=0}^{\infty}\left(\frac{1}{(m+2)^{q+1}} + \frac{1}{(m+3)^{q+1}} + \cdots\right) = \frac{1}{2^{q+1}} + \frac{2}{3^{q+1}} + \frac{3}{4^{q+1}} + \cdots =$$

$$= \sum_{p=2}^{\infty}\frac{p-1}{p^{q+1}} = \zeta(q) - \zeta(q+1) \Rightarrow S = \zeta(q) - \zeta(q+1)$$

Therefore,

$$\Omega = \lim_{k\to\infty}\sum_{q=2}^{k+2}\big(\zeta(q) - \zeta(q+1)\big) = \lim_{k\to\infty}\big(\zeta(2) - \zeta(k+3)\big) = \zeta(2) - 1$$

$$\Omega = \frac{\pi^2}{6} - 1$$

A generalization of the above problem for $s \geq 2$:

$$\lim_{k\to\infty}\sum_{q=0}^{k}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{(m+n+2)^{q+s+1}} = \zeta(s) - 1$$

1887. For $0 < b < a$ find Ω such that

$$e^{\Omega} = \sum_{k=-\infty}^{\infty} \left(\frac{a}{b}\right)^{|k|} \sum_{n=0}^{\infty} \frac{(ix)^n k^n}{n!}, x \in \mathbb{R}$$

Proposed by Tobi Joshua-Nigeria

Solution 1 by Adrian Popa-Romania

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow \sum_{n=0}^{\infty} \frac{(ixk)^n}{n!} = e^{inx}$$

$$\sum_{k=-\infty}^{\infty} \left(\frac{a}{b}\right)^{|k|} \sum_{n=0}^{\infty} \frac{(ix)^n k^n}{n!} = \sum_{k=-\infty}^{\infty} \left(\frac{a}{b}\right)^{|k|} e^{inx} = \sum_{k=-\infty}^{\infty} \left(\frac{a}{b}\right)^{|k|} (\cos(kx) + i\sin(kx)) =$$

$$= \sum_{k=-\infty}^{\infty} \left(\frac{a}{b}\right)^{|k|} \cos(kx) = 2\sum_{k=-\infty}^{\infty} \left(\frac{a}{b}\right)^{k} \cos(kx) + 1$$

$$S_1 = \sum_{k=-\infty}^{\infty} \left(\frac{a}{b}\right)^{k} \cos(kx), S_2 = \sum_{k=-\infty}^{\infty} \left(\frac{a}{b}\right)^{k} \sin(kx)$$

$$S_1 + iS_2 = \sum_{k=1}^{\infty} \left(\frac{b}{a}(\cos x + i\sin x)\right)^k = \sum_{k=1}^{\infty} \left(\frac{b}{a}e^{ix}\right)^k = \frac{b}{a}e^{ix} \cdot \frac{1}{1 - \frac{b}{a}e^{ix}} = \frac{\frac{b}{a}e^{ix}}{1 - \frac{b}{a}e^{ix}} =$$

$$= \frac{\frac{b}{a}(\cos x + i\sin x)}{1 - \frac{b}{a}(\cos x + i\sin x)} = \frac{\frac{b}{a}\cos x + i\frac{b}{a}\sin x}{1 - \frac{b}{a}\cos x - i\frac{b}{a}\sin x} =$$

$$= \frac{\left(\frac{b}{a}\cos x + i\frac{b}{a}\sin x\right)\left(1 - \frac{b}{a}\cos x + i\frac{b}{a}\sin x\right)}{\left(1 - \frac{b}{a}\cos x\right)^2 + \frac{b^2}{a^2}\sin^2 x} = \frac{\frac{b}{a}\cos x - \frac{b^2}{a^2} - i\frac{b}{a}\sin x}{1 - 2\frac{b}{a}\cos x + \frac{b^2}{a^2}}$$

$$S_1 = \frac{\frac{b}{a}\cos x - \frac{b^2}{a^2}}{1 - 2\frac{b}{a}\cos x + \frac{b^2}{a^2}}$$

$$e^{\Omega} = \frac{2\frac{b}{a}\cos x - 2\frac{b^2}{a^2}}{1 - 2\frac{b}{a}\cos x + \frac{b^2}{a^2}} + 1 = \frac{2\frac{b}{a}\cos x - 2\frac{b^2}{a^2} + 1 - 2\frac{b}{a}\cos x + \frac{b}{a}}{1 - 2\frac{b}{a}\cos x + \frac{b^2}{a^2}} =$$

$$=\frac{1-\frac{b^2}{a^2}}{1-2\frac{b}{a}\cos x+\frac{b^2}{a^2}}\Rightarrow\Omega=\log\left(\frac{1-\frac{b^2}{a^2}}{1-2\frac{b}{a}\cos x+\frac{b^2}{a^2}}\right)$$

Solution 2 by Akerele Olofin-Nigeria

$$e^{\Omega}=\sum_{k=-\infty}^{\infty}\left(\frac{a}{b}\right)^{|k|}\sum_{n=0}^{\infty}\frac{(ix)^n k^n}{n!},x\in\mathbb{R},\qquad S=\sum_{n=0}^{\infty}\frac{(ix)^n k^n}{n!}=\sum_{n=0}^{\infty}\frac{(ikx)^n}{n!}=e^{ikx}$$

$$\Rightarrow e^{\Omega}=\sum_{k=-\infty}^{\infty}\left(\frac{b}{a}\right)^{|k|}e^{ikx}=\sum_{k=-\infty}^{\infty}\left(\frac{b}{a}\right)^{|k|}(\cos(kx)+i\sin(kx))=$$

$$=\sum_{k=-\infty}^{\infty}\left(\frac{b}{a}\right)^{|k|}\cos(kx)+\sum_{k=-\infty}^{\infty}\left(\frac{b}{a}\right)^{|k|}\sin(kx)$$

Recall $\cos(-x)=\cos x$ and $\sin(-x)=-\sin x$, we get:

$$e^{\Omega}=\sum_{k=-\infty}^{\infty}\left(\frac{b}{a}\right)^{|k|}\cos(kx)=2\sum_{k=1}^{\infty}\left(\frac{b}{a}\right)^{k}\cos(kx)+1$$

$$A=\sum_{k=1}^{\infty}\left(\frac{b}{a}\right)^{k}\cos(kx)=\Re\left(\sum_{k=1}^{\infty}\left(\frac{b}{a}\right)^{k}e^{ikx}\right)=\Re\left(\sum_{k=1}^{\infty}\left(\frac{be^{ix}}{a}\right)^{k}\right)=$$

$$=\Re\left(\frac{be^{ix}}{a-be^{ix}}\right)=\Re\left(\frac{b\cos x+ib\sin x}{(a-b\cos x)-i(b\sin x)}\right)=\frac{ab\cos x-b^2}{a^2+b^2-2ab\cos x}$$

$$e^{\Omega}=2\frac{ab\cos x-b^2}{a^2+b^2-2ab\cos x}+1=\frac{a^2+b^2-2b^2}{a^2+b^2-2ab\cos x}=\frac{a^2-b^2}{a^2+b^2-2ab\cos x}$$

$$\Omega=\log(a^2-b^2)-\log(a^2+b^2-2ab\cos x)$$

1888.

If $(x_n)_{n\geq 1}, x_n=\sum_{k=1}^{n}\tan^{-1}\left(\frac{1}{k^2-k+1}\right)$, then find:

$$\Omega=\lim_{n\to\infty}\left(\frac{\pi^2}{4}-x_n^2\right)\sqrt[n]{(2n-1)!!}$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

Solution by Marian Ursărescu-Romania

$$\Omega=\lim_{n\to\infty}\left(\frac{\pi^2}{4}-x_n^2\right)\sqrt[n]{(2n-1)!!}=\lim_{n\to\infty}\left(\frac{\pi}{2}+x_n\right)\left(\frac{\pi}{2}-x_n\right)n\cdot\frac{\sqrt[n]{(2n-1)!!}}{n};(1)$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{(2n-1)!!}{n^n}}\overset{C-D}{=}\lim_{n\to\infty}\frac{(2n+1)!!}{(n+1)^{n+1}}\cdot\frac{n^n}{(2n-1)!!}=$$

$$=\lim_{n\to\infty}\frac{2n+1}{n+1}\cdot\left(\frac{n}{n+1}\right)^n=\frac{2}{e};(2)$$

$$x_n=\tan^{-1}1+\sum_{k=1}^{n}\tan^{-1}\left(\frac{1}{1+k(k-1)}\right)=\frac{\pi}{4}+\sum_{k=1}^{n}\tan^{-1}\left(\frac{\frac{1}{k-1}-\frac{1}{k}}{1-\frac{1}{(k-1)k}}\right)=$$

$$=\frac{\pi}{4}+\sum_{k=2}^{n}\left(\tan^{-1}\left(\frac{1}{k-1}\right)-\tan^{-1}\left(\frac{1}{k}\right)\right)=\frac{\pi}{4}+\tan^{-1}1-\tan^{-1}\frac{1}{n}=\frac{\pi}{2}-\tan^{-1}\frac{1}{n}$$

$$\Rightarrow\lim_{n\to\infty}x_n=\frac{\pi}{2}\Rightarrow\lim_{n\to\infty}\left(\frac{\pi}{2}+x_n\right)=\pi;(3)$$

$$\lim_{n\to\infty}\left(\frac{\pi}{2}-x_n\right)n=\lim_{n\to\infty}\frac{\frac{\pi}{2}-x_n}{\frac{1}{n}}=\lim_{n\to\infty}\frac{\frac{\pi}{2}-x_{n+1}-\frac{\pi}{2}+x_n}{\frac{1}{n+1}-\frac{1}{n}}=$$

$$=\lim_{n\to\infty}\frac{-\frac{\pi}{2}-\tan^{-1}\frac{1}{n+1}+\frac{\pi}{2}-\tan^{-1}\frac{1}{n}}{\frac{n-n-1}{n(n+1)}}=\lim_{n\to\infty}\frac{-\tan^{-1}\frac{1}{n+1}+\tan^{-1}\frac{1}{n}}{\frac{1}{n(n+1)}}=$$

$$=\lim_{x\to\infty}\frac{-\tan^{-1}\frac{1}{x+1}+\tan^{-1}\frac{1}{x}}{\frac{1}{x(x+1)}}\overset{L'H}{=}\lim_{n\to\infty}\frac{\frac{-1}{1+\frac{1}{(x+1)^2}}\cdot\frac{-1}{(x+1)^2}+\frac{1}{1+\frac{1}{x^2}}\cdot\frac{-1}{x^2}}{-\frac{2x+1}{x^2(x+1)^2}}=$$

$$=\lim_{x\to\infty}\frac{\frac{1}{(x+1)^2+1}-\frac{1}{x^2+1}}{-\frac{2x+1}{x^2(x+1)^2}}=\lim_{x\to\infty}\frac{-(2x+1)}{((x+1)^2+1)(x^2+1)}\cdot\frac{x^2(x+1)^2}{-(2x+1)}=1;(4$$

From (1),(2),(3) and (4), it follows that:

$$\Omega=\lim_{n\to\infty}\left(\frac{\pi^2}{4}-x_n^2\right)\sqrt[n]{(2n-1)!!}=\pi\cdot\frac{2}{e}\cdot 1=\frac{2\pi}{e}$$

1889. Find:

$$\Omega = \lim_{n\to\infty} \frac{(n+1)\cdot\sqrt[n+1]{(2n+1)!!} - n\cdot\sqrt[n]{(2n-1)!!}}{\sqrt[n]{n!}}$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

Solution by Marian Ursărescu-Romania

$$\Omega = \lim_{n\to\infty} \frac{(n+1)\cdot\sqrt[n+1]{(2n+1)!!} - n\cdot\sqrt[n]{(2n-1)!!}}{\sqrt[n]{n!}} =$$

$$= \lim_{n\to\infty} \frac{\sqrt[n]{(2n-1)!!}}{\sqrt[n]{n!}}\cdot n\left[\frac{n+1}{n}\cdot\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}} - 1\right]; (1)$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!}}{n} = \lim_{n\to\infty}\sqrt[n]{\frac{(2n-1)!!}{n^n}} \overset{C-D'A}{=} \lim_{n\to\infty}\frac{(2n+1)!!}{(n+1)^{n+1}}\cdot\frac{n^n}{(2n-1)!!} =$$

$$= \lim_{n\to\infty}\frac{2n+1}{n+1}\cdot\left(\frac{n}{n+1}\right)^n = \frac{2}{e}; (2)$$

$$\lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}} = \lim_{n\to\infty}\sqrt[n]{\frac{n^n}{n!}} \overset{C-D'A}{=} \lim_{n\to\infty}\frac{(n+1)^{n+1}}{(n+1)!}\cdot\frac{n!}{n^n} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = e; (3)$$

$$\lim_{n\to\infty} n\left[\frac{n+1}{n}\cdot\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}} - 1\right] = \lim_{n\to\infty} n\left[e^{\log\left(\frac{n+1}{n}\cdot\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}}\right)} - 1\right] =$$

$$= \lim_{n\to\infty}\frac{\left[e^{\log\left(\frac{n+1}{n}\cdot\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}}\right)} - 1\right]}{\log\left(\frac{n+1}{n}\cdot\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}}\right)}\cdot n\log\left(\frac{n+1}{n}\cdot\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}}\right) =$$

$$= \lim_{n\to\infty}\log\left(\left(\frac{n+1}{n}\right)^n\cdot\left(\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}}\right)^n\right) =$$

$$= \lim_{n\to\infty}\log\left(\left(1+\frac{1}{n}\right)^n\cdot\frac{(2n+1)!!}{(2n-1)!!}\cdot\frac{1}{\sqrt[n+1]{(2n+1)!!}}\right) =$$

$$= \log\left(\lim_{n\to\infty} e\cdot\frac{2n+1}{n+1}\cdot\frac{n+1}{\sqrt[n+1]{(2n+1)!!}}\right) \overset{(1)}{=} \log\left(e\cdot 2\cdot\frac{e}{2}\right) = 2; (4)$$

From (1), (2), (3) and (4), it follows that:

$$\Omega = \lim_{n\to\infty} \frac{(n+1)\cdot \sqrt[n+1]{(2n+1)!!} - n\cdot \sqrt[n]{(2n-1)!!}}{\sqrt[n]{n!}} = 4$$

1890. If we define the function $S(n)$ for $n > 0$

$$S(n) = \int_{-\infty}^{\infty} \sin(\pi x^2)\sin^2\left(\frac{1}{2}\pi x(n+x)\right)dx$$

then prove the sum:

$$\sum_{n=1}^{\infty} \frac{S(n)}{n^2} = \frac{\pi^2}{384}\left(1 + 16\sqrt{2} - 3\sqrt{2-\sqrt{2}}\right)$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Rana Ranino-Setif-Algerie

$$S(n) = \int_{-\infty}^{\infty} \sin(\pi x^2)\sin^2\left(\frac{1}{2}\pi x(n+x)\right)dx =$$

$$= \frac{1}{2}\underbrace{\int_{-\infty}^{\infty} \sin(\pi x^2)\,dx}_{\frac{1}{\sqrt{2}}} - \frac{1}{2}\int_{-\infty}^{\infty} \sin(\pi x^2)\cos(\pi x62 + n\pi x)\,dx =$$

$$= \frac{1}{2\sqrt{2}} - \frac{1}{4}\underbrace{\int_{-\infty}^{\infty} \sin(2\pi x^2 + n\pi x)\,dx}_{A} + \frac{1}{4}\underbrace{\int_{-\infty}^{\infty} \sin(n\pi x)\,dx}_{=0-\text{odd function}}$$

$$A = \int_{-\infty}^{\infty} \sin\left(2\pi\left(x+\frac{n}{4}\right)^2 - \frac{\pi n^2}{8}\right)dx \overset{y=x+\frac{n}{4}}{=}$$

$$= \cos\left(\frac{\pi n^2}{8}\right)\int_{-\infty}^{\infty} \sin(2\pi y^2)\,dy - \sin\left(\frac{\pi n^2}{8}\right)\int_{-\infty}^{\infty} \cos(2\pi y^2)\,dy =$$

$$= \frac{1}{2}\left(\cos\left(\frac{\pi n^2}{8}\right) - \sin\left(\frac{\pi n^2}{8}\right)\right) = \frac{1}{\sqrt{2}}\sin\left(\frac{\pi}{4} - \frac{\pi n^2}{8}\right)$$

$$S(n) = \frac{1}{2\sqrt{2}} - \frac{1}{4\sqrt{2}}\sin\left(\frac{\pi}{4} - \frac{\pi n^2}{8}\right)$$

$$\sum_{n=1}^{\infty}\frac{S(n)}{n^2}=\frac{1}{2\sqrt{2}}\sum_{n=1}^{\infty}\frac{1}{n^2}-\frac{1}{4\sqrt{2}}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{\pi}{4}-\frac{\pi n^2}{8}\right)}{n^2}=\frac{\pi^2}{12\sqrt{2}}-\frac{\Omega}{4\sqrt{2}}$$

$$\Omega=\frac{\sin\frac{\pi}{8}}{1^2}-\frac{\frac{1}{\sqrt{2}}}{2^2}+\frac{\sin\frac{\pi}{8}}{3^2}-\frac{\frac{1}{\sqrt{2}}}{4^2}+\frac{\sin\frac{\pi}{8}}{5^2}-\frac{\frac{1}{\sqrt{2}}}{6^2}+\frac{\sin\frac{\pi}{8}}{7^2}-\frac{\frac{1}{\sqrt{2}}}{8^2}+\overset{(\text{period}=8)}{\cdots}=$$

$$=\frac{\sin\frac{\pi}{8}}{64}\underbrace{\left(\psi^{(1)}\left(\frac{1}{8}\right)-\psi^{(1)}\left(\frac{3}{8}\right)-\psi^{(1)}\left(\frac{5}{8}\right)+\psi^{(1)}\left(\frac{7}{8}\right)\right)}_{4\pi^2\sqrt{2}}-$$

$$-\frac{1}{64\sqrt{2}}\underbrace{\left(\psi^{(1)}\left(\frac{1}{4}\right)-\psi^{(1)}\left(\frac{1}{2}\right)+\psi^{(1)}\left(\frac{3}{4}\right)-\psi^{(1)}(1)\right)}_{\frac{4\pi^2}{3}}$$

$$\Omega=\frac{\sqrt{2}\pi^2\sqrt{2-\sqrt{2}}}{32}-\frac{\pi^2}{48\sqrt{2}}$$

$$\sum_{n=1}^{\infty}\frac{S(n)}{n^2}=\frac{\pi^2}{12\sqrt{2}}-\frac{1}{4\sqrt{2}}\left(\frac{\sqrt{2}\pi^2\sqrt{2-\sqrt{2}}}{32}-\frac{\pi^2}{48\sqrt{2}}\right)=$$

$$=\frac{\pi^2}{48\sqrt{2}}-\frac{\sqrt{2}\pi^2\sqrt{2-\sqrt{2}}}{32}+\frac{\pi^2}{348}$$

Therefore,

$$\sum_{n=1}^{\infty}\frac{S(n)}{n^2}=\frac{\pi^2}{384}\left(1+16\sqrt{2}-3\sqrt{2-\sqrt{2}}\right)$$

1891. Find:

$$\Omega=\int_1^{21}\frac{dx}{e^{\left[2x+\frac{1}{4}\right]}},\ [*]-\text{GIF}.$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Asmat Qatea-Afghanistan

$$\text{Let } 2x+\frac{1}{4}=n\Rightarrow x=\frac{4n-1}{8},\ a_n=\frac{4n-1}{8}+1=\frac{4n+7}{8}$$

$$\Omega=\int_1^{\frac{11}{8}}\frac{1}{e^2}dx+\sum_{n=1}^{39}\int_{a_n}^{a_{n+1}}\frac{1}{e^{\left[2x+\frac{1}{4}\right]}}dx+\int_{\frac{4\cdot 40+7}{8}}^{21}\frac{1}{e^{\left[2x+\frac{1}{4}\right]}}dx=$$

$$=\left(\frac{11}{8}-1\right)\cdot\frac{1}{e^2}+\sum_{n=1}^{39}(a_{n+1}-a_n)\cdot\frac{1}{e^{n+2}}+\int_{\frac{167}{8}}^{21}\frac{1}{e^{42}}dx=$$

$$=\frac{3}{8e^2}+\sum_{n=1}^{39}\left(\frac{4n+4+7}{8}-\frac{4n+7}{8}\right)\cdot\frac{1}{e^{n+2}}+\left(21-\frac{167}{8}\right)\cdot\frac{1}{e^{42}}=$$

$$=\frac{3}{8e^2}+\sum_{n=1}^{39}\frac{1}{2}\cdot\frac{1}{e^2\cdot e^n}+\frac{1}{8e^{42}}=\frac{3}{8e^2}+\frac{1}{2e^2}\cdot\frac{\frac{1}{e^{40}}-\frac{1}{e}}{\frac{1}{e}-1}+\frac{1}{8e^{42}}$$

Solution 2 by Florentin Vișescu-Romania

$$\Omega=\int_1^{21}\frac{dx}{e^{\left[2x+\frac{1}{4}\right]}}\overset{\left(u=2x+\frac{1}{4}\right)}{=}\frac{1}{2}\int_{\frac{9}{4}}^{\frac{169}{4}}e^{-[u]}\,du=$$

$$=\frac{1}{2}\int_{\frac{9}{4}}^{3}e^{-2}du+\frac{1}{2}\int_3^4 e^{-3}du+\cdots+\frac{1}{2}\int_{41}^{42}e^{-41}\,du+\frac{1}{2}\int_{42}^{\frac{169}{4}}e^{-42}\,du=$$

$$=\frac{1}{2}\left(\left(3-\frac{9}{4}\right)e^{-2}+e^{-3}+e^{-4}+\cdots+e^{-41}+\left(\frac{169}{4}-42\right)e^{-42}\right)=$$

$$=\frac{1}{2}\left(\frac{3}{4}e^{-2}+\frac{1}{4}e^{-42}+e^{-3}\cdot\frac{e^{-40}-1}{e^{-1}-1}\right)=\frac{1}{2}\left(\frac{3}{4}e^{-2}+\frac{1}{4}e^{-42}+\frac{e^{-43}-e^{-3}}{e^{-1}-1}\right)=$$

$$=\frac{5e^{-42}-e^{-41}-4e^{-3}+3e^{-2}-3e^{-1}}{8(1-e)}$$

Solution 3 by Ravi Prakash-New Delhi-India

$$\Omega=\int_1^{21}\frac{dx}{e^{\left[2x+\frac{1}{4}\right]}}\overset{\left(t=2x+\frac{1}{4}\right)}{=}\frac{1}{2}\int_{2+\frac{1}{4}}^{42+\frac{1}{4}}\frac{dt}{e^{[t]}}=\frac{1}{2}\left[\int_{2+\frac{1}{4}}^{3}\frac{dt}{e^2}+\sum_{k=3}^{41}\int_k^{k+1}\frac{dt}{e^k}+\int_{42}^{42+\frac{1}{4}}\frac{dt}{e^{42}}\right]=$$

$$=\frac{1}{2}\left[\frac{1}{e^2}\cdot\frac{3}{4}+\sum_{k=3}^{41}\frac{1}{e^k}+\frac{1}{e^{42}}\cdot\frac{1}{4}\right]=\frac{1}{2}\left[\frac{3}{4}\cdot\frac{1}{e^2}+\frac{\frac{1}{e^3}\left(1-\left(\frac{1}{e}\right)^{39}\right)}{1-\frac{1}{e}}+\frac{1}{4e^{42}}\right]=$$

$$=\frac{3}{8e^2}+\frac{1}{8e^{42}}+\frac{1}{2e^2(e-1)}\left[1-\frac{1}{e^{39}}\right]$$

1892. For $\Re(n) > 0$ prove that:

$$\int_0^1 \frac{\log x}{x^n + x^{n-1} + \cdots + x + 1} dx = \frac{1}{n^2}\left[\psi^{(1)}\left(\frac{2}{n}\right) - \psi^{(1)}\left(\frac{1}{n}\right)\right]$$

Proposed by Muhammad Afzal-Pakistan

Solution 1 by Daniel Immarube-Nigeria

$$\Omega = \int_0^1 \frac{\log x}{x^n + x^{n-1} + \cdots + x + 1} dx = \int_0^1 \frac{(1-x)\log x}{(1-x)(x^n + x^{n-1})} dx =$$

$$= \int_0^1 \frac{(1-x)\log x}{x^{n-1}(1-x^2)} dx \overset{(x \to x^n)}{=} \frac{1}{n}\int_0^1 \frac{\log x}{x^{n-1}(1-x)} dx =$$

$$= \frac{1}{n}\int_0^1 \frac{\log x}{x^{n-1}(1+x)} dx = \frac{1}{n^2}(-1)^n \sum_{n=0}^{\infty} \int x^{a+n-1}\, dx =$$

$$= \frac{1}{n^2}(-1)^n \sum_{n=0}^{\infty} \frac{1}{a+n} = \frac{d}{da}(-1)^n \sum_{n=0}^{\infty} \frac{1}{a+n} =$$

$$= \frac{2}{n^2}\cdot\frac{1}{2}\left(\psi^{(1)}\left(\frac{2}{n}\right) - \psi^{(1)}\left(\frac{1}{n}\right)\right) = \frac{1}{n^2}\left(\psi^{(1)}\left(\frac{2}{n}\right) - \psi^{(1)}\left(\frac{1}{n}\right)\right)$$

Solution 2 by Samar Das-India

$$\Omega = \int_0^1 \frac{\log x}{x^n + x^{n-1} + \cdots + x + 1} dx = \int_0^1 \frac{\log x}{\frac{1-x^n}{1-x}} dx =$$

$$= \int_0^1 \left(\frac{1}{1-x^n} - \frac{x}{1-x^n}\right)\log x\, dx = \int_0^1 \frac{(1-x)\log x}{1-x^n} dx =$$

$$= \int_0^1 \left(\sum_{k=0}^{\infty} r^k - x\sum_{k=0}^{\infty} r^k\right)\log x\, dx \overset{(r=x^n)}{=}$$

$$= \int_0^1 \left(\sum_{k=0}^{\infty} x^{kn} - \sum_{k=0}^{\infty} x^{kn+1}\right)\log x\, dx = \sum_{k=0}^{\infty}\int_0^1 \left(x^{nk} - x^{nk+1}\right)\log x\, dx =$$

$$= \sum_{k=0}^{\infty}\left[\left(\frac{x^{nk+1}}{nk+1}\log x - \frac{x^{nk+1}}{(nk+1)^2}\right) - \left(\frac{x^{nk+2}}{nk+2}\log x - \frac{x^{nk+2}}{(nk+2)^2}\right)\right]_0^1 =$$

$$=\frac{1}{n^2}\sum_{k=0}^{\infty}\left(\frac{1}{\left(k+\frac{2}{n}\right)^2}-\frac{1}{\left(k+\frac{1}{n}\right)^2}\right)=\frac{1}{n^2}\left(\psi^{(1)}\left(\frac{2}{n}\right)-\psi^{(1)}\left(\frac{1}{n}\right)\right)$$

Solution 3 by Hikmat Mammadov-Azerbaijan

$$\Omega=\int_0^1\frac{\log x}{x^n+x^{n-1}+\cdots+x+1}dx=\int_0^1\frac{(1-x)\log x}{1-x^{n+1}}dx-\int_0^1\frac{x\log x}{1-x^{n+1}}dx=M-N$$

$$M=\int_0^1\frac{\log x}{1-x^{n+1}}dx=\int_0^1\sum_{j=0}^{\infty}x^{j(n+1)}\log x\,dx=$$

$$=\sum_{j=0}^{\infty}\int_0^1 x^{j(n+1)}\left(\frac{d}{dt}x^t|_{t=0_+}x^t\right)dx=\frac{d}{dt}|_{t=0_+}\sum_{j=0}^{\infty}\int_0^1 x^{(n+1)rj+t}dx=$$

$$=\frac{d}{dt}|_{t=0_+}\sum_{j=0}^{\infty}\frac{1}{(n+1)j+t+1}=-\sum_{j=0}^{\infty}\frac{1}{[(n+1)j+1]^2}$$

$$N=\int_0^1\frac{x\log x}{1-x^{n+1}}dx=\int_0^1\sum_{j=0}^{\infty}x^{j(n+1)+1}\log x\,dx=$$

$$=\sum_{j=0}^{\infty}\int_0^1 x^{j(n+1)+1}\left(\frac{d}{dt}x^t|_{t=0_+}x^t\right)dx=\frac{d}{dt}|_{t=0_+}\sum_{j=0}^{\infty}\int_0^1 x^{(n+1)rj+t+1}dx=$$

$$=\frac{d}{dt}|_{t=0_+}\sum_{j=0}^{\infty}\frac{1}{(n+1)j+t+2}=-\sum_{j=0}^{\infty}\frac{1}{[(n+1)j+2]^2}$$

Hence,

$$\Omega=\int_0^1\frac{\log x}{x^n+x^{n-1}+\cdots+x+1}dx=-\sum_{j=0}^{\infty}\frac{1}{[(n+1)j+1]^2}\pm\sum_{j=0}^{\infty}\frac{1}{[(n+1)j+2]^2}=$$

$$=-\sum_{j=0}^{\infty}\frac{1}{(n+1)^2\left(j+\frac{1}{n+1}\right)^2}+\sum_{j=0}^{\infty}\frac{1}{(n+1)^2\left(j+\frac{2}{n+1}\right)^2}=$$

$$=\frac{1}{(n+1)^2}\left[\psi^{(1)}\left(\frac{2}{n+1}\right)+\psi^{(1)}\left(\frac{1}{n+1}\right)\right]$$

$$\because\ \psi^{(m)}(z)=(-1)^{m+1}m!\sum_{k=0}^{\infty}\frac{1}{(z+k)^{m+1}}$$

1893. Find:

$$\Omega=\int_0^\infty \frac{x\log(1+x)}{(x+1)(x^2+1)}dx$$

Proposed by Vasile Mircea Popa-Romania

Solution 1 by Ankush Kumar Parcha-India

$$\Omega=\int_0^\infty \frac{x\log(1+x)}{(x+1)(x^2+1)}dx \overset{x=\frac{1}{y}}{=} \int_0^\infty \frac{\log(1+x)}{(1+x)(1+x^2)}dx-\int_0^\infty \frac{\log x}{(1+x)(1+x^2)}dx$$

$$2\Omega=\int_0^\infty \frac{\log(1+x)}{1+x^2}dx-\int_0^\infty \frac{\log x}{(1+x)(1+x^2)}dx\,;(1)$$

$$I_1=\int_0^\infty \frac{\log(1+x)}{1+x^2}dx\,;I_2=\int_0^\infty \frac{\log x}{(1+x)(1+x^2)}dx$$

$$I_1=\int_0^\infty \frac{\log(1+x)}{1+x^2}dx \overset{x=\tan y}{=} \int_0^{\frac{\pi}{2}} \frac{\log(1+\tan y)}{1+\tan^2 y}\sec^2 y\,dy=$$

$$=\int_0^{\frac{\pi}{2}}\log(\sin x+\cos x)\,dx-\int_0^{\frac{\pi}{2}}\log(\cos x)\,dx=$$

$$=\int_0^{\frac{\pi}{2}}\log\left[\sqrt{2}\cos\left(\frac{\pi}{4}-x\right)\right]dx-\int_0^{\frac{\pi}{2}}\log\cos x\,dx=$$

$$=\frac{\pi}{4}\log 2+\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\log\cos x\,dx-\int_0^{\frac{\pi}{2}}\log\cos x\,dx=$$

$$=\frac{\pi}{4}\log 2+\Re\left\{\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\log\left(\frac{1+e^{-2ix}}{2e^{-ix}}\right)dx\right\}-\Re\left\{\int_0^{\frac{\pi}{2}}\log\left(\frac{1+e^{-2ix}}{2e^{ix}}\right)dx\right\}=$$

$$=\frac{\pi}{4}\log 2-\log 2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}dx+\Re\left\{\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\log(1+e^{-2ix})\,dx\right\}+\Re\left\{\int_0^{\frac{\pi}{2}}\log(1+e^{-2ix})\,dx\right\}+\frac{\pi}{2}\log 2$$

$$=\frac{\pi}{4}\log 2+\Re\left\{\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}e^{-2nix}dx\right\}-\Re\left\{\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n}\int_0^{\frac{\pi}{2}}e^{-2inx}dx\right\}=$$

$$=\frac{\pi}{4}\log 2+\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n}\left(\frac{e^{in\frac{\pi}{2}}-e^{-in\frac{\pi}{2}}}{2in}\right)-\Re\left\{\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n}\left(\frac{e^{-inx}}{-2in}\right)\Bigg|_0^{\frac{\pi}{2}}\right\}=$$

$$=\frac{\pi}{4}\log 2+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\sin\frac{n\pi}{2}\Rightarrow$$

$$I_1=\int_0^{\infty}\frac{\log(1+x)}{1+x^2}dx=\frac{\pi}{4}\log 2+G$$

$$I_2=\int_0^{\infty}\frac{\log x}{(1+x)(1+x^2)}dx=\int_0^1\frac{\log x}{(1+x)(1+x^2)}dx+\int_1^{\infty}\frac{\log x}{(1+x)(1+x^2)}dx=$$

$$=\int_0^1\frac{\log x}{2}\left(\frac{1}{1+x}+\frac{1-x}{1+x^2}\right)dx+\int_1^{\infty}\frac{\log\left(\frac{1}{y}\right)}{\frac{1+y}{y}\frac{1+y^2}{y^2}}\left(-\frac{dy}{y^2}\right)=$$

$$=\frac{1}{2}\int_0^1\frac{\log x}{1+x}dx+\frac{1}{2}\int_0^1\frac{\log x}{1+x^2}dx-\frac{1}{2}\int_0^1\frac{\log(\sqrt{x})}{1+x}\frac{dx}{2}-\frac{1}{2}\int_0^1\log x\left(\frac{1+x}{1+x^2}-\frac{1}{1+x}\right)dx$$

$$=$$

$$=\frac{3}{8}\int_0^1\frac{\log x}{1+x}dx+\frac{1}{2}\int_0^1\frac{\log x}{1+x}dx-\frac{1}{2}\int_0^1\frac{x\log x}{1+x^2}dx-\frac{1}{2}\int_0^1\frac{\log x}{1+x}dx+\frac{1}{2}\int_0^1\frac{\log x}{1+x}dx$$

$$=\frac{7}{8}\int_0^1\frac{\log x}{1+x}dx-\frac{1}{2}\int_0^1\frac{\log(\sqrt{x})}{1+x}\frac{dx}{2}=\frac{3}{4}\int_0^1\frac{\log x}{1+x}dx=$$

$$=\frac{3}{4}\sum_{n=0}^{\infty}(-1)^n\int_0^1 x^n\log x\,dx$$

$$\because\int_0^1 x^m\log^n x\,dx=\frac{(-1)^n n!}{(m+1)^{n+1}},m\neq-1,n>-1$$

$$I_2=-\frac{3}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^2}=-\frac{3}{4}\eta(2)=-\frac{3}{4}(1-2^{-1})\zeta(2)=-\frac{\pi^2}{16}$$

By adding I_1, I_2 and put the value in (1), we get:

$$2\Omega=\frac{\pi}{4}\log 2+G+\frac{\pi^2}{16}$$

$$\Omega=\frac{\pi^2+16G+\pi\log 16}{32}$$

Solution 2 by Togrul Ehmedov-Azerbaijan

$$\Omega=\int_0^{\infty}\frac{x\log(1+x)}{(x+1)(x^2+1)}dx=\int_0^{\infty}\frac{\log(1+x)}{(x+1)(x^2+1)}dx-\int_0^{\infty}\frac{\log x}{(x+1)(x^2+1)}dx$$

$$2\Omega=\int_0^\infty\frac{x\log(1+x)}{(x+1)(x^2+1)}dx+\int_0^\infty\frac{\log(1+x)}{(x+1)(x^2+1)}dx-\int_0^\infty\frac{\log x}{(x+1)(x^2+1)}dx$$

$$2\Omega=\int_0^\infty\frac{\log(1+x)}{x^2+1}dx-\int_0^\infty\frac{\log x}{(x+1)(x^2+1)}dx=I_1-I_2$$

$$I_1=\int_0^\infty\frac{\log(1+x)}{x^2+1}dx=G+\frac{\pi}{4}\log 2$$

$$I_2=\int_0^\infty\frac{\log x}{(x+1)(x^2+1)}dx=\int_0^\infty\frac{\log x}{x^2+1}dx-\int_0^\infty\frac{x\log x}{(x+1)(x^2+1)}dx=I_{2a}-I_{2b}$$

$$I_{2a}=\int_0^\infty\frac{\log x}{x^2+1}dx=0$$

$$I_{2b}=\int_0^\infty\frac{x\log x}{(x+1)(x^2+1)}dx=\frac{\pi^2}{16},\qquad 2\Omega=I_1-I_2=G+\frac{\pi}{4}\log 2+\frac{\pi^2}{16}$$

$$\Omega=\frac{\pi}{8}\log 2+\frac{\pi^2}{32}+\frac{G}{2}$$

Solution 3 by Daniel Immarube-Nigeria

$$\Omega=\int_0^\infty\frac{x\log(1+x)}{(x+1)(x^2+1)}dx=\int_0^\infty\frac{\log(1+x)}{(x+1)(x^2+1)}dx-\int_0^\infty\frac{\log x}{(x+1)(x^2+1)}dx=$$

$$=\Omega_1-\Omega_2$$

$$\Omega_1=\int_0^\infty\frac{\log(1+x)}{(x+1)(x^2+1)}dx\overset{x=\tan x}{=}\int_0^{\frac{\pi}{2}}\log(1+\tan x)\,dx=$$

$$=\int_0^{\frac{\pi}{2}}\sqrt{2}\log\left(cos\left(x-\frac{\pi}{4}\right)\right)dx-\int_0^{\frac{\pi}{2}}\log(\cos x)\,dx=$$

$$=\int_0^{\frac{\pi}{2}}\sqrt{2}dx+\int_0^{\frac{\pi}{2}}\log\left(cos\left(x-\frac{\pi}{4}\right)\right)dx-\int_0^{\frac{\pi}{2}}\log(\cos x)\,dx=$$

$$=\frac{\pi}{4}\log 2+2\int_0^{\frac{\pi}{4}}\log(\cos x)\,dx-\int_0^{\frac{\pi}{2}}\log(\cos x)\,dx=$$

$$=\frac{\pi}{4}\log 2+2\int_0^{\frac{\pi}{4}}\log(\cos x)\,dx-\int_0^{\frac{\pi}{2}}\log(\cos x)\,dx=$$

$$=\frac{\pi}{4}\log 2+2\left(\frac{1}{2}Cl_2\left(2\left(\frac{\pi}{4}\right)\right)-\frac{\pi}{4}\log 2\right)+\frac{\pi}{2}\log 2=$$

$$=\frac{\pi}{4}\log 2+Cl_2\left(\frac{\pi}{2}\right)-\frac{\pi}{2}\log 2+\frac{\pi}{2}\log 2=\frac{\pi}{4}\log 2+Cl_2\left(\frac{\pi}{2}\right)$$

$$\Omega_2=\int_0^\infty\frac{\log x}{(x+1)(x^2+1)}dx=\int_0^\infty\frac{\log(1+x)}{1+x^2}dx-\int_0^\infty\frac{x\log(1+x)}{(1+x)(1+x^2)}dx=$$

$$=\frac{1}{2}\int_0^\infty\frac{\log(1+x)}{1+x^2}dx+\frac{1}{2}\int_0^\infty\frac{\log x}{(1+x)(1+x^2)}dx=$$

$$=\frac{1}{2}\left(\frac{\pi}{4}\log 2+Cl_2\left(\frac{\pi}{2}\right)\right)+\frac{1}{2}\left(-\int_0^1\frac{x\log x}{1+x}dx+\int_0^1\frac{\log x}{1+x}dx\right)=$$

$$=\frac{1}{2}\left(\frac{\pi}{4}\log 2+Cl_2\left(\frac{\pi}{2}\right)\right)+\frac{1}{2}\left(-\sum_{n=0}^\infty\frac{(-1)^n}{(2n+2)^2}+\sum_{n=0}^\infty\frac{(-1)^{n-1}}{(n+1)^2}\right)=$$

$$=\frac{1}{2}\left(\frac{\pi}{4}\log 2+Cl_2\left(\frac{\pi}{2}\right)\right)+\frac{1}{2}\left(\frac{\pi^2}{48}-\frac{\pi^2}{12}\right)=\frac{1}{2}\left(\frac{\pi}{4}\log 2+Cl_2\left(\frac{\pi}{2}\right)\right)-\frac{1}{2}\cdot\frac{\pi^2}{16}=$$

$$=\frac{\pi}{8}\log 2+\frac{1}{2}Cl_2\left(\frac{\pi}{2}\right)-\frac{\pi^2}{32}$$

$$\Omega=\frac{\pi}{4}\log 2+Cl_2\left(\frac{\pi}{2}\right)-\frac{\pi}{8}\log 2-\frac{1}{2}Cl_2\left(\frac{\pi}{2}\right)+\frac{\pi^2}{32}=$$

$$=\frac{\pi}{8}\log 2+\frac{1}{2}Cl_2\left(\frac{\pi}{2}\right)+\frac{\pi^2}{32}$$

1894. Let $t\geq 0$ and $(a_n)_{n\geq 1},(b_n)_{n\geq 1}$ sequence of real numbers strictly positive such that:

$$\lim_{n\to\infty}\frac{a_{n+1}}{n^t\cdot a_n}=a>0,\ \lim_{n\to\infty}\frac{b_{n+1}}{n^{t+1}\cdot b_n}=b>0.\ \text{Find:}$$

$$\Omega=\lim_{n\to\infty}\frac{\sqrt[n+1]{b_{n+1}}-\sqrt[n]{b_n}}{\sqrt[n]{a_n}}$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

Solution by Marian Ursărescu-Romania

$$\Omega=\lim_{n\to\infty}\frac{\sqrt[n+1]{b_{n+1}}-\sqrt[n]{b_n}}{\sqrt[n]{a_n}}=\lim_{n\to\infty}\frac{\sqrt[n]{b_n}}{n\cdot\sqrt[n]{a_n}}\cdot n\left(\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}}-1\right);(1)$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{b_n}}{n\cdot\sqrt[n]{a_n}}=\lim_{n\to\infty}\frac{\sqrt[n]{b_n}}{n^{t+1}}\cdot\frac{n^t}{\sqrt[n]{a_n}};(2)$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{b_n}}{n^{t+1}}=\lim_{n\to\infty}\sqrt[n]{\frac{b_n}{n^{n(t+1)}}}\overset{C-D'A}{=}\lim_{n\to\infty}\frac{b_{n+1}}{(n+1)^{(n+1)(t+1)}}\cdot\frac{n^{n(t+1)}}{b_n}=$$

$$=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{n(t+1)}\cdot\frac{b_{n+1}}{n^{t+1}\cdot b_n}\cdot\frac{n^{t+1}}{(n+1)^{t+1}}=\frac{1}{e^{t+1}}\cdot b\cdot 1=\frac{b}{e^{t+1}};(3)$$

$$\lim_{n\to\infty}\frac{n^t}{\sqrt[n]{a_n}}=\lim_{n\to\infty}\sqrt[n]{\frac{n^{nt}}{a_n}}\overset{C-D'A}{=}\lim_{n\to\infty}\frac{(n+1)^{(n+1)t}}{a_{n+1}}\cdot\frac{a_n}{n^{nt}}=$$

$$=\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{nt}\cdot\frac{a_n n^t}{a_{n+1}}\cdot\frac{(n+1)^t}{n^t}=e^t\cdot\frac{1}{a}\cdot 1=\frac{e^t}{a};\quad(4)$$

From (2), (3) and (4) we get:

$$\lim_{n\to\infty}\frac{\sqrt[n]{b_n}}{n\cdot\sqrt[n]{a_n}}=\frac{b}{ae};(5)$$

$$\lim_{n\to\infty}n\left(\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}}-1\right)=\lim_{n\to\infty}\frac{n\left(e^{\log\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}}}-1\right)}{\log\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}}}\cdot\log\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}}=$$

$$=\lim_{n\to\infty}n\log\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}}=\lim_{n\to\infty}\log\frac{\sqrt[n+1]{b_{n+1}^n}}{b_n}=$$

$$=\log\left(\lim_{n\to\infty}\frac{b_{n+1}}{b_n}\cdot\frac{1}{\sqrt[n+1]{b_{n+1}}}\right)=\log\left(\lim_{n\to\infty}\frac{b_{n+1}}{n^{t+1}\cdot b_n}\cdot\frac{n^{t+1}}{\sqrt[n+1]{b_{n+1}}}\right)=$$

$$=\log\left(\lim_{n\to\infty}b\cdot\frac{n^{t+1}}{(n+1)^{t+1}}\cdot\frac{(n+1)^{t+1}}{\sqrt[n+1]{b_{n+1}}}\right)\overset{(3)}{=}\log\left(b\cdot 1\cdot\frac{e^{t+1}}{b}\right)=t+1;(6)$$

From (1), (5) and (6), we get:

$$\Omega=\frac{b}{ae}(t+1)=\frac{t+1}{e}\cdot\frac{b}{a}$$

1895. Let $a_n = \sqrt{2!} \cdot \sqrt[3]{3!} \cdot \ldots \cdot \sqrt[n]{n!} \in \mathbb{N}^* - \{1\}$ and $(b_n)_{n\geq 1}$ a sequence of real numbers strictly positive such that $\lim_{n\to\infty} \frac{b_{n+1}}{b_n \cdot \sqrt[n]{a_n}} = b > 0$. Find:

$$\Omega = \lim_{n\to\infty}\left(\sqrt[n+1]{b_{n+1}} - \sqrt[n]{b_n}\right)$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

Solution by Marian Ursărescu-Romania

$$\Omega = \lim_{n\to\infty}\left(\sqrt[n+1]{b_{n+1}} - \sqrt[n]{b_n}\right) = \lim_{n\to\infty} \sqrt[n]{b_n}\left(\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}} - 1\right) =$$

$$= \lim_{n\to\infty} \frac{\sqrt[n]{b_n}}{n} \cdot n\left(\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}} - 1\right); (1)$$

$$\lim_{n\to\infty} \frac{\sqrt[n]{b_n}}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{b_n}{n^n}} \overset{C-D'A}{=} \lim_{n\to\infty} \frac{b_{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{b_n} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n \cdot \frac{b_{n+1}}{b_n} \cdot \frac{1}{n+1} =$$

$$= \frac{1}{e}\lim_{n\to\infty} \frac{b_{n+1}}{b_n\sqrt[n]{a_n}} \cdot \frac{n}{n+1} \cdot \frac{\sqrt[n]{a_n}}{n} = \frac{1}{e} \cdot b \lim_{n\to\infty} \sqrt[n]{\frac{a_n}{n^n}} = \frac{b}{e}\lim_{n\to\infty} \frac{a_{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{a_n} =$$

$$= \frac{b}{e}\lim_{n\to\infty} \frac{a_{n+1}}{a_n} \cdot \frac{n^n}{(n+1)^{n+1}} = \frac{b}{e}\lim_{n\to\infty} \frac{\sqrt[n+1]{(n+1)!}}{n+1} \cdot \left(\frac{n}{n+1}\right)^n =$$

$$= \frac{b}{e} \cdot \frac{1}{e}\lim_{n\to\infty} \frac{\sqrt[n]{n}}{n!} = \frac{b}{e^2}\lim_{n\to\infty} \sqrt[n]{\frac{n!}{n^n}} = \frac{b}{e^2}\lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} =$$

$$= \frac{b}{e^2}\lim_{n\to\infty}\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \frac{b}{e^3}; (2)$$

$$\lim_{n\to\infty} n\left(\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}} - 1\right) = \lim_{n\to\infty} \frac{n\left(e^{\log\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}}} - 1\right)}{\log\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}}} \cdot \log\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}} =$$

$$= \lim_{n\to\infty} n\log\left(\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}}\right) = \lim_{n\to\infty} \log\left(\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}}\right)^n =$$

$$=\log\left(\lim_{n\to\infty}\frac{b_{n+1}}{b_n}\cdot\frac{1}{\sqrt[n+1]{b_{n+1}}}\right)=\log\left(\lim_{n\to\infty}\frac{b_{n+1}}{b_n\sqrt[n]{a_n}}\cdot\frac{\sqrt[n]{a_n}}{\sqrt[n+1]{b_{n+1}}}\right)=$$

$$=\log\left(b\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n}\cdot\frac{n}{n+1}\cdot\frac{n+1}{\sqrt[n]{b_{n+1}}}\right)\overset{(2)}{=}\log\left(b\cdot\frac{1}{e^2}\cdot 1\cdot\frac{e^3}{b}\right)=$$

$\log e=1$; (3). From (1), (2) and (3), it follows that $\Omega=\frac{b}{e^3}$.

1896.

Let $(a_n)_{n\geq 1}$ be a sequence of real numbers strictly positive such that

$$\lim_{n\to\infty}\frac{a_{n+1}}{n\cdot a_n}=a \text{ and let } b_n(x)=n^{\sin^2 x}\left(\sqrt[n+1]{a_{n+1}^{\cos^2 x}}-\sqrt[n]{a_n^{\cos^2 x}}\right),$$

$$\forall x\in\mathbb{R}, n\in\mathbb{N}^*. \text{ Find: } \lim_{n\to\infty}b_n(x).$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

Solution by Adrian Popa-Romania

$$b_n(x)=n^{\sin^2 x}\left(\sqrt[n+1]{a_{n+1}^{\cos^2 x}}-\sqrt[n]{a_n^{\cos^2 x}}\right)=n^{\sin^2 x}\cdot\sqrt[n]{a_n^{\cos^2 x}}\left(\frac{\sqrt[n+1]{a_{n+1}^{\cos^2 x}}}{\sqrt[n]{a_n^{\cos^2 x}}}-1\right)=$$

$$=n^{1-\cos^2 x}\cdot\sqrt[n]{a_n^{\cos^2 x}}\left(\frac{\sqrt[n+1]{a_{n+1}^{\cos^2 x}}}{\sqrt[n]{a_n^{\cos^2 x}}}-1\right)=\sqrt[n]{\frac{a_n^{\cos^2 x}}{n^{n\cos^2 x}}}\cdot n\left(\frac{\sqrt[n+1]{a_{n+1}^{\cos^2 x}}}{\sqrt[n]{a_n^{\cos^2 x}}}-1\right)$$

$$\lim_{n\to\infty}\sqrt[n]{\frac{a_n^{\cos^2 x}}{n^{n\cos^2 x}}}\overset{C-D}{=}\lim_{n\to\infty}\frac{a_{n+1}^{\cos^2 x}}{(n+1)^{(n+1)\cos^2 x}}\cdot\frac{n^{n\cos^2 x}}{a_n^{\cos^2 x}}=$$

$$=\lim_{n\to\infty}\left(\frac{a_{n+1}}{n\cdot a_n}\cdot\left(\frac{n}{n+1}\right)^n\cdot\frac{n}{n+1}\right)^{\cos^2 x}=\left(\frac{a}{e}\right)^{\cos^2 x}$$

$$\lim_{n\to\infty}n\left(\frac{\sqrt[n+1]{a_{n+1}^{\cos^2 x}}}{\sqrt[n]{a_n^{\cos^2 x}}}-1\right)=\lim_{n\to\infty}n\left(\frac{e^{\log\frac{\sqrt[n+1]{a_{n+1}^{\cos^2 x}}}{\sqrt[n]{a_n^{\cos^2 x}}}}-1}{\log\frac{\sqrt[n+1]{a_{n+1}^{\cos^2 x}}}{\sqrt[n]{a_n^{\cos^2 x}}}}\cdot\log\frac{\sqrt[n+1]{a_{n+1}^{\cos^2 x}}}{\sqrt[n]{a_n^{\cos^2 x}}}\right)=$$

$$= \lim_{n\to\infty} n\cdot\log\frac{\sqrt[n+1]{a_{n+1}^{\cos^2 x}}}{\sqrt[n]{a_n^{\cos^2 x}}} = \lim_{n\to\infty}\log\left(\frac{\sqrt[n+1]{a_{n+1}^{\cos^2 x}}}{\sqrt[n]{a_n^{\cos^2 x}}}\right)^n =$$

$$= \log\left(\lim_{n\to\infty}\frac{a_{n+1}^{\cos^2 x}}{a_n^{\cos^2 x}\left(a_{n+1}^{\cos^2 x}\right)^{\frac{1}{n+1}}}\right) = \log\left(\lim_{n\to\infty}\left(\frac{a_{n+1}}{n\cdot a_n}\right)^{\cos^2 x}\cdot\frac{n^{\cos^2 x}}{\left(a_{n+1}^{\cos^2 x}\right)^{\frac{1}{n+1}}}\right) =$$

$$= \log\left(\lim_{n\to\infty}\left(\frac{a_{n+1}}{n\cdot a_n}\right)^{\cos^2 x}\cdot\sqrt[n+1]{\frac{n^{(n+1)\cos^2 x}}{a_{n+1}^{\cos^2 x}}}\right) = \log\left(a^{\cos^2 x}\cdot\left(\frac{e}{a}\right)^{\cos^2 x}\right) = \cos^2 x$$

Therefore,

$$\Omega = \left(\frac{a}{e}\right)^{\cos^2 x}\cdot\cos^2 x.$$

1897. Find:

$$\Omega = \lim_{n\to\infty}\frac{H_n}{n(H_{2n-1} - 2H_{n-1})}$$

Proposed by Daniel Sitaru-Romania

Solution by Syed Shahabudeen-Kerala-India

$$\Omega = \lim_{n\to\infty}\frac{H_n}{n(H_{2n-1} - 2H_{n-1})} = \lim_{n\to\infty}\frac{H_n}{n\left(H_{2n-1} - 2\left(H_n - \frac{1}{n}\right)\right)} =$$

$$= \lim_{n\to\infty}\frac{H_n}{nH_{2n-1} - 2nH_n - 1} = \lim_{n\to\infty}\frac{1}{n\frac{H_{2n-1}}{H_n} - 2n - \frac{1}{H_n}}$$

$$\text{Here: } \lim_{n\to\infty} n\left(\frac{H_{2n-1}}{H_n} - 2\right) = \lim_{n\to\infty}\frac{\frac{H_{2n-1}}{H_n} - 2}{\frac{1}{n}}$$

$$\lim_{n\to\infty}\frac{H_{2n-1}}{H_n} \overset{Cesaro-S}{=} \lim_{n\to\infty}\frac{H_{2n+1} - H_{2n-1}}{H_{n+1} - H_n} = \lim_{n\to\infty}\frac{\frac{1}{2n+1} + \frac{1}{2n}}{\frac{1}{n}} = 1$$

$$\Rightarrow \lim_{n\to\infty} n\left(\frac{H_{2n-1}}{H_n} - 2\right) = -\infty$$

$$\lim_{n\to\infty}\frac{1}{H_n}=0$$

Therefore,

$$\lim_{n\to\infty}\left(n\frac{H_{2n-1}}{H_n}-2n-\frac{1}{H_n}\right)=0\Rightarrow\Omega=\lim_{n\to\infty}\frac{1}{n\frac{H_{2n-1}}{H_n}-2n-\frac{1}{H_n}}=0$$

1898. Find:

$$\Omega=\int_0^1\int_0^1(x^2+2xy+x)\log\left(1+\frac{1}{x+y}\right)dxdy$$

Proposed by Asmat Qatea-Afghanistan

Solution by Yen Tung Chung-Taichung-Taiwan

$$(*)\begin{cases}u=\log\left(1+\frac{1}{x+y}\right)\\ du=\frac{-1}{(x+y)(1+x+y)}\end{cases}\Rightarrow\begin{cases}dv=(x^2+2xy+x)dx\\ v=xy(1+x+y)\end{cases}$$

$$\Omega=\int_0^1(x^2+2xy+y^2)\log\left(1+\frac{1}{x+y}\right)dy\overset{(*)}{=}$$

$$=xy(1+x+y)\log\left(1+\frac{1}{x+y}\right)\Big|_0^1+\int_0^1\frac{xy}{x+y}dy=$$

$$=x(2+x)\log\left(\frac{x+2}{x+1}\right)+(xy-y^2\log(x+y))\Big|_0^1=$$

$$=x(x+2)\log\left(\frac{x+2}{x+1}\right)+x-x^2\log(x+1)=$$

$$=(x^2+2x)\log(x+2)-2(x^2+x)\log(x+1)+x$$

So, we have:

$$\Omega=\int_0^1\int_0^1(x^2+2xy+y^2)\log\left(1+\frac{1}{x+y}\right)dxdy=$$

$$=\int_0^1[(x^2+2x)\log(x+2)-2(x^2+x)\log(x+1)+x]dx=$$

$$=\left(\frac{1}{3}x^3+x^2\right)\log(x+2)\Big|_0^1-\int_0^1\frac{\frac{1}{3}x^3+x^2}{x+2}dx-2\left(\frac{1}{3}x^3+\frac{1}{2}x^2\right)\log(x+1)\Big|_0^1-$$

$$-2\int_0^1 \frac{\frac{1}{3}x^3+\frac{1}{2}x^2}{x+1}dx+\frac{1}{2}x^2\Big|_0^1=$$

$$=\frac{4}{3}\log 3-\int_0^1\left(\frac{1}{3}x^2+\frac{1}{3}x-\frac{2}{3}+\frac{4}{3}\cdot\frac{1}{x+2}\right)dx-\frac{5}{3}\log 2-$$

$$-2\int_0^1\left(\frac{1}{3}x^2+\frac{1}{6}x-\frac{1}{6}+\frac{1}{6}\cdot\frac{1}{x+1}\right)dx+\frac{1}{2}=$$

$$=\frac{4}{3}\log 3-\left(\frac{1}{9}x^3+\frac{1}{6}x^2-\frac{2}{3}x+\frac{4}{3}\log(x+2)\right)\Big|_0^1-\frac{5}{3}\log 2-$$

$$-2\left(\frac{1}{9}x^3+\frac{1}{12}x^2-\frac{1}{6}x+\frac{1}{6}\log(x+1)\right)\Big|_0^1+\frac{1}{2}=$$

$$=\frac{4}{3}\log 3+\frac{7}{18}-\frac{4}{3}\log 2-\frac{5}{3}\log 2-\frac{1}{18}+\frac{1}{3}\log 2+\frac{1}{2}=\frac{5}{6}$$

1899. Find:

$$\Omega(a)=\int_0^\infty \frac{x}{(x^2+x+1)(1+a^2x^2)}dx, a\in R\backslash\{0\}$$

Proposed by Vasile Mircea Popa-Romania

Solution by Togrul Ehmedov-Azerbaijan

$$\Omega(a)=I(a)=\int\frac{x}{(x^2+x+1)(1+a^2x^2)}dx=$$

$$=\frac{a^2(a^2-1)}{a^4-a^2+1}\int\frac{x}{1+a^2x^2}dx+\frac{a^2}{a^4-a^2+1}\int\frac{1}{1+a^2x^2}dx$$

$$-\frac{a^2-1}{a^4-a^2+1}\int\frac{x}{x^2+x+1}dx-\frac{a^2}{a^4-a^2+1}\int\frac{1}{x^2+x+1}dx=$$

$$=\frac{1}{2}\frac{a^2-1}{a^4-a^2+1}\log(1+a^2x^2)+\frac{|a|}{a^4-a^2+1}\arctan(|a|x)-$$

$$-\frac{a^2-1}{a^4-a^2+1}\left[\frac{\log(x^2+x+1)}{2}-\frac{\arctan\left(\frac{2x+1}{\sqrt{3}}\right)}{\sqrt{3}}\right]-\frac{2}{\sqrt{3}}\frac{a^2}{a^4-a^2+1}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)$$

$$=$$

$$=\frac{1}{2}\frac{a^2-1}{a^4-a^2+1}[\log(1+a^2x^2)-\log(x^2+x+1)]+\frac{|a|}{a^4-a^2+1}\arctan(|a|x)$$

$$-\frac{1}{\sqrt{3}}\frac{a^2+1}{a^4-a^2+1}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)+C$$

$$I(a)=\int_0^\infty\frac{x}{(x^2+x+1)(1+a^2x^2)}dx=\lim_{x\to\infty}I(x)-I(0)$$

Hence,

$$\lim_{x\to\infty}I(x)=\frac{1}{2}\frac{a^2-1}{a^4-a^2+1}\lim_{x\to\infty}[\log(1+a^2x^2)-\log(x^2+x+1)]+$$

$$+\frac{|a|}{a^4-a^2+1}\lim_{x\to\infty}\arctan(|a|x)-\frac{1}{\sqrt{3}}\frac{a^2+1}{a^4-a^2+1}\lim_{x\to\infty}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)=$$

$$=\frac{1}{2}\frac{a^2-1}{a^4-a^2+1}\lim_{x\to\infty}\left[\log\left(\frac{1+a^2x^2}{a^2x^2}\right)-\log\left(\frac{x^2+x+1}{x^2}\right)+2\log|a|\right]+\frac{\pi}{2}\frac{|a|}{a^4-a^2+1}-$$

$$-\frac{\pi}{2\sqrt{3}}\frac{a^2+1}{a^4-a^2+1}=\frac{a^2-1}{a^4-a^2+1}\log|a|+\frac{\pi}{2}\frac{|a|}{a^4-a^2+1}-\frac{\pi}{2\sqrt{3}}\frac{a^2+1}{a^4-a^2+1}$$

$$I(0)=I_{x=0}=-\frac{\pi}{6\sqrt{3}}\frac{a^2+1}{a^4-a^2+1}$$

$$I(a)=\int_0^\infty\frac{x}{(x^2+x+1)(1+a^2x^2)}dx=\lim_{x\to\infty}I(x)-I(0)=$$

$$=\frac{a^2-1}{a^4-a^2+1}\log|a|+\frac{\pi}{2}\frac{|a|}{a^4-a^2+1}-\frac{\pi}{2\sqrt{3}}\frac{a^2+1}{a^4-a^2+1}+\frac{\pi}{6\sqrt{3}}\frac{a^2+1}{a^4-a^2+1}=$$

$$=\frac{a^2-1}{a^4-a^2+1}\log|a|+\frac{\pi}{2}\frac{|a|}{a^4-a^2+1}-\frac{\pi}{3\sqrt{3}}\frac{a^2+1}{a^4-a^2+1}=$$

$$=\frac{9\pi|a|-2\sqrt{3}(1+a^2)\pi+19(-1+a^2)\log|a|}{18(1-a^2+a^4)},a\in R\setminus\{0\}$$

1900. If $0<a\le b$ then:

$$2\int_a^b\int_a^b\int_a^b\left(\frac{y+x}{y+z}+\frac{y+z}{y+x}\right)dxdydz+2(b-a)^3\le 3(b+a)(b-a)^2\log\left(\frac{b}{a}\right)$$

Proposed by Daniel Sitaru-Romania

Solution by Ruxandra Daniela Tonilă-Romania

$$2\int_a^b\int_a^b\int_a^b\left(\frac{y+x}{y+z}+\frac{y+z}{y+x}\right)dxdydz+2(b-a)^3=$$

$$=2(b-a)^3+\int_a^b\int_a^b\int_a^b\left((y+x)\frac{1}{y+z}+(y+z)\frac{1}{y+x}\right)dxdydz\overset{AHM}{\leq}$$

$$\leq 2(b-a)^3+\int_a^b\int_a^b\int_a^b\left((y+x)\cdot\frac{\frac{1}{y}+\frac{1}{z}}{2}+(y+z)\cdot\frac{\frac{1}{y}+\frac{1}{x}}{2}\right)dxdydz=$$

$$=2(b-a)^3+\frac{1}{2}\int_a^b\int_a^b\int_a^b\left(2+\frac{y}{z}+\frac{x}{y}+\frac{x}{z}+\frac{y}{x}+\frac{z}{y}+\frac{z}{x}\right)dxdydz=$$

$$=2(b-a)^3\int_a^b\int_a^b\int_a^b dxdydz+\frac{1}{2}\int_a^b\int_a^b\int_a^b\left(x\left(\frac{1}{y}+\frac{1}{z}\right)+(y+z)\frac{1}{x}+\frac{y}{z}+\frac{z}{y}\right)dxdydz$$

$$=3(b-a)^3+\frac{1}{2}\int_a^b\int_a^b\left(\frac{b^2-a^2}{2}\left(\frac{1}{y}+\frac{1}{z}\right)+(y+z)\log\frac{b}{a}+(b-a)\left(\frac{y}{z}+\frac{z}{y}\right)\right)dydz=$$

$$=3(b-a)^3+\frac{1}{2}\int_a^b\left(\frac{b^2-a^2}{2}\log\frac{b}{a}+\frac{b-a}{z}\cdot\frac{b^2-a^2}{2}+\frac{b^2-a^2}{2}\log\frac{b}{a}\right)dz+$$

$$+\int_a^b\left((b-a)z\log\frac{b}{a}+\frac{b-a}{z}\cdot\frac{b^2-a^2}{2}+(b-a)z\log\frac{b}{a}\right)dz=$$

$$=3(b-a)^3+\frac{3}{2}(b+a)(b-a)^2\log\frac{b}{a}$$

Therefore, we have to prove:

$$3(b-a)^3+\frac{3}{2}(b+a)(b-a)^2\log\frac{b}{a}\leq 3(b+a)(b-a)^2\log\left(\frac{b}{a}\right)\Leftrightarrow$$

$$3(b-a)^3\leq\frac{3}{2}(b+a)(b-a)^2\log\frac{b}{a}\Leftrightarrow$$

$$b-a\leq\frac{b+a}{2}\log\frac{b}{a}\Leftrightarrow\frac{2}{b+a}\leq\frac{\log\frac{b}{a}}{b-a}\Leftrightarrow$$

$$\frac{b+a}{2}\geq\frac{b-a}{\log b-\log a};\forall a,b>0$$

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru