RMM - Geometry Marathon 501 - 600

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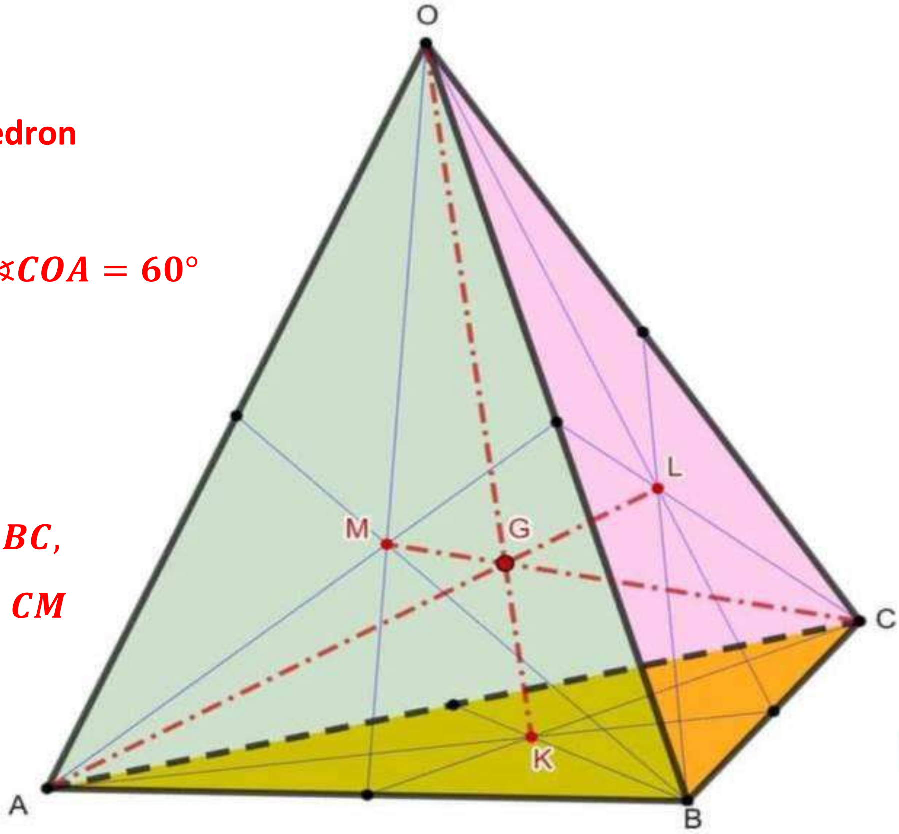
501.



$OABC$ –tetrahedron

$\sphericalangle AOB = \sphericalangle BOC = \sphericalangle COA = 60^\circ$

$OA = OC$

K, L, M –centroids of ΔABC, $\Delta OBC, \Delta OAB$ and $AL \perp CM$

Find: OK

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by proposer

Plagiogonal system: $OA \equiv Ox, OB \equiv Oy, OC \equiv Oz$.

Let $OB = 3b, O(0,0,0), A(3,0,0), B(0,3b,0), C(0,0,3)$

$K(1,b,1), L(0,b,1), M(1,b,0), \overrightarrow{AL}(-3,b,1), \overrightarrow{CM}(1,b,-3)$

$$AL \perp CM \Rightarrow \overrightarrow{AL} \cdot \overrightarrow{CM} = 0 \Rightarrow b^2 - 2b - 1 = 0; b > 0 \Rightarrow b = 1 + \sqrt{3}$$

$$OK^2 = 1^2 + b^2 + 1^2 + 1 \cdot b + b \cdot 1 + 1 = 8 + 4\sqrt{3} \Rightarrow OK = 2\sqrt{2 + \sqrt{2}}.$$

502. In ΔABC the following relationship holds:

$$\sum_{cyc} \sqrt[3]{\frac{{m_a}^2 + m_b m_c}{{m_b}^2 + {m_c}^2}} \geq \frac{3}{2}\left(1 + \frac{4r^2}{R^2}\right).$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By GM − HM inequality we have :

$$\sqrt[3]{\frac{m_a^2+m_bm_c}{m_b^2+m_c^2}}=\sqrt[3]{1.1.\frac{m_a^2+m_bm_c}{m_b^2+m_c^2}}\ge\frac{3}{\frac{1}{1}+\frac{1}{1}+\frac{m_b^2+m_c^2}{m_a^2+m_bm_c}}$$

$$=\frac{3(m_a^2+m_bm_c)}{2m_a^2+(m_b+m_c)^2}\overset{CBS}{\ge}\frac{3(m_a^2+m_bm_c)}{2m_a^2+2(m_b^2+m_c^2)}$$

$$\text{Then : }\sqrt[3]{\frac{m_a^2+m_bm_c}{m_b^2+m_c^2}}\ge\frac{3(m_a^2+m_bm_c)}{2(m_a^2+m_b^2+m_c^2)}\quad(\text{And analogs})$$

$$\text{Thus, }\sum_{cyc}\sqrt[3]{\frac{m_a^2+m_bm_c}{m_b^2+m_c^2}}\ge\frac{3}{2}.\frac{(m_a^2+m_b^2+m_c^2)+(m_am_b+m_bm_c+m_cm_a)}{m_a^2+m_b^2+m_c^2}$$

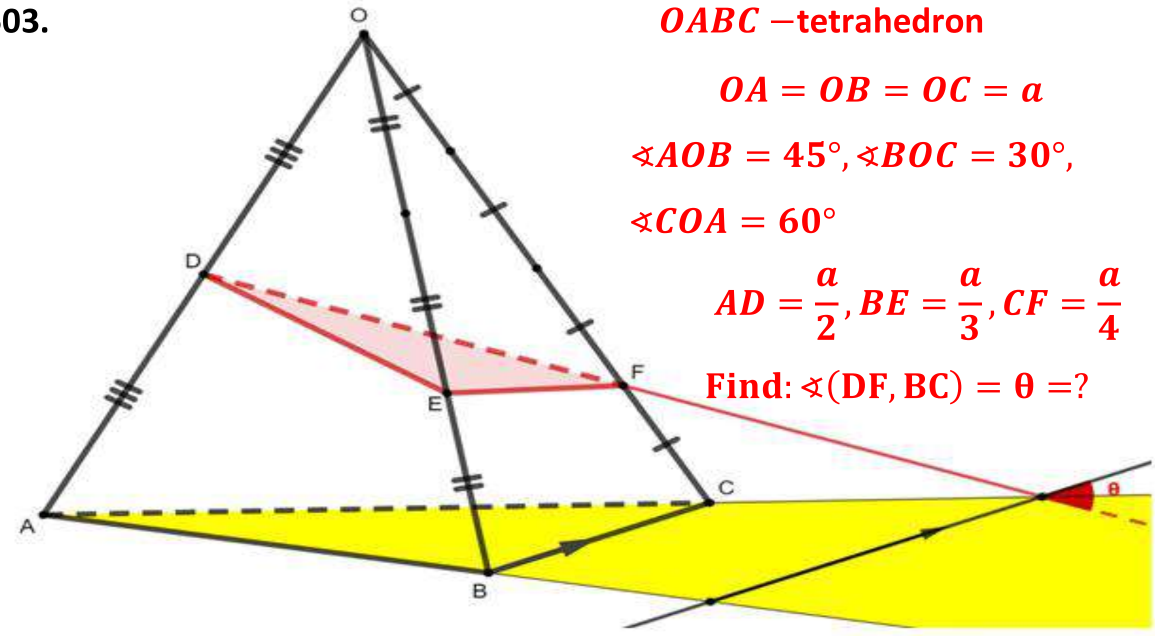
$$=\frac{3}{2}\left(1+\frac{m_am_b+m_bm_c+m_cm_a}{m_a^2+m_b^2+m_c^2}\right)$$

$$\text{We have : }\sum_{cyc}m_bm_c\ge\sum_{cyc}h_bh_c=\frac{2s^2r}{R}\ge 27r^2\ \text{ and}$$

$$\sum_{cyc}m_a^2=\frac{3}{4}\sum_{cyc}a^2\overset{Leibniz}{\le}\frac{3}{4}.9R^2=\frac{27R^2}{4}$$

$$\text{Therefore, }\sum_{cyc}\sqrt[3]{\frac{m_a^2+m_bm_c}{m_b^2+m_c^2}}\ge\frac{3}{2}\left(1+\frac{27r^2}{\frac{27R^2}{4}}\right)=\frac{3}{2}\left(1+\frac{4r^2}{R^2}\right).$$

503.



$OABC$ −tetrahedron

$OA=OB=OC=a$

$\sphericalangle AOB=45°,\sphericalangle BOC=30°,$

$\sphericalangle COA=60°$

$AD=\frac{a}{2},BE=\frac{a}{3},CF=\frac{a}{4}$

Find: $\sphericalangle(DF,BC)=\theta=?$

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by proposer

Plagiogonal 3rd system $OA \equiv Ox, OB \equiv Oy, OC \equiv Oz$

$$O(0,0,0), A(a,0,0), B(0,a,0), C(0,0,a)$$

$$D\left(\frac{a}{2},0,0\right), E\left(\frac{0,2a}{3},0\right), F\left(0,0,\frac{3a}{4}\right)$$

$$\overrightarrow{DF}\left(-\frac{a}{2},0,\frac{3a}{4}\right), \overrightarrow{BC}(0,-a,a)$$

$$\overrightarrow{DF}\cdot\overrightarrow{BC} = 0+90+\frac{3a^2}{4}+\frac{a^2}{2}+\frac{\sqrt{2}}{-2}+\left(-\frac{3a^2}{4}\right)\frac{\sqrt{3}}{2}+\left(-\frac{a^2}{2}\right)\frac{1}{2} = \frac{4+2\sqrt{2}-3\sqrt{3}}{8}a^2$$

$$\left|\overrightarrow{DF}\right|^2 = \frac{a^2}{4}+\frac{9a^2}{16}+2\left(-\frac{a}{2}\right)\frac{3a}{4}\cdot\frac{1}{2} = \frac{7a^2}{16} \Rightarrow \left|\overrightarrow{DF}\right| = \frac{a\sqrt{7}}{4}$$

$$\left|\overrightarrow{BC}\right|^2 = a^2+a^2+\frac{2(-a)(a)\sqrt{3}}{2} = a^2(2-\sqrt{3}) \Rightarrow \left|\overrightarrow{BC}\right| = a\sqrt{2-\sqrt{3}}$$

$$\cos\theta = \frac{\overrightarrow{DF}\cdot\overrightarrow{BC}}{\left|\overrightarrow{DF}\right|\cdot\left|\overrightarrow{BC}\right|} = \frac{\frac{4+2\sqrt{2}-3\sqrt{3}}{8}}{\frac{\sqrt{7}}{4}\sqrt{2-\sqrt{3}}} \Rightarrow \cos\theta = \frac{4-5\sqrt{2}+4\sqrt{3}+\sqrt{6}}{4\sqrt{7}} \Rightarrow \theta \cong 53.42^\circ$$

504. **In $\Delta ABC, K$ −Lemoine's point. Prove that:**

$$AK\cdot BC+BK\cdot CA+CK\cdot AB \le \frac{4(R+r)(4R+r)}{3\sqrt{3}}$$

Proposed by Marian Ursărescu-Romania

Solution by Ertan Yildirim-Turkiye

$$AK = \frac{2bc}{a^2+b^2+c^2}\cdot m_a; (,K - \text{Lemoine's point}); (1)$$

$$\sum a^2 = 2(s^2-r^2-4Rr); (2)$$

$$\sum m_a \le 4R+r; (3)$$

$$s\sqrt{3} \le 4R+r\ (Doucet); (4)$$

$$AK\cdot BC+BK\cdot CA+CK\cdot AB = \sum_{cyc}\frac{2bc}{a^2+b^2+c^2}\cdot am_a =$$

$$=\frac{2abc}{a^2+b^2+c^2}\sum_{cyc}m_a\overset{(?)}{\leq}\frac{4(R+r)(4R+r)}{3\sqrt{3}}$$

$$\frac{2abc}{a^2+b^2+c^2}\sum_{cyc}m_a\overset{(3)}{\leq}\frac{2abc}{a^2+b^2+c^2}(4R+r)\overset{(?)}{\leq}\frac{4(R+r)(4R+r)}{3\sqrt{3}}$$

$$\frac{2abc}{a^2+b^2+c^2}\leq\frac{4(R+r)}{3\sqrt{3}}\Rightarrow\frac{2\cdot 4Rr\cdot 3s\sqrt{3}}{2(s^2-r^2-4Rr)}\leq 4(R+r)$$

$$\frac{Rr\cdot 3s\sqrt{3}}{s^2-r^2-4Rr}\overset{((4),Gerretsen)}{\leq}\frac{Rr\cdot 3(4R+r)}{16Rr-5r^2-r^2-4Rr}\leq R+r$$

$$12R^2r+3Rr^2\leq(R+r)(12Rr-6r^2)$$

$$12R^2r+3Rr^2\leq 12R^2r-6Rr^2+12Rr^2-6r^3$$

$$6r^3\leq 3Rr^2\Leftrightarrow R\geq 2r(Euler)$$

505. Let ΔDEF be the pedal triangle of incenter in ΔABC. Prove that:

$$\sqrt{\frac{EF}{h_a}}+\sqrt{\frac{FD}{h_b}}+\sqrt{\frac{DE}{h_c}}\leq\sqrt[4]{27}\cdot\frac{R}{2r}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $a=BC, b=CA,$

$c=AB$ be the side lengths of ΔABC and F be the area of ΔABC.

We know that : $AE=\frac{bc}{a+c}$ and $AF=\frac{bc}{a+b}$. By the Law of cosines in ΔAEF :

$$EF^2=AE^2+AF^2-2.AE.AF.\cos A$$

$$=\left(\frac{bc}{a+c}\right)^2+\left(\frac{bc}{a+b}\right)^2-2\left(\frac{bc}{a+c}\right)\left(\frac{bc}{a+b}\right).\frac{b^2+c^2-a^2}{2bc}=$$

$$=\frac{b^2c^2}{(a+c)^2}+\frac{b^2c^2}{(a+b)^2}-\frac{bc[(b-c)^2+2bc-a^2]}{(a+b)(a+c)}$$

$$=b^2c^2\left(\frac{1}{a+c}-\frac{1}{a+b}\right)^2-\frac{bc(b-c)^2}{(a+b)(a+c)}+\frac{a^2bc}{(a+b)(a+c)}=$$

$$=\frac{b^2c^2(b-c)^2}{(a+b)^2(a+c)^2}-\frac{bc(b-c)^2}{(a+b)(a+c)}+\frac{a^2bc}{(a+b)(a+c)}$$

$$=\frac{a^2bc}{(a+b)(a+c)}-\frac{bc(b-c)^2[(a+b)(a+c)-bc]}{(a+b)^2(a+c)^2}=$$

$$=\frac{a^2bc}{(a+b)(a+c)}-\frac{2sabc(b-c)^2}{(a+b)^2(a+c)^2}\le\frac{a^2bc}{(a+b)(a+c)}\overset{AM-GM}{\le}\left[\frac{a}{2}\left(\frac{b}{a+b}+\frac{c}{a+c}\right)\right]^2$$

$$Then:\ EF\le\frac{a}{2}\left(\frac{b}{a+b}+\frac{c}{a+c}\right)\quad (And\ analogs)$$

$$Now,by\ CBS\ inequality\ we\ have:\sum_{cyc}\sqrt{\frac{EF}{h_a}}\le\sum_{cyc}\sqrt{\frac{a}{2F}\cdot\frac{a}{2}\left(\frac{b}{a+b}+\frac{c}{a+c}\right)}$$

$$\le\sqrt{\left(\sum\frac{a^2}{4F}\right)\left[\sum\left(\frac{b}{a+b}+\frac{c}{a+c}\right)\right]}$$

$$With:\sum\left(\frac{b}{a+b}+\frac{c}{a+c}\right)=\sum\left(\frac{b}{a+b}+\frac{a}{b+a}\right)=\sum 1=3$$

$$And:\sum a^2\le 9R^2\ (Leibniz's\ inequality)\ and\ F=sr$$

$$\ge 3\sqrt{3}r^2\ (Mitrinovic's\ inequality)$$

$$Therefore,\quad \sum_{cyc}\sqrt{\frac{EF}{h_a}}\le\sqrt{\frac{9R^2}{4.3\sqrt{3}r^2}.3}=\sqrt[4]{27}.\frac{R}{2r}.$$

506. In ΔABC the following relationship holds:

$$\frac{\sqrt{3}}{2R^2r}\le\left(\frac{1}{a}+\frac{1}{b}\right)\left(\frac{1}{b}+\frac{1}{c}\right)\left(\frac{1}{c}+\frac{1}{a}\right)+\frac{1}{abc}\le\frac{\sqrt{3}}{4Rr^2}$$

Proposed by Ertan Yildirim-Izmir-Turkiye

Solution 1 by Avishek Mitra-West Bengal-India

$$\frac{1}{abc}+\prod_{cyc}\left(\frac{1}{a}+\frac{1}{b}\right)=\frac{1}{abc}+\frac{1}{a^2b^2c^2}\prod_{cyc}(a+b)=\frac{abc+2abc+\sum ab(a+b)}{a^2b^2c^2}=$$

$$=\frac{3abc+\sum ab(2s-c)}{a^2b^2c^2}=\frac{3abc-3abc+2s\sum ab}{a^2b^2c^2}\le$$

$$\le\frac{(\sum a)\cdot\frac{(\sum a)^2}{3}}{a^2b^2c^2}=\frac{(2s)^2}{3\cdot 16R^2r^2s^2}=\frac{s}{6R^2r^2}\overset{Mitrinovic}{\le}\frac{3\sqrt{3}R}{12R^2r^2}=\frac{\sqrt{3}}{4Rr^2}$$

$$\frac{1}{abc}+\prod_{cyc}\left(\frac{1}{a}+\frac{1}{b}\right)=2s\frac{\sum ab}{\prod a^2}=\frac{2s(s^2+r^2+4Rr)}{16R^2r^2s^2}=\frac{s^2+r^2+4Rr}{8R^2\,r^2s}$$

We need to show:

$$\frac{s^2+r^2+4Rr}{8R^2\,r^2s}\ge\frac{\sqrt{3}}{2R^2r}\Leftrightarrow s^2+r^2+4Rr\ge 4\sqrt{3}rs$$

$$4\sqrt{3}rs\overset{Mitrinovic}{\le}4\sqrt{3}r\cdot\frac{3\sqrt{3}R}{2}=18Rr$$

$$s^2+r62+4Rr\ge 18Rr\Leftrightarrow s^2\ge 14Rr-r^2$$

$$s^2\ge 16Rr-5r^2(Gerretsen)$$

$16Rr-5r^2\ge 14Rr-r^2\Leftrightarrow 2Rr\ge 4r^2\Leftrightarrow 2r(R-2r)\ge 0$ true from $R\ge 2r$ $(Euler)$.

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\frac{1}{abc}+\prod_{cyc}\left(\frac{1}{a}+\frac{1}{b}\right)=\frac{\sum ab}{8R^2r^2s};(*),\sum ab\le 2\sqrt{3}Rs;(1),\sum ab\ge 4\sqrt{3}F;(2)$$

$$(*),(1)\Rightarrow\frac{\sum ab}{8R^2r^2s}\le\frac{2\sqrt{3}Rs}{8R^2r^2s}=\frac{\sqrt{3}}{4Rr^2},\qquad (*),(2)\Rightarrow\frac{\sum ab}{8R^2r^2s}\ge\frac{4\sqrt{3}Rs}{8R^2r^2s}=\frac{\sqrt{3}}{2R^2r}$$

507.

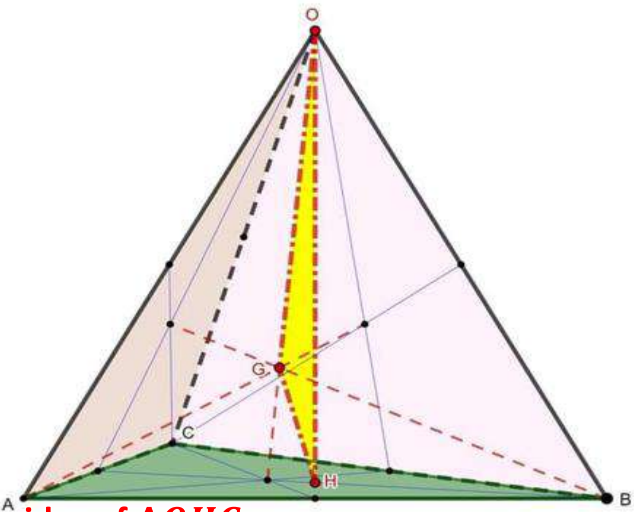
$(OABC)$ –tetrahedron

$\sphericalangle AOB=\sphericalangle BOC=\sphericalangle COA=\theta, 0<\theta\le\frac{\pi}{2}$

$OA=OB=8, OC=5$

$(ABC)\equiv(P), H\in(P), OH\perp(P)$

G –centroid of $(OABC)$



$V_{(OABC)}=\frac{80\sqrt{2}}{3}(volume)$. Find the lengths sides of ΔOHG

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by proposer

Plagiogonal 3[rd] system: $OA \equiv Ox, OB \equiv Oy, OC \equiv Oz$

$$V = \frac{80\sqrt{2}}{3} = \frac{8 \cdot 5 \cdot 5}{6}\sqrt{1 - 3cos^2\theta + 2\cos^2\theta} \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = 60^\circ$$

$$O(0,0,0), A(8,0,0), B(0,8,0), C(0,0,5), G\left(2,2,\frac{5}{4}\right), H(h_1,h_2,h_3)$$

$$\overrightarrow{AB}(-8,8,0);\ \overrightarrow{AC}(-8,0,5); \overrightarrow{BC}(0,-8,5), \overrightarrow{OH}(h_1,h_2,h_3)$$

$$(P): \frac{x}{8} + \frac{y}{8} + \frac{z}{5} = 1 \Rightarrow 5x + 5y + 8z = 1$$

$$H \in (P): 5p_1 + 5p_2 + 8p_3 = 40; (1), OH \perp (P) \Rightarrow \overrightarrow{OH} \cdot \overrightarrow{AB} = 0 \Rightarrow$$

$$-8h_1 + 8h_2 + 0 \cdot h_3 + (-8h_2 + 8h_1) \cdot \frac{1}{2} + (8h_3) \cdot \frac{1}{2} + (-8h_2) \cdot \frac{1}{2} = 0; (2)$$

$$OH \perp (P): \overrightarrow{OH} \cdot \overrightarrow{AC} = 0 \Rightarrow$$

$$-8h_1 + 0h_2 + 5h_3 + (-8h_2) \cdot \frac{1}{2} + (5h_2) \cdot \frac{1}{2} + (-8h_3 + 5h_1) \cdot \frac{1}{2} = 0; (3)$$

From (1),(2),(3) we get:

$$OH^2 = h_1^2 + h_2^2 + h_3^2 + h_1h_2 + h_2h_3 + h_3h_1 \Rightarrow OH^2 = \frac{800}{33} \Rightarrow OH = 20\sqrt{\frac{2}{33}}$$

$$OG^2 = 2^2 + 2^2 + \left(\frac{5}{4}\right)^2 + 4 + 2 \cdot 2 \cdot \frac{5}{4} = \frac{287}{16} \Rightarrow OG = \frac{3\sqrt{33}}{4}$$

$$G\left(2,2,\frac{5}{4}\right), g_1 = 2; g_2 = 2; g_3 = \frac{5}{4}$$

$$HG^2 = (g_1 - h_1)^2 + (g_2 - h_2)^2 + (g_3 - h_3)^2 + (g_1 - h_1)(g_2 - h_2) + (g_2 - h_2)(g_3 - h_3) + (g_3 - h_3)(g_1 - h_1)$$

$$HG^2 = \frac{3401}{528} \Rightarrow HG = \frac{1}{4}\sqrt{\frac{3401}{33}}$$

508. In any Δ ABC the following relationship holds:

$$3\left(\sqrt[5]{\sin A} + \sqrt[5]{\sin B} + \sqrt[5]{\sin C}\right) \le \pi\left(\frac{\sqrt[5]{\sin A}}{\mu(A)} + \frac{\sqrt[5]{\sin B}}{\mu(B)} + \frac{\sqrt[5]{\sin C}}{\mu(C)}\right)$$

Proposed by Neculai Stanciu-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$:\text{Let } f(x)=\frac{\pi.\sqrt[5]{\sin x}}{x}-3.\sqrt[5]{\sin x}\ \forall\, x\in(0,\pi)\ \therefore f''(x)$$
$$=\frac{\left(3-\frac{\pi}{x}\right).\sqrt[5]{\sin x}}{5}+\frac{4\left(3-\frac{\pi}{x}\right)\cos^2 x}{25(\sin x)^{\frac{9}{5}}}+\frac{2\pi}{x^2}\left(\frac{\sqrt[5]{\sin x}}{x}-\frac{\cos x}{5(\sin x)^{\frac{4}{5}}}\right)$$
$$=\frac{5\left(3-\frac{\pi}{x}\right)\sin^2 x+4\left(3-\frac{\pi}{x}\right)\cos^2 x}{25(\sin x)^{\frac{9}{5}}}+\frac{2\pi}{x^2}.\frac{5\sin x-x\cos x}{5x(\sin x)^{\frac{4}{5}}}$$
$$=\frac{4\left(3-\frac{\pi}{x}\right)+\left(3-\frac{\pi}{x}\right)\sin^2 x}{25(\sin x)^{\frac{9}{5}}}+\frac{2\pi}{x^2}.\frac{5\sin x-x\cos x}{5x(\sin x)^{\frac{4}{5}}}$$
$$\therefore f''(x)\overset{(*)}{=}\frac{\left(3-\frac{\pi}{x}\right)(4+\sin^2 x)}{25(\sin x)^{\frac{9}{5}}}+\frac{2\pi}{x^2}.\frac{5\sin x-x\cos x}{5x(\sin x)^{\frac{4}{5}}}$$

$\boxed{\text{Case 1}}$ $x\in\left[\frac{\pi}{3},\pi\right)\ \therefore 3\geq\frac{\pi}{x}\Rightarrow 3-\frac{\pi}{x}\overset{(i)}{\geq}0$ *and* if $F(x)=\sin x-x\cos x\ \forall\, x\in[0,\pi)$, then : $F'(x)$
$=x\sin x\geq 0\ \because 0\leq x<\pi\Rightarrow F(x)$ is $\uparrow$ on $[0,\pi)\Rightarrow F(x)\geq F(0)=0$

$\therefore \sin x-x\cos x\overset{(ii)}{\geq}0\ \forall\, x\in[0,\pi)$ with equ*al*ity occurring iff $x=0\Rightarrow\forall\, x$
$\in(0,\pi),\ \sin x-x\cos x>0\Rightarrow 5\sin x-x\cos x>4\sin x>0$

$$\therefore 5\sin x-x\cos x\overset{(iii)}{>}0$$

$$\therefore (i),(iii)\Rightarrow\frac{\left(3-\frac{\pi}{x}\right)(4+\sin^2 x)}{25(\sin x)^{\frac{9}{5}}}+\frac{2\pi}{x^2}.\frac{5\sin x-x\cos x}{5x(\sin x)^{\frac{4}{5}}}>0\Rightarrow via\ (*),\ f''(x)>0$$

$\boxed{\text{Case 2}}$ $x\in\left(0,\frac{\pi}{3}\right)$ *and via* $(*)$, $f''(x)=\dfrac{\left(3-\frac{\pi}{x}\right)x^3(4+\sin^2 x)+2\pi\sin x(5\sin x-x\cos x)}{25x^3(\sin x)^{\frac{9}{5}}}$

$$\Rightarrow f''(x)\overset{(**)}{=}\frac{(3x^3-\pi x^2+50\pi)\sin^2 x-10\pi x\cos x\sin x+12x^3-4\pi x^2}{25x^3(\sin x)^{\frac{9}{5}}}$$

Let $h(x)=(3x^3-\pi x^2+50\pi)\sin^2 x-10\pi x\cos x\sin x+12x^3-4\pi x^2\ \forall\, x\in\left[0,\frac{\pi}{3}\right)\ \therefore h'(x)$
$=2(3x^3-\pi x^2+45\pi)\cos x\sin x-9x(2\pi+x)\cos^2 x+45x^2$

Now, $\because 0\leq x<\frac{\pi}{3}<\frac{3}{2}\ \therefore x^2\overset{(\bullet)}{<}\frac{9}{4}<45\Rightarrow 45-x^2>0\Rightarrow\pi(45-x^2)>0\Rightarrow 3x^3-\pi x^2+45\pi$
$>0\ \therefore h'(x)=2(3x^3-\pi x^2+45\pi)\cos x\sin x-9x(2\pi+x)\cos^2 x+45x^2$
$\overset{\text{via (ii)}}{\geq}2(3x^3-\pi x^2+45\pi)\cos x.x\cos x-9x(2\pi+x)\cos^2 x$
$+45x^2\overset{\frac{1}{4}<\cos^2 x\leq 1\ \because\ 0\leq x<\frac{\pi}{3}}{\geq}2(3x^3-\pi x^2+45\pi)x.\frac{1}{4}-9x(2\pi+x)+45x^2$
$=x.\dfrac{3x^3+72x+\pi(9-x^2)}{2}$

$$= x.\frac{3x^3+72x+\pi\left(\frac{27}{4}+\frac{9}{4}-x^2\right)}{2} \overset{\text{via }(\bullet)}{\geq} x.\frac{3x^3+72x+\frac{27\pi}{4}}{2} \overset{x\geq 0}{\geq} 0 \Rightarrow h'(x)\geq 0\ \forall\, x\in\left[0,\frac{\pi}{3}\right)$$

$$\Rightarrow h(x) \text{ is } \uparrow \text{ on } \left[0,\frac{\pi}{3}\right) \Rightarrow h(x)\geq h(0)=0 \text{ with equality occurring iff } x=0$$

$$\Rightarrow \forall\, x\in\left(0,\frac{\pi}{3}\right), h(x)=(3x^3-\pi x^2+50\pi)\sin^2 x-10\pi x\cos x\sin x+12x^3-4\pi x^2>0$$

$$\Rightarrow \textit{via } (**), f''(x)>0 \therefore \text{ combining both cases}, \forall\, x\in(0,\pi), f''(x)>0$$

$$\Rightarrow f(x) \text{ is convex} \Rightarrow \sum_{cyc}\left(\frac{\pi.\sqrt[5]{\sin A}}{\mu(A)}-3.\sqrt[5]{\sin A}\right) \overset{\text{Jensen}}{\geq} 3\left(\frac{\pi.\sqrt[5]{\sin\frac{\pi}{3}}}{\frac{\pi}{3}}-3.\sqrt[5]{\sin\frac{\pi}{3}}\right)=0$$

$$\Rightarrow \sum_{cyc}\frac{\pi.\sqrt[5]{\sin A}}{\mu(A)}\geq 3\sum_{cyc}\sqrt[5]{\sin A}$$

$$\Rightarrow 3\left(\sqrt[5]{\sin A}+\sqrt[5]{\sin B}+\sqrt[5]{\sin C}\right)\leq \pi\left(\frac{\sqrt[5]{\sin A}}{\mu(A)}+\frac{\sqrt[5]{\sin B}}{\mu(B)}+\frac{\sqrt[5]{\sin C}}{\mu(C)}\right) \text{ (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\text{Let } f(x)=\frac{\sqrt[5]{\sin x}}{x}, x\in(0,\pi). \text{ We have: } f'(x)=\frac{x\cos x-5\sin x}{5x^2\sqrt[5]{\sin^4 x}}, \forall x\in(0,\pi)$$

If $x\in\left[\frac{\pi}{2},\pi\right)$ we have: $\cos x\leq 0$ then $f'(x)\leq 0$

If $x\in\left(0,\frac{\pi}{2}\right)$ we have: $x\leq \tan x\leq 5\tan x$, then $x\cos x-5\sin x\leq 0$ and $f'(x)\leq 0$. Thus, f – is decreasing on $(0,\pi)$.

WLOG we may assume that $\mu(A)\geq\mu(B)\geq\mu(C)$. We have: $\frac{\sqrt[5]{\sin A}}{\mu(A)}\leq\frac{\sqrt[5]{\sin B}}{\mu(B)}\leq\frac{\sqrt[5]{\sin C}}{\mu(C)}$

By Chebishev's inequality, we have:

$$3\left(\sqrt[5]{\sin A}+\sqrt[5]{\sin B}+\sqrt[5]{\sin C}\right)=3\left(\mu(A)\cdot\frac{\sqrt[5]{\sin A}}{\mu(A)}+\mu(B)\cdot\frac{\sqrt[5]{\sin B}}{\mu(B)}+\mu(C)\cdot\frac{\sqrt[5]{\sin C}}{\mu(C)}\right)\leq$$

$$\leq\left(\mu(A)+\mu(B)+\mu(C)\right)\left(\frac{\sqrt[5]{\sin A}}{\mu(A)}+\frac{\sqrt[5]{\sin B}}{\mu(B)}+\frac{\sqrt[5]{\sin C}}{\mu(C)}\right)=$$

$$=\pi\left(\frac{\sqrt[5]{\sin A}}{\mu(A)}+\mu(B)\cdot\frac{\sqrt[5]{\sin B}}{\mu(B)}+\mu(C)\cdot\frac{\sqrt[5]{\sin C}}{\mu(C)}\right)$$

509. In ΔABC the following relationship holds:

$$\sqrt{(a+b)(a+c)bc}+\sqrt{(b+c)(b+a)ca}+\sqrt{(c+a)(c+b)ab}\leq 3s^2-r^2-4Rr$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Avishek Mitra-West Bengal-India

$$\sum_{cyc}\sqrt{(a+b)(a+c)bc}\overset{AGM}{\leq}\sum_{cyc}\frac{b+c+2a}{2}\cdot\frac{b+c}{2}=$$

$$=\frac{1}{4}\left[\sum_{cyc}(b+c)^2+2\sum_{cyc}a(b+c)\right]=\frac{1}{4}\left[2\sum_{cyc}a^2+6\sum_{cyc}ab\right]=$$

$$=\frac{1}{4}[2(2s^2-8Rr-2r^2)+6(s^2+r^2+4Rr)]=\frac{1}{4}(10s^2+8Rr+2r^2)$$

Need to show:

$$\frac{1}{4}(10s^2+8Rr+2r^2)\le 3s^2-r^2-4Rr$$

$$10s^2+8Rr+2r^2\le 12s^2-4r^2-16Rr\Leftrightarrow 2s^2\ge 24Rr+2r^2$$

But $s^2\ge 16Rr-5r^2(Gerretsen)$. **We must show:**

$$16Rr-5r^2\ge 12Rr+3r^2\Leftrightarrow 4Rr-8r^2\ge 0\Leftrightarrow R\ge 2r(Euler).$$

Solution 2 by Ertan Yildirim-Turkiye

$$ab+bc+ca=s^2+r^2+4Rr,\ \sum\nolimits_{cyc}\sqrt{(a+b)(a+c)bc}\le$$

$$\le\sqrt{(a+b)c+(b+c)a+(c+a)b}\cdot\sqrt{(a+c)b+(b+a)c+(c+b)a}=$$

$$=\sqrt{2(ab+bc+ca)}\cdot\sqrt{2(ab+bc+ca)}=2(ab+bc+ca)=2(s^2+R^2+4rR)$$

We must to prove that:

$$2s^2+2r^2+8Rr\le 3s^2-r^2-4Rr\Leftrightarrow 3r^2+12Rr\le s^2$$

$$3r^2+12Rr\overset{Gerretsen}{\le}16Rr-5r^2\le s^2\Leftrightarrow R\ge 2r\ (Euler).$$

510. In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{a}{6a+b+c}\le\frac{3}{16}\cdot\frac{R}{r}$$

Proposed by Kostas Geronikolas-Greece

Solution by Marian Ursărescu-Romania

We must show that:

$$\sum_{cyc}\frac{a}{5a+2s}\le\frac{3}{16}\cdot\frac{R}{r}\Leftrightarrow\sum_{cyc}\frac{5a}{5a+2s}\le\frac{15}{16}\cdot\frac{R}{r}$$

$$\sum_{cyc}\frac{5a+2s-2s}{5a+2s}\le\frac{15}{16}\cdot\frac{R}{r}\Leftrightarrow\sum_{cyc}\left(1-\frac{2s}{5a+2s}\right)\le\frac{15}{16}\cdot\frac{R}{r}$$

$$3-2s\sum_{cyc}\frac{1}{5a+2s}\le\frac{15}{16}\cdot\frac{R}{r}\Leftrightarrow 2s\sum_{cyc}\frac{1}{5a+2s}\ge 3-\frac{15}{16}\cdot\frac{R}{r};(1)$$

$$\sum_{cyc}\frac{1}{5a+2s}\overset{Bergstrom}{\ge}\frac{9}{5(a+b+c)+6s}=\frac{9}{16s};(2)$$

From (1) and (2) we must show:

$$2s\cdot\frac{9}{16p}\ge 3-\frac{15}{16}\cdot\frac{R}{r}\Leftrightarrow\frac{15}{16}\cdot\frac{R}{r}\ge 3-\frac{18}{16}\Leftrightarrow\frac{15}{16}\cdot\frac{R}{r}\ge\frac{30}{16}\Leftrightarrow R\ge 2r(Euler)$$

511.

$OABC$ –tetrahedron,

$OA=OB=OC=4$

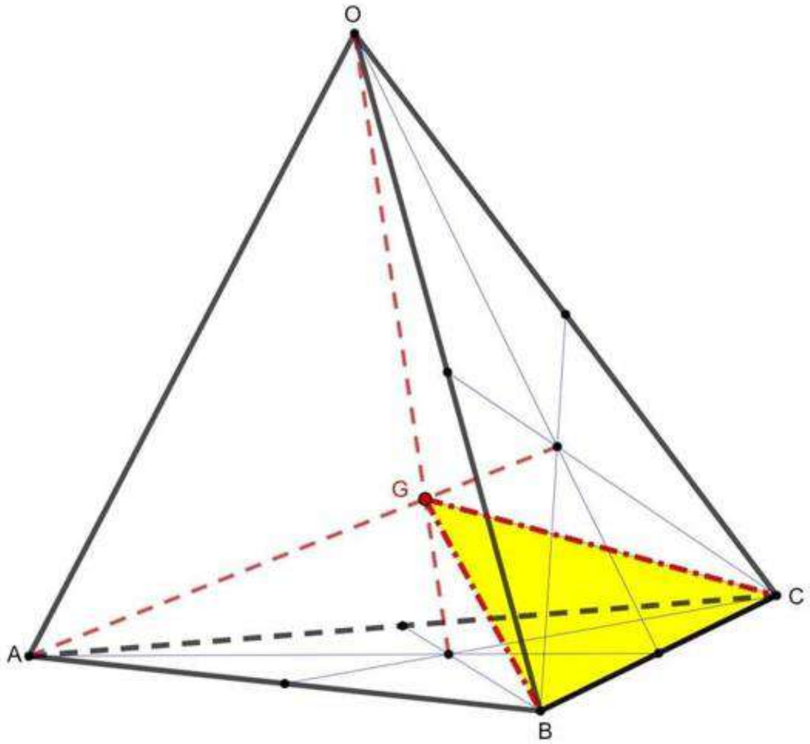
$\sphericalangle APB=45°,\sphericalangle BOC=30°,$

$\sphericalangle COA=60°$

G –centroid

Find:

$[GBC]=?\,(area)$



Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by proposer

Plagiogonal 3[rd] system: $OA\equiv Ox, OB\equiv Oy, OC\equiv Oz$

$$O(0,0,0),A(4,0,0),B(0,4,0),C(0,0,4),G(1,1,1),H(0,h_2,h_3)$$

$$BC:\frac{y-4}{4-0}=\frac{z-0}{0-4},H\in(BC)\Rightarrow h_2-4=-h_3\Rightarrow h_2+h_3=4;(1)$$

$$\overrightarrow{GH}(-1,h_2-1,h_3-1),\overrightarrow{BC}(0,-4,4),\qquad\overrightarrow{GH}\cdot\overrightarrow{BC}=0\Rightarrow$$

$$4(1-h_2)+4(h_3-1)+4\cdot\frac{\sqrt{2}}{2}+[4(h_2-1)-4(h_4-1)]\cdot\frac{\sqrt{3}}{2}+(-4)\cdot\frac{1}{2}$$

$$GH^2=(0-1)^2+(h_2-1)^2+(h_3-2)^2+2(-1)(h_2-1)\cdot\frac{\sqrt{2}}{2}+$$

$$+2(h_2-1)(h_3-1)\cdot\frac{\sqrt{3}}{2}+2(-1)(h_3-1)\cdot\frac{1}{2}$$

$$GH^2=\frac{2+\sqrt{3}+2\sqrt{6}}{4},\qquad GH=\frac{1}{2}\sqrt{2+\sqrt{3}+2\sqrt{6}}$$

$$BC^2=4^2+4^2-2\cdot4\cdot4\cdot\frac{\sqrt{3}}{2}=16(2-\sqrt{3})\Rightarrow BC=4\sqrt{2-\sqrt{3}}$$

$$[GBC]=\frac{1}{2}GH\cdot BC=\sqrt{1-6\sqrt{3}+4\sqrt{6}}$$

512. In acute $\Delta ABC, \mu(A)\le\mu(B)\le\mu(C)$. Prove that : $2\sin B\ge\cot\frac{C}{2}$.

Proposed by Cristian Miu-Romania

Solution Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:\ 2\sin B\ge\cot\frac{C}{2}\leftrightarrow 2\sin B.\sin C\ge\cot\frac{C}{2}.\sin C=2\cos^2\frac{C}{2}$$

$$\leftrightarrow\ \cos(C-B)-\cos(C+B)\ge1+\cos C\ \leftrightarrow\ \cos(C-B)+\cos A\ge1+\cos C$$

$$\leftrightarrow\ \cos(C-B)-\cos C\ge1-\cos A\ \leftrightarrow\ 2\sin\left(\frac{B}{2}\right).\sin\left(\frac{2C-B}{2}\right)\ge2\sin^2\frac{A}{2}$$

$$Which\ is\ true\ because\ \mu\left(\frac{A}{2}\right)\le\mu\left(\frac{B}{2}\right)\le\mu\left(\frac{2C-B}{2}\right)<\frac{\pi}{2}\ \ and\ \ x$$

$$\rightarrow\sin x\ is\ increasing\ on\ \left(0,\frac{\pi}{2}\right)$$

$$Therefore,\ 2\sin B\ge\cot\frac{C}{2}.\ Equality\ holds\ iff\ \mu(A)=\mu(B)=\mu(C)=\frac{\pi}{3}.$$

513. In ΔABC the following relationship holds:

$$\frac{21}{2}+\sum_{cyc}\frac{\sin^4 A}{\sin^4 B+\sin^4 C}\le12\left(\frac{R}{2r}\right)^4$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution by Marian Ursărescu-Romania

From sines law: $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2R$

We must show that

$$\frac{21}{2}+\sum_{cyc}\frac{a^4}{b^4+c^4}\le 12\cdot\frac{R^4}{16r^4}\Leftrightarrow\frac{21}{2}+\sum_{cyc}\frac{a^4+b^4+c^4-b^4-c^4}{b^4+c^4}\le\frac{3R^4}{4r^4}$$

$$\frac{21}{2}+(a^4+b^4+c^4)\sum_{cyc}\frac{1}{b^4+c^4}-3\le\frac{3}{4}\cdot\frac{R^4}{r^4}$$

$$(a^4+b^4+c^4)\sum_{cyc}\frac{1}{b^4+c^4}\le\frac{3}{4}\cdot\frac{R^4}{r^4}-\frac{15}{2}=\frac{3(R^4-10r^4)}{4r^4};(1)$$

$$\because a^4+b^4+c^4\le 54R^3(R-r);(2)$$

$$\sum_{cyc}\frac{1}{b^4+c^4}\le\sum_{cyc}\frac{1}{2b^2c^2}=\frac{1}{2}\cdot\frac{a^2+b^2+c^2}{a^2b^2c^2}\le\frac{9R^2}{2\cdot(4Rrs)^2}=$$

$$=\frac{9R^2}{32R^2r^2s^2}=\frac{9}{32r^2s^2}\overset{Mitrinovic}{\le}\frac{9}{32\cdot 27r^4};;(3)$$

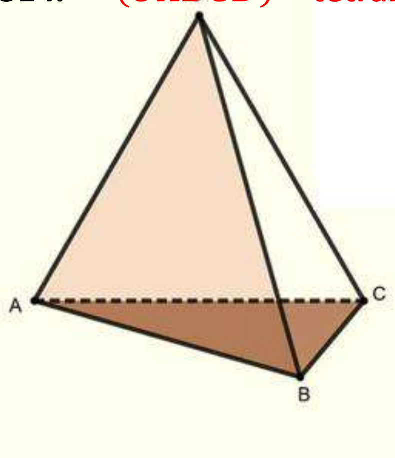
From (1),(2) and (3) we must show:

$$54R^3(R-r)\cdot\frac{9}{32\cdot 27r^4}\le\frac{3(R^4-10r^4)}{4r^4}\Leftrightarrow$$

$$3R^4-3R^3r\le 4R^4-40r^4\Leftrightarrow R^4+3R^3r-40r^4\ge 0\Leftrightarrow$$

$$(R-2r)(R^3+5R^2r+5Rr^2+20r^3)\ge 0 \text{ true from } R\ge 2r(Euler)$$

514. $(OABCD)$ −tetrahedron

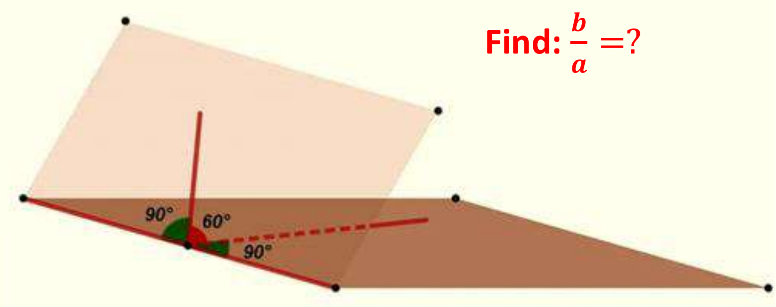


$OA=OC=a, OB=b$

$\sphericalangle AOB=\sphericalangle BOC=\sphericalangle COA=60^\circ$

$(OAB)=P,(OBC)=Q,\sphericalangle(P,Q)=\theta=60^\circ.$

Find: $\frac{b}{a}=?$



Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by proposer

Plagiogonal 3[rd] system: $OA \equiv Ox, OB \equiv Oy, OC \equiv Oz$

Let $\vec{u}(u_1, u_2, u_3), \vec{u} \perp P$. Is $\overrightarrow{OA}(a, 0, 0), \overrightarrow{OB}(0, b, 0)$

$$\vec{u} \cdot \overrightarrow{OA} = 0 \Rightarrow au_1 + \frac{au_2}{2} + \frac{au_3}{2} = 0 \Rightarrow 2u_1 + u_2 + u_3 = 0$$

$$\vec{u} \cdot \overrightarrow{OB} = 0 \Rightarrow bu_2 + \frac{bu_1}{2} + \frac{bu_3}{2} = 0 \Rightarrow u_1 + 2u_2 + u_3 = 0$$

$u_2 = u_1, u_3 = -u_1$. So, $\overrightarrow{u_0}(1, 1, -3) \perp P$

Let $\vec{v}(v_1, v_2, v_3), \vec{v} \perp Q,\ \overrightarrow{AB}(-a, b, 0), \overrightarrow{AC}(-a, 0, a)$

$$\vec{v} \cdot \overrightarrow{AB} = 0 \Rightarrow av_1 + bv_2 + \frac{bv_1 - av_2}{2} + \frac{bv_3}{2} - \frac{av_3}{2} = 0$$

$$\vec{v} \cdot \overrightarrow{AB} = 0 \Rightarrow -av_1 + av_3 - \frac{av_2}{2} + \frac{av_2}{2} + \frac{av_1 - av_3}{2} = 0$$

$$v_1 = v_3, (a - 2b)v_2 = (2b - 3a)v_1$$

So, $\overrightarrow{v_0}(a - 2b, 2b - 3a, a - 2b) \perp Q$

Is $\overrightarrow{u_0} \cdot \overrightarrow{v_0} = (a - 2b) + (2b - 3a) - 3(a - 2b) + [(2b - 3a) + (a - 2b) -$

$$-2(2b - 3a) + (a - 2b) - 3(a - 2b) + (a - 2b)] \cdot \frac{1}{2} = 2(b - 2a)$$

$$|\overrightarrow{u_0}|^2 = 1 + 1 + 9 + 1 - 3 - 3 = 6 \Rightarrow |\overrightarrow{u_0}| = \sqrt{6}$$

$$|\overrightarrow{v_0}|^2 = (a - 2b)^2 \cdot 2 + (2b - 3a)^2 + 2(a - 2b)(2b - 3a) + (a - 2b)^2$$

$$|\overrightarrow{v_0}| = \sqrt{2} \cdot \sqrt{3a^2 - 4ab + 4b^2}$$

$$\cos 60^\circ = \frac{|\overrightarrow{u_0 \cdot v_0}|}{|\overrightarrow{u_0}| \cdot |\overrightarrow{v_0}|} = \frac{1}{2} \Rightarrow \frac{1}{2} = \frac{2|b - a|}{\sqrt{6} \cdot \sqrt{2} \cdot \sqrt{3a^2 - 4ab + 4b^2}}$$

Therefore,

$$\frac{b}{a} = \frac{1 + \sqrt{6}}{2}$$

515. In ΔABC the following relationship holds:

$$3 \cdot \sqrt{\frac{m_a}{m_b} + \frac{2m_b}{m_c} + \frac{6m_c}{m_a}} \geq \sqrt{\frac{h_a}{m_b}} + 2 \cdot \sqrt{\frac{h_c}{m_c}} + 6 \cdot \sqrt{\frac{h_c}{m_a}}$$

Proposed by Daniel Sitaru-Romania

Solution by Adrian Popa-Romania

$$\left(\sqrt{\frac{h_a}{m_b}}+2\cdot\sqrt{\frac{h_c}{m_c}}+6\cdot\sqrt{\frac{h_c}{m_a}}\right)^2\le\left(\sqrt{\frac{m_a}{m_b}}+\sqrt{2}\cdot\sqrt{\frac{2m_b}{m_c}}+\sqrt{6}\cdot\sqrt{\frac{6m_c}{m_a}}\right)^2\overset{CBS}{\le}$$

$$\le\left(1^2+\sqrt{2}^2+\sqrt{6}^2\right)\left(\frac{m_a}{m_b}+\frac{2m_b}{m_c}+\frac{6m_c}{m_a}\right)=9\left(\frac{m_a}{m_b}+\frac{2m_b}{m_c}+\frac{6m_c}{m_a}\right)$$

Therefore,

$$3\cdot\sqrt{\frac{m_a}{m_b}+\frac{2m_b}{m_c}+\frac{6m_c}{m_a}}\ge\sqrt{\frac{h_a}{m_b}}+2\cdot\sqrt{\frac{h_c}{m_c}}+6\cdot\sqrt{\frac{h_c}{m_a}}$$

516. In ΔABC the following relationship holds:

$$\frac{2bc}{(b+c)^2}+\frac{1}{2}\ge\frac{16Rr}{s^2+r^2+2Rr}.$$

Proposed by Daniel Sitaru-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$By\ AM-GM\ inequality\ we\ have:$$

$$\frac{2bc}{(b+c)^2}+\frac{1}{2}\ge 2\sqrt{\frac{2bc}{(b+c)^2}.\frac{1}{2}}=\frac{2\sqrt{bc}}{b+c}=\frac{8abc}{2\sqrt{ab}.2\sqrt{ca}.(b+c)}$$

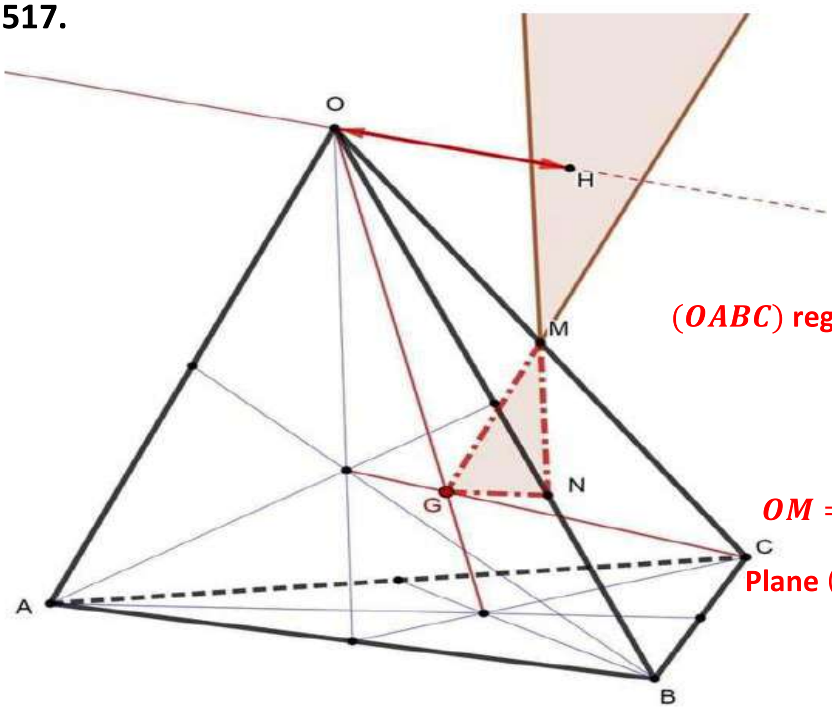
$$\ge\frac{8abc}{(a+b)(c+a)(b+c)},\quad with:$$

$$(a+b)(c+a)(b+c)=(a+b+c)(ab+bc+ca)-abc$$
$$=2s(s^2+r^2+4Rr)-4sRr=2s(s^2+r^2+2Rr)$$

$$Therefore,$$

$$\frac{2bc}{(b+c)^2}+\frac{1}{2}\ge\frac{8.4sRr}{2s(s^2+r^2+2Rr)}=\frac{16Rr}{s^2+r^2+2Rr}$$

517.



$(OABC)$ regular tetrahedron with side a

G –centroid of tetrahedron

$M \in OC$,

$OM = \frac{OC}{2}, N \in OB, 4ON = 3OB$

Plane $(GMN) \equiv P, OH \perp P$

Find: $OH = f(a) = ?$

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by proposer

Let $OA = OB = OC = 4 = a, \sphericalangle AOB = \sphericalangle BOC = \sphericalangle COA = \theta = 60^\circ$

Plagiogonal 3rd system: $OA \equiv Ox, OB \equiv Oy, OC \equiv Oz$

$O(0,0,0), A(4,0,0), M(0,0,2), N(0,3,0), G(1,1,1)$

$$(GMN) \equiv P: \begin{vmatrix} 1 & x & y & z \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 3 & 0 \\ 1 & 0 & 0 & 2 \end{vmatrix} = 0 \Rightarrow x + 2y + 3z - 6 = 0$$

Let $\vec{u}(u_1, u_2, u_3) \perp P$. Is $u_1 = 1 \cdot \frac{3}{4} + 2 \cdot \left(-\frac{1}{4}\right) + 3 \cdot \left(-\frac{1}{4}\right) = -\frac{1}{2}$

$$u_2 = 2 \cdot \frac{3}{4} + 3 \cdot \left(-\frac{1}{4}\right) + 1 \cdot \left(-\frac{1}{4}\right) = \frac{1}{2}, \qquad u_3 = 3 \cdot \frac{3}{4} + 1 \cdot \left(-\frac{1}{4}\right) + 2 \cdot \left(-\frac{1}{4}\right) = \frac{3}{2}$$

$$\vec{u}\left(-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}\right), |u|^2 = \left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + \left(-\frac{1}{2} \cdot \frac{1}{2}\right) + \frac{1}{2} \cdot \frac{3}{2} + \left(-\frac{1}{2} \cdot \frac{3}{2}\right) = \frac{5}{2}$$

$$OH^2 = \frac{(1 \cdot 0 + 2 \cdot 0 + 3 \cdot 0 - 6)^2}{\left[1 \cdot \left(-\frac{1}{2}\right) + 2 \cdot \frac{1}{2} + 3 \cdot \frac{3}{2}\right]^2} \cdot \frac{5}{2} \Rightarrow OH = \frac{3}{5}\sqrt{10}$$

$$OH=\frac{3}{20}\sqrt{10}\cdot 4\Rightarrow OH=\frac{3}{2\sqrt{10}}\cdot a$$

518. In ΔABC the following relationship holds:

$$3\left(\frac{R}{2r}\right)^2\geq\frac{s_a}{s_b}+\frac{s_b}{s_c}+\frac{s_c}{s_a}.$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ know\ that:\ \ h_a\leq s_a\leq m_a\ \ (and\ analogs)$$

$$Then:$$

$$\frac{s_a}{s_b}+\frac{s_b}{s_c}+\frac{s_c}{s_a}\leq\frac{m_a}{h_b}+\frac{m_b}{h_c}+\frac{m_c}{h_a}\overset{CBS}{\leq}\sqrt{({m_a}^2+{m_b}^2+{m_c}^2)\left(\frac{1}{{h_b}^2}+\frac{1}{{h_c}^2}+\frac{1}{{h_a}^2}\right)}=$$

$$=\sqrt{\frac{3(a^2+b^2+c^2)}{4}\cdot\frac{a^2+b^2+c^2}{(2sr)^2}}=\frac{\sqrt{3}(a^2+b^2+c^2)}{4sr}$$

$$From\ Leibniz's\ inequality\ \ a^2+b^2+c^2\leq 9R^2$$

$$Mitrinovic's\ inequality\ \ s\geq 3\sqrt{3}r\ \ we\ get:$$

$$\frac{s_a}{s_b}+\frac{s_b}{s_c}+\frac{s_c}{s_a}\leq\frac{\sqrt{3}.9R^2}{4.3\sqrt{3}r.r}=3\left(\frac{R}{2r}\right)^2$$

519. In ΔABC the following relationship holds:

$$\frac{4R}{r}-\frac{r}{R}-\frac{9}{2}\leq\sum_{cyc}\frac{{m_a}^2}{{r_a}^2}\leq\frac{4R^2}{r^2}-\frac{15R}{2r}+2.$$

Proposed by Marin Chirciu-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:\ \sum_{cyc}\frac{{m_a}^2}{{r_a}^2}=\sum_{cyc}\frac{(s-a)^2[3(b^2+c^2)-(a^2+b^2+c^2)]}{4F^2}$$

$$=\frac{3}{4F^2}\sum_{cyc}(b^2+c^2)(s-a)^2-\frac{a^2+b^2+c^2}{4F^2}\sum_{cyc}(s-a)^2=$$

$$= \frac{3}{4F^2}\left(2s^2\sum_{cyc} a^2 - 2s\sum_{sym} a^2 b + 2\sum_{cyc} a^2 b^2\right) - \frac{(s^2 - r^2 - 4Rr)(s^2 - 2r^2 - 8Rr)}{2F^2} =$$

$$= \frac{3}{4F^2}[2s^2.2(s^2 - r^2 - 4Rr) - 2s.2s(s^2 + r^2 - 2Rr)$$

$$+ 2(s^4 - 2s^2(4Rr - r^2) + (4Rr + r^2)^2)]$$

$$- \frac{s^4 - s^2(12Rr + 3r^2) + 2(4Rr + r^2)^2}{2F^2}$$

$$\text{After expanding and simplifying we get}: \sum_{cyc} \frac{{m_a}^2}{{r_a}^2}$$

$$= \frac{s^2}{r^2} - \frac{3(8R + r)}{2r} + \frac{(4R + r)^2}{2s^2}$$

Using Gerretsen's inequality

$s^2 \ge 16Rr - 5r^2$ *and Blundon Gerretsen's inequality*

$$s^2 \le \frac{R(4R + r)^2}{2(2R - r)} \text{ we get}:$$

$$\sum_{cyc} \frac{{m_a}^2}{{r_a}^2} \ge \frac{16Rr - 5r^2}{r^2} - \frac{3(8R + r)}{2r} + \frac{2R - r}{R} = \frac{4R}{r} - \frac{r}{R} - \frac{9}{2}.$$

Using Gerretsen's inequality $16Rr - 5r^2 \le s^2 \le 4R^2 + 4Rr + 3r^2$ *we get*:

$$\sum_{cyc} \frac{{m_a}^2}{{r_a}^2} \le \frac{4R^2 + 4Rr + 3r^2}{r^2} - \frac{3(8R + r)}{2r} + \frac{(4R + r)^2}{2(16Rr - 5r^2)}$$

$$= \frac{4R^2}{r^2} - \frac{15R}{2r} + 2 - \frac{3(R - 2r)}{2(16R - 5r)} \overset{Euler}{\lesseqgtr} \frac{4R^2}{r^2} - \frac{15R}{2r} + 2.$$

$$\text{Therefore, } \frac{4R}{r} - \frac{r}{R} - \frac{9}{2} \le \sum_{cyc} \frac{{m_a}^2}{{r_a}^2} \le \frac{4R^2}{r^2} - \frac{15R}{2r} + 2.$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}(s - a)^2 = \left(\sum_{cyc}(s - a)\right)^2 - 2\sum_{cyc}(s - b)(s - c) = s^2 - 2(4Rr + r^2)$$

$$\Rightarrow \sum_{cyc}(s - a)^2 \overset{(i)}{=} s^2 - 8Rr - 2r^2$$

$$\sum_{cyc} a^2(s-a)^2 = \left(\sum_{cyc} a(s-a)\right)^2 - 2\sum_{cyc} bc(s-b)(s-c)$$

$$= \left(s(2s) - 2(s^2-4Rr-r^2)\right)^2 - 2\sum_{cyc} bc\left(s^2 - s(b+c) + bc\right)$$

$$= 4r^2(4R+r)^2 - 2\left(s^2\sum_{cyc} bc - s\sum_{cyc} bc(2s-a) + \sum_{cyc} b^2c^2\right)$$

$$= 4r^2(4R+r)^2 - 2\left(-s^2\sum_{cyc} bc + \left(\sum_{cyc} bc\right)^2 + 3sabc - 4sabc\right)$$

$$= 4r^2(4R+r)^2 - 2\left((s^2+4Rr+r^2)(s^2+4Rr+r^2-s^2) - 4Rrs^2\right)$$

$$= 4r^2(4R+r)^2 - 2r^2s^2 - 2r^2(4R+r)^2$$

$$\Rightarrow \sum_{cyc} a^2(s-a)^2 \overset{(ii)}{=} 2r^2\left((4R+r)^2 - s^2\right)$$

Now, $\sum_{cyc} \frac{m_a^2}{r_a^2} = \sum_{cyc} \frac{(4m_a^2)(r_br_c)^2}{4r_a^2r_b^2r_c^2} = \frac{1}{4r^2s^4}\cdot\sum_{cyc}(2b^2+2c^2+2a^2-3a^2)s^2(s-a)^2$

$$= \frac{1}{4r^2s^2}\cdot\left(2\left(\sum_{cyc} a^2\right)\cdot\sum_{cyc}(s-a)^2 - 3\sum_{cyc} a^2(s-a)^2\right)$$

$$\overset{\text{via (i) and (ii)}}{=} \frac{1}{2r^2s^2}\left(2(s^2-8Rr-2r^2)(s^2-4Rr-r^2) - 3r^2\left((4R+r)^2 - s^2\right)\right)$$

$$\le \frac{4R^2}{r^2} - \frac{15R}{2r} + 2 = \frac{8R^2-15Rr+4r^2}{2r^2}$$

$$\Leftrightarrow 2s^4 - (8R^2+9Rr+7r^2)s^2 + r^2(4R+r)^2 \overset{(*)}{\le} 0$$

Now, LHS of $(*) \overset{\text{Gerretsen}}{\le} 2(4R^2+4Rr+3r^2)s^2 - (8R^2+9Rr+7r^2)s^2 + r^2(4R+r)^2 \overset{?}{\le} 0$

$$\Leftrightarrow (R+r)s^2 \underset{(**)}{\overset{?}{\ge}} r(4R+r)^2$$

Again, LHS of $(**) \overset{\text{Gerretsen}}{\ge} (R+r)(16Rr-5r^2) \overset{?}{\ge} r(4R+r)^2 \Leftrightarrow 3r(R-2r) \overset{?}{\ge} 0 \to$ true

$\because R \overset{\text{Euler}}{\ge} 2r \Rightarrow (**) \Rightarrow (*)$ is true $\Rightarrow \sum_{cyc} \frac{m_a^2}{r_a^2} \le \frac{4R^2}{r^2} - \frac{15R}{2r} + 2$

$$\text{Moreover, } \sum_{cyc} \frac{m_a^2}{r_a^2} \ge \frac{4R}{r} - \frac{r}{R} - \frac{9}{2}$$

$$\Leftrightarrow \frac{1}{2r^2s^2}\left(2(s^2-8Rr-2r^2)(s^2-4Rr-r^2)-3r^2((4R+r)^2-s^2)\right) \ge \frac{8R^2-9Rr-2r^2}{2Rr}$$

$$\Leftrightarrow 2Rs^4-(32R^2-6Rr-2r^2)rs^2+Rr^2(4R+r)^2 \overset{(***)}{\ge} 0$$

$$\text{Now, LHS of } (***) \overset{\text{Gerretsen}}{\ge} 2R(16Rr-5r^2)s^2-(32R^2-6Rr-2r^2)rs^2+Rr^2(4R+r)^2 \overset{?}{\ge} 0$$

$$\Leftrightarrow R(4R+r)^2 \underbrace{\overset{?}{\ge}}_{(****)} (4R-2r)s^2$$

$$\text{Now, RHS of } (4R-2r)s^2 \overset{\text{Rouche}}{\le} (4R-2r)\left(2R^2+10Rr-r^2+2(R-2r)\sqrt{R^2-2Rr}\right) \overset{?}{\le} R(4R+r)^2$$

$$\Leftrightarrow R(4R+r)^2-(2R^2+10Rr-r^2)(4R-2r) \overset{?}{\ge} 2(4R-2r)(R-2r)\sqrt{R^2-2Rr}$$

$$\Leftrightarrow (R-2r)(8R^2-12Rr+r^2) \underbrace{\overset{?}{\ge}}_{(\bullet)} 2(4R-2r)(R-2r)\sqrt{R^2-2Rr}$$

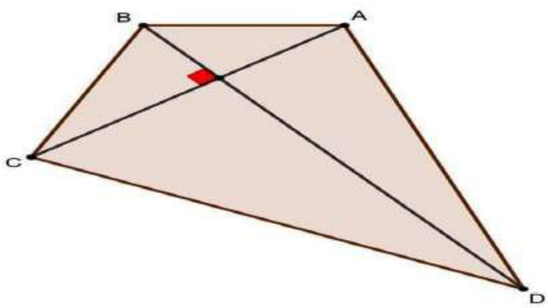
$$\because R-2r \overset{\text{Euler}}{\ge} \therefore \text{in order to prove } (\bullet)\text{, it suffices to prove: } 8R^2-12Rr+r^2 > 2(4R-2r)\sqrt{R^2-2Rr}$$

$$\Leftrightarrow (8R^2-12Rr+r^2)^2-4(R^2-2Rr)(4R-2r)^2 > 0 \Leftrightarrow r^2(4R+r)^2 > 0 \to true$$

$$\Rightarrow (\bullet) \text{ is true} \therefore (4R-2r)s^2 \le R(4R+r)^2 \Rightarrow (****) \Rightarrow (***) \text{ is true}$$

$$\therefore \sum_{cyc} \frac{m_a^2}{r_a^2} \ge \frac{4R}{r} - \frac{r}{R} - \frac{9}{2} \text{ (QED)}$$

520.



$$\overrightarrow{BA} = b, \overrightarrow{BC} = a, BD \perp AC$$

$$\overrightarrow{BD} = m \cdot \overrightarrow{BC} + n \cdot \overrightarrow{BA},$$

$$m > 0, n > 0$$

$$[ABCD] = F = f(m,n,a,b) = ?$$

$[ABCD]$ –area

Application: $a = 5, b = 4, n = 3, m = 2.$

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by proposer

Plagiogonal system: $BC \equiv Bx, BA \equiv By$

$$BC = a, BE = ma, BA = b, BF = nb$$

Let $\widehat{ABC} = \theta$ and $B(0,0), C(a,0), E(ma,0), A(0,b), F(0,nb), D(ma,nb)$

$$\lambda_{AC} = \lambda_1 = -\frac{b}{a}, \lambda_{BD} = \lambda_2 = \frac{nb}{ma}$$

$$BD \perp AC \Rightarrow (\lambda_1 + \lambda_2)\cos\theta + \lambda_1 + \lambda_2 + 1 = 0$$

$$\cos\theta = \frac{ma^2 - nb^2}{(m-n)ab}; m \neq n$$

$$\sin\theta = \sqrt{\frac{a^2b^2(m^2+n^2) - (ma^2)^2 - (nb^2)^2}{a^2b^2(m-n)^2}}$$

$$F = \frac{\sin\theta}{2}(a\cdot nb + b\cdot ma) = \frac{1}{2}\cdot\frac{m+n}{|m-n|}\cdot\sqrt{(a^2-b^2)(n^2b^2-m^2a^2)}$$

$$(a^2-b^2)(n^2b^2-m^2a^2) > 0; m \neq n$$

If $a = 5, b = 4, n = 3, m = 2$, we have: $F = 15\sqrt{11}$

521. $M \in Int(\Delta ABC), R_a, R_b, R_c$ –circumradii of $\Delta BMC, \Delta CMA, \Delta AMB$. Prove that :

$$\frac{R_a}{m_b\sin\frac{A}{2}} + \frac{R_b}{m_c\sin\frac{B}{2}} + \frac{R_c}{m_a\sin\frac{C}{2}} \geq 2\left(\sqrt{\frac{R_aR_b}{m_bm_c}} + \sqrt{\frac{R_bR_c}{m_cm_a}} + \sqrt{\frac{R_cR_a}{m_am_b}}\right).$$

Proposed by Bogdan Fuştei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By CBS inequality we have :

$$\frac{R_a}{m_b\sin\frac{A}{2}} + \frac{R_b}{m_c\sin\frac{B}{2}} + \frac{R_c}{m_a\sin\frac{C}{2}} \geq \frac{\left(\sqrt{\frac{R_a}{m_b}} + \sqrt{\frac{R_b}{m_c}} + \sqrt{\frac{R_c}{m_a}}\right)^2}{\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}} \quad (1)$$

From the well known inequality $(x+y+z)^2 \geq 3(xy+yz+zx)$ *we have :*

$$\left(\sqrt{\frac{R_a}{m_b}}+\sqrt{\frac{R_b}{m_c}}+\sqrt{\frac{R_c}{m_a}}\right)^2\ge 3\left(\sqrt{\frac{R_aR_b}{m_bm_c}}+\sqrt{\frac{R_bR_c}{m_cm_a}}+\sqrt{\frac{R_cR_a}{m_am_b}}\right)\quad (2)$$

Also by Jensen's inequality we have :

$$\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\le 3\sin\left(\frac{A+B+C}{6}\right)=\frac{3}{2}\quad (3)$$

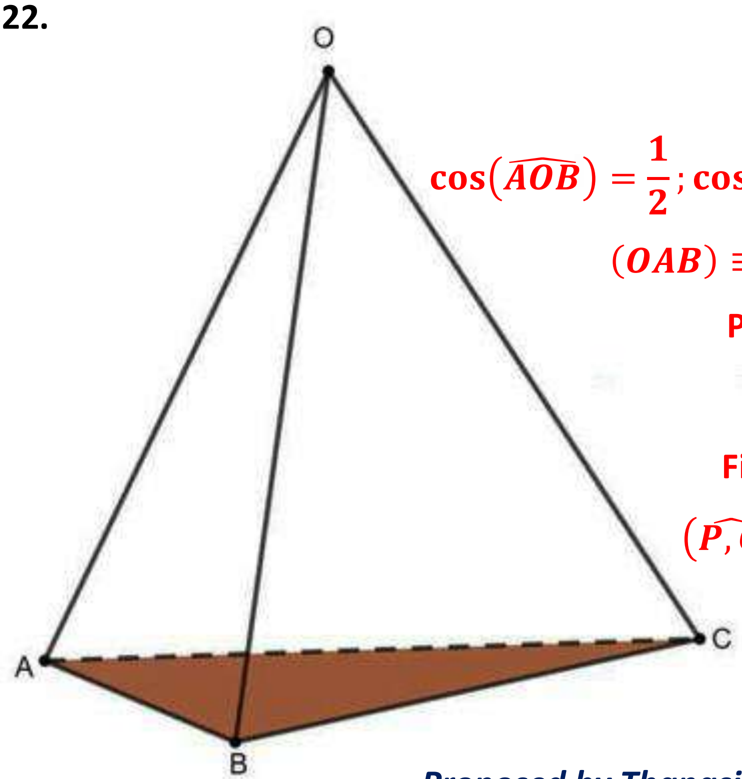
From (1), (2) and (3) we get :

$$\frac{R_a}{m_b\sin\frac{A}{2}}+\frac{R_b}{m_c\sin\frac{B}{2}}+\frac{R_c}{m_a\sin\frac{C}{2}}\ge\frac{3\left(\sqrt{\frac{R_aR_b}{m_bm_c}}+\sqrt{\frac{R_bR_c}{m_cm_a}}+\sqrt{\frac{R_cR_a}{m_am_b}}\right)}{\frac{3}{2}}$$

$$=2\left(\sqrt{\frac{R_aR_b}{m_bm_c}}+\sqrt{\frac{R_bR_c}{m_cm_a}}+\sqrt{\frac{R_cR_a}{m_am_b}}\right).$$

Equality holds iff ΔABC is equilateral and M is the center of ΔABC.

522.



$(OABC)$ –tetrahedron

$$\cos(\widehat{AOB})=\frac{1}{2};\cos(\widehat{BOC})=\frac{1}{3};\cos(\widehat{COA})=\frac{2}{3}$$

$$(OAB)\equiv P;(OBC)\equiv Q;(OCA)\equiv R$$

Prove :

$$(\widehat{P,R})=\theta_3=90^\circ$$

Find:

$$(\widehat{P,Q})=\theta_1=?;(\widehat{Q,R})=\theta_2=?$$

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by proposer

$$\text{PLagiogonal 3}^{\text{rd}}\text{ system: } OA \equiv Ox, OB \equiv Oy, OC \equiv Oz$$

$$\vec{u}(u_1, u_2, u_3) \perp P; \vec{v}(v_1, v_2, v_3) \perp Q, \vec{w}(w_1, w_2, w_3) \perp R$$

$$\vec{u}\left(-\frac{1}{2}, 0, \frac{3}{4}\right); \vec{v}\left(\frac{8}{9}, -\frac{5}{18}, -\frac{1}{2}\right); \vec{w}\left(-\frac{5}{18}, \frac{5}{9}, 0\right)$$

$$\vec{u}(-18, 0, 27); \vec{v}(32, -10, -18); \vec{w}(-10, 20, 0)$$

$$|\vec{u}| = 9\sqrt{5}; |\vec{v}| = 4\sqrt{30}; |\vec{w}| = 10\sqrt{3}$$

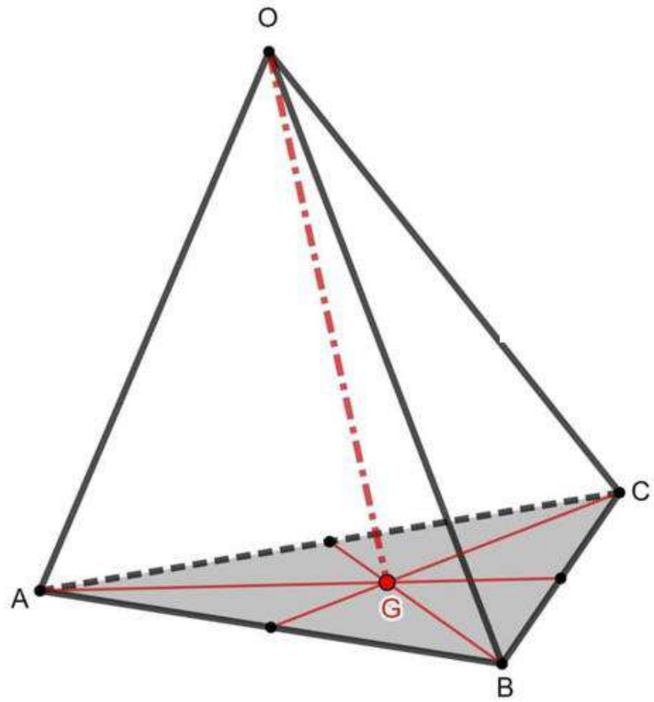
$$|\vec{u} \cdot \vec{v}| = 270; |\vec{v} \cdot \vec{w}| = 150; |\vec{u} \cdot \vec{w}| = 0$$

$$\cos\theta_1 = \frac{|\vec{u} \cdot \vec{v}|}{|\vec{u}| \cdot |\vec{v}|} = \frac{\sqrt{6}}{4} \Rightarrow \theta_1 \cong 52,24°$$

$$\cos\theta_2 = \frac{|\vec{v} \cdot \vec{w}|}{|\vec{v}| \cdot |\vec{w}|} = \frac{\sqrt{3}}{8} \Rightarrow \theta_2 \cong 77,50°$$

$$\cos\theta_3 = \frac{|\vec{u} \cdot \vec{w}|}{|\vec{u}| \cdot |\vec{w}|} = 0 \Rightarrow \theta_1 = 90°$$

523.



$(OABC)$ −tetrahedron

G −centroid of ΔABC

$OA = OB = OC = 3d$

$\sphericalangle BOC = \theta_1, \sphericalangle COA = \theta_2, \sphericalangle AOB = \theta_3$

Find:

$OG = ?$ If $\theta_1 = \theta_2 = \theta_3 = \theta$

$OG = ?$ If $\theta_1 \neq \theta_2 \neq \theta_3 \neq \theta_1$.

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by proposer

Plagiogonal 3rd system: $OA\equiv Ox, OB\equiv Oy, OC\equiv Oz$

$$O(0,0,0), A(3d,0,0), B(0,3d,0), C(0,0,3d), G(d,d,d)$$

$$OG^2=(d-0)^2+(d-0)^2+(d-0)^2+2d\cdot d\cdot\cos\theta_1+2d\cdot d\cdot\cos\theta_2+2d\cdot d\cdot\cos\theta_3$$

$$OG=d\sqrt{|3+2(\cos\theta_1+\cos\theta_2+\cos\theta_3)|}$$

If $\theta_1=\theta_2=\theta_3=\theta, OG=d\sqrt{3}\sqrt{|1+\cos\theta|}$

If $\theta_1=\theta_2=\theta_3=90^\circ, OG=d\sqrt{3}$

If $\theta_1=\theta_2=\theta_3=60^\circ, OG=d\sqrt{3}, 3d=a\Rightarrow OG=a\cdot\frac{\sqrt{6}}{3}=h$

the altitude of regular tetrahedron with side a.

524. In ΔABC the following relationship holds:

$$\sum_{cyc}\csc^2\frac{A}{2}\sin\frac{C}{2}\le 6\left(\frac{R}{2r}\right)^5$$

Proposed by Marin Chirciu-Romania

Solution by Marian Ursărescu-Romania

We must show:

$$\sum_{cyc}\frac{\sin\frac{C}{2}}{\sin^2\frac{A}{2}}\le 6\cdot\frac{R^5}{32r^5}\Leftrightarrow \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\cdot\sum_{cyc}\frac{1}{\sin^3\frac{A}{2}\sin\frac{B}{2}}\le\frac{3R^5}{16r^5};(1)$$

But $3(x^3y+y^3z+z^3x)\le(x^2+y^2+z^2)^2$.

$$\sum_{cyc}\frac{1}{\sin^3\frac{A}{2}\sin\frac{B}{2}}\le\frac{1}{3}\left(\sum_{cyc}\frac{1}{\sin^2\frac{A}{2}}\right)^2;(2)$$

From (1) and (2) we must show:

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\cdot\frac{1}{3}\left(\sum_{cyc}\frac{1}{\sin^2\frac{A}{2}}\right)^2\le\frac{3R^5}{16r^5};(3)$$

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\frac{r}{4R}\text{ and }\sum_{cyc}\frac{1}{\sin^2\frac{A}{2}}=\frac{s^2+r^2-8Rr}{r^2};(4)$$

From (3) and (4) we must show:

$$\frac{r}{4R}\cdot\frac{1}{3}\cdot\frac{(s^2+r^2-8Rr)^2}{r^4}\le\frac{3R^5}{16r^5}\Leftrightarrow(s^2+r^2-8Rr)^2\le\frac{9R^6}{4r^2}$$

$$s^2+r^2-8Rr\le\frac{3R^3}{2r};(5)$$

$$s^2\le 4R^2+4Rr+3r^2(Gerretsen);(6)$$

From (5) and (6) we must show:

$$4R^2-4Rr+4r^2\le\frac{3R^3}{2r}\Leftrightarrow 8R^2r-8Rr^2+8r^2\le 3R^3$$

$$3R^3-8R^2r+8Rr^2-9-8r^3\ge 0$$

$(R-2r)(3R^2-2Rr+4r^2)\ge 0$ true from $R\ge 2r$ $(Euler)$.

525. In ΔABC the following relationship holds:

$$16\left(\frac{2R^2}{r^2}-\frac{3R}{r}+1\right)\le\sum_{cyc}\csc^4\frac{A}{2}\le 16\left(\frac{R^4}{r^4}+\frac{R^2}{r^2}-17\right)$$

Proposed by Marin Chirciu-Romania

Solution by Marian Ursărescu-Romania

Because $\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}$ are the roots of the equation

$$16R^2x^3-8R(2R-r)x^2+(s^2+r^2+8Rr)x-r^2=0$$

then from Viete's relationships we have

$$\sum_{cyc}\frac{1}{\sin^4\frac{A}{2}}=\left(\frac{s^2+r^2-8Rr}{r^2}\right)^2-\frac{16R(2R-r)}{r^2}$$

We must show:

$$16\left(\frac{2R^2}{r^2}-\frac{3R}{r}+1\right)\le\left(\frac{s^2+r^2-8Rr}{r^2}\right)^2-\frac{16R(2R-r)}{r^2}\le 16\left(\frac{R^4}{r^4}+\frac{R^2}{r^2}-17\right)$$

For the left side $s^2\ge 16Rr-5r^2(Gerretsen)$ we have:

$$\left(\frac{s^2+r^2-8Rr}{r^2}\right)^2-\frac{16R(2R-r)}{r^2}\ge\left(\frac{8Rr-4r^2}{r^2}\right)^2-\frac{16R(2R-r)}{r^2}=$$

$$=16\left(\frac{2R-2}{r}\right)^2-\frac{16R(2R-r)}{r^2}=\frac{16(2R-r)}{r^2}(2R-r-R)=\frac{16(2R-r)(R-r)}{r^2}=$$

$$=16\left(\frac{2R^2}{r^2}-\frac{3R}{r}+1\right)$$

For the right side $s^2 \le 4R^2 + 4Rr + 3r^2 (Gerretsen)$ we have:

$$\left(\frac{s^2+r^2-8Rr}{r^2}\right)^2 - \frac{16R(2R-r)}{r^2} \le \left(\frac{4R^2-4Rr+4r^2}{r^2}\right)^2 - \frac{16R(2R-r)}{r^2} =$$

$$= 16\left[\frac{(R^2-Rr+r^2)^2}{r^2} - \frac{R(2R-r)}{r^2}\right] =$$

$$= 16\left(\frac{R^4+R^2r^2+r^4-2R63\,r+2R^2r^2-2Rr^3}{r^4} - \frac{2R^2-Rr}{r^2}\right) =$$

$$= 16\left(\frac{R^4}{r^4} - \frac{2R^3}{r^3} + \frac{3R^2}{r^2} - \frac{2R}{r} + 1 - \frac{2R^2}{r^2} + \frac{R}{r}\right) =$$

$$= 16\left(\frac{R^4}{r^4} - \frac{2R^3}{r^3} + \frac{R^2}{r^2} - \frac{R}{r} + 1\right)$$

We must show:

$$16\left(\frac{R^4}{r^4} - \frac{2R^3}{r^3} + \frac{R^2}{r^2} - \frac{R}{r} + 1\right) \le 16\left(\frac{R^4}{r^4} + \frac{R^2}{r^2} - 17\right)$$

$$\frac{R^4}{r^4} - \frac{2R^3}{r^3} + \frac{R^2}{r^2} - \frac{R}{r} + 1 \le \frac{R^4}{r^4} + \frac{R^2}{r^2} - 17$$

$$-\frac{2R^3}{r^3} - \frac{R}{r} + 1 \le -17$$

$$\frac{2R^3}{r^3} + \frac{R}{r} \ge 18 \text{ true, because: } \frac{R^3}{r^3} \ge 8 \text{ and } \frac{R}{r} \ge 2$$

526. For any ΔABC the following relationship holds:

$$\frac{c(a+b)}{a+b-c} + \frac{a(b+c)}{b+c-a} + \frac{b(c+a)}{c+a-b} \ge 4\left(\frac{(a+b-c)^2}{a+b} + \frac{(b+c-a)^2}{b+c} + \frac{(c+a-b)^2}{c+a}\right).$$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have: \frac{c(a+b)}{a+b-c} = \frac{4(a+b)c^2}{4c(a+b-c)} \overset{AM-GM}{\ge} \frac{4(a+b)c^2}{[(a+b-c)+c]^2} = \frac{4c^2}{a+b}$$

$$= 4\left(\frac{(a+b-c)^2}{a+b} + 2c - a - b\right)$$

$$\textit{Similarly we have}: \frac{a(b+c)}{b+c-a} \geq 4\left(\frac{(b+c-a)^2}{b+c}+2a-b-c\right) \;\&\; \frac{b(c+a)}{c+a-b}$$
$$\geq 4\left(\frac{(c+a-b)^2}{c+a}+2b-c-a\right)$$

Summing up these inequalities we get :

$$\frac{c(a+b)}{a+b-c}+\frac{a(b+c)}{b+c-a}+\frac{b(c+a)}{c+a-b} \geq 4\left(\frac{(a+b-c)^2}{a+b}+\frac{(b+c-a)^2}{b+c}+\frac{(c+a-b)^2}{c+a}\right)$$

Equality holds iff ΔABC is equilateral.

Solution 2 by Nguyen Van Canh-Ben Tre-Vietnam

Let $z=a+b-c>0, x=b+c-a>0, y=c+a-b>0$

$$\Rightarrow x+y+z=a+b+c, a=\frac{y+z}{2}; b=\frac{x+z}{2}; c=\frac{x+y}{2}$$

We have:

$$\frac{c(a+b)}{a+b-c}+\frac{a(b+c)}{b+c-a}+\frac{b(c+a)}{c+a-b} \geq 4\left(\frac{(a+b-c)^2}{a+b}+\frac{(b+c-a)^2}{b+c}+\frac{(c+a-b)^2}{c+a}\right);$$

$$\Leftrightarrow \frac{(x+y)(x+y+2z)}{4z}+\frac{(y+z)(y+z+2x)}{4x}+\frac{(x+z)(x+z+2y)}{4y}$$
$$\geq 4\left(\frac{z^2}{\frac{x+y+2z}{2}}+\frac{x^2}{\frac{y+z+2x}{2}}+\frac{y^2}{\frac{x+z+2y}{2}}\right);$$

$$\Leftrightarrow \frac{xy(x+y)(x+y+2z)+yz(y+z)(y+z+2x)+xz(x+z)(x+z+2y)}{xyz}$$
$$\geq \frac{32\left(z^2(y+z+2x)(x+z+2y)+x^2(x+y+2z)(x+z+2y)+y^2(x+y+2z)(y+z+2x)\right)}{(x+y+2z)(y+z+2x)(x+z+2y)};$$

$$\Leftrightarrow \frac{(x+y)(x+z)(y+z)(x+y+z)}{xyz} \geq \frac{32(x+y+z)\left(\sum x^3+2\sum xy(x+y)+xyz\right)}{(x+y+2z)(y+z+2x)(x+z+2y)};$$

$$\Leftrightarrow (x+y)(x+z)(y+z)(x+y+2z)(y+z+2x)(x+z+2y)$$
$$\geq 32xyz\left(\sum x^3+2\sum xy(x+y)+xyz\right); (*)$$

Without loss of generality, suppose that $x+y+z=1$. Let $q=\sum xy, r=xyz$

$$\leq \frac{1}{27}$$

$$(*) \Leftrightarrow 1-r^2+q^2-2r-1+2q \geq 32r(1-3q+3r+2q-6r+r);$$

$$\Leftrightarrow q^2+2q-r^2-2r\ge 32r(1-q-2r);$$
$$\Leftrightarrow q^2+2q+32rq+63r^2-34r\ge 0;$$

More, $\left(\sum yz\right)^2\ge 3xyz(x+y+z)=3r\Rightarrow q\ge\sqrt{3r}\Rightarrow q^2+2q+32rq+63r^2-34r$
$$\ge 3r+2\sqrt{3r}+32r\sqrt{3r}+63r^2-34r;$$

We need to prove that:

$$3r+2\sqrt{3r}+32r\sqrt{3r}+63r^2-34r\ge 0;$$
$$\Leftrightarrow t^2+2t+\frac{32t^3}{3}+\frac{63t^4}{9}-\frac{34t^2}{3}\ge 0;\left(\therefore t=\sqrt{3r},0<t\le\frac{1}{3}\right)$$
$$\Leftrightarrow\frac{1}{3}t(3t-1)(7t^2+13t-6)\ge 0;$$

Which is true since: $0<t\le\frac{1}{3}\Rightarrow t(3t-1)\le 0;\ 7t^2+13t-6\le-\frac{8}{9}<0$
$$\Rightarrow(*)\text{ true. Proved.}$$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$\frac{(a+b-c)^2}{a+b}+\frac{(b+c-a)^2}{b+c}+\frac{(c+a-b)^2}{c+a}=4\sum_{cyc}\frac{(s-a)^2}{b+c}=4\sum_{cyc}\frac{(2s-a-s)^2}{b+c}$$
$$=4\left(\sum_{cyc}\frac{(2s-a)^2}{2s-a}-2s\sum_{cyc}\frac{2s-a}{2s-a}+s^2\sum_{cyc}\frac{1}{b+c}\right)$$
$$=4\left(\sum_{cyc}(2s-a)-6s+\frac{s^2}{2s(s^2+2Rr+r^2)}\sum_{cyc}(c+a)(a+b)\right)$$
$$=4\left(-2s+\frac{s}{2(s^2+2Rr+r^2)}\left(\left(\sum_{cyc}a^2+2\sum_{cyc}ab\right)+\sum_{cyc}ab\right)\right)$$
$$=4\left(-2s+\frac{s(4s^2+s^2+4Rr+r^2)}{2(s^2+2Rr+r^2)}\right)$$
$$=4s.\frac{5s^2+4Rr+r^2-4(s^2+2Rr+r^2)}{2(s^2+2Rr+r^2)}$$
$$\Rightarrow 4\left(\frac{(a+b-c)^2}{a+b}+\frac{(b+c-a)^2}{b+c}+\frac{(c+a-b)^2}{c+a}\right)\overset{(*)}{=}\frac{16s(s^2-4Rr-3r^2)}{2(s^2+2Rr+r^2)}$$

Again, $\frac{c(a+b)}{a+b-c}+\frac{b(c+a)}{c+a-b}+\frac{a(b+c)}{b+c-a}=\frac{1}{2}\sum_{cyc}\frac{a(s+s-a)}{s-a}=\frac{1}{2}\left(s\sum_{cyc}\frac{a-s+s}{s-a}+\sum_{cyc}a\right)$
$$=\frac{1}{2}\left(s(-3)+\frac{s^2}{r^2s}\sum_{cyc}(s-b)(s-c)+2s\right)=\frac{1}{2}\left(-s+s.\frac{4R+r}{r}\right)$$

$$\Rightarrow \frac{c(a+b)}{a+b-c}+\frac{b(c+a)}{c+a-b}+\frac{a(b+c)}{b+c-a}\overset{(**)}{=}\frac{4Rs}{2r} \therefore (*),(**) \Rightarrow \frac{c(a+b)}{a+b-c}+\frac{b(c+a)}{c+a-b}+\frac{a(b+c)}{b+c-a}$$

$$\geq 4\left(\frac{(a+b-c)^2}{a+b}+\frac{(b+c-a)^2}{b+c}+\frac{(c+a-b)^2}{c+a}\right)$$

$$\Leftrightarrow \frac{4Rs}{2r} \geq \frac{16s(s^2-4Rr-3r^2)}{2(s^2+2Rr+r^2)} \Leftrightarrow R(s^2+2Rr+r^2) \geq 4r(s^2-4Rr-3r^2)$$

$$\Leftrightarrow (R-4r)s^2+R(2Rr+r^2)+4r(4Rr+3r^2) \overset{(\bullet)}{\geq} 0$$

Case 1 $R-4r \geq 0$ and then, LHS of $(\bullet) \geq R(2Rr+r^2)+4r(4Rr+3r^2)>0$

$\Rightarrow (\bullet)$ is true (strict inequality)

Case 2 $R-4r<0$ and then, LHS of $(\bullet)$

$$=-(4r-R)s^2+R(2Rr+r^2)+4r(4Rr+3r^2) \overset{\text{Gerretsen}}{\geq}$$

$$-(4r-R)(4R^2+4Rr+3r^2)+R(2Rr+r^2)+4r(4Rr+3r^2) \overset{?}{\geq} 0$$

$$\Leftrightarrow 2R^2-5Rr+2r^2 \overset{?}{\geq} 0 \Leftrightarrow (R-2r)(2R-r) \overset{?}{\geq} 0 \rightarrow \text{true} \because R \overset{\text{Euler}}{\geq} 2r \Rightarrow (\bullet) \text{is true}$$

$$\Rightarrow \frac{c(a+b)}{a+b-c}+\frac{b(c+a)}{c+a-b}+\frac{a(b+c)}{b+c-a}$$

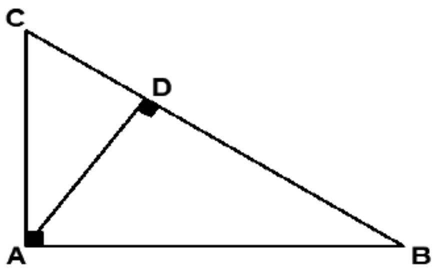
$$\geq 4\left(\frac{(a+b-c)^2}{a+b}+\frac{(b+c-a)^2}{b+c}+\frac{(c+a-b)^2}{c+a}\right) \text{ (QED)}$$

527. R, R_1, R_2 **−circumradii of** $\Delta ABC, \Delta ABD, \Delta ACD$. **Prove that :**

$$\frac{R-R_1}{R+R_1}+\frac{R-R_2}{R+R_2} \geq \frac{2h_a}{2p+h_a}.$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco



Let F be the area of ΔABC. *We have* : $2F = ah_a = bc$.

$$\text{Then}: \frac{2h_a}{2p+h_a}=\frac{2ah_a}{2pa+ah_a}=\frac{2bc}{a(a+b+c)+bc}=\frac{2bc}{(a+b)(a+c)} \quad (1)$$

$$\text{Now, we have}: \ R=\frac{a}{2}, \ R_1=\frac{c}{2}, \ R_2=\frac{b}{2} \text{ and } a^2=b^2+c^2 \text{ then}:$$

$$\frac{R-R_1}{R+R_1}+\frac{R-R_2}{R+R_2}=\frac{a-c}{a+c}+\frac{a-b}{a+b}=\frac{(a-c)(a+b)+(a-b)(a+c)}{(a+b)(a+c)}=\frac{2(a^2-bc)}{(a+b)(a+c)}$$

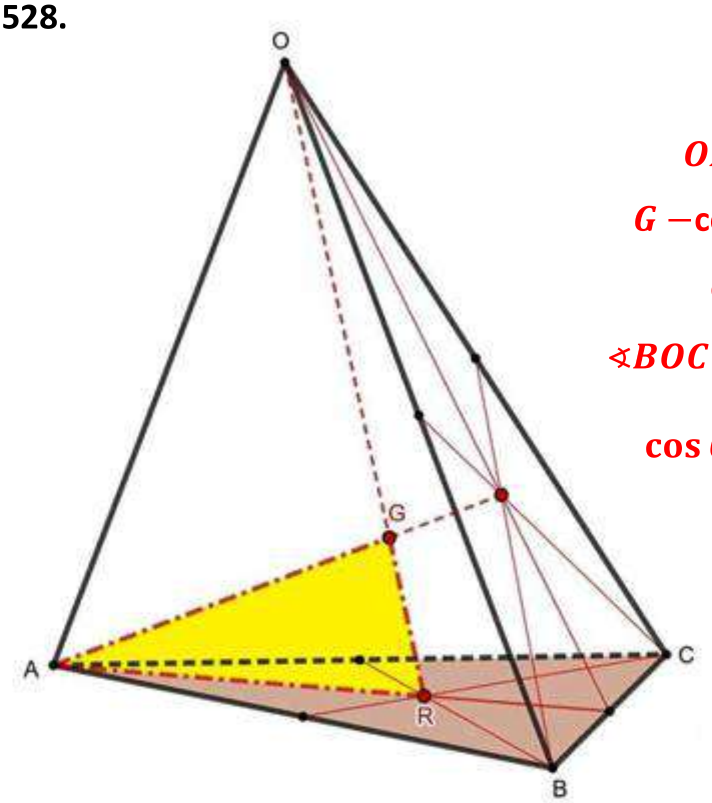
$$=\frac{2(b^2+c^2-bc)}{(a+b)(a+c)}\overset{AM-GM}{\geq}\frac{2(2bc-bc)}{(a+b)(a+c)}\overset{(1)}{=}\frac{2h_a}{2p+h_a}.$$

Therefore,

$$\frac{R-R_1}{R+R_1}+\frac{R-R_2}{R+R_2}\geq\frac{2h_a}{2p+h_a}.$$

Equality holds iff ΔABC is isosceles right on A.

528.



$(OABC)$ −tetrahedron

$OA=OB=OC=12$

G −centroid of $(OABC)$, R −centroid of ΔABC

$\measuredangle BOC=\theta_1, \measuredangle COA=\theta_2, \measuredangle AOB=\theta_3$

$\cos\theta_1=\frac{1}{4}, \cos\theta_2=\frac{1}{3}, \cos\theta_3=\frac{1}{2}$

Find $[AGR]=?$, $[*]$ −area.

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by proposer

Plagiogonal 3rd system: $AO\equiv Ox, OB\equiv Oy, OC\equiv Oz$.

$O(0,0,0), A(12,0,0), R(4,4,4), Q(3,3,3)$

$\overrightarrow{AG}(-9,3,3), \overrightarrow{AR}(-8,4,4)$

$$\left|\overrightarrow{AG}\right|^2=81+9+9+2(-9)\cdot 3\cdot\frac{1}{2}+2\cdot 3\cdot 3\cdot\frac{1}{4}+2\cdot(-9)\cdot 3\cdot\frac{1}{3}=\frac{117}{2}$$

$$\left|\overrightarrow{AR}\right|^2 = 64 + 16 + 16 + 2(-8)\cdot 4\cdot\frac{1}{2} + 2\cdot 4\cdot 4\cdot\frac{1}{4} + 2(-8)\cdot 4\cdot\frac{1}{3} = \frac{152}{3}$$

$$\overrightarrow{AG}\cdot\overrightarrow{AR} = (-9)(-8) + 3\cdot 4 + 3\cdot 4 + (-9\cdot 4 - 3\cdot 8)\cdot\frac{1}{2} + (3\cdot 4 + 3\cdot 4)\cdot\frac{1}{4} +$$

$$+(-3\cdot 8 - 4\cdot 9)\cdot\frac{1}{3} = 58$$

$$[AGR] = \frac{1}{2}\cdot\sqrt{\left(\left|\overrightarrow{AG}\right|\cdot\left|\overrightarrow{AR}\right|\right)^2 - \left(\overrightarrow{AG}\cdot\overrightarrow{AR}\right)^2} = \frac{1}{2}\sqrt{\frac{117}{2}\cdot\frac{152}{3} - 52^2}$$

$$[AGR] = \sqrt{65}$$

529. In ΔABC the following relationship holds:

$$\frac{16}{3}\left(\frac{2r}{R}\right) \le \sum_{cyc}\sec^4\frac{A}{2} \le \frac{16}{3}\left(\frac{R}{2r}\right)^3$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} r_a^4 = \left(\sum_{cyc} r_a^2\right)^2 - 2\sum_{cyc} r_a^2 r_b^2 = \left((4R+r)^2 - 2s^2\right)^2 - 2\left(\left(\sum_{cyc} r_a r_b\right)^2 - 2r_a r_b r_b\left(\sum_{cyc} r_a\right)\right)$$

$$= \left((4R+r)^2 - 2s^2\right)^2 - 2\left(s^4 - 2rs^2(4R+r)\right)$$

$$= (4R+r)^4 + 2s^4 - \left(64R^2 + 16Rr\right)s^2 \therefore \sum_{cyc} r_a^4 \overset{(i)}{=} (4R+r)^4 + 2s^4 - \left(64R^2 + 16Rr\right)s^2$$

$$\text{Now,} \sum_{cyc}\sec^4\frac{A}{2} = \sum_{cyc}\left(1 + \tan^2\frac{A}{2}\right)^2 = \sum_{cyc}\left(1 + 2\tan^2\frac{A}{2} + \tan^4\frac{A}{2}\right)$$

$$= 3 + \frac{2}{s^2}\cdot\sum_{cyc} r_a^2 + \frac{1}{s^4}\cdot\sum_{cyc} r_a^4 \overset{\text{via (i)}}{=} 3 + \frac{(4R+r)^2 - 2s^2}{s^2}$$

$$+\frac{(4R+r)^4 + 2s^4 - \left(64R^2 + 16Rr\right)s^2}{s^4}$$

$$\Rightarrow \sum_{cyc}\sec^4\frac{A}{2} \overset{(ii)}{=} \frac{(4R+r)^4 + s^4 - \left(32R^2 - 2r^2\right)s^2}{s^4} \therefore \sum_{cyc}\sec^4\frac{A}{2} \le \frac{16}{3}\left(\frac{R}{2r}\right)^3$$

$$\Leftrightarrow \frac{(4R+r)^4 + s^4 - \left(32R^2 - 2r^2\right)s^2}{s^4} \le \frac{2R^3}{3r^3}$$

$$\Leftrightarrow \left(2R^3 - 3r^3\right)s^4 + 3r^3\left(32R^2 - 2r^2\right)s^2 \overset{(*)}{\ge} 3r^3(4R+r)^4$$

$$\text{Now, LHS of } (*) \overset{\text{Gerretsen}}{\geq} \left((2R^3 - 3r^3)(16Rr - 5r^2)\right.$$
$$\left.+ 3r^3(32R^2 - 2r^2)\right)s^2 \overset{\text{Gerretsen}}{\geq} \left((2R^3 - 3r^3)(16Rr - 5r^2)\right.$$
$$\left.+ 3r^3(32R^2 - 2r^2)\right)(16Rr - 5r^2) \overset{?}{\geq} 3r^3(4R + r)^4$$
$$\Leftrightarrow 256t^5 - 544t^4 + 409t^3 - 768t^2 + 168t - 24 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right)$$
$$\Leftrightarrow (t - 2)(240t^4 + 16t^3(t - 2) + 306t^2 + 39t(t - 2) + 12) \overset{?}{\geq} 0 \to \text{true}$$
$$\because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (*) \text{is true}$$
$$\therefore \boxed{\sum_{cyc} \sec^4\frac{A}{2} \leq \frac{16}{3}\left(\frac{R}{2r}\right)^3} \textit{ and again, via (ii), } \frac{16}{3}\left(\frac{2r}{R}\right) \leq \sum_{cyc} \sec^4\frac{A}{2}$$
$$\Leftrightarrow \frac{(4R + r)^4 + s^4 - (32R^2 - 2r^2)s^2}{s^4} \geq \frac{32r}{3R}$$
$$\Leftrightarrow (3R - 32r)s^4 + 3R(4R + r)^4 - 3R(32R^2 - 2r^2)s^2 \overset{(**)}{\geq} 0$$
$$\text{Now, LHS of } (**) = (3R - 6r)s^4 - 26rs^4 + 3R(4R + r)^4 - 3R(32R^2 - 2r^2)s^2$$
$$\overset{\text{Gerretsen}}{\geq} (3R - 6r)(16Rr - 5r^2)s^2 - 26r(4R^2 + 4Rr + 3r^2)s^2 - 3R(32R^2 - 2r^2)s^2$$
$$+ 3R(4R + r)^4 \overset{?}{\geq} 0 \Leftrightarrow 3R(4R + r)^4 \underbrace{\overset{?}{\geq}}_{(***)} (96R^3 + 56R^2r + 209Rr^2 + 48r^3)s^2$$
$$\text{Now, RHS of } (***) \overset{\text{Gerretsen}}{\leq} (96R^3 + 56R^2r + 209Rr^2 + 48r^3)(4R^2 + 4Rr$$
$$+ 3r^2) \overset{?}{\leq} 3R(4R + r)^4 \Leftrightarrow 96t^5 + 40t^4 - 265t^3 - 287t^2 - 204t - 36 \overset{?}{\geq} 0$$
$$\Leftrightarrow (t - 2)(96t^4 + 232t^3 + 199t^2 + 111t + 18) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (***) \Rightarrow (**) \text{is true}$$
$$\therefore \boxed{\frac{16}{3}\left(\frac{2r}{R}\right) \leq \sum_{cyc} \sec^4\frac{A}{2}} \text{(QED)}$$

530. In ΔABC the following relationship holds:

$$\left(\frac{m_a}{h_a}\right)^2 + \left(\frac{m_b}{h_b}\right)^2 + \left(\frac{m_c}{h_c}\right)^2 \geq 3$$

Proposed by Kostas Geronikolas-Greece

Solution by Ertan Yildirim-Turkiye

$$m_a \geq \sqrt{s(s - a)}; \ (1), \qquad \sum a^2 = 2(s^2 - r^2 - 4Rr); \ (2)$$
$$\sum a^3 = 2s(s^2 - 3r^2 - 6Rr); \ (3)$$
$$\sum_{cyc}\left(\frac{m_a}{h_a}\right)^2 \overset{(1)}{\geq} \sum_{cyc} \frac{s(s - a)}{\left(\frac{bc}{2R}\right)^2} = 4R^2s\sum_{cyc}\frac{s - a}{b^2c^2} = \frac{4R^2s}{a^2b^2c^2} \cdot \sum_{cyc} a^2(s - a) =$$

$$= 4R^2 \frac{s}{16R^2s^2r^2} \sum_{cyc} (a^2s - a^3) = \frac{1}{4r^2s} \sum_{cyc} (a^2s - a^3) =$$

$$\overset{(2),(3)}{=} \frac{1}{4r^2s} [s \cdot 2(s^2 - r^2 - 4Rr) - 2s(s^2 - 3r^2 - 6Rr)] =$$

$$= \frac{2s}{4r^2s} (s^2 - r^2 - 4Rr - s^2 + 3r^2 + 6Rr) =$$

$$= \frac{1}{2r^2} (2Rr + 2r^2) = \frac{2r}{2r^2} (R + r) = \frac{1}{r} (R + r) \geq 3 \Leftrightarrow R \geq 2r \ (Euler)$$

531. In ΔABC the following relationship holds:

$$\frac{1}{R} \left(\frac{2r}{R}\right)^{\frac{10}{3}} \leq \sum_{cyc} \frac{h_a}{r_b^2 + r_c^2} \leq \frac{1}{2} \left(\frac{1}{R} + \frac{1}{2r}\right)$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have: \sum_{cyc} \frac{h_a}{r_b^2 + r_c^2} \overset{AM-GM}{\leq} \sum_{cyc} \frac{h_a}{2r_br_c} = \sum_{cyc} \frac{sr}{a.s(s-a)} = \frac{r}{s} \sum_{cyc} \left(\frac{1}{a} + \frac{1}{s-a}\right)$$

$$= \frac{r}{s} \left(\frac{s^2 + r^2 + 4Rr}{4sRr} + \frac{4R + r}{sr}\right) =$$

$$= \frac{1}{4R} + \frac{(4R + r)^2}{4Rs^2} \overset{Gerretsen}{\leq} \frac{1}{4R} + \frac{(4R + r)^2}{4R(16Rr - 5r^2)} \overset{?}{\leq} \frac{1}{2} \left(\frac{1}{R} + \frac{1}{2r}\right) \leftrightarrow$$

$$\frac{(4R + r)^2}{4Rr(16R - 5r)} \leq \frac{R + r}{4Rr}$$

$$\leftrightarrow (4R + r)^2 \leq (R + r)(16R - 5r) \leftrightarrow 3r(R - 2r)$$

$$\geq 0 \ which\ is\ true\ from\ Euler's\ inequality.$$

$$Now, we\ have: \sum_{cyc} \frac{h_a}{r_b^2 + r_c^2} \overset{AM-GM}{\geq} 3 \sqrt[3]{\frac{h_ah_bh_c}{\prod_{cyc}(r_b^2 + r_c^2)}} \overset{AM-GM}{\geq} \frac{9\sqrt[3]{h_ah_bh_c}}{2(r_a^2 + r_b^2 + r_c^2)}$$

$$With: r_a^2 + r_b^2 + r_c^2 = (4R + r)^2 - 2s^2 \overset{Gerretsen}{\leq} (4R + r)^2 - 2(16Rr - 5r^2)$$

$$= 16R^2 - 24Rr + 11r^2 \overset{?}{\leq} \frac{27R^3}{8r}$$

$$\leftrightarrow \frac{(R-2r)[(R-2r)(27R-20r)+4r^2]}{8r} \geq 0$$

which is true from Euler's inequality.

$$Then: {r_a}^2+{r_b}^2+{r_c}^2 \leq \frac{27R^3}{8r}.$$

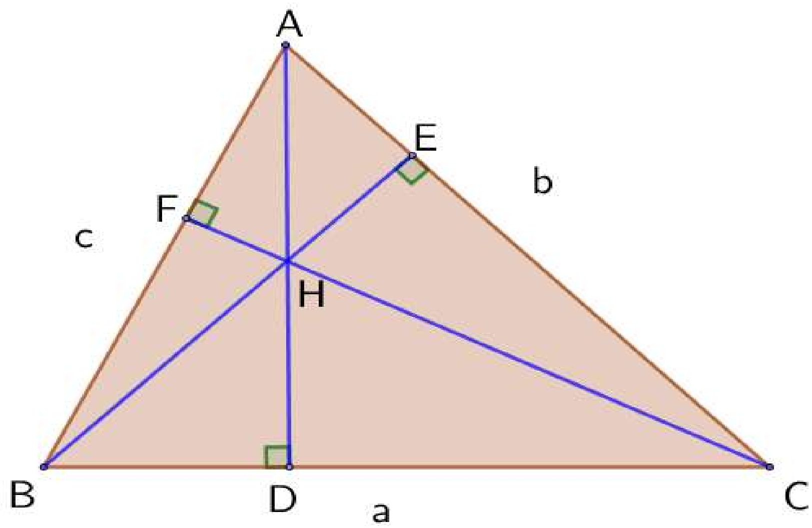
$$We\ also\ have: h_ah_bh_c = \frac{2s^2r^2}{R} \overset{Mitrinovic}{\geq} \frac{2.27r^2.r^2}{R} = \frac{54r^4}{R}.$$

$$Then: \sum_{cyc} \frac{h_a}{{r_b}^2+{r_c}^2} \geq \frac{9\sqrt[3]{\frac{54r^4}{R}}}{2.\frac{27R^3}{8r}} = \frac{1}{R}\left(\frac{2r}{R}\right)^{\frac{7}{3}} \overset{Euler}{\geq} \frac{1}{R}\left(\frac{2r}{R}\right)^{\frac{10}{3}}.$$

$$Therefore,\ \frac{1}{R}\left(\frac{2r}{R}\right)^{\frac{10}{3}} \leq \sum_{cyc} \frac{h_a}{{r_b}^2+{r_c}^2} \leq \frac{1}{2}\left(\frac{1}{R}+\frac{1}{2r}\right).$$

532. In acute $\Delta ABC, AD, BE, CF$ −altitudes, H −orthocenter.Prove that:

$$\left(\frac{AH}{HD}\right)^4+\left(\frac{BH}{HE}\right)^4+\left(\frac{CH}{HF}\right)^4 \geq 48$$



Proposed by George Apostolopoulos-Messolonghi-Greece

Solution 1 by Avishek Mitra-West Bengal-India

$$AH = AE = \frac{2}{\cos(DAC)} = AB \cdot \cos A \cdot \sec\left(\frac{\pi}{2}-C\right) = c \cdot \cos A \cdot \frac{1}{\sin C} =$$

$$= 2R\cos A\,;\left(\because \frac{c}{\sin C} = 2R\right)$$

Again, $HD = BD \cdot \tan(HBD) = BD \cdot \tan\left(\frac{\pi}{2} - C\right) =$

$$= AB \cdot \cos B \cdot \cot C = c \cdot \cos B \cdot \frac{\cos C}{\sin C} = 2R \cdot \cos B \cos C$$

$$\sum_{cyc}\left(\frac{AH}{HD}\right)^4 = \sum_{cyc}\left(\frac{2R\cos A}{2R\cos B\cos C}\right)^4 \overset{Power\ Mean}{\geq} 3\cdot\left(\frac{1}{3}\sum_{cyc}\frac{\cos A}{\cos B\cos C}\right)^4 =$$

$$= \frac{1}{27}\left(\frac{1}{\prod\cos A}\sum_{cyc}\cos^2 A\right)^4 \overset{AGM}{\geq} \frac{1}{27}\left(\frac{1}{\frac{(\sum\cos A)^3}{27}}\sum_{cyc}(1-\sin^2 A)\right)^4 =$$

$$= \frac{1}{27}\left(\frac{27}{\left(1+\frac{r}{R}\right)^3}\left(3 - \sum_{cyc}\sin^2 A\right)\right)^4 \overset{Euler}{\geq} \frac{1}{27}\left(\frac{27}{\left(1+\frac{1}{2}\right)^3}\left(3 - \frac{1}{4R^2}\sum_{cyc}a^2\right)\right)^4 \overset{Leibniz}{\geq}$$

$$\geq \frac{1}{27}\left(8\left(3 - \frac{1}{4R^2}\cdot 9R^2\right)\right)^4 = \frac{1}{27}\left(8\cdot\frac{3}{4}\right)^4 = 48$$

Solution 2 by Eldeniz Hesenov-Azerbaijan

$\frac{AE}{c} = \cos A \Rightarrow AE = c\cos A$

$EC = a\cos C\,, BD = c\cos B$

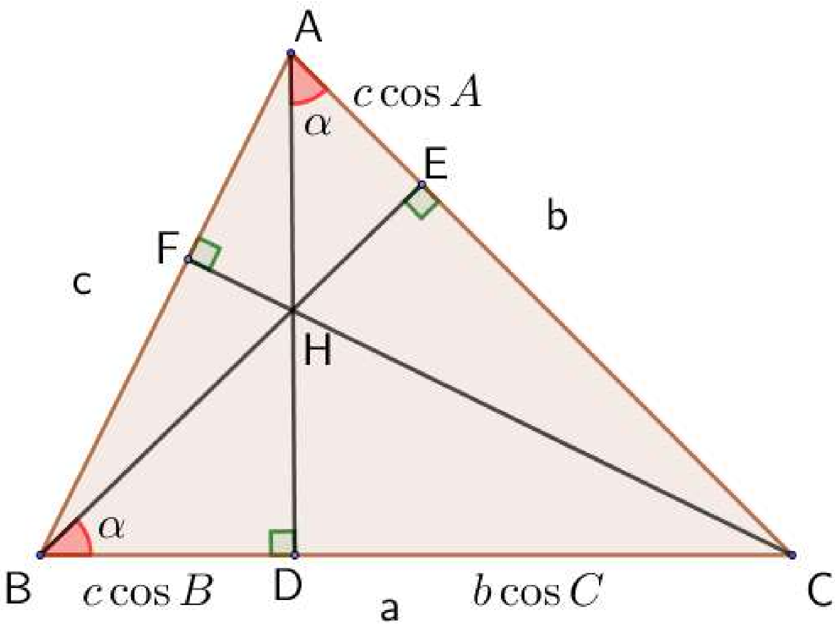
$$\cos\alpha = \frac{c\cos A}{AH} = \frac{BE}{a}$$

$$AH = \frac{ac\cos A}{BE} = \frac{abc\cos A}{2F}$$

$$AH = 2R\cos A\,;(1)$$

$$\tan\alpha = \frac{HD}{c\cos B} = \frac{b\cos C}{AD}$$

$$HD = 2R\cos B\cos C\,;(2)$$



From (1) and (2) we get:

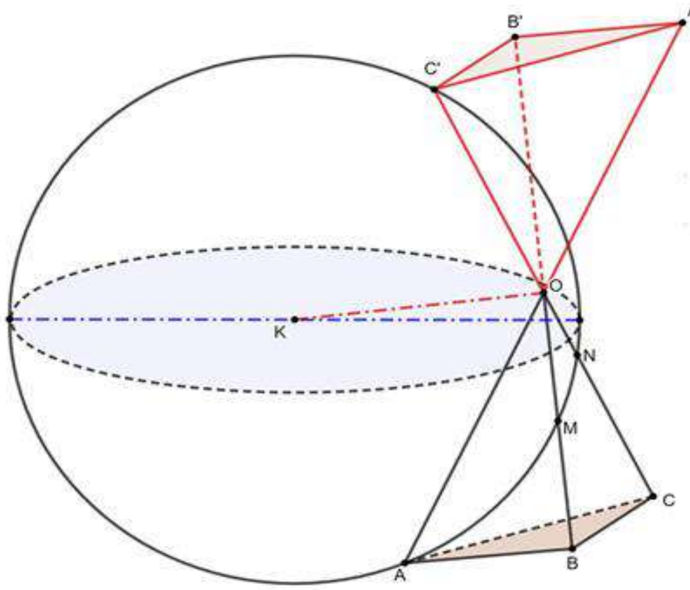
$$\frac{AH}{HD}=\frac{\cos A}{\cos B\cos C}(and\ analogs)$$

$$\sum_{cyc}\left(\frac{AH}{HD}\right)^4=\sum_{cyc}\left(\frac{\cos A}{\cos B\cos C}\right)^4\overset{\sum x^2\ge\sum xy}{\ge}\sum_{cyc}\left(\frac{\cos A}{\cos B\cos C}\right)^2\left(\frac{\cos B}{\cos C\cos A}\right)^2\ge$$

$$\overset{\sum x^2\ge\sum xy}{\ge}\sum_{cyc}\cos^2 A\cdot\frac{1}{(\prod\cos A)^2}\overset{\left(\prod\cos A\ge\frac{r^2}{2R^2}\right)}{\ge}\sum_{cyc}(1-\sin^2 A)\cdot\frac{4R^4}{r^4}=$$

$$=\frac{4R^4}{r^4}\sum_{cyc}\left(1-\frac{a^2}{4R^2}\right)=\frac{4R^4}{r^4}\left(3-\frac{1}{4R^2}\sum_{cyc}a^2\right)\overset{Leibniz}{\ge}\frac{4R^4}{r^4}\cdot\frac{3}{4}=\frac{3R^4}{r^4}\overset{Euler}{\ge}\frac{48r^4}{r^4}=48.$$

533.



$OABC$ –regular tetrahedron with side a, $OA'B'C'$ –symmetric tetrahedron to the point O, $M\in OB$, $2MO=OB, N\in OC$ $3ON=OC$ $(A,M,N,C')\in(S)$, (S) –sphere, K –center of (S), R –radius of (S). Find: $R=f(a)=?$, $OK=f(a)=?$

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by proposer

Plagiogonal 3rd system: $OA\equiv Ox, OB\equiv Oy, OC\equiv Oz$.

Let $OA=1, O(0,0,0), A(1,0,0), M\left(0,\frac{1}{2},0\right), N\left(0,0,\frac{1}{3}\right), C'(0,0,-1)$

$$u=x^2+y^2+z^2+xy+yx+zx$$

$$(S):\begin{vmatrix} u & x & y & z & 1\\ 1 & 1 & 0 & 0 & 1\\ \frac{1}{4} & 0 & \frac{1}{2} & 0 & 1\\ \frac{1}{9} & 0 & 0 & \frac{1}{3} & 1\\ 1 & 0 & 0 & -1 & 1\end{vmatrix}=0$$

$$(S): x^2+y^2+z^2+xy+yz+zx-\frac{2}{3}x+\frac{1}{6}y+\frac{2}{3}z-\frac{1}{3}=0;(1)$$

Let $K(k_1,k_2,k_3)$.

$$\begin{cases}-2k_1-k_2-k_3=-\frac{2}{3}\\ -k_1-2k_2-k_3=\frac{1}{6}\\ -k_1-k_2-2k_3=\frac{2}{3}\end{cases}\Rightarrow\begin{cases}k_1=\frac{17}{24}\\ k_2=-\frac{1}{8}\\ k_3=-\frac{5}{8}\end{cases}$$

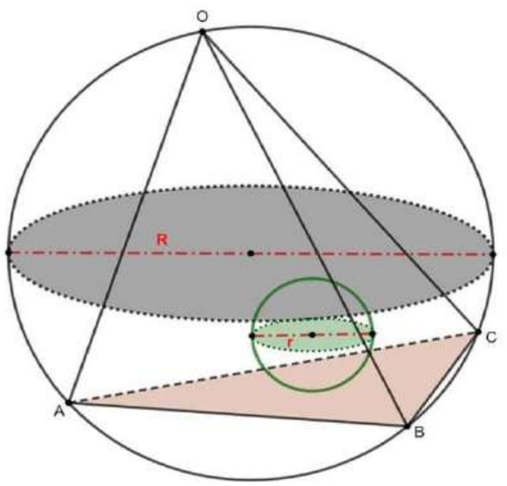
$$k_1^2+k_2^2+k_3^2+k_1k_2+k_2k_3+k_3k_1-R^2=-\frac{1}{3}\Rightarrow R^2=\frac{227}{288}$$

$$R=\frac{\sqrt{154}}{24}a$$

$$OK^2=\left(0-\frac{17}{24}\right)^2+\left(0+\frac{1}{8}\right)^2+\left(0+\frac{5}{8}\right)^2+\left(-\frac{17}{24}\right)\left(\frac{1}{8}\right)+\left(\frac{1}{8}\right)\left(\frac{5}{8}\right)+\left(\frac{5}{8}\right)\left(-\frac{17}{24}\right)$$

$$OK^2=\frac{131}{288}\Rightarrow OK=\frac{\sqrt{262}}{24}$$

534.



$OABC$ – tetrahedron with volume V

$OA=a, OB=b, OC=c,$

a,b,c – const.

$\measuredangle BOC, \measuredangle COA, \measuredangle AOB$ variable

r – radius of insphere,

R – radius of circumsphere

If $V = V_{max}$ find: $r = r(a,b,c) = ?$, $R = R(a,b,c) = ?$

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by proposer

$$V = \frac{abc}{6}\sqrt{1 - \cos^2\theta_1 - \cos^2\theta_2 - \cos^2\theta_3 + 2\cos^2\theta_1\cos^2\theta_2\cos^2\theta_3}$$

$$\frac{\partial V}{\partial(\cos\theta_1)} = 0 \Rightarrow \cos\theta_1 = \cos\theta_2\cos\theta_3$$

$$\frac{\partial V}{\partial(\cos\theta_2)} = 0 \Rightarrow \cos\theta_2 = \cos\theta_3\cos\theta_1$$

$$\frac{\partial V}{\partial(\cos\theta_3)} = 0 \Rightarrow \cos\theta_3 = \cos\theta_1\cos\theta_1\,, 0 < \theta_1, \theta_2, \theta_3 < \pi$$

Hence: $\theta_1 = \theta_2 = \theta_3$

$$V_{max} = \frac{abc}{6}, S_1 = \frac{bc}{2}, S_2 = \frac{ca}{2}, S_3 = \frac{ab}{2}, S_4 = \frac{1}{2}\sqrt{a^2b^2 + b^2c^2 + c^2a^2}$$

$$V = \frac{3V_{max}}{S_1 + S_2 + S_3 + S_4} \Rightarrow V = \frac{abc}{ab + bc + ca + \sqrt{a^2b^2 + b^2c^2 + c^2a^2}}$$

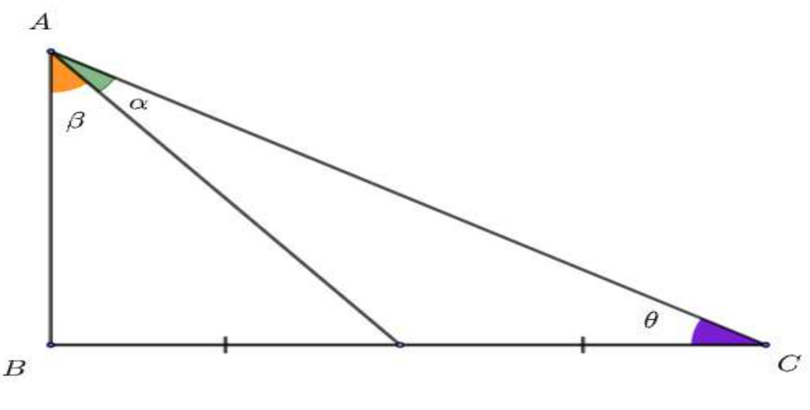
Let $BC = a_1 = \sqrt{b^2 + c^2}, AC = b_2 = \sqrt{a^2 + c^2}, AB = c_1 = \sqrt{a^2 + b^2}$

$$s = \frac{aa_1 + bb_1 + cc_1}{2} \Rightarrow s = \frac{1}{2}\left(a\sqrt{b^2 + c^2} + b\sqrt{c^2 + a^2} + c\sqrt{a^2 + b^2}\right)$$

$$R^2 = \frac{s(s - aa_1)(s - bb_1)(s - cc_1)}{(6V)^2} = \frac{\frac{1}{4}a^2b^2c^2(a^2 + b^2 + c^2)}{(abc)^2}$$

$$R = \frac{\sqrt{a^2 + b^2 + c^2}}{2}$$

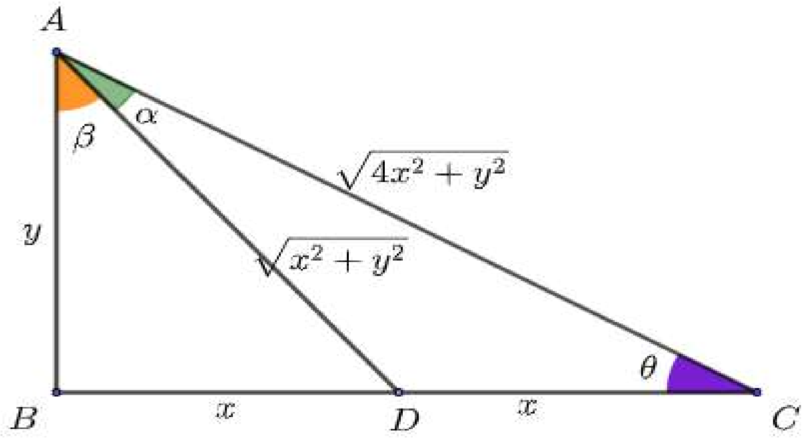
535.



In $\Delta ABC, \mu(\sphericalangle ABC) = 90°, AD$ –median, $\mu(\sphericalangle BAD) = \beta, \mu(\sphericalangle DAC) = \alpha$ $\mu(\sphericalangle BCA) = \theta$. Find the maximum value of α in radians and prove that for this value of α we have $\beta = \theta$.

Proposed by Mehmet Şahin, Alican Gullu-Turkiye

Solution 1 by Adrian Popa-Romania



$$\frac{x}{\sin\alpha} = \frac{\sqrt{x^2+y^2}}{\sin\theta} \Rightarrow \frac{x}{\sqrt{x^2+y^2}} = \frac{\sin\alpha}{\sin\theta} \Rightarrow \sin\beta = \frac{\sin\alpha}{\sin\theta}$$

$$\alpha+\beta+\theta = \frac{\pi}{2} \Rightarrow \beta = \frac{\pi}{2} - (\alpha+\beta)$$

$$\cos\alpha\cos\theta - \sin\alpha\sin\theta = \frac{\sin\alpha}{\sin\theta}, \qquad \cos\theta - \tan\alpha\sin\theta = \frac{\tan\alpha}{\sin\theta}$$

$$\cos\theta\sin\theta - \tan\alpha\sin^2\theta = \tan\alpha$$

$$\tan\alpha = \frac{\cos\theta\sin\theta}{1+\sin^2\theta} = \frac{\sin 2\theta}{2(1+\sin^2\theta)} = f(\theta)$$

$$f'(\theta) = \frac{4\cos 2\theta\,(1+\sin^2\theta) - 2\sin^2 2\theta}{4(1+\sin^2\theta)^2},$$

$$f'(\theta) = 0 \Rightarrow \sin^2 2\theta = 2\cos 2\theta\,(1+\sin^2\theta)$$

$$\sin^2 2\theta = 2\cos 2\theta\left(1+\frac{1-\cos 2\theta}{2}\right)$$

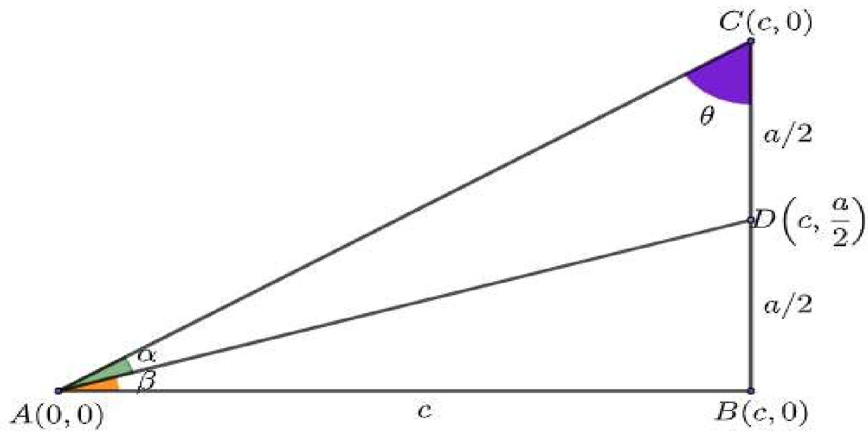
$$\sin^2 2\theta = \cos 2\theta\,(3-\cos 2\theta)$$

$$\cos 2\theta = \frac{1}{3} \Rightarrow 2\cos^2\theta - 1 = \frac{1}{3} \Rightarrow \cos^2\theta = \frac{2}{3} \Rightarrow \sin\theta = \frac{\sqrt{3}}{3}$$

$$\tan\alpha = \frac{2\sin\theta\cos\theta}{2(1+\sin^2\theta)} = \frac{\frac{1}{\sqrt{3}}\cdot\frac{\sqrt{2}}{\sqrt{3}}}{1+\frac{1}{3}} = \frac{\sqrt{2}}{4}$$

$$\alpha_{max} = \tan^{-1}\left(\frac{\sqrt{2}}{4}\right) \Rightarrow \sin\beta = \frac{\sin\alpha}{\sin\theta} = \frac{\frac{1}{3}}{\frac{1}{\sqrt{3}}} = \frac{\sqrt{3}}{3} = \sin\theta \Rightarrow \beta = \theta.$$

Solution 2 by Bedri Hajrizi-Mitrovica-Kosovo



$$AC: y = \frac{a}{c}x; AD: y = \frac{a}{2c}x$$

$$\tan\alpha = \frac{\frac{a}{c}-\frac{a}{2c}}{1+\frac{a^2}{2c^2}} = \frac{a}{2c}\cdot\frac{2c^2}{2c^2+a^2} = \frac{ac}{a^2+2c^2}$$

$$\tan\alpha \le \frac{ac}{2\sqrt{a^2\cdot 2c^2}} = \frac{ac}{2\sqrt{2}ac} = \frac{1}{2\sqrt{2}}$$

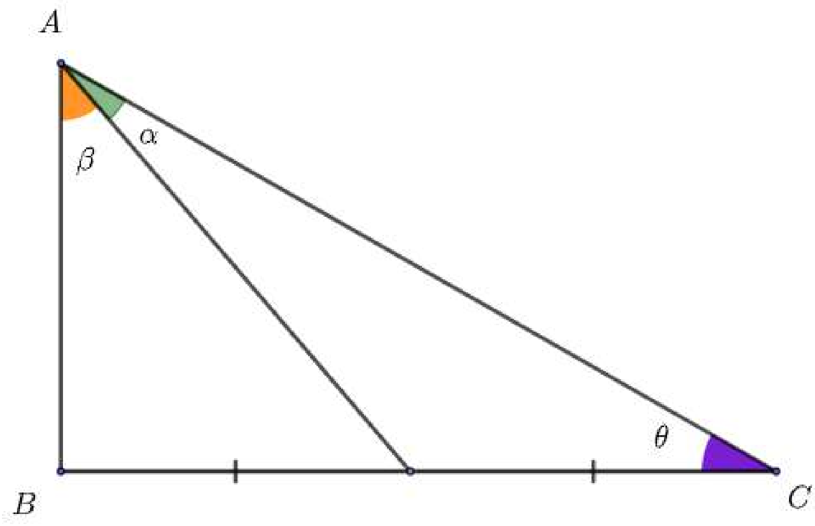
Equality for $c = \frac{a}{\sqrt{2}} = \frac{a\sqrt{2}}{2}$.

$$\tan\beta = \frac{1}{\tan(\alpha+\beta)} = \frac{1-\tan\alpha\tan\theta}{\tan\alpha+\tan\theta} = \frac{1-\frac{1}{2\sqrt{2}}\tan\theta}{\frac{1}{2\sqrt{2}}+\tan\theta} =$$

$$= \frac{2\sqrt{2}-\tan\theta}{1+2\sqrt{2}\tan\theta} \overset{\left(\tan\theta=\frac{c}{a}=\frac{\sqrt{2}}{2}\right)}{=} \frac{2\sqrt{2}-\frac{\sqrt{2}}{2}}{1+2\sqrt{2}\cdot\frac{\sqrt{2}}{2}} = \frac{\sqrt{2}}{2} = \tan\theta$$

$$\tan\theta = \tan\beta \Leftrightarrow \theta = \beta.$$

Solution 3 by Ertan Yildirim-Izmir-Turkiye



$$\cot\beta = 2\cot(\alpha+\beta) = 2\tan\theta, \qquad \cot\beta = 2\cot(\alpha+\beta) \Rightarrow \frac{1}{\tan\beta} = \frac{2}{\tan(\alpha+\beta)}$$

$$2\tan\beta = \tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

Let $x = \tan\beta$, $y = \tan\alpha$, then $2x = \frac{y+x}{1-xy}$

$$2x - 2x^2y = y + x \Rightarrow x = y(2x^2+1) \Rightarrow y = \frac{x}{2x^2+1}$$

$$y' = \frac{2x^2+1-4x^2}{(1+2x^2)^2} = \frac{1-2x^2}{(1+2x^2)^2} = 0$$

$$x = \frac{1}{\sqrt{2}} \Rightarrow \tan\beta = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\cot\beta = \sqrt{2} = 2\tan\theta \Rightarrow \tan\theta = \frac{\sqrt{2}}{2} \Rightarrow \tan\beta = \tan\theta \Rightarrow \beta = \theta.$$

536. In ΔABC the following relationship holds:

$$\sqrt[3]{\frac{a^4}{b^2+c(a+b)}} + \sqrt[3]{\frac{b^4}{c^2+a(b+c)}} + \sqrt[3]{\frac{c^4}{a^2+b(c+a)}} \geq 3\cdot\sqrt[3]{4r^2}$$

Proposed by Marin Chirciu-Romania

Solution by Ertan Yildirim-Izmir-Turkiye

$$\sqrt[3]{\frac{a^4}{b^2+c(a+b)}}+\sqrt[3]{\frac{b^4}{c^2+a(b+c)}}+\sqrt[3]{\frac{c^4}{a^2+b(c+a)}}=$$

$$=\frac{a^{\frac{4}{3}}}{\left(b^2+c(a+b)\right)^{\frac{1}{3}}}+\frac{b^{\frac{4}{3}}}{\left(c^2+a(b+c)\right)^{\frac{1}{3}}}+\frac{c^{\frac{4}{3}}}{\left(a^2+b(c+a)\right)^{\frac{1}{3}}}\overset{Radon}{\geq}$$

$$\geq\frac{(a+b+c)^{\frac{4}{3}}}{\left(a^2+b^2+c^2+2(ab+bc+ca)\right)^{\frac{1}{3}}}=\frac{(a+b+c)^{\frac{4}{3}}}{(a+b+c)^{\frac{2}{3}}}=(a+b+c)^{\frac{2}{3}}=(2s)^{\frac{2}{3}}$$

$$(2s)^{\frac{2}{3}}\geq 3\cdot\sqrt[3]{4r^2}\Leftrightarrow(2s)^2\geq 27\cdot 4r^2\Leftrightarrow 4s^2\geq 27\cdot 4r^2\Leftrightarrow s^2\geq 27r^2\Leftrightarrow$$

$$s^2\geq 3\sqrt{3}r\ (Mitrinovic)$$

537. In ΔABC the following relationship holds:

$$\frac{a}{(b+\lambda c)^{n+1}{w_a}^n}+\frac{b}{(c+\lambda a)^{n+1}{w_b}^n}+\frac{c}{(c+\lambda a)^{n+1}{w_c}^n}\geq\frac{3}{(1+\lambda)^{n+1}}\left(\frac{1}{pR}\right)^n,$$

$$\lambda\geq 0, n\in N.$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:\ \sum_{cyc}\frac{a}{(b+\lambda c)^{n+1}{w_a}^n}=\sum_{cyc}\frac{\left(\frac{a}{b+\lambda c}\right)^{n+1}}{(aw_a)^n}\overset{Hölder}{\geq}\frac{\left(\sum_{cyc}\frac{a}{b+\lambda c}\right)^{n+1}}{\left(\sum_{cyc}aw_a\right)^n}\quad(1)$$

$$Also\ we\ have:\ \sum_{cyc}\frac{a}{b+\lambda c}\overset{CBS}{\geq}\frac{(a+b+c)^2}{\sum_{cyc}a(b+\lambda c)}\geq\frac{3(ab+bc+ca)}{(1+\lambda)(ab+bc+ca)}=\frac{3}{1+\lambda}\quad(2)$$

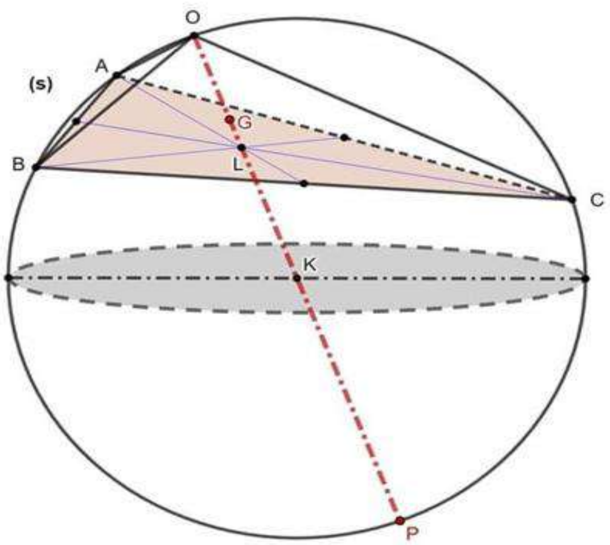
$$And:\ \sum_{cyc}aw_a\overset{w_a\leq\sqrt{p(p-a)}}{\leq}\sum_{cyc}a\sqrt{p(p-a)}\overset{CBS}{\leq}\sqrt{\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}p(p-a)\right)}\overset{Leibniz}{\leq}$$

$$\leq\sqrt{9R^2.p^2}=3pR\quad(3)$$

$$From\ (1),(2)\ and\ (3)\ we\ get:$$

$$\frac{a}{(b+\lambda c)^{n+1}{w_a}^n}+\frac{b}{(c+\lambda a)^{n+1}{w_b}^n}+\frac{c}{(c+\lambda a)^{n+1}{w_c}^n}\geq\frac{\left(\frac{3}{1+\lambda}\right)^{n+1}}{(3pR)^n}$$
$$=\frac{3}{(1+\lambda)^{n+1}}\left(\frac{1}{pR}\right)^n$$

538.



$OABC$ **–tetrahedron,** G **–centroid of** $OABC$, $OA=a, OB=b, OC=c$
$\sphericalangle BOC=\theta_1=90°$, $\sphericalangle COA=\theta_2=90°$
$\sphericalangle AOB=\theta_3=90°$, (s) **–circumsphere of** $OABC$, $OG\cap(s)=P$, K **–center of** (s)
L **–centroid of** ΔABC**. Prove:**

$$\frac{OG}{OP}=\frac{1}{4}$$

(O,G,L,K) **–is harmonic range.**

Proposed by Thanasis Gakopoulos-Farsala-Greece

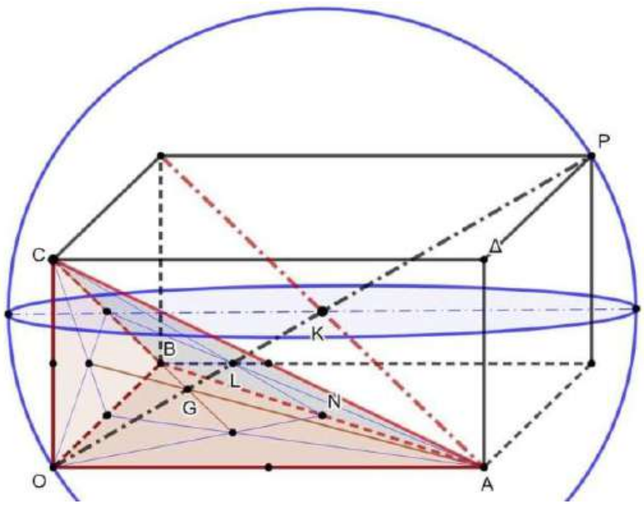
Solution by proposer

$O(0,0,0), A(a,0,0), B(0,b,0), C(0,0,c)$

$$G\left(\frac{a}{4},\frac{b}{4},\frac{c}{4}\right), L\left(\frac{a}{3},\frac{b}{3},\frac{c}{3}\right), K\left(\frac{a}{2},\frac{b}{2},\frac{c}{2}\right)$$
$$P(a,b,c)$$

So, $A,G,L,K.P$ **–are collinear.**

$$\frac{OG}{OP}=\frac{1}{4},\frac{OK}{OP}=\frac{1}{2},\frac{GL}{OP}=\frac{1}{3}-\frac{1}{4}=\frac{1}{12}$$
$$\frac{LK}{OP}=\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$$
$$2OK=12GL=4OG=6LK$$



$$\frac{OK}{6}=\frac{GL}{1}=\frac{OG}{3}=\frac{LK}{2}$$

Therefore,

$\frac{OG}{OP}=\frac{1}{4}$ and (O,G,L,K) are in harmonic range.

539. Let I −be the centre of ΔABC and ΔMNP the median triangle of ΔABC. Let A_1,B_1,C_1 −be the symmetrics of I to M,N,P and R_1,R_2,R_3 the circumradies of $\Delta BCA_1,\Delta CAB_1,\Delta ABC_1$. Prove that: $R_1^2+R_2^2+R_3^2\ge 3R^2$.

Proposed by Marian Ursărescu-Romania

Solution by Adrian Popa-Romania

$BM\equiv MC, A_1$ −symmetric point of I to M, then $IM\equiv MA_1$

$BICA$ −parallelogram, then $\sphericalangle BCA_1=\sphericalangle BCI=\frac{1}{2}\sphericalangle B$ and $\sphericalangle CBA_1=\sphericalangle ICB=\frac{1}{2}\sphericalangle C$

From Law of sines in ΔCBA_1 we have

$$\frac{BC}{\sin A_1}=2R_1$$

$$A_1=\pi-\frac{B}{2}-\frac{C}{2}=\pi-\frac{\pi-A}{2}=\frac{\pi}{2}+\frac{A}{2}$$

$$\sin A_1=\sin\left(\frac{\pi}{2}+\frac{A}{2}\right)=\cos\frac{A}{2}$$

$$\frac{a}{\cos\frac{A}{2}}=2R_1\Rightarrow\frac{2R\sin A}{\cos\frac{A}{2}}=2R_1\Rightarrow\frac{2R\sin\frac{A}{2}\cos\frac{A}{2}}{\cos\frac{A}{2}}=R_1\Rightarrow R_1=2R\sin\frac{A}{2}$$

Similarly, we get

$$R_2=2R\sin\frac{B}{2}\text{ and }R_3=2R\sin\frac{C}{2}$$

So, we must show:

$$4R^2\left(\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}\right)\ge 3R^2\,\Big|:4R^2$$

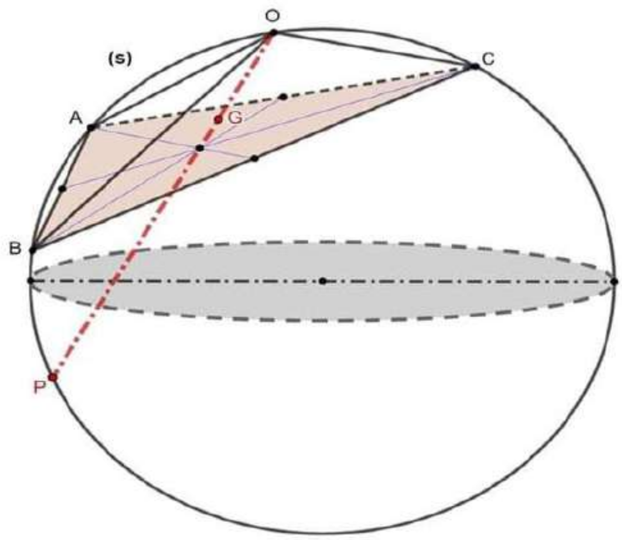
$$\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}\ge\frac{3}{4}(*)$$

$$\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}=1-\cos A+1-\cos B+\sin^2\frac{C}{2}\geq\frac{3}{4}\Leftrightarrow$$

$$2-2\cos\frac{A+B}{2}\cos\frac{A-B}{2}+\sin^2\frac{C}{2}\geq\frac{3}{4}\Leftrightarrow\sin^2\frac{C}{2}-2\cos\frac{A-B}{2}\sin\frac{C}{2}+\frac{5}{4}\geq 0$$

$$\Delta=4\cos^2\frac{A-B}{2}-5<0\Rightarrow(*)\text{ is true!}$$

540.



$OABC$ **tetrahedron,**

G **centroid of** $OABC$

$OA=a, OB=b, OC=c$

$\sphericalangle BOC=\theta_1=120^\circ$

$\sphericalangle COA=\theta_2, \sphericalangle AOB=\theta_3=60^\circ$

(s) **circumsphere of** $OABC$,

$OG\cap(s)=P$

Prove:

$$\frac{OG}{OP}=\frac{1}{4}\Leftrightarrow\cos\theta_2=\frac{b(c-a)}{2ac}$$

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by proposer

$$\text{Let } k=a^2+b^2+c^2+2ab\cdot\cos\theta_3+2bc\cdot\cos\theta_1+2ca\cdot\cos\theta_2$$

Plagiogonal 3rd system: $OA\equiv Ox, OB\equiv Oy, OC\equiv Oz$

$$O(0,0,0), A(a,0,0), B(0,b,0), C(0,0,c)$$

$$G\left(\frac{a}{4},\frac{b}{4},\frac{c}{4}\right), P\left(\frac{a(a^2+b^2+c^2)}{k},\frac{b(a^2+b^2+c^2)}{k},\frac{c(a^2+b^2+c^2)}{k}\right)$$

$$\frac{OG}{OP}=\frac{\frac{a}{4}}{\frac{a(a^2+b^2+c^2)}{k}}=\frac{k}{4(a^2+b^2+c^2)}$$

$$\frac{OG}{OP}=\frac{1}{4}\left(1+\frac{2ab\cdot\cos\theta_3+2bc\cdot\cos\theta_1+2ca\cdot\cos\theta_2}{a^2+b^2+c^2}\right);\ (1)$$

$$\frac{OG}{OP}=\frac{1}{4}\overset{(1)}{\Leftrightarrow}2ab\cdot\cos\theta_3+2bc\cdot\cos\theta_1+2ca\cdot\cos\theta_2=0$$

$$\theta_3=60^\circ,\theta_1=120^\circ\Rightarrow ab-bc+2ac\cos\theta_2=0\Leftrightarrow$$

$$\cos\theta_2=\frac{b}{2ac}(c-a)$$

Must $-1<\frac{b}{2ac}(b-a)<1$

If $c=a\Leftrightarrow\theta_2=90^\circ$.

541. In ΔABC the following relationship holds:

$$\frac{I_bI_c\cdot II_a}{AI(AI_a-AI)}+\frac{I_aI_c\cdot II_b}{BI(BI_b-BI)}+\frac{I_aI_b\cdot II_c}{CI(CI_c-CI)}=\frac{a+b+c}{r}$$

Proposed by Ertan Yildirim-Izmir-Turkiye

Solution by Jose Ferreira Queiroz-Olinda-Brazil

$$2s=a+b+c,IM=r,M\in(AB)$$

$$I_aL=r_a,L\in(AB),AL=s$$

$$r_a=AI_a\cdot\sin\frac{A}{2};r=AI\cdot\sin\frac{A}{2};II_a=\frac{a}{s-a}\cdot AI$$

$$I_bI_c=\frac{a}{\sin\frac{A}{2}}$$

$$\frac{I_bI_c\cdot II_a}{AI(AI_a-AI)}=\frac{\frac{a}{\sin\frac{A}{2}}\cdot\frac{a}{s-a}\cdot AI}{AI\left(\frac{r_a}{\sin\frac{A}{2}}-\frac{r}{\sin\frac{A}{2}}\right)}=\frac{a^2AI}{(s-a)\sin\frac{A}{2}}\cdot\frac{\sin\frac{A}{2}}{AI(r_a-r)}=$$

$$=\frac{a^2}{(s-a)(r_a-r)}=\frac{a^2}{(s-a)\left(\frac{sr}{s-a}-r\right)}=\frac{a^2}{ar}=\frac{a}{r}$$

Similarly, we have:

$$\frac{I_aI_c\cdot II_b}{BI(BI_b-BI)}=\frac{b}{r}\text{ and }\frac{I_aI_b\cdot II_c}{CI(CI_c-CI)}=\frac{c}{r}$$

By adding, we get:

$$\frac{I_bI_c\cdot II_a}{AI(AI_a-AI)}+\frac{I_aI_c\cdot II_b}{BI(BI_b-BI)}+\frac{I_aI_b\cdot II_c}{CI(CI_c-CI)}=\frac{a+b+c}{r}$$

542. Let I –be the centre of ΔABC and ΔMNP the median triangle of ΔABC. Let A_1, B_1, C_1 –be the symmetrics of I to M, N, P and R_1, R_2, R_3 the circumradies of $\Delta BCA_1, \Delta CAB_1, \Delta ABC_1$. Prove that:

$$\frac{1}{R_1^2}+\frac{1}{R_2^2}+\frac{1}{R_3^2}\geq\frac{3}{R^2}$$

Proposed by Marian Ursărescu-Romania

Solution by Adrian Popa-Romania

$BM\equiv MC, A_1$ –symmetric point of I to M, then $IM\equiv MA_1$

$BICA$ –parallelogram, then $\sphericalangle BCA_1=\sphericalangle BCI=\frac{1}{2}\sphericalangle B$ and $\sphericalangle CBA_1=\sphericalangle ICB=\frac{1}{2}\sphericalangle C$

From Law of sines in ΔCBA_1 we have

$$\frac{BC}{\sin A_1}=2R_1$$

$$A_1=\pi-\frac{B}{2}-\frac{C}{2}=\pi-\frac{\pi-A}{2}=\frac{\pi}{2}+\frac{A}{2}$$

$$\sin A_1=\sin\left(\frac{\pi}{2}+\frac{A}{2}\right)=\cos\frac{A}{2}$$

$$\frac{a}{\cos\frac{A}{2}}=2R_1\Rightarrow\frac{2R\sin A}{\cos\frac{A}{2}}=2R_1\Rightarrow\frac{2R\sin\frac{A}{2}\cos\frac{A}{2}}{\cos\frac{A}{2}}=R_1$$

$$\Rightarrow R_1=2R\sin\frac{A}{2}$$

Similarly, we get

$$R_2=2R\sin\frac{B}{2}\text{ and }R_3=2R\sin\frac{C}{2}$$

So, we must show:

$$\frac{1}{4R^2\sin^2\frac{A}{2}}+\frac{1}{4R^2\sin^2\frac{B}{2}}+\frac{1}{4R^2\sin^2\frac{C}{2}}\geq\frac{3}{R^2}\Leftrightarrow\frac{1}{\sin^2\frac{A}{2}}+\frac{1}{\sin^2\frac{B}{2}}+\frac{1}{\sin^2\frac{C}{2}}\geq 12$$

$$\text{Let } f(x)=\frac{1}{\sin^2\frac{x}{2}}, x\in(0,\pi), f'(x)=\frac{-\sin x}{2\sin^4\frac{x}{2}}$$

$$f''(x)=\frac{6-4\sin^2\frac{x}{2}}{4\sin^4\frac{x}{2}}>0\Rightarrow f-\text{concave function.}$$

$$f(A)+f(B)+f(C)\ge 3f\left(\frac{A+B+C}{3}\right)\Leftrightarrow\frac{1}{\sin^2\frac{A}{2}}+\frac{1}{\sin^2\frac{B}{2}}+\frac{1}{\sin^2\frac{C}{2}}\ge\frac{3}{\sin^2\frac{\frac{A}{2}+\frac{B}{2}+\frac{C}{2}}{3}}$$

$$\frac{1}{\sin^2\frac{A}{2}}+\frac{1}{\sin^2\frac{B}{2}}+\frac{1}{\sin^2\frac{C}{2}}\ge\frac{3}{\sin^2\frac{\pi}{6}}\Leftrightarrow\frac{1}{\sin^2\frac{A}{2}}+\frac{1}{\sin^2\frac{B}{2}}+\frac{1}{\sin^2\frac{C}{2}}\ge 12$$

543. If $M,P\in Int(\Delta ABC)$ then prove that :

$$\frac{(PB.PC)^2}{MB.MC}+\frac{(PC.PA)^2}{MC.MA}+\frac{(PA.PB)^2}{MA.MB}\ge\left(\frac{a.PA+b.PB+c.PC}{MA+MB+MC}\right)^2.$$

Proposed by Bogdan Fuştei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Let\ x,y,z>0.\ We\ have:\left(x\overrightarrow{PA}+y\overrightarrow{PB}+z\overrightarrow{PC}\right)^2\ge 0$$

$$\leftrightarrow\sum_{cyc}x^2PA^2+\sum_{cyc}yz.2\overrightarrow{PB}.\overrightarrow{PC}\ge 0$$

$$\leftrightarrow\sum_{cyc}x^2PA^2+\sum_{cyc}yz.(PB^2+PC^2-BC^2)\ge 0\leftrightarrow$$

$$\sum_{cyc}(x^2+xy+xz)PA^2\ge\sum_{cyc}yza^2$$

$$\leftrightarrow(x+y+z)\left(\sum_{cyc}xPA^2\right)\ge\sum_{cyc}yza^2\leftrightarrow$$

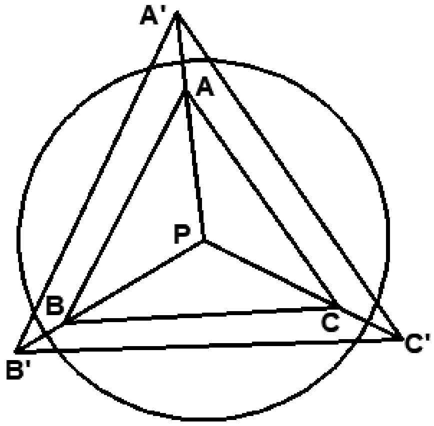
$$(x+y+z).xyz\sum_{cyc}\frac{PA^2}{yz}\ge xyz\sum_{cyc}\frac{a^2}{x}$$

$$\leftrightarrow(x+y+z)^2\left(\frac{PA^2}{yz}+\frac{PB^2}{zx}+\frac{PC^2}{xy}\right)\ge(x+y+z)\left(\frac{a^2}{x}+\frac{b^2}{y}+\frac{c^2}{z}\right)$$

$$\textit{By CBS inequality we have}: \quad (x+y+z)\left(\frac{a^2}{x}+\frac{b^2}{y}+\frac{c^2}{z}\right)\geq (a+b+c)^2$$

$$\textit{Then}, \quad \frac{PA^2}{yz}+\frac{PB^2}{zx}+\frac{PC^2}{xy}\geq\left(\frac{a+b+c}{x+y+z}\right)^2 \quad (1)$$

Now, let A', B', C' be the inverse of points A, B, C by means of a circle of center P and radius $\rho=\sqrt{PA.PB.PC}$.



By the definition of inversion we have: $PA'.PA=\rho^2$ *or* $PA' = PB.PC$ *(and analogs)*

By the Law of Cosines in $\Delta B'PC'$ we have:

$$B'C'^2=a'^2=PB'^2+PC'^2-2PB'.PC'.\cos\widehat{B'PC'}$$

$$=(PC.PA)^2+(PA.PB)^2-2(PC.PA).(PA.PB).\cos\widehat{B'PC'}=$$

$$=PA^2.\left(PB^2+PC^2-2PB.PC.\cos\widehat{BPC}\right)=PA^2.a^2.$$

Then: $a'=a.PA$ *(and analogs)*

(*See*: *D. Mitrinović, J. E. Pečarić and V. Volenec, Recent Advances in Geometric Inequalities[M], Kluwer Academic Publishers, 1989, 293 − 294.*)

Applying the inequality (1) in $\Delta A'B'C'$, we obtain:

$$\frac{PA'^2}{yz}+\frac{PB'^2}{zx}+\frac{PC'^2}{xy}\geq\left(\frac{a'+b'+c'}{x+y+z}\right)^2$$

$$\textit{Or} \quad \frac{(PB.PC)^2}{yz}+\frac{(PC.PA)^2}{zx}+\frac{(PA.PB)^2}{xy}\geq\left(\frac{a.PA+b.PB+c.PC}{x+y+z}\right)^2$$

Taking $x=MA$, $y=MB$, $z=MC$ we get:

$$\frac{(PB.PC)^2}{MB.MC}+\frac{(PC.PA)^2}{MC.MA}+\frac{(PA.PB)^2}{MA.MB}\ge\left(\frac{a.PA+b.PB+c.PC}{MA+MB+MC}\right)^2.$$

544. In ΔABC the following relationship holds:

$$a^3+b^3+c^3\ge 8\sqrt[4]{3}\cdot\left(\sqrt{F}\right)^3+\frac{1}{2}\cdot\sum_{cyc}\left(a^{\frac{3}{2}}-b^{\frac{3}{2}}\right)^2$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

Solution by Adrian Popa-Romania

$$\frac{1}{2}\cdot\sum_{cyc}\left(a^{\frac{3}{2}}-b^{\frac{3}{2}}\right)^2=\sum_{cyc}a^3-\sum_{cyc}b^{\frac{3}{2}}c^{\frac{3}{2}}$$

We must show that:

$$\sum_{cyc}b^{\frac{3}{2}}c^{\frac{3}{2}}\ge 8\sqrt[4]{3}\cdot\left(\sqrt{F}\right)^3$$

$$\sum_{cyc}b^{\frac{3}{2}}c^{\frac{3}{2}}\overset{AGM}{\ge}3abc=3\cdot 4RF=12RF\overset{?}{\ge}8\sqrt[4]{3}\cdot\left(\sqrt{F}\right)^3$$

$$3RF\ge 2\sqrt[4]{3}\cdot\left(\sqrt{F}\right)^3\Leftrightarrow 9R^2F^2\ge 4\sqrt{3}F^3\Leftrightarrow 9R^2\ge 4\sqrt{3}F\Leftrightarrow R^2\ge\frac{4\sqrt{3}}{9}\cdot sr$$

$$\text{But } s\le\frac{3\sqrt{3}}{2}R\ (Mitrinovic)\Rightarrow R>\frac{2s}{3\sqrt{3}}=\frac{2s\sqrt{3}}{9}\Rightarrow R^2>\frac{12s^2}{81}$$

So, we must show:

$$\frac{12s^2}{81}\ge\frac{4\sqrt{3}}{9}\cdot sr\Leftrightarrow s\ge 3\sqrt{3}r(Mitrinovic)$$

545. If $m,n\in\mathbb{N}^*$ and $x,y,z>0$ then in ΔABC holds:

$$\frac{mx+ny}{z}\cdot a^4+\frac{my+nz}{x}\cdot b^4+\frac{mz+nx}{y}\cdot c^4\ge 16(m+n)\cdot F^2$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

Solution 1 by Avishek Mitra-West Bengal-India

$$\sum_{cyc}\frac{mx+ny}{z}\cdot a^4=m\sum_{cyc}\frac{x}{z}\cdot a^4+n\sum_{cyc}\frac{y}{z}\cdot a^4\overset{AGM}{\ge}$$

$$\geq 3m\left(\frac{x}{z}\cdot\frac{y}{x}\cdot\frac{z}{y}\prod_{cyc}a^4\right)^{\frac{1}{3}}+3n\left(\frac{x}{z}\cdot\frac{y}{x}\cdot\frac{z}{y}\prod_{cyc}a^4\right)^{\frac{1}{3}}=3(m+n)(4Rrs)^{\frac{4}{3}}$$

Need to show:

$$3(m+n)(4Rrs)^{\frac{4}{3}}\geq 16(m+n)F^2\Leftrightarrow$$

$$27(4Rrs)^4\geq 4^6(rs)^6\Leftrightarrow\frac{27}{16}R^4\geq r^2s^2(true!)$$

$R\geq 2r(Euler)\Rightarrow R^2\geq 4r^2$ **and** $3\sqrt{3}R\geq 2s(Mitrinovic)\Rightarrow 27R^2\geq 4s^2$

$$\Rightarrow 27R^2\cdot R^2\geq 4s^2\cdot 4r^2\Rightarrow 27R^4\geq 16r^2s^2$$

Solution 2 by Tapas Das-India

$$\sum_{cyc}\frac{mx+ny}{z}\cdot a^4=m\sum_{cyc}\frac{x}{z}\cdot a^4+n\sum_{cyc}\frac{y}{z}\cdot a^4\overset{AGM}{\geq}$$

$$\geq 3m\sqrt[3]{\frac{x}{z}\cdot\frac{y}{x}\cdot\frac{z}{y}\prod_{cyc}a^4}+3n\sqrt[3]{\frac{x}{z}\cdot\frac{y}{x}\cdot\frac{z}{y}\prod_{cyc}a^4}=$$

$$=3(m+n)4Rrs\sqrt[3]{4Rrs}\overset{?}{\geq}\geq 16(m+n)F^2$$

Now, we know that

$abc=4RF$ **and** $a+b+c\leq 3R\sqrt{3}\Rightarrow\frac{4F}{\sqrt{3}}=\frac{abc}{R\sqrt{3}}\leq\frac{abc}{2s}=\frac{3abc}{a+b+c}\leq(abc)^{\frac{2}{3}}\Rightarrow$

$$(abc)^2\geq\left(\frac{4F}{\sqrt{3}}\right)^3\Rightarrow abc\geq\left(\frac{4F}{\sqrt{3}}\right)^{\frac{3}{2}}$$

$$\Rightarrow(abc)^{\frac{4}{3}}\geq\left(\frac{4F}{\sqrt{3}}\right)^2=\frac{16F^2}{3}\Rightarrow 3(abc)^{\frac{4}{3}}\geq 16F^2$$

$$\sin^2A+\sin^2B+\sin^2C=2+2\cos A\cos B\cos C$$

$$a=2R\sin A,b=2R\sin B,c=2R\sin C\Rightarrow$$

$$a^2+b^2+c^2=4R^2(\sin^2A+\sin^2B+\sin^2C)=$$

$$=4R^2(2+2\cos A\cos B\cos C)$$

$$\cos A\cos B\cos C\leq\frac{1}{8}\Rightarrow\sin^2A+\sin^2B+\sin^2C\leq\frac{9}{4}$$

$$a^2+b^2+c^2\leq 9R^2$$

$$\frac{(a+b+c)^2}{3} \le a^2+b^2+c^2 \Rightarrow \frac{(a+b+c)^2}{3} \le 9R^2 \Leftrightarrow$$

$$(a+b+c)^2 \le 27R^2 \Leftrightarrow a+b+c \le 3\sqrt{3}R$$

546. If $m \ge 0$ and $t,u,x,y,z > 0$ then in ΔABC holds:

$$\frac{tx+uy}{z}\cdot a^m+\frac{ty+uz}{x}\cdot b^m+\frac{tz+ux}{y}\cdot c^m \ge 2^m\left(\sqrt[4]{4}\right)^{4-m}\cdot(t+u)\cdot\left(\sqrt{F}\right)^m$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

Solution by Tapas Das-India

$$\frac{tx+uy}{z}\cdot a^m+\frac{ty+uz}{x}\cdot b^m+\frac{tz+ux}{y}\cdot c^m =$$

$$= t\left(\frac{x}{z}a^m+\frac{y}{x}b^m+\frac{z}{y}c^m\right)+u\left(\frac{y}{z}a^m+\frac{z}{x}b^m+\frac{x}{y}c^m\right) \ge$$

$$\ge 3t\cdot(abc)^{\frac{m}{3}}+3u\cdot(abc)^{\frac{m}{3}} = 3(u+t)(abc)^{\frac{m}{3}}$$

We need to show:

$$3(u+t)(abc)^{\frac{m}{3}} \ge 2^m\left(\sqrt[4]{4}\right)^{4-m}\cdot(t+u)\cdot\left(\sqrt{F}\right)^m$$

$$8(abc)^{\frac{m}{3}} \ge 2^m\left(\sqrt[4]{4}\right)^{4-m}\cdot\left(\sqrt{F}\right)^m; (1)$$

$$(abc)^2 \ge \left(\frac{4F}{\sqrt{3}}\right)^3 (Carlitz) \Rightarrow abc \ge \left(\frac{4F}{\sqrt{3}}\right)^{\frac{3}{2}}$$

$$(abc)^{\frac{m}{3}} \ge \left(\frac{4F}{\sqrt{3}}\right)^{\frac{3}{2}\cdot\frac{m}{3}} = \left(\frac{4F}{\sqrt{3}}\right)^{\frac{m}{2}} = \frac{4^{\frac{m}{2}}\cdot F^{\frac{m}{2}}}{\left(\sqrt{3}\right)^{\frac{m}{2}}} = \frac{2^m\cdot F^m}{\left(\sqrt[4]{3}\right)^m}$$

$$3(abc)^{\frac{m}{3}} \ge 3\cdot 2^m\cdot\left(\sqrt{F}\right)^m\cdot 3^{-\frac{m}{4}} = 2^m\cdot 3^{\frac{4-m}{4}}\cdot\left(\sqrt{F}\right)^m = 2^m\cdot\left(\sqrt[4]{3}\right)^{4-m}\cdot\left(\sqrt{F}\right)^m$$

Therefore,

$$\frac{tx+uy}{z}\cdot a^m+\frac{ty+uz}{x}\cdot b^m+\frac{tz+ux}{y}\cdot c^m \ge 2^m\left(\sqrt[4]{4}\right)^{4-m}\cdot(t+u)\cdot\left(\sqrt{F}\right)^m$$

547. In ΔABC the following relationship holds:

$$\frac{2a^3+3b^3+5c^3}{2a^2+3b^2+5c^2}+\frac{3a^3+5b^3+2c^3}{3a^2+5b^2+2c^2}+\frac{5a^3+2b^3+3c^3}{5a^2+2b^2+3c^2} \ge 6\sqrt{3}r$$

Proposed by Daniel Sitaru-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\textit{We have}:$$

$$(2a^3+3b^3+5c^3)(2a+3b+5c)\overset{CBS}{\geq}(2a^2+3b^2+5c^2)^2\overset{CBS}{\geq}$$

$$\geq(2a^2+3b^2+5c^2).\frac{(2a+3b+5c)^2}{2+3+5}$$

$$\textit{Then}:\ \frac{2a^3+3b^3+5c^3}{2a^2+3b^2+5c^2}\geq\frac{2a+3b+5c}{10}.$$

$$\textit{Similarly we have}:$$

$$\frac{3a^3+5b^3+2c^3}{3a^2+5b^2+2c^2}\geq\frac{3a+5b+2c}{10}\quad\&\quad\frac{5a^3+2b^3+3c^3}{5a^2+2b^2+3c^2}\geq\frac{5a+2b+3c}{10}$$

$$\textit{Summing up these inequalities we get}:$$

$$\frac{2a^3+3b^3+5c^3}{2a^2+3b^2+5c^2}+\frac{3a^3+5b^3+2c^3}{3a^2+5b^2+2c^2}+\frac{5a^3+2b^3+3c^3}{5a^2+2b^2+3c^2}\geq a+b+c=$$

$$=2s\overset{Mitrinovic}{\geq}2.3\sqrt{3}r=6\sqrt{3}r.$$

548. In ΔABC the following relationship holds:

$$\sum\frac{{r_a}^2}{r_c}\leq\frac{9R^2}{4r}\left(\frac{R}{r}-1\right)^2$$

Proposed by Marin Chirciu-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\textit{Lemma}:\ \textit{if } x,y,z\geq 0\ \textit{then we have}:\ 3(x^3y+y^3z+z^3x)\leq(x^2+y^2+z^2)^2$$

$$\textit{Proof}:\ \textit{Using the well known inequality}\ (u+v+w)^2\geq 3(uv+vw+wu)$$

$$\textit{With}:\ u=x^2+yz-xy,\ v=y^2+zx-yz,\ w=z^2+xy-zx:$$

$$(x^2+y^2+z^2)^2=(u+v+w)^2\geq 3(uv+vw+wu)$$
$$=3\sum_{cyc}(x^2+yz-xy)(y^2+zx-yz)=3(x^3y+y^3z+z^3x).$$

$$Now, we\ have:\ \sum \frac{{r_a}^2}{r_c} = \frac{{r_a}^3 r_b + {r_b}^3 r_c + {r_c}^3 r_a}{r_a r_b r_c} \overset{Lemma}{\leq} \frac{({r_a}^2 + {r_b}^2 + {r_c}^2)^2}{3 r_a r_b r_c}$$

$$= \frac{[(4R+r)^2 - 2s^2]^2}{3s^2 r} \leq$$

$$\overset{Gerretsen}{\leq} \frac{[(4R+r)^2 - 2(16Rr - 5r^2)]^2}{3(16Rr - 5r^2)r}$$

$$= \frac{256R^4 - 768R^3 r + 928R^2 r^2 - 528Rr^3 + 121r^4}{3(16R - 5r)r^2} \overset{?}{\leq} \frac{9R^2}{4r}\left(\frac{R}{r} - 1\right)^2$$

$$\leftrightarrow 4r(256R^4 - 768R^3 r + 928R^2 r^2 - 528Rr^3 + 121r^4)$$

$$\leq 27R^2(R^2 - 2Rr + r^2)(16R - 5r)$$

$$\leftrightarrow 432R^5 - 2023R^4 r + 3774R^3 r^2 - 3847R^2 r^3 + 2112Rr^4 - 484r^5 \geq 0$$

$$\leftrightarrow (R - 2r)[(R - 2r)(432R^3 - 295R^2 r + 866Rr^2 + 797r^3) + 1836r^4] \geq 0$$

Which is true from Euler's inequality $R \geq 2r$.

$$Therefore,\ \sum \frac{{r_a}^2}{r_c} \leq \frac{9R^2}{4r}\left(\frac{R}{r} - 1\right)^2.$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \frac{r_a^2}{r_c} = \sum_{cyc} \frac{\sum_{cyc} r_a^2 - (r_b^2 + r_c^2)}{r_c} = \left(\sum_{cyc} r_a^2\right)\left(\sum_{cyc} \frac{1}{r_a}\right) - \sum_{cyc} r_a - \sum_{cyc} \frac{r_b^2}{r_c}$$

$$= \frac{(4R+r)^2 - 2s^2}{r} - (4R+r) - \sum_{cyc} \frac{r_b^3}{r_b r_c} \overset{Holder}{\leq} \frac{(4R+r)^2 - 2s^2}{r} - (4R+r)$$

$$- \frac{\left(\sum_{cyc} r_a\right)^3}{3\sum_{cyc} r_b r_c}$$

$$\overset{?}{\leq} \frac{9R^2}{4r}\left(\frac{R}{r} - 1\right)^2 \Leftrightarrow \frac{(4R+r)^3}{3s^2} + 4R + r - \frac{(4R+r)^2 - 2s^2}{r} + \frac{9R^2(R-r)^2}{4r^3} \overset{?}{\geq} 0$$

$$\Leftrightarrow \frac{4r^3(4R+r)^3 + 12s^2 r^3(4R+r) - 12r^2 s^2\left((4R+r)^2 - 2s^2\right) + 27s^2 R^2(R-r)^2}{12s^2 r^3} \overset{?}{\geq} 0$$

$$\Leftrightarrow 24r^2 s^4 + \left(27R^4 - 54R^3 r - 165R^2 r^2 - 48Rr^3\right) + 4r^3(4R+r)^3 \underset{(*)}{\overset{?}{\geq}} 0\ and$$

$$\because 24r^2\left(s^2 - 16Rr + 5r^2\right)^2 \overset{Gerretsen}{\geq} 0$$

$\therefore$ **in order to prove** $(*)$**, it suffices to prove :**

$$24r^2 s^4 + \left(27R^4 - 54R^3 r - 165R^2 r^2 - 48Rr^3\right)s^2 + 4r^3(4R+r)^3 \geq 24r^2\left(s^2 - 16Rr + 5r^2\right)^2$$

$$\Leftrightarrow \left(27R^4 - 54R^3 r - 165R^2 r^2 + 720Rr^3 - 240r^4\right)s^2$$

$$+ r^3\left(256R^3 - 5952R^2 r + 3888Rr^2 - 596r^3\right) \overset{(**)}{\geq} 0$$

$$\because 27R^4 - 54R^3 r - 165R^2 r^2 + 720Rr^3 - 240r^4$$

$$= (R - 2r)\left((R - 2r)\left(27R^2 + 25Rr + 29r(R - 2r) + r^2\right) + 276r^3\right)$$

$$+ 540r^4 \overset{Euler}{\geq} 540r^4 > 0 \therefore LHS\ of\ (**) \overset{Rouche}{\geq}$$

$$(27R^4-54R^3r-165R^2r^2+720Rr^3-240r^4)\left(2R^2+10Rr-r^2-2(R-2r)\sqrt{R^2-2Rr}\right)$$
$$+r^3(256R^3-5952R^2r+3888Rr^2-596r^3)\overset{?}{\geq}0$$
$$\Leftrightarrow(27R^4-54R^3r-165R^2r^2+720Rr^3-240r^4)(2R^2+10Rr-r^2)$$
$$+r^3(256R^3-5952R^2r+3888Rr^2-596r^3)$$
$$\overset{?}{\geq}2(R-2r)(27R^4-54R^3r-165R^2r^2+720Rr^3-240r^4)\sqrt{R^2-2Rr}$$
$$\Leftrightarrow(R-2r)(54R^5+270R^4r-357R^3r^2-614R^2r^3-295Rr^4+178r^5)\overset{?}{\geq}2(R-2r)(27R^4$$
$$-54R^3r-165R^2r^2+720Rr^3-240r^4)\sqrt{R^2-2Rr}$$
$$\Leftrightarrow 54R^5+270R^4r-357R^3r^2-614R^2r^3-295Rr^4$$
$$+178r^5\underbrace{\overset{?}{\geq}}_{(***)}2(27R^4-54R^3r-165R^2r^2+720Rr^3-240r^4)\sqrt{R^2-2Rr}$$
$$\because 54R^5+270R^4r-357R^3r^2-614R^2r^3-295Rr^4+178r^5$$
$$=(R-2r)(54R^4+378R^3r+399R^2r^2+184Rr^3+73r^4)$$
$$+324r^5\overset{\text{Euler}}{\geq}324r^5>0\ \therefore(***)\Leftrightarrow$$
$$(54R^5+270R^4r-357R^3r^2-614R^2r^3-295Rr^4+178r^5)^2$$
$$\geq 4(R^2-2Rr)(27R^4-54R^3r-165R^2r^2+720Rr^3-240r^4)^2$$
$$\Leftrightarrow 46656t^9+34992t^8-533844t^7+471609t^6+637080t^5-3400094t^4+6398368t^3$$
$$-3126759t^2+355780t+31684\geq 0\ \left(t=\frac{R}{r}\right)$$
$$\Leftrightarrow(t-2)\Big((t-2)(466656t^7+221616t^6+165996t^5+249129t^4+710531t^3$$
$$+259081t^2(t-2)+447272t+734977)+1506600\Big)+104976\geq 0$$
$$\rightarrow\text{true}\because t\overset{\text{Euler}}{\geq}2$$
$$\therefore(***)\Rightarrow(**)\Rightarrow(*)\text{ is true}\Rightarrow\text{in any }\Delta ABC,\ \sum_{cyc}\frac{r_a^2}{r_c}\leq\frac{9R^2}{4r}\left(\frac{R}{r}-1\right)^2\ (QED)$$

549. In ΔABC the following relationship holds:

$$w_a^3r_a+w_b^3r_b+w_c^3r_c\leq 3^5\left(\frac{R}{2}\right)^4$$

Proposed by Kostas Geronikolas-Greece

Solution by Marian Ursărescu-Romania

$$w_ar_a\leq\sqrt{s(s-a)}\cdot\frac{F}{s-a}=\sqrt{s(s-a)}\cdot\frac{\sqrt{s(s-a)(s-b)(s-c)}}{s-a}=$$
$$=s\sqrt{(s-b)(s-c)}\leq\frac{s(s-b+s-c)}{2}=\frac{as}{2}$$
$$\Rightarrow w_ar_a\leq\frac{as}{2}$$

We must show:

$$\sum_{cyc} w_a^2 \cdot \frac{as}{2} \le \frac{243}{4} R^4; (1)$$

$$w_a \le \sqrt{s(s-a)}; (2)$$

From (1) and (2) we must show that

$$\frac{s^2}{2} \sum_{cyc} a(s-a) \le \frac{243}{16} R^4; (3)$$

$$\text{But } \sum_{cyc} a(s-a) = 2r(4R+r); (4)$$

From (3) and (4) we must show

$$s^2 r(4R+r) \le \frac{243}{16} R^4; (5)$$

$$s^2 \le \frac{27R^2}{4} (Mitrinovic) \text{ and } r \le \frac{R}{2} (Euler); (6)$$

From (5) and (6) we must show

$$4R + r \le \frac{9}{2} R \Leftrightarrow 8R + 2r \le 9R \Leftrightarrow 2r \le R(Euler).$$

550. If $m \ge 0$ then in ΔABC holds:

$$a^{2m+2} + b^{2m+2} + c^{2m+2} \ge$$

$$\ge 2^{2m+2} \cdot (\sqrt{3})^{1-m} \cdot F^{m+1} + \frac{1}{2}((a^{m+1} - b^{m+1})^2 + (b^{m+1} - c^{m+1})^2 + (c^{m+1} - a^{m+1})^2)$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

Solution by Avishek Mitra-West Bengal-India

Need to show:

$$\sum_{cyc} a^{2m+2} \ge 2^{2m+2} (\sqrt{3})^{1-m} F^{m+1} + \frac{1}{2} \sum_{cyc} (a^{m+1} - b^{m+1})^2$$

$$\sum_{cyc} a^{2m+2} \ge 2^{2m+2} (\sqrt{3})^{1-m} F^{m+1} + \frac{1}{2} \left(2 \sum_{cyc} a^{2m+2} - 2 \sum_{cyc} a^{m+1} b^{m+1} \right)$$

$$\sum_{cyc} a^{m+1} b^{m+1} \ge 2^{2m+2} (\sqrt{3})^{1-m} F^{m+1}$$

Now, we have:

$$\sum_{cyc} a^{m+1}b^{m+1} \overset{AGM}{\geq} 3\sqrt[3]{\prod_{cyc} a^{2m+2}} = 3\sqrt[3]{(4Rrs)^{2m+2}}$$

Need to show:

$$3\sqrt[3]{(4Rrs)^{2m+2}} \geq 2^{2m+2}(\sqrt{3})^{1-m}F^{m+1}, \quad 27(4Rrs)^{2m+2} \geq 2^{3(2m+2)}(\sqrt{3})^{3(1-m)}(rs)^{3m+3}$$

$$27R^{2m+2} \geq 2^{2m+2}(\sqrt{3})^{3(1-m)}(rs)^{m+1}\ (true!), \quad R \geq 2r\ (Euler) \Rightarrow R^{m+1} \geq (2r)^{m+1}$$

$$3\sqrt{3}R \geq 2s \Rightarrow (3\sqrt{3}R)^{m+1} \geq (2s)^{m+1}, \quad (3\sqrt{3})^{m+1}R^{2m+2} \geq 2^{2m+2}(rs)^{m+1}$$

$$27R^{2m+2} \geq 2^{2m+2}(\sqrt{3})^{3(1-m)}(rs)^{m+1}$$

551. In ΔABC the following relationship holds:

$$9r \leq \frac{m_b m_c}{m_a} + \frac{m_c m_a}{m_b} + \frac{m_a m_b}{m_c} \leq \frac{R}{2r}(4R + r)$$

Proposed by Marin Chirciu-Romania

Solution by Marian Ursărescu-Romania

For the left side we have:

$$\frac{m_b m_c}{m_a} + \frac{m_c m_a}{m_b} + \frac{m_a m_b}{m_c} \overset{AGM}{\geq} 3\sqrt[3]{m_a m_b m_c}$$

We must show that:

$$3\sqrt[3]{m_a m_b m_c} \geq 9r \Leftrightarrow m_a m_b m_c \geq 27r^3; (1)$$

But $m_a \geq \sqrt{s(s-a)} \Rightarrow m_a m_b m_c \geq sF = s^2 r; (2)$

From (1) and (2) we must show that: $s^2 r \geq 27r^3 \Leftrightarrow s^2 \geq 27r^2 (Mitrinovic)$

For the right side we must show:

$$m_a m_b m_c \left(\frac{1}{m_a^2} + \frac{1}{m_b^2} + \frac{1}{m_c^2}\right) \leq \frac{R}{2r}(4R + r); (3)$$

$$m_a m_b m_c \leq \frac{Rs^2}{2}; (4)$$

$$m_a \geq \sqrt{s(s-a)} \Rightarrow \frac{1}{m_a^2} \leq \frac{1}{s(s-a)} \Rightarrow \sum_{cyc} \frac{1}{m_a^2} \leq \frac{1}{p}\sum_{cyc} \frac{1}{s-a}$$

$$\sum_{cyc} \frac{1}{m_2^2} \leq \frac{1}{s} \cdot \frac{4R + r}{sr} = \frac{4R + r}{s^2 r}; (5)$$

From (4) and (5) we must show:

$$m_a m_b m_c\left(\frac{1}{m_a^2}+\frac{1}{m_b^2}+\frac{1}{m_c^2}\right)\le\frac{Rs^2}{2}\cdot\frac{4R+r}{s^2r}=\frac{R(4R+r)}{2r}\Rightarrow(3)\text{ is true.}$$

552. In ΔABC the following relationship holds:

$$cotA+cotB+cotC\ge\frac{r(r_a+r_b+r_c)}{F}$$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution by Daniel Sitaru-Romania

$$\sum_{cyc}cotA=\sum_{cyc}\frac{cosA}{sinA}=\sum_{cyc}\frac{b^2+c^2-a^2}{2bcsinA}=\sum_{cyc}\frac{b^2+c^2-a^2}{4F}=$$

$$=\frac{a^2+b^2+c^2}{4F}=\frac{2(s^2-r^2-4Rr)}{4F}=\frac{s^2-r^2-4Rr}{2F}\ge$$

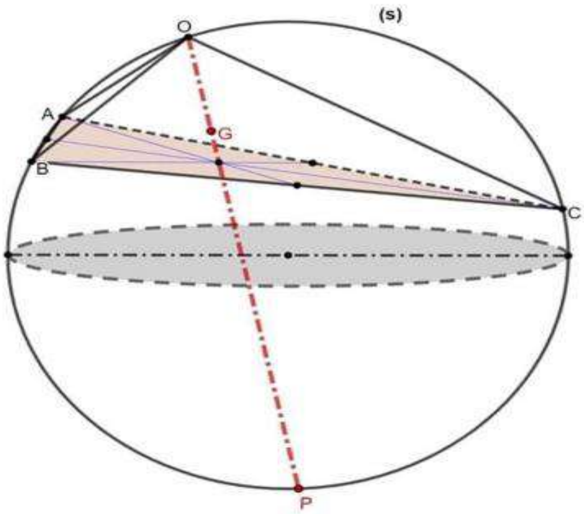
$$\overset{GERRETSEN}{\ge}\frac{16Rr-5r^2-r^2-4Rr}{2F}=\frac{12Rr-6r^2}{2F}=$$

$$=\frac{6Rr-3r^2}{F}=\frac{r(6R-3r)}{F}=\frac{r(4R+2R-3r)}{F}\ge$$

$$\overset{EULER}{\ge}\frac{r(4R+2\cdot 2r-3r)}{F}=\frac{r(4R+r)}{F}=\frac{r(r_a+r_b+r_c)}{F}$$

Equality for $a=b=c$.

553.



$OABC$-tetrahedron, G –centroid of $OABC$, $OA=a, OB=b, OC=c$

$\sphericalangle BOC = \theta_1, \sphericalangle COA = \theta_2, \sphericalangle AOB = \theta_3, (s)$ circumsphere of $OABC, OG \cap (s) = P$

Prove:

$$\frac{OG}{OP} = \frac{1}{4} \Leftrightarrow \frac{\cos\theta_1}{a} + \frac{\cos\theta_2}{b} + \frac{\cos\theta_3}{c} = 0$$

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by proposer

Plagiogonal 3D system: $OA \equiv Ox, OB \equiv Oy, OC \equiv Oz$

$$O(0,0,0), A(a,0,0), B(0,b,0), C(0,0,c)$$

$$K = a^2 + b^2 + c^2 + 2bc\cos\theta_1 + 2ca\cos\theta_2 + 2ab\cos\theta_3$$

$$G\left(\frac{a}{4},\frac{b}{4},\frac{c}{4}\right), P\left(\frac{a(a^2+b^2+c^2)}{K},\frac{b(a^2+b^2+c^2)}{K},\frac{c(a^2+b^2+c^2)}{K}\right)$$

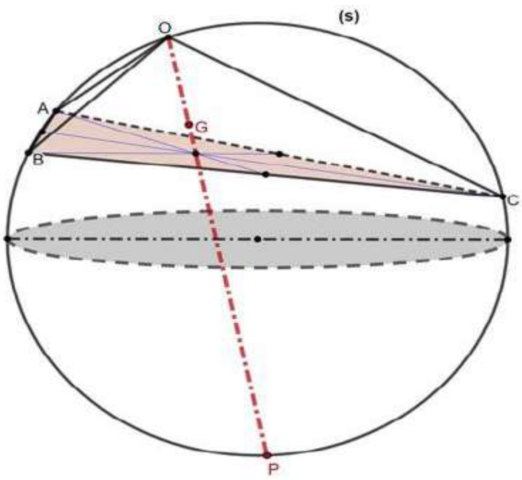
$$\frac{OG}{OP} = \frac{\frac{a}{4}}{\frac{a(a^2+b^2+c^2)}{K}} = \frac{K}{4(a^2+b^2+c^2)}$$

$$\frac{OG}{OP} = \frac{1}{4}\left(1 + \frac{a^2+b^2+c^2+2bc\cos\theta_1+2ca\cos\theta_2+2ab\cos\theta_3}{a^2+b^2+c^2}\right)$$

$$\frac{OG}{OP} = \frac{1}{4} \Leftrightarrow a^2+b^2+c^2+2bc\cos\theta_1+2ca\cos\theta_2+2ab\cos\theta_3 = 0$$

$$\frac{\cos\theta_1}{a} + \frac{\cos\theta_2}{b} + \frac{\cos\theta_3}{c} = 0$$

554.



$OABC$ tetrahedron, G centroid of $OABC, OA = a, OB = b, OC = c,$

$\sphericalangle BOC = \theta_1, \sphericalangle COA = \theta_2 \sphericalangle AOB = \theta_3, (s)$ –circumsphere of $OABC$,

$OG \cap (s) = P$. Prove that: $\frac{OG}{OP} = \frac{1}{4}\left[1 + \frac{2bc\cos\theta_1 + 2ca\cos\theta_2 + 2ab\cos\theta_3}{a^2+b^2+c^2}\right]$

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by proposer

Plagiogonal 3D system: $OA \equiv Ox, OB \equiv Oy, OC \equiv Oz$

$$O(0,0,0), A(a,0,0), B(0,b,0), C(0,0,c)$$

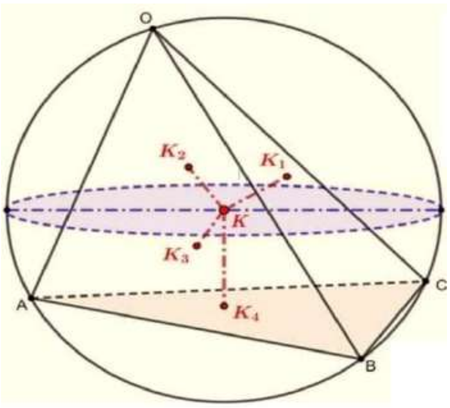
$$K = a^2 + b^2 + c^2 + 2bc\cos\theta_1 + 2ca\cos\theta_2 + 2ab\cos\theta_3$$

$$G\left(\frac{a}{4}, \frac{b}{4}, \frac{c}{4}\right), P\left(\frac{a(a^2+b^2+c^2)}{K}, \frac{b(a^2+b^2+c^2)}{K}, \frac{c(a^2+b^2+c^2)}{K}\right)$$

$$\frac{OG}{OP} = \frac{\frac{a}{4}}{\frac{a(a^2+b^2+c^2)}{K}} = \frac{K}{4(a^2+b^2+c^2)}$$

$$\frac{OG}{OP} = \frac{1}{4}\left(1 + \frac{2bc\cos\theta_1 + 2ca\cos\theta_2 + 2ab\cos\theta_3}{a^2+b^2+c^2}\right)$$

555.



$OABC$ –tetrahedron, K –center of circumsphere,

$OA = a, OB = b, OC = c,$

$\sphericalangle BOC = \theta_1 = 60°, \sphericalangle COA = \theta_2 = 60°,$

$\sphericalangle AOB = \theta_3 = 60°$

$(BOC) = P_1, (COA) = P_2,$

$(AOB) = P_3, (ABC) = P_4$

$K_1 \in P_1, K_2 \in P_2, K_3 \in P_3, K_4 \in P_4,$

$KK_1 \perp P_1, KK_2 \perp P_2, KK_3 \perp P_3, KK_4 \perp P_4$. Prove:

$$KK_1 = \frac{\sqrt{6}}{12}|3a - b - b|, KK_2 = \frac{\sqrt{6}}{12}|3b - c - a|, KK_3 = \frac{\sqrt{6}}{12}|3c - a - b|,$$

$$KK_4 = \frac{\sqrt{2}}{4} \cdot \frac{|(a+b+c)(ab+bc+ca) - 8abc)|}{\sqrt{3(a^2b^2 + b^2c^2 + c^2a^2) - 2abc(a+b+c)}}$$

Note: If $a = b = c$, then $KK_1 = KK_2 = KK_3 = KK_4 = \frac{\sqrt{6}}{12}a = r$.

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by proposer

Plagiogonal 3D system: $SA \equiv Sx, SB \equiv Sy, SC \equiv Sz$

$$K(K_1, K_2, K_3), K_1 = \frac{3a-b-c}{4}, K_2 = \frac{3b-c-a}{4}, K_3 = \frac{3c-a-b}{4}$$

$$P_1: x = 0. \text{ Let } \vec{u} \perp P_1, \vec{u}(u_1, u_2, u_3), u_1 = \frac{3}{4}, u_2 - \frac{1}{4}, u_3 = -\frac{1}{4}$$

$$|\vec{u}|^2 = \left(\frac{3}{4}\right)^2 + \left(-\frac{1}{4}\right)^2 + \left(-\frac{1}{4}\right)^2 + \frac{3}{4}\left(-\frac{1}{4}\right) + \left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right) + \left(-\frac{1}{4}\right)\left(\frac{3}{4}\right)$$

$$|\vec{K}| = \frac{\sqrt{6}}{4}$$

$$KK_1 = \left|\frac{1 \cdot \frac{3a-b-c}{4}}{1 \cdot \frac{3}{4}}\right| \cdot \frac{\sqrt{6}}{4} \Rightarrow KK_1 = \frac{\sqrt{6}}{4}|3a-b-c|$$

$$P_2: y = 0. \text{ Let } \vec{v} \perp P_2, \vec{v}(v_1, v_2, v_3), v_1 = -\frac{1}{4}, v_2 = \frac{3}{4}, v_3 = -\frac{1}{4}$$

$$|\vec{v}|^2 = \frac{6}{16} \Rightarrow |\vec{v}| = \frac{\sqrt{6}}{4} \Rightarrow KK_2 = \left|\frac{1 \cdot K_2}{1 \cdot \frac{3}{4}}\right| \cdot \frac{\sqrt{6}}{4} \Rightarrow KK_2 = \frac{\sqrt{6}}{12}|3b-a-c|$$

$$P_3: z = 0. \text{ Let } \vec{w} \perp P_3, \vec{w}(w_1, w_2, w_3), w_1 = -\frac{1}{4}, w_2 = -\frac{1}{4}, w_3 = \frac{3}{4}, |\vec{w}| = \frac{\sqrt{6}}{4}$$

$$KK_3 = \left|\frac{1 \cdot K_3}{1 \cdot \frac{3}{4}}\right| \cdot |\vec{w}| \Rightarrow KK_3 = \frac{\sqrt{6}}{12}|3c-a-b|$$

$$P_4: \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \Rightarrow bcx + acy + abz - abc = 0$$

$$\vec{q} \perp PV, \vec{q}(q_1, q_2, q_3)$$

$$q_1 = bc\left(\frac{3}{4}\right) + ac\left(-\frac{1}{4}\right) + ab\left(-\frac{1}{4}\right)$$

$$q_2 = ac\left(\frac{3}{4}\right) + ab\left(-\frac{1}{4}\right) + bc\left(-\frac{1}{4}\right)$$

$$q_3 = ab\left(\frac{3}{4}\right) + bc\left(-\frac{1}{4}\right) + ac\left(-\frac{1}{4}\right)$$

$$|\vec{q}|^2 = q_1^2 + q_2^2 + q_3^2 + q_1q_2 + q_2q_3 + q_3q_1$$

$$|\vec{q}|^2 = \frac{1}{8}[3(a^2b^2 + b^2c^2 + c^2a^2) - 2abc(a+b+c)]$$

$$KK_4^2 = \frac{[bc\cdot K_1 + ca\cdot K_2 + ab\cdot K_3 + (-abc)]^2}{(bc\cdot q_1 + ca\cdot q_2 + ab\cdot q_3)^2}\cdot |q|^2$$

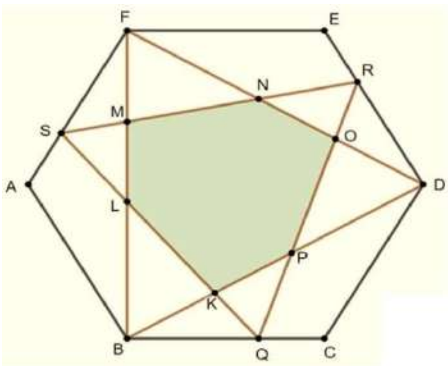
$$KK_4^2 = \frac{[(a+b+c)(ab+bc+ca) - 8abc]^2}{[3(a^2b^2+b^2c^2+c^2a^2) - 2abc(a+b+c)]^2}$$

$$\cdot\frac{1}{8}[3(a^2b^2+b^2c^2+c^2a^2 - 2abc(a+b+c)]$$

Therefore,

$$KK_4 = \frac{\sqrt{2}}{4}\cdot\frac{|(a+b+c)(ab+bc+ca) - 8abc|}{\sqrt{3(a^2b^2+b^2c^2+c^2a^2) - 2abc(a+b+c)}}$$

556.



$ABCDEF$ –regular hexagon,

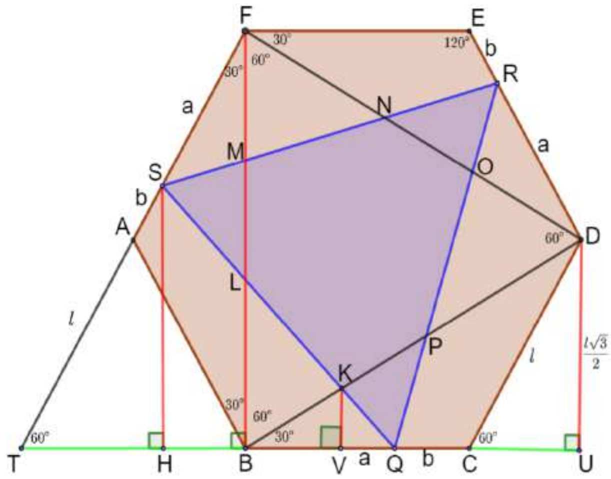
$$\frac{BQ}{QC} = \frac{DR}{RE} = \frac{FS}{SA} = \omega$$

$$\frac{[KLMNOP]}{[ABCDEF]} = \frac{49}{162};$$

$([*] - \text{area of } *)$ Find $\omega = ?$

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by Jose Ferreira Queiroz-Olinda-Brazil



$AB = l$(regular hexagon), ΔFBD is equilateral

$$\frac{FB}{2l} = \sin 60° = \frac{\sqrt{3}}{2} \Rightarrow FB = l\sqrt{3}$$

$$\Delta BLK \equiv \Delta DPO \equiv FNM$$

$$\begin{cases} BQ = a \\ QC = b \end{cases} \Rightarrow \begin{cases} BQ + QC = l \\ \dfrac{BQ}{QC} = \omega \end{cases} \Rightarrow \begin{cases} a + b = l \\ \dfrac{a}{b} = \omega \end{cases} \Rightarrow \begin{cases} a = \dfrac{\omega l}{\omega} \\ b = \dfrac{l}{\omega + 1} \end{cases}$$

Similarly, $DR = FS = a, RE = SA = b$. Applying Menelaus' theorem in the ΔSOT, points F, L and B are collinear, we have:

$$\frac{BQ}{BT} \cdot \frac{TF}{FS} \cdot \frac{SL}{LQ} = 1 \Rightarrow \frac{a}{l} \cdot \frac{2a}{l} \cdot \frac{SL}{LQ} = 1 \Rightarrow LQ = 2SL \Rightarrow SQ = 3SL$$

$$\Delta STH: \frac{SH}{ST} = \frac{SH}{l + b} = \sin 60° \Rightarrow SH = (l + b) \cdot \frac{\sqrt{3}}{2}$$

$$\Delta SHQ \sim \Delta LBQ: \frac{SQ}{LQ} = \frac{SH}{LB} \Rightarrow \frac{3SL}{2SL} = \frac{\sqrt{3}}{2} \cdot \frac{l + b}{LB} \Rightarrow (l + b)\sqrt{3} = 3LB$$

$$\Rightarrow LB = (l + b) \cdot \frac{\sqrt{3}}{3} \Rightarrow LB = \frac{\sqrt{3}}{3} \cdot \frac{l(\omega + 2)}{\omega + 1}$$

Now, $\Delta DBU \sim \Delta KBV$, then: $\frac{DU}{KV} = \frac{UB}{VB} \Rightarrow \frac{\frac{l\sqrt{3}}{2}}{KV} = \frac{\frac{3l}{2}}{VB} \Rightarrow VB = KV\sqrt{3}$

$$\Delta LBQ \sim \Delta KVQ \Rightarrow \frac{LB}{KV} = \frac{BQ}{VQ} \Rightarrow \frac{\frac{\sqrt{3}}{3}(l + b)}{KV} = \frac{a}{a - VB}$$

$$\frac{\sqrt{3}}{3}(l + b)(a - VB) = a \cdot KV \Rightarrow \frac{\sqrt{3}}{3}(l + b)\left(a - \sqrt{3}KV\right) = a \cdot KV$$

$$\frac{\sqrt{3}}{3}a(l + b) - (l + b)KV = a \cdot KV$$

$$KV(a + b + l) = \frac{\sqrt{3}}{3}a(l + b) \Rightarrow KV = \frac{\sqrt{3}}{3} \cdot \frac{a(l + b)}{2l}$$

In $\Delta KVB: \frac{KV}{KB} = \sin 30° = \frac{1}{2} \Rightarrow KB = 2KV$

$$KB = \frac{\sqrt{3}}{3} \cdot \frac{a(l + b)}{l} = \frac{\sqrt{3}}{3l} \cdot \frac{\omega l}{\omega + 1} \cdot \frac{l(\omega + 2)}{\omega + 1} \Rightarrow KB = \frac{l\sqrt{3}}{3} \cdot \frac{\omega(\omega + 2)}{(\omega + 1)^2}$$

Now,

$$[FBD] = \frac{(l\sqrt{3})^2\sqrt{3}}{4} = \frac{3l^2\sqrt{3}}{4}$$

$$[BLK] = \frac{1}{2}\cdot KB\cdot LB\cdot \sin 60^\circ = \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\cdot\frac{l\omega(\omega+2)}{(\omega+1)^2}\cdot\frac{\sqrt{3}}{3}\cdot\frac{l(\omega+2)}{\omega+1}\cdot\frac{\sqrt{3}}{2} =$$

$$= \frac{\sqrt{3}}{12}\cdot\frac{l^2\omega(\omega+2)^2}{(\omega+1)^3}$$

$$[KLMNOP] = [FBD] - 3[BLK]$$

$$[ABCDEF] = \frac{6l^2\sqrt{3}}{4}$$

So, we have:

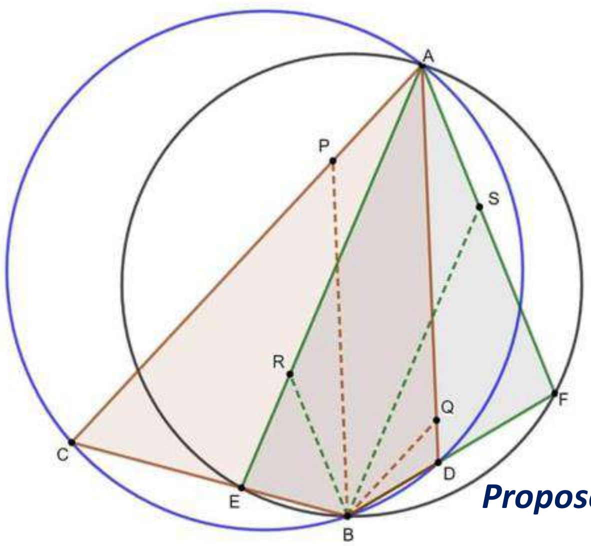
$$\frac{\frac{3l^3\sqrt{3}}{4} - \frac{3\sqrt{3}}{12}\cdot\frac{l^2\omega(\omega+2)^2}{(\omega+1)^3}}{\frac{6l^2\sqrt{3}}{4}} = \frac{49}{162} \Leftrightarrow \frac{3 - \frac{\omega(\omega+2)^2}{(\omega+1)^3}}{6} = \frac{49}{162}$$

$$\frac{\omega(\omega+2)^2}{(\omega+1)^3} = 3 - 6\cdot\frac{49}{162} = \frac{192}{162} = \frac{32}{27}$$

$$\frac{\omega(\omega+2)^2}{(\omega+1)^3} = \frac{32}{27} \Rightarrow (\omega-2)^2(5\omega+8) = 0$$

Therefore: $\omega = 2$.

557.



ABCD −cyclic

BP ∥ AD, BQ ∥ AC

AEBF −cyclic

BR ∥ AF, BS ∥ AE

Prove:

$$AP\cdot AC - AR\cdot AE = AS\cdot AF - AQ\cdot AD$$

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by Jose Ferreira Queiroz-Olinda-Brazil

$ABCD$ −cyclic and $BP \parallel AD, BQ \parallel AC$, then:

$$AB^2 = AP \cdot AC + AD \cdot AQ$$

$AEBF$ −cyclic and $BR \parallel AF, BS \parallel AE$, then:

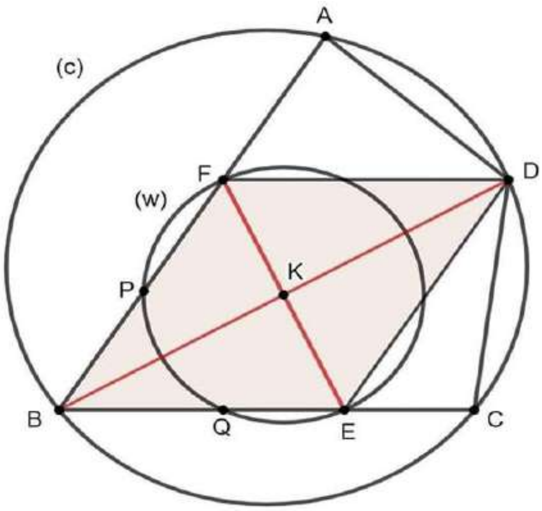
$$AB^2 = AE \cdot AR + AF \cdot AS$$

Thus,

$$AP \cdot AC + AD \cdot AQ = AE \cdot AR + AF \cdot AS$$

$$AP \cdot AC - AR \cdot AE = AS \cdot AF - AQ \cdot AD$$

558.



$BDEF$ −parallelogram with center $K, \{B, D\} \in \odot(c), (c) \cap BE = C, (c) \cap BF = A \odot(K, KE) = (w), (w) \cap BA = \{F, P\}, (w) \cap BC = \{E, Q\}$. Prove that:

$$\frac{BP - FA}{BE} + \frac{BQ - EC}{BF} = 0$$

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by Manolis Nikoloudakis-Greece

$$\frac{BP - FA}{BE} + \frac{BQ - EC}{BF} = 0 \Leftrightarrow \frac{BP}{BE} + \frac{BQ}{BF} = \frac{FA}{BE} + \frac{EC}{BF}$$

$$2\cos B = \frac{FA}{BE} + \frac{EC}{BF}$$

But $ABCD$ –cyclic, then $2\cos B = \frac{EC}{BF} + \frac{FA}{BE}$

$$\frac{EC}{BF} + \frac{FA}{BE} = \frac{FA}{BE} + \frac{EC}{BF} \Leftrightarrow 1 = 1.$$

559. In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{sinB + sinC}{h_a} \le \sum_{cyc} \frac{sinB + sinC}{r_a}$$

Proposed by Marin Chirciu-Romania

Solution by Daniel Sitaru-Romania

$$\sum_{cyc} \frac{sinB + sinC}{h_a} \le \sum_{cyc} \frac{sinB + sinC}{r_a} \leftrightarrow \sum_{cyc} \frac{2RsinB + 2RsinC}{h_a} \le \sum_{cyc} \frac{2RsinB + 2RsinC}{r_a}$$

$$\sum_{cyc} \frac{b+c}{h_a} \le \sum_{cyc} \frac{b+c}{r_a} \leftrightarrow \sum_{cyc} \frac{b+c}{\frac{2F}{a}} \le \sum_{cyc} \frac{b+c}{\frac{F}{s-a}}$$

$$\frac{1}{2}\sum_{cyc} a(b+c) \le \sum_{cyc} (s-a)(b+c) \leftrightarrow \sum_{cyc} a(b+c) \le \sum_{cyc} (2s-2a)(b+c)$$

$$2\sum_{cyc} ab \le \sum_{cyc} (b+c-a)(b+c) \leftrightarrow 2\sum_{cyc} ab \le \sum_{cyc} (b+c)^2 - \sum_{cyc} a(b+c)$$

$$2\sum_{cyc} ab \le 2\sum_{cyc} a^2 + 2\sum_{cyc} ab - 2\sum_{cyc} ab$$

$$2\sum_{cyc} ab \le 2\sum_{cyc} a^2 \leftrightarrow (a-b)^2 + (b-c)^2 + (c-a)^2 \ge$$

560. In $\Delta ABC, n_a$ –Nagel's cevian, p_a –Speiker's cevian. Prove that:

$$\sum n_a + 3\sum m_a \le 4\sum p_a + \frac{s^2(R^4 - 16r^4)}{3r^5 + sr^4}$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:\ {n_a}^2 = s(s-a) + \frac{s(b-c)^2}{a} = s^2 - \frac{s[a^2-(b-c)^2]}{a}$$

$$= s^2 - \frac{4s(s-b)(s-c)}{a} = s^2 - \frac{4s.sr^2}{a(s-a)} = s^2 - 2h_a r_a\ \ (i)$$

$$\textit{Then, we have}: \frac{R}{r}-1\overset{?}{\geq}\frac{n_a}{h_a} \leftrightarrow \frac{2(R-r)s}{a}\geq n_a \leftrightarrow$$

$$\frac{4(R-r)^2s^2}{a^2}\geq {n_a}^2 \overset{(i)}{=} s^2-2h_ar_a=s^2\left[1-2\left(\frac{2r}{a}\right)\left(\tan\frac{A}{2}\right)\right]$$

$$\leftrightarrow\ 4(R-r)^2\geq(2R\sin A)^2-4R\sin\frac{A}{2}\cos\frac{A}{2}.4r\tan\frac{A}{2}\ \leftrightarrow$$

$$(R-r)^2\geq R^2(1-\cos^2A)-2Rr(1-\cos A)$$

$$\leftrightarrow\ r^2\geq -R^2\cos^2A+2Rr\cos A\ \leftrightarrow\ (R\cos A-r)^2\geq 0, \textit{which is true},\ \ \textit{then}$$

$$\frac{n_a}{h_a}\leq\frac{R}{r}-1\ \ or\ \ n_a\leq\left(\frac{R}{r}-1\right)h_a$$

$$\textit{Also we have}:\ m_a\overset{Panaitopol}{\leq}\frac{Rh_a}{2r},\quad p_a\geq h_a,$$

$$s\overset{Mitrinovic}{\leq}\frac{3\sqrt{3}R}{2}\leq 3R.\ \ \textit{Then it suffices to prove}:$$

$$\sum\left(\frac{R}{r}-1\right)h_a+3\sum\frac{Rh_a}{2r}\leq 4\sum h_a+\frac{s^2(R^4-16r^4)}{3r^5+3Rr^4}\quad or$$

$$5\left(\frac{R}{2r}-1\right)\sum h_a\leq\frac{s^2(R^4-16r^4)}{3r(r^4+Rr^3)}$$

$$Or\ \ 5\left(\frac{R}{2r}-1\right).\frac{s^2+r(4R+r)}{2R}\leq\frac{s^2(R^4-16r^4)}{3r(r^4+Rr^3)}$$

$$\textit{By Doucet's inequality we have}:\ \ 3r(4R+r)\leq s^2\ \ \textit{so it suffices to prove}$$

$$:\ \ 5\left(\frac{R}{2r}-1\right).\frac{3+1}{3.2R}\leq\frac{R^4-16r^4}{3r(r^4+Rr^3)}$$

$$Or\ \ 5(r^4+Rr^3)(R-2r)\leq R(R^4-16r^4)=R(R-2r)(R^3+2R^2r+4Rr^2+8r^3)$$

$$Or$$

$$(R-2r)(R^4+2R^3r+4R^2r^2+3Rr^3-5r^4)\geq 0$$

which is true by Euler's inequality.

$$\textit{Therefore},\ \ \sum n_a+3\sum m_a\leq 4\sum p_a+\frac{s^2(R^4-16r^4)}{3r^5+sr^4}.$$

561. In $\Delta ABC, \omega$ −Brocard's angle, holds :

$$4F(\cot\omega-\sqrt{3})\geq\sum_{cyc}(a-b)^2+16Rr\sum_{cyc}\left(\cos^2\frac{A}{2}-\cos\frac{B}{2}\cos\frac{C}{2}\right).$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\text{From the identity } \cot\omega=\cot A+\cot B+\cot C=\frac{a^2+b^2+c^2}{4F},$$

$$\text{the desired inequality is succesively equivalent to:}$$

$$a^2+b^2+c^2-4\sqrt{3}F\geq 2\sum_{cyc}(a^2-bc)+\frac{4abc}{s}\sum_{cyc}\left(\frac{s(s-a)}{bc}-\sqrt{\frac{s^2(s-b)(s-c)}{a^2bc}}\right)$$

$$\sum_{cyc}(ab+ca-a^2)-4\sqrt{3}F\geq 4\sum_{cyc}a(s-a)-4\sum_{cyc}\sqrt{bc(s-b)(s-c)}$$

$$2\sum_{cyc}a(s-a)\geq 2\sum_{cyc}\left(\sqrt{b(s-b)}-\sqrt{c(s-c)}\right)^2+4\sqrt{3}F$$

$$\sum_{cyc}\sqrt{a(s-a)}^2\geq\sum_{cyc}\left(\sqrt{b(s-b)}-\sqrt{c(s-c)}\right)^2+2\sqrt{3}F \quad (1)$$

$$\text{Now let's prove that}$$

$$a'=\sqrt{a(s-a)}, b'=\sqrt{b(s-b)}, c'=\sqrt{c(s-c)} \text{ can be the sides of a triangle:}$$

$$\text{Let } x=(s-b)(s-c),\ \ y=(s-c)(s-a),\ \ z=(s-a)(s-b)$$

$$\text{Since: } a'^2=y+z \text{ (and analogs) we have: } a'^2+b'^2-c'^2=2z>0 \text{ then}$$

$$: \ a'+b'>c' \text{ (and analogs)}$$

$$\text{So } a',b',c' \text{ can be the sides of a triangle } \Delta' \text{ with area } F' \text{ such that:}$$

$$16F'^2=2\sum_{cyc}a'^2b'^2-\sum_{cyc}a'^4=2\sum_{cyc}(y+z)(z+x)-\sum_{cyc}(y+z)^2=4\sum_{cyc}yz=$$

$$=4(s-a)(s-b)(s-c)\sum_{cyc}(s-a)=4sr^2.s=4F^2 \text{ then: } F'=\frac{F}{2}.$$

$$\text{Applying now Hadwiger}-\text{Finsler inequality in } \Delta' \text{ we get:}$$

$$\sum_{cyc}a'^2\geq\sum_{cyc}(b'-c')^2+4\sqrt{3}F'$$

$$\Leftrightarrow\sum_{cyc}\sqrt{a(s-a)}^2\geq\sum_{cyc}\left(\sqrt{b(s-b)}-\sqrt{c(s-c)}\right)^2+2\sqrt{3}F$$

which is (1) and the proof is complete.

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\mathrm{s}\sec\frac{A}{2}\ge w_a+r_a\Leftrightarrow s\ge\frac{2bc}{b+c}\cos^2\frac{A}{2}+s\tan\frac{A}{2}\cos\frac{A}{2}=\frac{2bc}{b+c}\cdot\frac{s(s-a)}{bc}+s\sin\frac{A}{2}$$
$$=\frac{(b+c+a)(s-a)}{b+c}+s\sin\frac{A}{2}=s-a+\frac{a(s-a)}{b+c}+s\sin\frac{A}{2}\Leftrightarrow a\left(1-\frac{s-a}{b+c}\right)\ge s\sin\frac{A}{2}$$
$$\Leftrightarrow a\left(\frac{2s-a-s+a}{b+c}\right)\ge s\sin\frac{A}{2}\Leftrightarrow\frac{a}{b+c}\ge\sin\frac{A}{2}\Leftrightarrow 4R\sin\frac{A}{2}\cos\frac{A}{2}\ge 4R\cos\frac{A}{2}\cos\frac{B-C}{2}.\sin\frac{A}{2}$$
$$\Leftrightarrow\cos\frac{B-C}{2}\le 1\to\text{true}\therefore\boxed{s\sec\frac{A}{2}\ge w_a+r_a\text{ and analogs}}\to(1)$$

$$\text{Now, }4F(\cot\omega-\sqrt{3})\ge\sum_{cyc}(a-b)^2+16Rr\sum_{cyc}\left(\cos^2\frac{A}{2}-\cos\frac{B}{2}\cos\frac{C}{2}\right)\Leftrightarrow 4F\left(\frac{\sum_{cyc}a^2}{4F}-\sqrt{3}\right)$$
$$\ge 2\sum_{cyc}a^2-2\sum_{cyc}ab+8Rr\sum_{cyc}(1+\cos A)-16Rr\left(\prod_{cyc}\cos\frac{A}{2}\right)\sum_{cyc}\sec\frac{A}{2}$$
$$\Leftrightarrow 2\sum_{cyc}ab-\sum_{cyc}a^2-4\sqrt{3}F+16Rr.\frac{s}{4R}.\sum_{cyc}\sec\frac{A}{2}\ge 8Rr\left(3+1+\frac{r}{R}\right)$$
$$\Leftrightarrow 2(s^2+4Rr+r^2)-2(s^2-4Rr-r^2)-4\sqrt{3}rs+4rs\sum_{cyc}\sec\frac{A}{2}\ge 8r(4R+r)$$
$$\Leftrightarrow s\sum_{cyc}\sec\frac{A}{2}\ge 4R+r+\sqrt{3}s\overset{\text{squaring}}{\Leftrightarrow}s^2\left(\sum_{cyc}\sec^2\frac{A}{2}+2\sum_{cyc}\sec\frac{B}{2}\sec\frac{C}{2}\right)$$
$$\ge(4R+r)^2+3s^2+2\sqrt{3}s(4R+r)$$
$$\Leftrightarrow s^2.\frac{(4R+r)^2+s^2}{s^2}+2s^2\sum_{cyc}\sec\frac{A}{2}\sec\frac{B}{2}\ge(4R+r)^2+3s^2+2\sqrt{3}s(4R+r)$$
$$\Leftrightarrow s^2\sum_{cyc}\sec\frac{A}{2}\sec\frac{B}{2}\ge s^2+\sqrt{3}s(4R+r)$$
$$\overset{\text{squaring}}{\Leftrightarrow}s^4\left(\sum_{cyc}\sec^2\frac{A}{2}\sec^2\frac{B}{2}+2\left(\prod_{cyc}\sec\frac{A}{2}\right)\sum_{cyc}\sec\frac{A}{2}\right)$$
$$\ge s^4+3s^2(4R+r)^2+2\sqrt{3}s^3(4R+r)$$
$$\Leftrightarrow s^4.\frac{16R^2}{s^2}.\sum_{cyc}\cos^2\frac{A}{2}+2s^4.\frac{4R}{s}.\sum_{cyc}\sec\frac{A}{2}\ge s^4+3s^2(4R+r)^2+2\sqrt{3}s^3(4R+r)$$
$$\Leftrightarrow 8R^2s^2\left(\frac{4R+r}{R}\right)+8Rs^3\sum_{cyc}\sec\frac{A}{2}\ge s^4+3s^2(4R+r)^2+2\sqrt{3}s^3(4R+r)$$
$$\Leftrightarrow\boxed{8Rs^2(4R+r)+8Rs^3\sum_{cyc}\sec\frac{A}{2}\overset{(\bullet)}{\ge}s^4+3s^2(4R+r)^2+2\sqrt{3}s^3(4R+r)}$$

$$\text{Now},(1)\Rightarrow s\sum_{cyc}\sec\frac{A}{2}\ge\sum_{cyc}w_a+\sum_{cyc}r_a\ge\sum_{cyc}h_a+4R+r=\frac{s^2+4Rr+r^2}{2R}+4R+r$$

$$\Rightarrow 8Rs^3\sum_{cyc}\sec\frac{A}{2}\overset{(i)}{\ge}4s^2(s^2+4Rr+r^2)+8Rs^2(4R+r)$$

$$\therefore (i)\Rightarrow \text{LHS of }(\bullet)\ge 8Rs^2(4R+r)+4s^2(s^2+4Rr+r^2)+8Rs^2(4R+r)\overset{?}{\ge}\text{RHS of }(\bullet)$$
$$=s^4+3s^2(4R+r)^2+2\sqrt{3}s^3(4R+r)$$

$$\Leftrightarrow 16Rs^2(4R+r)+4s^2(s^2+4Rr+r^2)-s^4-3s^2(4R+r)^2-2\sqrt{3}s^3(4R+r)\overset{?}{\ge}0$$

$$\Leftrightarrow 3s^4+s^2(4R+r)^2-2\sqrt{3}s^3(4R+r)\overset{?}{\ge}0\Leftrightarrow s^2\left(\sqrt{3}s-(4R+r)\right)^2\overset{?}{\ge}0$$

$\rightarrow$ true

$$\Rightarrow(\bullet)\text{ is true}\Rightarrow 4F(\cot\omega-\sqrt{3})$$

$$\ge\sum_{cyc}(a-b)^2+16Rr\sum_{cyc}\left(\cos^2\frac{A}{2}-\cos\frac{B}{2}\cos\frac{C}{2}\right)\text{ with equality iff }\sqrt{3}s$$

$$=(4R+r)\text{ and }\because\sqrt{3}s\overset{\text{Trucht}}{\le}4R+r$$

with equality iff Δ ABC is equilateral $\therefore$ equality iff Δ ABC is equilateral (QED)

562. If in $\Delta ABC, I_a, I_b, I_c$ −excenters, then :

$$\frac{p}{4R^2}(8R^2-3Rr-2r^2)\le\sum_{cyc}r_a\cdot\frac{AI_a}{I_bI_c}\le\frac{p}{4Rr}(4R^2-3Rr+2r^2).$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Using the identities : $AI_a=\dfrac{p}{\cos\frac{A}{2}}$ *and* $I_bI_c=4R\cos\frac{A}{2}$ *we get :*

$$r_a\cdot\frac{AI_a}{I_bI_c}=r_a\cdot\frac{p}{4R\cos^2\frac{A}{2}}=\frac{pr_a}{4R}\left(1+\tan^2\frac{A}{2}\right)=\frac{pr_a}{4R}\left(1+\frac{r_a^{\,2}}{p^2}\right)$$

$$=\frac{p}{4R}\left(r_a+\frac{r_a^{\,3}}{p^2}\right)\ (\text{and analogs})$$

$$\text{Then}:\ \sum_{cyc}r_a\cdot\frac{AI_a}{I_bI_c}=\frac{p}{4R}\sum_{cyc}\left(r_a+\frac{r_a^{\,3}}{p^2}\right)=\frac{p}{4R}\left((4R+r)+\frac{(4R+r)^3-12Rp^2}{p^2}\right)$$

$$=\frac{p}{4R}\left(\frac{(4R+r)^3}{p^2}-8R+r\right)$$

$$\text{By Blundon}-\text{Gerretsen's inequality we have}:\ \frac{(4R+r)^2}{p^2}\ge\frac{2(2R-r)}{R}$$

$$Then: \sum_{cyc} r_a \cdot \frac{AI_a}{I_bI_c} \geq \frac{p}{4R}\left(\frac{2(2R-r)(4R+r)}{R} - 8R + r\right)$$

$$= \frac{p}{4R^2}(8R^2 - 3Rr - 2r^2) \ \ (1)$$

$$Also\ we\ have: \frac{(4R+r)^2}{p^2} \overset{Gerretsen}{\leq} \frac{(4R+r)^2}{r(16R-5r)} = \frac{R+r}{r} - \frac{3(R-2r)}{16R-5r} \overset{Euler}{\leq} \frac{R+r}{r}$$

$$Then: \sum_{cyc} r_a \cdot \frac{AI_a}{I_bI_c} \leq \frac{p}{4R}\left(\frac{(R+r)(4R+r)}{r} - 8R + r\right)$$

$$= \frac{p}{4Rr}(4R^2 - 3Rr + 2r^2) \ \ (2)$$

From (1) and (2) yields the desired inequality.

563. In Δ ABC the following relationship holds:

$$\sum_{cyc} \frac{a}{\lambda a + b + c} \leq \frac{3}{\lambda + 2} \leq \frac{3}{\lambda + 2}\left(\frac{R}{2r}\right) \ \forall \lambda \geq 1$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Proof}: \boxed{\text{Case 1}}\ \lambda = 1\ and\ then: \sum_{cyc} \frac{a}{\lambda a + b + c} = \sum_{cyc} \frac{a}{a+b+c} = \frac{1}{2s}\sum_{cyc} a = \frac{2s}{2s}$$

$$= 1\ and\ also, \frac{3}{\lambda+2} = \frac{3}{1+2} = 1 \therefore \sum_{cyc} \frac{a}{\lambda a + b + c} = \frac{3}{\lambda+2}$$

$$\boxed{\text{Case 2}}\ \lambda > 1\ and\ then: \sum_{cyc} \frac{a}{\lambda a + b + c} = \sum_{cyc} \frac{a}{\lambda a + 2s - a} = \sum_{cyc} \frac{a}{2s + a(\lambda - 1)}$$

$$= \frac{1}{\lambda - 1}\sum_{cyc} \frac{a(\lambda-1) + 2s - 2s}{2s + a(\lambda-1)}$$

$$= \frac{1}{\lambda-1}\sum_{cyc} \frac{a(\lambda-1)+2s}{2s+a(\lambda-1)} - \frac{2s}{\lambda-1}\sum_{cyc} \frac{1}{2s+a(\lambda-1)}$$

$$\overset{Bergstrom}{\leq} \frac{3}{\lambda-1} - \frac{2s}{\lambda-1} \cdot \frac{9}{6s + (\lambda-1)(a+b+c)} = \frac{3}{\lambda-1} - \frac{2s}{\lambda-1} \cdot \frac{9}{6s + 2s(\lambda-1)}$$

$$= \frac{3}{\lambda-1} - \frac{9}{(\lambda-1)(3+(\lambda-1))} = \frac{3}{\lambda-1} - \frac{9}{(\lambda+2)(\lambda-1)} = \frac{3((\lambda+2)-3)}{(\lambda+2)(\lambda-1)}$$

$$= \frac{3}{\lambda+2}$$

$$\therefore \sum_{cyc}\frac{a}{\lambda a+b+c}\le\frac{3}{\lambda+2}\text{ equality iff }\Delta ABC\text{ is equilateral}$$

$$\therefore\text{ combining cases 1 and 2, }\sum_{cyc}\frac{a}{\lambda a+b+c}\le\frac{3}{\lambda+2}\left(\frac{R}{2r}\right)\ \forall\lambda$$

$$\ge 1,\text{ equality iff }\lambda=1\text{ or iff }\Delta ABC\text{ is equilateral}$$

$$\text{and }\because\frac{3}{\lambda+2}\overset{\text{Euler}}{\le}\frac{3}{\lambda+2}\left(\frac{R}{2r}\right)\therefore\text{ in any }\Delta ABC,\sum_{cyc}\frac{a}{\lambda a+b+c}\le\frac{3}{\lambda+2}\le\frac{3}{\lambda+2}\left(\frac{R}{2r}\right)\ \forall\lambda$$

$$\ge 1\text{ with equality iff }\lambda=1\text{ or iff }\Delta ABC\text{ is equilateral (QED)}$$

564. In any Δ ABC, the following relationship holds:

$$\frac{a^3}{b^3}+\frac{b^3}{c^3}+\frac{c^3}{a^3}+\frac{R^5}{8r^5}\ge\frac{9(a^4+b^4+c^4)}{(a^2+b^2+c^2)^2}+\frac{36(a^3+b^3+c^3)}{(a+b+c)^3}$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a^3}{b^3}+\frac{b^3}{c^3}+\frac{c^3}{a^3}\overset{\text{Chebyshev}}{\ge}\frac{1}{3}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right)\ge\frac{1}{3}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\cdot\frac{9(a^4+b^4+c^4)}{(a^2+b^2+c^2)^2}$$

$$\left(\because\sum_{cyc}\frac{x}{y}\ge\frac{9\sum_{cyc}x^2}{\left(\sum_{cyc}x\right)^2}\ \forall x,y,z>0\text{ and choosing }x=a^2,y=b^2,z=c^2\ \Rightarrow\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\ge\frac{9(a^4+b^4+c^4)}{(a^2+b^2+c^2)^2}\right)$$

$$\overset{A-G}{\ge}\frac{1}{3}\cdot 3\sqrt[3]{\frac{a}{b}\cdot\frac{b}{c}\cdot\frac{c}{a}}\cdot\frac{9(a^4+b^4+c^4)}{(a^2+b^2+c^2)^2}=\frac{9(a^4+b^4+c^4)}{(a^2+b^2+c^2)^2}\therefore\boxed{\frac{a^3}{b^3}+\frac{b^3}{c^3}+\frac{c^3}{a^3}\overset{(*)}{\ge}\frac{9(a^4+b^4+c^4)}{(a^2+b^2+c^2)^2}}$$

$$\text{Again, }\frac{36(a^3+b^3+c^3)}{(a+b+c)^3}$$

$$=\frac{72s(s^2-6Rr-3r^2)}{8s^3}\overset{\text{Mitrinovic}}{\le}\frac{72(s^2-6Rr-3r^2)}{8.27r^2}\overset{\text{Gerretsen}}{\le}\frac{8(4R^2+4Rr+3r^2-6Rr-3r^2)}{8.3r^2}\overset{?}{\le}\frac{R^5}{8r^5}$$

$$\Leftrightarrow 3R^4\overset{?}{\ge}8r^3(4R-2r)$$

$$\Leftrightarrow 3R^4-32Rr^3+16r^4\overset{?}{\ge}0\Leftrightarrow(R-2r)\left(3R^3+6R^2r+8Rr^2+4r^2(R-2r)\right)\overset{?}{\ge}0\to\text{true}$$

$$\because R\overset{\text{Euler}}{\ge}2r\therefore\boxed{\frac{R^5}{8r^5}\overset{(**)}{\ge}\frac{36(a^3+b^3+c^3)}{(a+b+c)^3}}$$

$$\therefore (*) + (**) \Rightarrow \text{in any } \Delta\,ABC, \frac{a^3}{b^3} + \frac{b^3}{c^3} + \frac{c^3}{a^3} + \frac{R^5}{8r^5}$$

$$\geq \frac{9(a^4+b^4+c^4)}{(a^2+b^2+c^2)^2} + \frac{36(a^3+b^3+c^3)}{(a+b+c)^3} \text{ (QED)}$$

565. In any Δ ABC the following relationship holds:

$$\frac{16r}{9R-2r} \leq \frac{2\csc\frac{B}{2}\csc\frac{C}{2}}{\left(\csc\frac{B}{2}+\csc\frac{C}{2}\right)^2} + \frac{1}{2} \leq 1$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{2\csc\frac{B}{2}\csc\frac{C}{2}}{\left(\csc\frac{B}{2}+\csc\frac{C}{2}\right)^2} + \frac{1}{2} = \frac{2\csc\frac{B}{2}\csc\frac{C}{2}}{\csc\frac{B}{2}+\csc\frac{C}{2}} \cdot \frac{1}{\csc\frac{B}{2}+\csc\frac{C}{2}} + \frac{1}{2} \overset{H_m \leq A_m}{\leq} \frac{\csc\frac{B}{2}+\csc\frac{C}{2}}{2} \cdot \frac{1}{\csc\frac{B}{2}+\csc\frac{C}{2}} + \frac{1}{2}$$

$$= \frac{1}{2} + \frac{1}{2} \Rightarrow \boxed{\frac{2\csc\frac{B}{2}\csc\frac{C}{2}}{\left(\csc\frac{B}{2}+\csc\frac{C}{2}\right)^2} + \frac{1}{2} \leq 1}$$

$$\text{Now, } \sqrt[3]{2\sin\frac{A}{2}\cdot\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)^2} \overset{G_m \leq A_m}{\leq} \frac{2\sin\frac{A}{2} + 2\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)}{3}$$

$$= \frac{2}{3}\sum_{cyc}\sin\frac{A}{2} \overset{Jensen}{\leq} \frac{2}{3}\cdot\frac{3}{2} \left(\because f(x) = \sin\frac{x}{2} \text{ is concave } \forall x \in (0,\pi)\right)$$

$$\Rightarrow 2\sin\frac{A}{2}\cdot\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)^2 \overset{(*)}{\leq} 1$$

$$\text{Now, } \frac{2\csc\frac{B}{2}\csc\frac{C}{2}}{\left(\csc\frac{B}{2}+\csc\frac{C}{2}\right)^2} + \frac{1}{2} = \frac{\frac{2}{\sin\frac{B}{2}\sin\frac{C}{2}}}{\left(\frac{1}{\sin\frac{B}{2}}+\frac{1}{\sin\frac{C}{2}}\right)^2} + \frac{1}{2} = \frac{2\sin\frac{B}{2}\sin\frac{C}{2}}{\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)^2} + \frac{1}{2}$$

$$= \frac{4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{2\sin\frac{A}{2}\cdot\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)^2} + \frac{1}{2} \overset{\text{via } (*)}{\geq} \frac{4\left(\frac{r}{4R}\right)}{1} + \frac{1}{2} = \frac{r}{R} + \frac{1}{2} = \frac{R+2r}{2R} \overset{?}{\geq} \frac{16r}{9R-2r}$$

$$\Leftrightarrow 9R^2+16Rr-4r^2 \overset{?}{\geq} 32Rr \Leftrightarrow 9R^2-16Rr-4r^2 \overset{?}{\geq} 0 \Leftrightarrow (9R+2r)(R-2r) \overset{?}{\geq} 0 \rightarrow \text{true}$$

$$\because R \overset{\text{Euler}}{\geq} 2r \therefore \boxed{\frac{2\csc\frac{B}{2}\csc\frac{C}{2}}{\left(\csc\frac{B}{2}+\csc\frac{C}{2}\right)^2}+\frac{1}{2} \geq \frac{16r}{9R-2r}} \text{ (QED)}$$

566. If $t, u \in (0,1)$ and $M \in Int(\Delta ABC)$, $x = MA$, $y = MB$, $z = MC$ then:

$$\sum_{cyc}\left(\frac{xy}{t(1-t^2)ab}+\frac{zx}{u(1-u^2)ca}\right)^4 \geq 27.$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$By\ AM-GM\ inequality\ we\ have:$$

$$t(1-t^2)=\sqrt{\frac{1}{2}.(2t^2)(1-t^2)^2} \leq \sqrt{\frac{1}{2}.\left(\frac{(2t^2)+2(1-t^2)}{3}\right)^3}=\sqrt{\frac{1}{2}.\left(\frac{2}{3}\right)^3}=\frac{2}{3\sqrt{3}}.$$

$$Then: \sum_{cyc}\left(\frac{xy}{t(1-t^2)ab}+\frac{zx}{u(1-u^2)ca}\right)^4 \geq \left(\frac{3\sqrt{3}}{2}\right)^4.\sum_{cyc}\left(\frac{xy}{ab}+\frac{zx}{ca}\right)^4 \geq$$

$$\overset{\text{Hölder}}{\geq} \frac{27}{16}\left[\sum_{cyc}\left(\frac{xy}{ab}+\frac{zx}{ca}\right)\right]^4=27\left(\frac{xy}{ab}+\frac{yz}{bc}+\frac{zx}{ca}\right)^4 \overset{\text{Hayashi}}{\geq} 27.1^4$$

$$=27.\ So\ the\ proof\ is\ completed.$$

$$Equality\ holds\ iff\ \Delta ABC\ is\ equilateral,\ M\ is\ the\ center\ of\ \Delta ABC,\ t=u=\frac{\sqrt{3}}{3}$$

567. If ABC is an acute triangle with the medians AM, BN, CP and

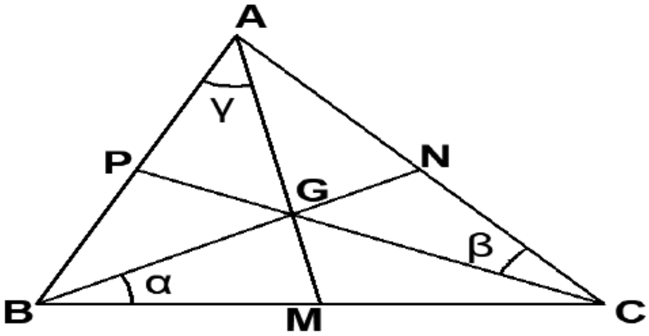
$\angle CBN = \alpha, \angle ACP = \beta, \angle BAM = \gamma$, then prove that

$$i)\ \cot\beta+\cot\gamma-\cot\alpha > \cot(C-\beta)-\cot(A-\gamma)-\cot(B-\alpha)$$

$$ii)\ \cot\beta+\cot\gamma-\cot\alpha > \cot C-\cot A-\cot B.$$

Proposed by Marius Drăgan, Neculai Stanciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco



Let G be the centroid of ΔABC and F be the area of ΔABC.

$$i)\ \text{In } \Delta BGC \text{ we have: } \cot\alpha + \cot(C-\beta) = \frac{a^2+BG^2-CG^2}{4S(\Delta BGC)} + \frac{a^2+CG^2-BG^2}{4S(\Delta BGC)}$$

$$= \frac{a^2}{2.S(\Delta BGC)} = \frac{a^2}{2.\frac{F}{3}} = \frac{3a^2}{2F}$$

$$\text{Similarly we have: } \cot\beta + \cot(A-\gamma) = \frac{3b^2}{2F} \text{ and } \cot\gamma + \cot(B-\alpha) = \frac{3c^2}{2F}$$

$$\text{Then: } (\cot\beta + \cot\gamma - \cot\alpha) - [\cot(C-\beta) - \cot(A-\gamma) - \cot(B-\alpha)] =$$

$$= [\cot\beta + \cot(A-\gamma)] + [\cot\gamma + \cot(B-\alpha)] - [\cot\alpha + \cot(C-\beta)]$$

$$= \frac{3(b^2+c^2-a^2)}{2F} \overset{\Delta ABC \text{ is acute}}{>} 0.$$

$$\text{Then: } \cot\beta + \cot\gamma - \cot\alpha > \cot(C-\beta) - \cot(A-\gamma) - \cot(B-\alpha)$$

$$ii)\ \text{Similarly to the part } i), \text{in } \Delta BNC \text{ we have: } \cot\alpha + \cot C = \frac{a^2}{2.S(\Delta BNC)} = \frac{a^2}{F}.$$

$$\text{Similarly we get: } \cot\beta + \cot A = \frac{b^2}{F} \text{ and } \cot\gamma + \cot B = \frac{c^2}{F}, \text{ then:}$$

$$(\cot\beta + \cot\gamma - \cot\alpha) - (\cot C - \cot A - \cot B)$$

$$= (\cot\beta + \cot A) + (\cot\gamma + \cot B) - (\cot\alpha + \cot C) =$$

$$= \frac{b^2+c^2-a^2}{F} \overset{\Delta ABC \text{ is acute}}{>} 0.$$

$$\text{Therefore, } \cot\beta + \cot\gamma - \cot\alpha > \cot C - \cot A - \cot B.$$

568. **In ΔABC, prove that the following inequality $\frac{\cos A}{x} + \frac{\cos B}{y} + \frac{\cos C}{z} \le \frac{x}{yz}$ holds for all positive real numbers $x^2 = y^2 + z^2$. When does equality holds?**

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\textit{The desired inequality is succesively equivalent to}$$

$$2yz\cos A + 2zx\cos B + 2xy\cos C \le 2x^2 = x^2 + y^2 + z^2$$

$$-2yz\cos(B+C) + 2zx\cos B + 2xy\cos C \le x^2 + y^2 + z^2$$

$$-2yz(\cos B\cos C - \sin B\sin C) + 2zx\cos B + 2xy\cos C$$
$$\le x^2 + y^2(\cos^2 C + \sin^2 C) + z^2(\cos^2 B + \sin^2 B)$$

$$(y\sin C - z\sin B)^2 + (y\cos C + z\cos B)^2 - 2x(y\cos C + z\cos B) + x^2 \ge 0$$

$$(y\sin C - z\sin B)^2 + (y\cos C + z\cos B - x)^2 \ge 0.\ \textit{Which is true}.$$

$$\textit{Equality holds iff } y\sin C = z\sin B\,(\textit{and analogs}) \leftrightarrow \frac{x}{\sin A} = \frac{y}{\sin B} = \frac{z}{\sin C}$$

$$\leftrightarrow \frac{x^2}{\sin^2 A} = \frac{y^2+z^2}{\sin^2 B + \sin^2 C} = \frac{x^2}{\sin^2 B + \sin^2 C} \leftrightarrow \sin^2 A$$
$$= \sin^2 B + \sin^2 C \textit{ so } \Delta ABC \textit{ is right on } A.$$

$$\textit{Therefore equality holds iff } \Delta ABC \textit{ is right on } A,\ y = x\sin B \textit{ and } z = x\cos B.$$

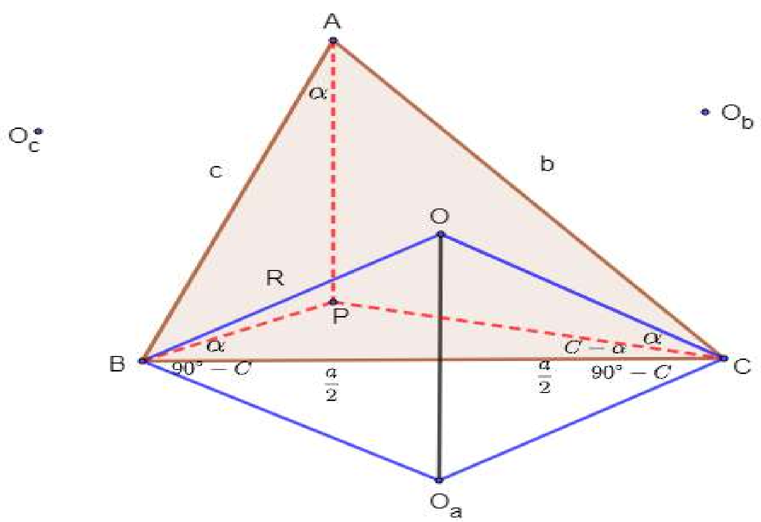
569. O **−circumcenter,** P **−Brocard's point in**

$\Delta ABC, O_a, O_b, O_c$ **−circumcenters of** $\Delta BPC, \Delta CPA, \Delta APB$**. Prove that:**

$$\frac{\sqrt{3}R^2}{4r^2} \ge \sum_{cyc} \frac{OO_a}{A} \ge \sqrt{3}$$

Proposed by Eldeniz Hesenov-Georgia

Solution by proposer



$$\measuredangle BPC = \pi - C \Rightarrow \measuredangle BO_aC = 2\widehat{C} \Rightarrow \measuredangle OO_aC = \widehat{C}$$

$$\measuredangle OBC = \measuredangle OCB = \frac{\pi}{2} - \widehat{A} \Rightarrow \measuredangle OCO_a = \pi \measuredangle APC = \widehat{B}$$

$$\text{In } \Delta OO_aC: \ \frac{R}{\sin C} = \frac{OO_a}{\sin B} \Rightarrow OO_a = \frac{R\sin B}{\sin C} \ \text{(and analogs)}$$

$$\sum_{cyc} \frac{R\sin B}{a\sin C} = R\sum_{cyc}\frac{b}{ac} = \frac{R}{4RF}\sum_{cyc} b^2 \geq \frac{1}{4F}\cdot 4F\sqrt{3} = \sqrt{3}; (1)$$

$$\sum_{cyc}\frac{OO_a}{a} = R\sum_{cyc}\frac{b}{ac} = \frac{1}{4F}\sum_{cyc} a^2 \overset{Leibniz}{\leq} \frac{1}{4F}\cdot 9R^2 \overset{Mitrinovic}{\leq} \frac{\sqrt{3}R^2}{4r^2}; (2)$$

From (1) and (2) it follows that:

$$\frac{\sqrt{3}R^2}{4r^2} \geq \sum_{cyc}\frac{OO_a}{A} \geq \sqrt{3}$$

570. In any acute Δ ABC, the following relationship holds:

$$\sqrt[2021]{\tan^n A} + \sqrt[2021]{\tan^n B} + \sqrt[2021]{\tan^n C} \geq 3\left(1 + \frac{n(\sqrt{3}-1)}{2021}\right)$$

$$\forall\, n \in \mathbb{N}, n \geq 2021$$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sqrt[2021]{\tan^n A} + \sqrt[2021]{\tan^n B} + \sqrt[2021]{\tan^n C} =$$

$$= \sum_{cyc}\left(1 + (\tan A - 1)\right)^{\frac{n}{2021}} \overset{\text{Bernoulli} \because \frac{n}{2021} \geq 1 \text{ and } \tan A - 1 > -1 \text{ and analogs}}{\geq}$$

$$\sum_{cyc}\left(1 + \frac{n}{2021}.(\tan A - 1)\right) = 3 + \frac{n}{2021}\sum_{cyc}\tan A - 3.\frac{n}{2021}$$

$$\overset{\text{Jensen}}{\geq} 3 + \frac{n}{2021}.3\tan\frac{\pi}{3} - 3.\frac{n}{2021}$$

$$\left(\because f(x) = \tan x\ \forall\, x \in \left(0, \frac{\pi}{2}\right) \text{ is convex as } f''(x) = 2\tan x\sec^2 x > 0\right)$$

$$= 3 + \frac{n}{2021}.3\sqrt{3} - 3.\frac{n}{2021} = 3\left(1 + \frac{n(\sqrt{3}-1)}{2021}\right) \text{ (QED)}$$

Solution 2 by Khaled Abd Imouti-Damascus-Syria

$$f(x)=\sqrt[2021]{\tan^n x}=(\tan x)^{\frac{n}{2021}}$$

$$f'(x)=\frac{n}{2021}(\tan x)^{\frac{n}{2021}-1}=\frac{n}{2021}\left[(\tan x)^{\frac{n}{2021}-1}+(\tan x)^{\frac{n}{2021}+1}\right]$$

$$f''(x)=\frac{n}{2021}\left[\left(\frac{n}{2021}-1\right)(\tan x)^{\frac{n}{2021}-2}+\left(\frac{n}{2021}+1\right)(\tan x)^{\frac{n}{2021}}\right](1+\tan^2 x)=$$

$$=\frac{n}{2021}\left[\frac{n-2021}{2021}\cdot\frac{1}{\tan^2 x}+\frac{n+2021}{2021}\right](\tan x)^{\frac{n}{2021}}(1+\tan^2 x)>0$$

f –convex function

$$\sqrt[2021]{\tan^n A}+\sqrt[2021]{\tan^n B}+\sqrt[2021]{\tan^n C}\geq 3\sqrt[2021]{\tan^n\left(\frac{A+B+C}{3}\right)}$$

$$\sqrt[2021]{\tan^n A}+\sqrt[2021]{\tan^n B}+\sqrt[2021]{\tan^n C}\geq 3(\sqrt{3})^{\frac{n}{2021}}\geq 3\left(1+\frac{n(\sqrt{3}-1)}{2021}\right)$$

571. If $x,y,z\in\mathbb{R}$ such that $x(y+z)>0, y(z+x)>0, z(x+y)>0,$

$xyz(x+y+z)>0$, then:

$$\sqrt{\frac{xy(y+z)(x+z)+yz(x+z)(x+y)+xz(x+y)(y+z)}{xyz(x+y+z)}}\geq$$

$$\geq\sqrt{\frac{4yz+x(y+z)}{4xz+y(x+z)}}+\sqrt{\frac{4xz+y(x+z)}{4yz+x(y+z)}}$$

Proposed by Bogdan Fuştei-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:\ xyz(x+y+z)>0\ \Leftrightarrow\ xyz(x+y)>-xyz^2\ \Leftrightarrow\ xy>-\frac{xyz^2}{z(x+y)}$$

$$Then\ we\ have:\ 4xy+z(x+y)>-\frac{4xyz^2}{z(x+y)}+z(x+y)=\frac{z^2(x-y)^2}{z(x+y)}\geq 0$$

$$Similarly\ we\ have:\ 4yz+x(y+z)>0\ and\ 4zx+y(z+x)>0.$$

$$Now\ let\ a=\sqrt{4xy+z(x+y)},\ b=\sqrt{4yz+x(y+z)},\ c=\sqrt{4zx+y(z+x)}.$$

We have:

$$a^2b^2+b^2c^2+c^2a^2=$$

$$=\sum_{cyc}[4xy+z(x+y)][4yz+x(y+z)]\underbrace{\overset{expanding\ \&\ simplifying}{\Longleftrightarrow}}_{(i)}$$

$$9\left(\sum_{cyc}(xy)^2+3xyz\sum_{cyc}x\right)=$$

$$=9\sum_{cyc}(xy+yz)(xy+zx)=9\sum_{cyc}xy(x+z)(y+z)\ \ then$$

$$:\sum_{cyc}xy(x+z)(y+z)=\frac{a^2b^2+b^2c^2+c^2a^2}{9}\ \ (1)$$

$$Also:\ a^4+b^4+c^4=\sum_{cyc}[4xy+z(x+y)]^2\underbrace{\overset{expanding\ \&\ simplifying}{\Longleftrightarrow}}_{(ii)}$$

$$18\left(\sum_{cyc}(xy)^2+xyz\sum_{cyc}x\right)$$

$$From\ (i)\ \&\ (ii)\ we\ get:\ xyz(x+y+z)$$

$$=\frac{2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)}{36}\ \ (2)$$

$$From\ (1)\ and\ (2)\ the\ problem\ becomes\ to\ prove\ that$$

$$2\sqrt{\frac{a^2b^2+b^2c^2+c^2a^2}{2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)}}\ge\frac{b}{c}+\frac{c}{b}\ (*)$$

$$We\ have:\ (*)\overset{squaring}{\Longleftrightarrow}\ 4b^2c^2(a^2b^2+b^2c^2+c^2a^2)$$

$$\ge[2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)](2b^2c^2+b^4+c^4)$$

$$\Leftrightarrow\ 0\ge 2(a^2b^2+b^2c^2+c^2a^2)(b^4+c^4)-a^4(b^2+c^2)^2-2b^2c^2(b^4+c^4)$$
$$-(b^4+c^4)^2$$

$$\Leftrightarrow\ 0\ge 2(a^2b^2+c^2a^2)(b^4+c^4)-a^4(b^2+c^2)^2-(b^4+c^4)^2$$

$$=-[a^2(b^2+c^2)-(b^4+c^4)]^2\ \ which\ is\ true.$$

$$So\ (*)\ is\ true\ and\ the\ proof\ is\ complete.$$

Solution 2 by proposer

In $\Delta ABC,\omega$ –Brocard's angle.

$$\frac{\sin(A+\omega)}{\sin\omega}=\frac{b}{c}+\frac{c}{b}(and\ analogs)$$

$$\frac{1}{\sin\omega}\geq\frac{b}{c}+\frac{c}{b}$$

m_a, m_b, m_c –can be the sides of a triangle and let ω_m –Brocard's angle in $\Delta m_a m_b m_c$.
The triangles ABC and $m_a m_b m_c$ are equivalency, hence, $\sin\omega=\sin\omega_m$

$$\frac{1}{\sin\omega}\geq\frac{m_b}{m_c}+\frac{m_c}{m_b}$$

Let $x, y, z\in\mathbb{R}$ such that $x(y+z), y(z+x), z(x+y)>0$ and $xyz(x+y+z)>0$, then
Now, $a_1=\sqrt{x(y+z)}, b_1=\sqrt{y(z+x)}, c_1=\sqrt{z(x+y)}$ are the sides of a triangle with area $F_1=\frac{1}{2}\sqrt{xyz(x+y+z)}$

$$m_{a_1}=\sqrt{\frac{2(b_1^2+c_1^2)-a_1^2}{4}}$$

$$m_{a_1}=\sqrt{\frac{2[y(z+x)+z(x+y)]-x(y+z)}{4}}=\sqrt{\frac{xy+yz+zx}{4}}$$

$$\frac{1}{\sin\omega}=\frac{\sqrt{a^2b^2+b^2c^2+c^2a^2}}{2F}=$$

$$=\sqrt{\frac{xy(y+z)(x+z)+yz(x+z)(x+y)+xz(x+y)(y+z)}{xyz(x+y+z)}}$$

$$m_{b_1}=\sqrt{\frac{4xz+y(z+x)}{4}}\Rightarrow\frac{m_{a_1}}{m_{b_1}}=\sqrt{\frac{4yz+x(y+z)}{4xz+y(x+z)}}$$

Therefore,

$$\sqrt{\frac{xy(y+z)(x+z)+yz(x+z)(x+y)+xz(x+y)(y+z)}{xyz(x+y+z)}}\geq$$

$$\geq\sqrt{\frac{4yz+x(y+z)}{4xz+y(x+z)}}+\sqrt{\frac{4xz+y(x+z)}{4yz+x(y+z)}}$$

572. Let a, b, c be sides in ΔABC and $a'=\sqrt{a}, b'=\sqrt{b}, c'=\sqrt{c}$ sides in $\Delta A'B'C'$. Prove that:

$$\frac{a'}{cosA'}+\frac{b'}{cosB'}+\frac{c'}{cosC'}=2\sqrt{\frac{R}{F}}(r_a+r_b+r_c)$$

Proposed by Mehmet Şahin-Ankara-Turkiye

Solution by Daniel Sitaru-Romania

$$\sum_{cyc}\frac{a'}{cosA'}=\sum_{cyc}\frac{a'}{\frac{(b')^2+(c')^2-(a')^2}{2b'c'}}=\sum_{cyc}\frac{2a'b'c'}{(b')^2+(c')^2-(a')^2}=2a'b'c'^{\sum_{cyc}\frac{1}{b+c-a}}$$

$$=2\sqrt{abc}\sum_{cyc}\frac{1}{2s-2a}=\sqrt{abc}\sum_{cyc}\frac{1}{s-a}=\sqrt{abc}\cdot\frac{4R+r}{F}$$

$$=\sqrt{4RF}\cdot\frac{4R+r}{F}=2\sqrt{\frac{R}{F}}(4R+r)=2\sqrt{\frac{R}{F}}(r_a+r_b+r_c)$$

Equality holds for : $a=b=c$.

Recall:

http://www.ssmrmh.ro/wp-content/uploads/2020/09/ABOUT-TRIANGLE-UVW-OF-MEHMET-SAHIN.pdf

In problem 11857 from A.M.M, 2015, pp. 700, Mehmet Şahin proves that: If UVW is a triangle with the sides $u=\sqrt{a}, v=\sqrt{b}, w=\sqrt{c}$ triangle ABC then denoting $\Delta=$ area UVW we have:

(M.S) $$\Delta=\frac{1}{2}\sqrt{r(r_a+r_b+r_c)}$$

Indeed, we have:

(1) $$r=\frac{F}{s}=\frac{\sqrt{s(s-a)(s-b)(s-c)}}{s}=\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

and

(2) $$r_a=\frac{F}{s-a}=\frac{\sqrt{s(s-a)(s-b)(s-c)}}{s-a}=\sqrt{\frac{s(s-b)(s-c)}{s-a}}$$

and the analogs. So,

$$r\cdot r_a=\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}\sqrt{\frac{s(s-b)(s-c)}{s-a}}=(s-b)(s-c)$$

and the analogs according to Heron's formula, we have:

$$\Delta=\sqrt{\frac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{2}\cdot\frac{-\sqrt{a}+\sqrt{b}+\sqrt{c}}{2}\cdot\frac{\sqrt{a}-\sqrt{b}+\sqrt{c}}{2}\cdot\frac{\sqrt{a}+\sqrt{b}-\sqrt{c}}{2}}=$$

$$=\frac{1}{4}\sqrt{(\sqrt{a}+\sqrt{b}+\sqrt{c})(-\sqrt{a}+\sqrt{b}+\sqrt{c})(\sqrt{a}-\sqrt{b}+\sqrt{c})(\sqrt{a}+\sqrt{b}-\sqrt{c})}=$$

$$=\frac{1}{4}\sqrt{((\sqrt{b}+\sqrt{c})^2-a)(a-(\sqrt{b}-\sqrt{c})^2)}=\frac{1}{4}\sqrt{(2\sqrt{bc}+(b+c-a))(2\sqrt{bc}-(b+c-a))}=$$

$$=\frac{1}{4}\sqrt{4bc-(b+c-a)^2}=\frac{1}{4}\sqrt{4bc-a^2-b^2+c^2+2ab+2ac-2bc}$$

$$=\frac{1}{4}\sqrt{2ab+2bc+2ca-a^2-b^2-c^2}=\frac{1}{4}\sqrt{a^2-(b-c)^2+b^2-(c-a)^2+c^2-(a-b)^2}=$$

$$=\frac{1}{4}\sqrt{4((s-b)(s-c)+(s-c)(s-a)+(s-a)(s-b))}=$$

$$(3)\qquad =\frac{1}{2}\sqrt{r\cdot r_a+r\cdot r_b+r\cdot r_c}=\frac{1}{2}\sqrt{r(r_a+r_b+r_c)}=\frac{1}{2}\sqrt{r(4R+r)}$$

573. Let a,b,c be sides in ΔABC and let $\Delta A'B'C'$ be the triangle with sides

$a'=\sqrt{a}, b'=\sqrt{b}, c'=\sqrt{c}$. Prove that:

$$\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}=\cos^2A'+\cos^2B'+\cos^2C'$$

Proposed by Mehmet Şahin-Ankara-Turkiye

Solution by Daniel Sitaru-Romania

$$\sum_{cyc}\sin^2\frac{A}{2}=\sum_{cyc}\frac{(s-b)(s-c)}{bc}=\frac{1}{abc}\sum_{cyc}a(s^2-sb-sc+bc)=$$

$$=\frac{1}{abc}\left(s^2\sum_{cyc}a-2s\sum_{cyc}ab+3abc\right)=$$

$$=\frac{1}{abc}(2s^3-2s(s^2+r^2+4Rr)+3abc)=$$

$$=\frac{1}{abc}(2s^3-2s^3-2sr^2-8Rrs+12Rrs)=\frac{1}{abc}(4Rrs-2sr^2)=$$

$$=\frac{2rs}{abc}(2R-r)=\frac{2F(2R-r)}{abc}$$

$$\sum_{cyc}\cos^2A'=\sum_{cyc}\left(\frac{b+c-a}{2\sqrt{bc}}\right)^2=\sum_{cyc}\frac{(2s-2a)^2}{4bc}=\sum_{cyc}\frac{(s-a)^2}{bc}=$$

$$=\frac{1}{abc}\sum_{cyc}a(s^2-2as+a^3)=$$

$$=\frac{1}{abc}\left(s^2\sum_{cyc}a-2s\sum_{cyc}a^2+\sum_{cyc}a^3\right)=$$

$$=\frac{1}{abc}\left(2s^3-4s(s^2-r^2-4Rr)+2s(s^2-3r^2-6Rr)\right)=$$

$$=\frac{1}{abc}(4Rrs-2sr^2)=\frac{2rs}{abc}(2R-r)=\frac{2F(2R-r)}{abc}$$

574. **If a,b,c are sides in ΔABC, $a'=\sqrt{a}, b'=\sqrt{b}, c'=\sqrt{c}$ are sides in $\Delta A'B'C'$ then:**

$$\sum_{cyc}\frac{1}{sin^2A'}=\frac{(a+b+c)^2}{4r(r_a+r_b+r_c)}+1$$

Proposed by Mehmet Şahin-Ankara-Turkey

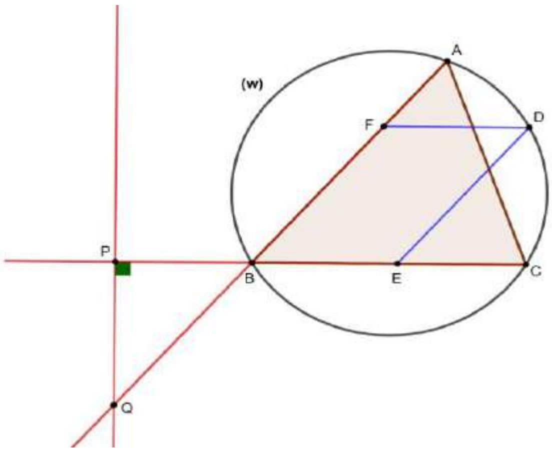
Solution by Daniel Sitaru-Romania

$$\sum_{cyc}\frac{1}{sin^2A'}=\sum_{cyc}\frac{1}{\frac{(a')^2}{4(R')^2}}=4\sum_{cyc}\frac{(a')^2(b')^2(c')^2}{16(F')^2\cdot(a')^2}=\frac{1}{4}\sum_{cyc}\frac{(b')^2(c')^2}{(F')^2}=\frac{1}{4}\sum_{cyc}\frac{bc}{\frac{1}{4}r(r_a+r_b+r_c)}=$$

$$=\frac{1}{r(r_a+r_b+r_c)}\sum_{cyc}bc=\frac{s^2+r^2+4Rr}{r(r_a+r_b+r_c)}=\frac{s^2}{r(r_a+r_b+r_c)}+\frac{r^2+4Rr}{r(r_a+r_b+r_c)}=$$

$$=\frac{(a+b+c)^2}{4r(r_a+r_b+r_c)}+\frac{r(4R+r)}{r(4R+r)}=\frac{(a+b+c)^2}{4r(r_a+r_b+r_c)}+1$$

575.



In ΔABC, (w) –circumcircle, $D\in\widehat{AC}$, $DE\parallel BA$, $DF\parallel BC$.

Prove that:

$$\frac{EC}{BF}+\frac{FA}{BE}=2\frac{BP}{BQ}\Leftrightarrow BP\perp PQ$$

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by Jose Ferreira Queiroz-Olinda-Brazil

Let $\theta=\sphericalangle PBQ=\sphericalangle ABC$. The triangle PBQ is a right, then

$$\cos\theta=\frac{BP}{BQ}$$

$ABCD$ –is cyclic and $DE\parallel BA, DF\parallel BC$, so

$$\cos\theta=\frac{1}{2}\left(\frac{EC}{BF}+\frac{FA}{BE}\right)$$

Therefore,

$$\frac{EC}{BF}+\frac{FA}{BE}=2\cdot\frac{BP}{BQ}$$

576. In $\Delta ABC,\Delta A'B'C'$ the following relationship holds:

$$\frac{aa'}{a+a'}+\frac{bb'}{b+b'}+\frac{cc'}{c+c'}\le\frac{3\sqrt{3}RR'}{2(r+r')}$$

Proposed by Daniel Sitaru-Romania

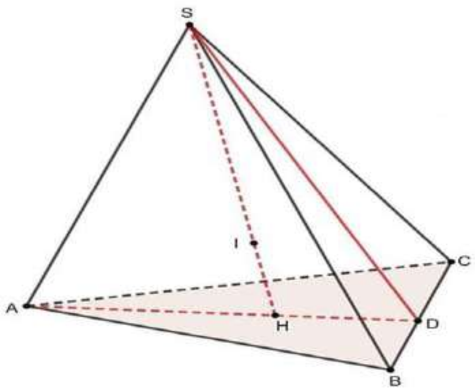
Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:\ \frac{aa'}{a+a'}+\frac{bb'}{b+b'}+\frac{cc'}{c+c'}=\left(a-\frac{a^2}{a+a'}\right)+\left(b-\frac{b^2}{b+b'}\right)+\left(c-\frac{c^2}{c+c'}\right)=$$

$$=2s-\left(\frac{a^2}{a+a'}+\frac{b^2}{b+b'}+\frac{c^2}{c+c'}\right)\overset{Bergström}{\leq}2s-\frac{(a+b+c)^2}{(a+b+c)+(a'+b'+c')}=$$

$$=2s-\frac{4s^2}{2s+2s'}=\frac{2s.2s'}{2(s+s')}\overset{Mitrinovic}{\leq}\frac{3\sqrt{3}R.3\sqrt{3}R'}{2(3\sqrt{3}r+3\sqrt{3}r')}=\frac{3\sqrt{3}RR'}{2(r+r')}.$$

$$Therefore,\ \frac{aa'}{a+a'}+\frac{bb'}{b+b'}+\frac{cc'}{c+c'}\leq\frac{3\sqrt{3}RR'}{2(r+r')}.$$



577.

$SABCD$ tetrahedron, I –insphere center, $SA = a, SB = b, SC = c,$

$\sphericalangle BSC = \theta 1 = 60°, \sphericalangle CSA = \theta_2 = 60°, \sphericalangle ASB = \theta_3 = 60°$

$(SBC) = P_1, (SCA) = P_2, (SAB) = P_1, (ABC) = P_4, OI \cap P_4 = H,$

$AH \cap BC = D$. Prove that:

$$i)\ \frac{BD}{DC}=\frac{b}{c},\quad ii)\ SD=\frac{bc}{b+c}\sqrt{3}\quad iii)\ d_{(D,P_2)}=d_{(D,P_3)}=\frac{bc}{b+c}\sqrt{\frac{2}{3}}$$

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution 1 by proposer

Plagiogonal 3d system: $SA \equiv Sx, SB \equiv Sy, SC \equiv Sz$

$S(0,0,0), A(a,0,0), B(0,b,0), C(0,0,c), D(0,d_1,d_2)$

Is (SAD) bisector plane of dihedral $(SAB - SAC)$

$$(SAD): y = 2, BC: \tfrac{y}{b}+\tfrac{z}{c}=1 \Rightarrow \begin{cases} y = d_2 = \frac{bc}{b+c} \\ z = d_3 = \frac{bc}{b+c} \end{cases} \Rightarrow SD \text{ bisector of } \sphericalangle BSC \Rightarrow \frac{BD}{DC}=\frac{b}{c}$$

$$OD^2 = d_2^2 + d_3^2 + 2d_2d_3\cos 60° \Rightarrow OD^2 = 3d_2^2 \Rightarrow OD = \frac{bc}{b+c}\sqrt{3}$$

$D(0, d_2, d_3), P_3: z = 0$. Let $\vec{u}(u_1, u_2, u_3) \perp P_3$

Is $u_1 = -\frac{1}{4}, u_2 = -\frac{1}{4}, u_3 = \frac{3}{4}, |\vec{u}|^2 = \frac{3}{8} \Rightarrow |\vec{u}| = \frac{\sqrt{3}}{2\sqrt{2}}$

$$d_{(D,P_3)} = \frac{|1 \cdot d_3|}{\left|1 \cdot \frac{3}{4}\right|} \cdot \frac{\sqrt{3}}{2\sqrt{2}} = \frac{bc}{b+c} \cdot \sqrt{\frac{2}{3}}$$

$$\text{So, } d_{(D,P_2)} = d_{(D,P_3)} = \frac{bc}{b+c} \cdot \sqrt{\frac{2}{3}}$$

Solution 2 by proposer

We denote $d_{(D,P_2)} = d_2, d_{(D,P_3)} = d_3$

I is insphere center, plane (SAD) is bisector of dihedral $(SAB) - (SAC) \Rightarrow d_2 = d_3$.

$$\text{Is: } \frac{V_{SABD}}{V_{SACD}} = \frac{[SAB]}{[SAC]} \cdot \frac{d_3}{d_2} \overset{d_2=d_3}{=} \frac{ab \cdot \sin\theta_3}{ac \cdot \sin\theta_2} = \frac{b}{c} \text{ and } \frac{V_{SABD}}{V_{SACD}} = \frac{[ABD]}{[ACD]} = \frac{BD}{DC} \Rightarrow \frac{BD}{DC} = \frac{b}{c}; (1)$$

$$(1) \Rightarrow SD - \text{bisector of } \sphericalangle BSD \Rightarrow BD = \frac{2bc}{b+c} \cdot \frac{\cos(\widehat{BSC})}{2} \Rightarrow BD = \frac{bc}{b+c} r_3$$

$$\text{Is } \frac{V_{SABD}}{V_{SACD}} = \frac{V_1}{V_2} = \frac{b}{c} \Rightarrow \frac{\frac{1}{3}(SAB) \cdot d_3}{V_1 + V_2} = \frac{b}{b+c}$$

$$\frac{\frac{1}{6} ab \cdot \frac{\sqrt{3}}{2} \cdot d_3}{\frac{abc}{6} \cdot \frac{1}{\sqrt{8}}} = \frac{b}{b+c} \Rightarrow d_3 = \frac{bc}{b+c} \cdot \sqrt{\frac{2}{3}}$$

$$d_{(D,P_3)} = d_{(D,P_3)} = \frac{bc}{b+c} \cdot \sqrt{\frac{2}{3}}$$

578. In ΔABC the following relationship holds:

$$\frac{m_a}{s_a} + \frac{m_b}{s_b} + \frac{m_c}{s_c} + \frac{w_a w_b w_c}{h_a h_b h_c} \geq 4$$

Proposed by Marin Chirciu-Romania

Solution by Avishek Mitra-West Bengal-India

$$\prod_{cyc} w_a = \prod_{cyc} \frac{2}{b+c}\sqrt{bcs(s-a)} = \frac{8abc\sqrt{s^3(s-a)(s-b)(s-c)}}{(b+c)(c+a)(a+b)} =$$

$$= \frac{8abc \cdot sF}{2abc + \sum ab(a+b)} = \frac{32Rrs}{2bac + \sum ab(2s-c)} \cdot s^2 r = \frac{32Rrs}{2s\sum ab - abc} s^2 r =$$

$$= \frac{32Rrs}{2s(s^2+r^2+4Rr) - 4Rrs} s^2 r = \frac{32Rrs}{2s(s^2+r^2+2Rr)} s^2 r = \frac{16Rr^2 s^2}{s^2+r^2+2Rr}$$

$$\prod_{cyc} h_a = \frac{8F^3}{abc} = \frac{8r^3s^3}{4Rrs} = \frac{2r^2s^2}{R}$$

$$\prod_{cyc} \frac{w_a}{h_a} = \frac{16Rr^2s^2}{s^2+r^2+2Rr} \cdot \frac{R}{2r^2s^2} = \frac{8R^2}{s^2+r^2+2Rr}$$

$$\sum_{cyc} \frac{m_a}{s_a} = \sum_{cyc} \frac{\sqrt{\frac{2b^2+2c^2-a^2}{4}}}{\frac{bc\sqrt{2b^2+2c^2-a^2}}{b^2+c^2}} = \sum_{cyc} \frac{b^2+c^2}{2bc} \overset{AGM}{\geq} \sum_{cyc} \frac{2bc}{2bc} = 3$$

Need to show:

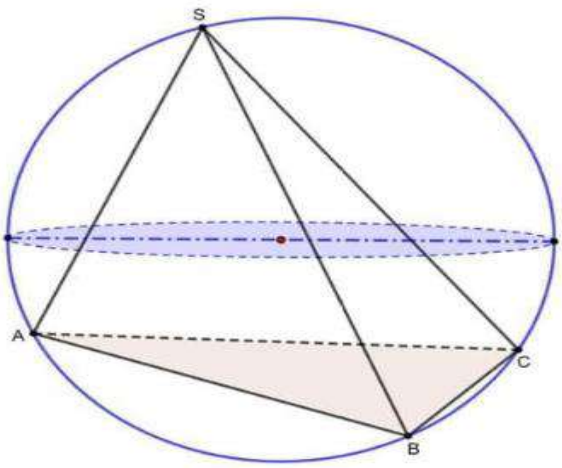
$$3 + \frac{8R^2}{s^2+r^2=2Rr} \geq 4 \Rightarrow \frac{8R^2}{s^2+r^2+2Rr} \geq 1$$

$8R^2 \geq s^2+r^2+2Rr$. But $s^2 \leq 4R^2+4Rr+3r^2$ $(Gerretsen)$, hence,

$$8R^2 \geq 4R^2+4Rr+3r^2+r^2+2Rr \Rightarrow 4R^2-6Rr-4r^2 \geq 0$$

$2(R-2r)(2R+r) \geq 0$ true from $R \geq 2r$ $(Euler)$.

579.



$SABC$ –tetrahedron, $SA = a, SB = c, SC = c, \sphericalangle BSC = \theta_1 = 60°$,

$\sphericalangle CSA = \theta_2 = 60°, \sphericalangle ASB = \theta_3 = 60°, R$ –circumsphere radius, r –insphere radius

Prove that:

$$R = \frac{\sqrt{2}}{4}\sqrt{(a-b)^2+(b-c)^2+(c-a)^2+a^2+b^2+c^2}$$

$$r = \frac{abc\sqrt{2}}{\sqrt{3}(ab+bc+ca)+\sqrt{3(a^2b^2+b^2c^2+c^2a^2)-2abc(a+b+c)}}$$

Note: If $a=b=c$ then $R = \frac{a\sqrt{6}}{4}, r = \frac{a\sqrt{6}}{12}$.

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by proposer

$$V_{(SABC)}=V=\frac{abc}{6}\cdot\frac{1}{\sqrt{2}},S_1=[SBC]=bc\cdot\frac{\sqrt{3}}{4},S_2=[SAC]=ca\cdot\frac{\sqrt{3}}{4}$$

$$S_3=[SAB]=ab\cdot\frac{\sqrt{3}}{4},S_4=[ABC]=\frac{1}{4}\sqrt{3(a^2b^2+b^2c^2+c^2a^2)-2abc(a+b+c)}$$

$$r=\frac{3V}{S_1+S_2+S_3+S_4}=$$

$$=\frac{abc\sqrt{2}}{\sqrt{3}(ab+bc+ca)+\sqrt{3(a^2b^2+b^2c^2+c^2a^2)-2abc(a+b+c)}}$$

Let K the center of circumsphere

Plagiogonal system: $SA\equiv Sx, SB\equiv Sy, SC\equiv Sz$

$$S(0,0,0),A(a,0,0),B(0,b,0),C(0,0,c),K(K_1,K_2,K_3)$$

$$K_1=\frac{3a-b-c}{4},K_2=\frac{3b-c-a}{4},K_3=\frac{3c-a-b}{4}$$

$$SK^2=R^2=K_1^2+K_2^2+K_3^2+K_1K_2+K_2K_3+K_3K_1$$

$$R=\frac{\sqrt{2}}{4}\sqrt{(a-b)^2+(b-c)^2+(c-a)^2+a^2+b^2+c^2}$$

If $a=b=c$ then $R=\frac{a\sqrt{6}}{4},r=\frac{a\sqrt{6}}{12}$.

580. In ΔABC the following relationship holds:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{R^3-8r^3}{2r^3}\geq\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{3}{2}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution 1 by Aggeliki Papaspyropoulou-Greece

$$R\geq 2r\ (Euler)\Rightarrow R^3\geq 8r^3$$

So, it is enough to prove:

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{3}{2}\leq\frac{a}{b}+\frac{b}{c}+\frac{c}{a}$$

Lemma: If $a,b,c>0$ then $\frac{a}{b+c}\leq\frac{2a^2+\frac{a(b+c)}{2}}{2(ab+bc+ca)}$

Proof. $2a(ab+bc+ca)\leq 2a^2(b+c)+\frac{a(b+c)^2}{2}\Leftrightarrow$

$$4a^2b+4a^2c+4abc\le 4a^2b+4a^2c+a(b+c)^2 \Leftrightarrow$$

$4abc\le a(b+c)^2 \Leftrightarrow bc\le (b+c)^2$ **true! So, we have:**

$$\frac{a}{b+c}\le\frac{2a^2+\frac{b+c}{2}}{2(ab+bc+ca)};(1),\quad \frac{b}{c+a}\le\frac{2b^2+\frac{b(c+a)}{2}}{2(ab+bc+ca)};(2)$$

$$\frac{c}{a+b}\le\frac{2c^2+\frac{c(a+b)}{2}}{2(ab+bc+ca)};(3)$$

We have:

$$\sum_{cyc}\frac{a}{b+c}\le\frac{2(a^2+b^2+c^2)+ab+bc+ca}{2(ab+bc+ca)}=\frac{1}{2}+\frac{a^2+b^2+c^2}{ab+bc+ca}\Leftrightarrow$$

$$\sum_{cyc}\frac{a}{b+c}\le\frac{1}{2}+\frac{3}{2}+\frac{a^2+b^2+c^2}{ab+bc+ca}=2+\frac{a^2+b^2+c^2}{ab+bc+ca}\Leftrightarrow$$

$$\sum_{cyc}\frac{a}{b+c}+\frac{3}{2}\le\frac{a^2+b^2+c^2+2ab+2bc+2ca}{ab+bc+ca}=\frac{a^2+b^2+c^2}{ab+bc+ca}$$

So, it is enough to prove:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge\frac{(a+b+c)^2}{ab+bc+ca}\text{ which is true, because:}$$

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}=\frac{a^2}{ab}+\frac{b^2}{bc}+\frac{c^2}{ac}\ge\frac{(a+b+c)^2}{ab+bc+ca}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge\frac{9(a^2+b^2+c^2)}{(a+b+c)^2}=\frac{18(s^2-4Rr-r^2)}{4s^2}\Rightarrow\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\overset{(*)}{\ge}\frac{9(s^2-4Rr-r^2)}{2s^2}$$

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{3}{2}=\sum_{cyc}\frac{\sum_{cyc}a-(b+c)}{b+c}+\frac{3}{2}=2s\sum_{cyc}\frac{1}{b+c}-\frac{3}{2}$$

$$=\frac{2s}{2s(s^2+2Rr+r^2)}\left(\left(\sum_{cyc}a^2+2\sum_{cyc}ab\right)+\sum_{cyc}ab\right)-\frac{3}{2}$$

$$=\frac{4s^2+s^2+4Rr+r^2}{s^2+2Rr+r^2}-\frac{3}{2}$$

$$\Rightarrow\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{3}{2}\overset{(**)}{=}\frac{7s^2+2Rr-r^2}{2(s^2+2Rr+r^2)}\therefore(*),(**)\Rightarrow\text{in order to prove : }\frac{a}{b}+\frac{b}{c}+\frac{c}{a}$$

$$\ge\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{3}{2},\text{it suffices to prove :}$$

$$\frac{9(s^2-4Rr-r^2)}{2s^2} \geq \frac{7s^2+2Rr-r^2}{2(s^2+2Rr+r^2)} \Leftrightarrow 9(s^2-4Rr-r^2)(s^2+2Rr+r^2)$$

$$\geq s^2(7s^2+2Rr-r^2) \Leftrightarrow 2s^4-(20Rr-r^2)s^2-r^2(72R^2+54Rr+9r^2) \overset{(\bullet)}{\geq} 0$$

Now, LHS of $(\bullet) \overset{\text{Gerretsen}}{\geq} (32Rr-10r^2)s^2-(20Rr-r^2)s^2-r^2(72R^2+54Rr+9r^2)$

$$=(12Rr-9r^2)s^2-r^2(72R^2+54Rr+9r^2)$$

$$\overset{\text{Gerretsen}}{\geq} (12Rr-9r^2)(16Rr-5r^2)-r^2(72R^2+54Rr+9r^2) \overset{?}{\geq} 0$$

$$\Leftrightarrow 20R^2-43Rr+6r^2 \overset{?}{\geq} 0 \Leftrightarrow (R-2r)(20R-3r) \overset{?}{\geq} 0 \to \text{true} \because R \overset{\text{Euler}}{\geq} 2r$$

$\Rightarrow (\bullet)$ is true

$$\therefore \frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{3}{2}$$

$$\Rightarrow \frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{R^3-8r^3}{2r^3} \overset{R^3-8r^3 \geq 0 \text{ via Euler}}{\geq} \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{3}{2} \text{ (QED)}$$

581. In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{h_a}{a\sin A} \geq \frac{2r}{R} \sum_{cyc} \frac{r_a}{a\sin A}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \frac{h_a}{a\sin A} = \sum_{cyc} \frac{2rs}{a^2 \cdot \frac{a}{2R}} = \sum_{cyc} \frac{abc}{a^2 \cdot a} = \sum_{cyc} \frac{bc \cdot b^2c^2}{a^2b^2c^2}$$

$$= \frac{1}{16R^2r^2s^2}\left(\left(\sum_{cyc} ab\right)^3 - 3abc\prod_{cyc}(a+b)\right)$$

$$= \boxed{\frac{(s^2+4Rr+r^2)^3-24Rrs^2(s^2+2Rr+r^2)}{16R^2r^2s^2} \overset{(*)}{=} \sum_{cyc} \frac{h_a}{a\sin A}}$$

$$\frac{2r}{R}\sum_{cyc} \frac{r_a}{a\sin A} = \frac{2r}{R} \cdot \sum_{cyc}\left(\frac{rs}{s-a} \cdot \frac{2R}{a^2}\right) = \frac{r}{R} \cdot \sum_{cyc} \frac{abc(s-b)(s-c)}{a^2(s-a)(s-b)(s-c)}$$

$$= \frac{r}{Rr^2s} \cdot \sum_{cyc} \frac{b^2c^2(s-b)(s-c)}{abc} = \frac{r}{Rr^2s \cdot 4Rrs} \sum_{cyc}\left(b^2c^2(-s^2+as+bc)\right)$$

$$=\frac{1}{4R^2r^2s^2}\left(-s^2\sum_{cyc}b^2c^2+sabc\sum_{cyc}ab+\sum_{cyc}b^3c^3\right)$$

$$=\frac{1}{4R^2r^2s^2}\left(-s^2\left((s^2+4Rr+r^2)^2-16Rrs^2\right)+4Rrs^2(s^2+4Rr+r^2)\right.$$
$$\left.+(s^2+4Rr+r^2)^3-24Rrs^2(s^2+2Rr+r^2)\right)$$

$$=\boxed{\frac{r^2\left(s^4-(4Rr-2r^2)s^2+r(4R+r)^3\right)}{4R^2r^2s^2}\overset{(**)}{=}\frac{2r}{R}\sum_{cyc}\frac{r_a}{a\sin A}}\therefore(*),(**)\Rightarrow\sum_{cyc}\frac{h_a}{a\sin A}$$
$$\geq\frac{2r}{R}\sum_{cyc}\frac{r_a}{a\sin A}$$

$$\Leftrightarrow\frac{(s^2+4Rr+r^2)^3-24Rrs^2(s^2+2Rr+r^2)}{16R^2r^2s^2}\geq\frac{r^2\left(s^4-(4Rr-2r^2)s^2+r(4R+r)^3\right)}{4R^2r^2s^2}$$

$$\Leftrightarrow s^6-(12Rr+r^2)s^4+r^3s^2(16R-5r)-3r^3(4R+r)^3\overset{(\bullet)}{\geq}0$$

$$\text{Now, LHS of }(\bullet)\overset{\text{Gerretsen}}{\geq}(4Rr-6r^2)s^4+r^3s^2(16R-5r)$$
$$-3r^3(4R+r)^3\overset{\text{Gerretsen}}{\geq}\left((4Rr-6r^2)(16Rr-5r^2)+r^3(16R-5r)\right)s^2$$
$$-3r^3(4R+r)^3\overset{?}{\geq}0$$

$$\Leftrightarrow(64R^2-100Rr+25r^2)s^2-3r(4R+r)^3\underbrace{\overset{?}{\geq}}_{(\bullet\bullet)}0\ and\ \because 64R^2-100Rr+25r^2$$
$$=14R^2+50R(R-2r)+25r^2\overset{\text{Euler}}{\geq}14R^2+25r^2>0\therefore\text{LHS of }(\bullet\bullet)\overset{\text{Gerretsen}}{\geq}$$
$$(64R^2-100Rr+25r^2)(16Rr-5r^2)-3r(4R+r)^3\overset{?}{\geq}0$$
$$\Leftrightarrow 52t^3-129t^2+54t-8\overset{?}{\geq}0\left(t=\frac{R}{r}\right)$$
$$\Leftrightarrow(t-2)\left(39t^2+13t(t-2)+t+4\right)\overset{?}{\geq}0\to\text{true}\because t\overset{\text{Euler}}{\geq}2$$
$$\Rightarrow(\bullet\bullet)\Rightarrow(\bullet)\text{ is true}\therefore\sum_{cyc}\frac{h_a}{a\sin A}\geq\frac{2r}{R}\sum_{cyc}\frac{r_a}{a\sin A}\ \text{(QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum\nolimits_{cyc}\frac{h_a}{a\sin A}\geq\frac{2r}{R}\sum\nolimits_{cyc}\frac{r_a}{a\sin A}\Leftrightarrow\sum\nolimits_{cyc}\frac{2F.2R}{a.a.a}$$
$$\geq\frac{8(s-a)(s-b)(s-c)}{abc}\sum\nolimits_{cyc}\frac{F.2R}{a.(s-a).a}$$

$$\Leftrightarrow \sum_{cyc} \frac{abc}{a^3} \geq 4\sum_{cyc} \frac{(s-a)(s-b)(s-c)}{a^2(s-a)} \Leftrightarrow \sum_{cyc} \frac{bc}{a^2} \geq \sum_{cyc} \frac{4(s-b)(s-c)}{a^2}$$

By AM − GM inequality we have :

$$\sum_{cyc} \frac{bc}{a^2} \geq 3\sqrt[3]{\frac{bc.ca.ab}{a^2.b^2.c^2}} = 3 \ \ and \ \sum_{cyc} \frac{4(s-b)(s-c)}{a^2} \leq \sum_{cyc} \frac{[(s-b)+(s-c)]^2}{a^2}$$

$$= \sum_{cyc} 1 = 3.$$

$$Then: \sum_{cyc} \frac{bc}{a^2} \geq 3 \geq \sum_{cyc} \frac{4(s-b)(s-c)}{a^2} \ \ and\ the\ proof\ is\ complete.$$

$$Therefore, \sum_{cyc} \frac{h_a}{a\sin A} \geq \frac{2r}{R}\sum_{cyc} \frac{r_a}{a\sin A}.$$

Solution 3 by Marian Ursărescu-Romania

$$\sum_{cyc} \frac{h_a}{a\sin A} \geq 3\sqrt[3]{\prod_{cyc} \frac{h_a}{a\sin A}}; (1)$$

$$h_a h_b h_c = \frac{2s^2r^2}{R}, abc = 4Rrs \ \text{ and } \ \sin A \sin B \sin C = \frac{sr}{2R^2}$$

$$\prod_{cyc} \frac{h_a}{a\sin A} = 1; (2)$$

From (1) and (2) we get:

$$\sum_{cyc} \frac{h_a}{a\sin A} \geq 3; \ \ (3)$$

$$rr_a = \frac{S^2}{s(s-a)} = (s-b)(s-c) \leq \left(\frac{s-b+s-c}{2}\right)^2 = \frac{a^2}{4}$$

$$\frac{2r}{R}\sum_{cyc} \frac{r_a}{a\sin A} \leq \frac{2r}{R}\sum_{cyc} \frac{a^2}{4ra\sin A} = \frac{1}{2R}\sum_{cyc} \frac{a}{\sin A} = \frac{1}{2R}\cdot 6R = 3; \ (4)$$

From (3) and (4) it follows that:

$$\sum_{cyc} \frac{h_a}{a\sin A} \geq \frac{2r}{R}\sum_{cyc} \frac{r_a}{a\sin A}$$

582. In acute ΔABC the following relationship holds:

$$\sum_{cyc}\frac{bc}{\cot B\cot C}\ge 108r^2$$

Proposed by Marin Chirciu-Romania

Solution 1 by Marian Ursărescu-Romania

We must show that:

$$\sum_{cyc} bc\tan B\tan C\ge 108r^2;(1)$$

ΔABC acute, then $\tan A,\tan B,\tan C>0$

$$\sum_{cyc} bc\tan B\tan C\ge 3\sqrt[3]{(abc)^2(\tan A\tan B\tan C)^2};(2)$$

From (1) and (2) we must show:

$$\sqrt[3]{(abc)^2(\tan A\tan B\tan C)^2}\ge 36r^2\Leftrightarrow$$

$$(abc)^2(\tan A\tan B\tan C)^2\ge 2^6\cdot 3^6\cdot r^6;(3)$$

$$\tan A\tan B\tan C=\frac{2sr}{s^2-(2R+r)^2}$$

$$s^2\overset{Gerretsen}{\le}4R62+4Rr+3r^2\Rightarrow s^2-4R^2-4Rr-r^2\le 2r^2$$

$$\tan A\tan B\tan C\ge\frac{2sr}{2r^2}=\frac{s}{r};(4)$$

From (3) and (4) we must show:

$$(abc)^2\cdot\frac{s^2}{r^2}\ge 2^6\cdot 3^6\cdot r^6;(5)$$

$$abc=4Rrs;(6)$$

From (5) and (6) we must show:

$$2^4\cdot s^2R^2r^2\cdot\frac{s^2}{r^2}\ge 2^6\cdot 3^6\cdot r^6\Leftrightarrow s^4R^2\ge 2^2\cdot 3^6r^6 \text{ true bacause}$$

$$R^2\ge 2^2r^2 \text{ and } s^2\ge 27r^2\Rightarrow s^4\ge 3^6r^4$$

Solution 2 by Avishek Mitra-West Bengal-India

In acute ΔABC: $\tan(A+B)=\tan(\pi-C)\Rightarrow\frac{\tan A+\tan B}{1-\tan A\tan B}=-\tan C$

$$\tan A + \tan B = -\tan C + \prod_{cyc} \tan A$$

$$\sum_{cyc} \tan A = \prod_{cyc} \tan A \overset{let}{=} x$$

$$\frac{1}{3}\sum_{cyc} \tan A \overset{AGM}{\geq} 3\sqrt[3]{\prod_{cyc} \tan A} \Rightarrow \frac{1}{3}\prod_{cyc} \tan A \geq \sqrt[3]{\prod_{cyc} \tan A}$$

$$\frac{x}{3} \geq \sqrt[3]{x} \Rightarrow \frac{x^3}{27} \geq x \Rightarrow x^2 \geq 27 \Rightarrow x \geq 3\sqrt{3} \Leftrightarrow \prod_{cyc} \tan A \geq 3\sqrt{3}$$

$$\sum_{cyc} \frac{bc}{\cot B \cot C} = \sum_{cyc} bc \tan B \tan C \overset{AGM}{\geq} 3\sqrt[3]{\prod_{cyc} a^2 \cdot \prod_{cyc} \tan^2 A} \geq$$

$$\geq 3\sqrt[3]{16R^2r^2s^2 \cdot \left(3\sqrt{3}\right)^2} \overset{Euler}{\geq} 3\sqrt[3]{16 \cdot 27 \cdot (2r)^2 r^2 \left(3\sqrt{3}r\right)^2} = 108r^2$$

Solution 3 by Tapas Das-India

$$\cot(A+B) = \cot(\pi - C)$$

$$\frac{\cot A \cot B - 1}{\cot A + \cot B} = -\cot C \Rightarrow \cot A \cot B - 1 = -\cot B \cot C + \cot C \cot A$$

$$\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$$

$$\sum_{cyc} \frac{bc}{\cot B \cot C} = \sum_{cyc} \frac{\left(\sqrt{bc}\right)^2}{\cot B \cot C} \geq \frac{\left(\sum \sqrt{bc}\right)^2}{\sum \cot B \cot C} = \left(\sum_{cyc} \sqrt{bc}\right)^2 \geq \left[3\sqrt[3]{\prod_{cyc} \sqrt{bc}}\right]^2 =$$

$$= 9\sqrt[3]{(abc)^2} = 9\sqrt[3]{(4RF)^2} \geq 9\sqrt[3]{(4 \cdot 2r \cdot rs)^2} \geq 9\sqrt[3]{\left(4 \cdot 2r \cdot r3\sqrt{3}r\right)^2} = 108r^2$$

583. Let ABC is not acute triangle. Prove that :

$$A^2 + B^2 + C^2 \geq \frac{3\pi^2}{8}$$

Proposed by Phan Ngoc Chau-Vietnam

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $C = max\{A, B, C\}$. *Since ABC is not acute triangle we have* $C \geq \frac{\pi}{2}$.

By CBS inequality we have : $A^2 + B^2 \geq \frac{(A+B)^2}{2} = \frac{(\pi - C)^2}{2} = \frac{C^2}{2} - \pi C + \frac{\pi^2}{2}$

$$Then:\ A^2+B^2+C^2\ge\frac{3C^2}{2}-\pi C+\frac{\pi^2}{2}=\left(C-\frac{\pi}{2}\right)\left(\frac{3C}{2}-\frac{\pi}{4}\right)+\frac{3\pi^2}{8}\overset{C\ge\frac{\pi}{2}}{\ge}\frac{3\pi^2}{8}.$$

$$Equality\ holds\ iff\ (A,B,C)=\left(\frac{\pi}{4},\frac{\pi}{4},\frac{\pi}{2}\right)\ and\ their\ permutation.$$

Solution 2 by Soumava Chakraborty-Kolkata-India

WLOG, we may assume A is non − acute $\therefore \frac{\pi}{2}\le A<\pi\Rightarrow 0\le A-\frac{\pi}{2}<\frac{\pi}{2}$ and assigning $X=A-\frac{\pi}{2}$, we arrive at : $A=X+\frac{\pi}{2}$ with $0\le X<\frac{\pi}{2}$

$$\therefore A^2+B^2+C^2=\left(X+\frac{\pi}{2}\right)^2+B^2+C^2\ge\left(X+\frac{\pi}{2}\right)^2+\frac{1}{2}(B+C)^2=\left(X+\frac{\pi}{2}\right)^2+\frac{1}{2}(A-\pi)^2$$

$$=\left(X+\frac{\pi}{2}\right)^2+\frac{1}{2}\left(A-\frac{\pi}{2}-\frac{\pi}{2}\right)^2=\left(X+\frac{\pi}{2}\right)^2+\frac{1}{2}\left(X-\frac{\pi}{2}\right)^2$$

$\therefore A^2+B^2+C^2\ge\left(X+\frac{\pi}{2}\right)^2+\frac{1}{2}\left(X-\frac{\pi}{2}\right)^2=f(X)\ (say)\ \forall X\in\left[0,\frac{\pi}{2}\right)$ and then : $f'(X)=\frac{6X+\pi}{2}>0\Rightarrow f(X)$ is $\uparrow$ on $\left[0,\frac{\pi}{2}\right)\therefore f(X)\ge f(0)=\left(0+\frac{\pi}{2}\right)^2+\frac{1}{2}\left(0-\frac{\pi}{2}\right)^2=\frac{3\pi^2}{8}$

with equality iff $X=0\Leftrightarrow A-\frac{\pi}{2}=0\Leftrightarrow A=\frac{\pi}{2}$ and consequently $B=C=\frac{\pi}{4}$

$\therefore$ in any non − acute $\Delta ABC, A^2+B^2+C^2\ge\frac{3\pi^2}{8}$ with equality iff $\left(A=\frac{\pi}{2},B=C=\frac{\pi}{4}\right)$ and cyclic permutations (QED)

Solution 3 by Ravi Prakash-New Delhi-India

Let $C=\frac{\pi}{2}+2x, 0\le x<\frac{\pi}{4}\Rightarrow A+B=\frac{\pi}{2}-2x$. Let $A=\frac{\pi}{4}-x+y$ and $B=\frac{\pi}{4}-x-y$, where

$$-\frac{\pi}{4}+x<y<\frac{\pi}{4}-x$$

Now,

$$A^2+B^2+C^2=\left(\frac{\pi}{4}-x+y\right)^2+\left(\frac{\pi}{4}-x-y\right)^{2^2}+\left(\frac{\pi}{2}+2x\right)^2=$$

$$=2\left(\frac{\pi}{4}-x\right)^2+2y^2+\left(\frac{\pi}{2}+2x\right)^2=$$

$$=2\left(\frac{\pi^2}{16}-\frac{\pi}{2}x+x^2\right)+2y^2+\frac{\pi^2}{4}+4x^2+2\pi x=$$

$$=\frac{3\pi^2}{8}+6x^2+2y^2+\pi x\ge\frac{3\pi^2}{8};(\because x\ge 0)$$

Equality holds when $x=0, y=0\Leftrightarrow A=B=\frac{\pi}{4}, C=\frac{\pi}{2}$.

584. In ΔABC the following relationship holds:

$$\frac{a^4}{b^2+c^2}+\frac{b^4}{c^2+a^2}+\frac{c^4}{a^2+b^2}+\frac{R^4-16r^4}{r^2}\ge\frac{a^3}{b+c}+\frac{b^3}{c+a}+\frac{c^3}{a+b}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By Euler's inequality $R\ge 2r$ *we have:* $R^4-16r^4\ge 0$ *so it suffices to prove*

$$\frac{a^4}{b^2+c^2}+\frac{b^4}{c^2+a^2}+\frac{c^4}{a^2+b^2}\ge\frac{a^3}{b+c}+\frac{b^3}{c+a}+\frac{c^3}{a+b}$$

$$\Leftrightarrow\left(\frac{a^4}{b^2+c^2}-\frac{a^3}{b+c}\right)+\left(\frac{b^4}{c^2+a^2}-\frac{b^3}{c+a}\right)+\left(\frac{c^4}{a^2+b^2}-\frac{c^3}{a+b}\right)\ge 0$$

$$\Leftrightarrow\frac{a^3b(a-b)-ca^3(c-a)}{(b^2+c^2)(b+c)}+\frac{b^3c(b-c)-ab^3(a-b)}{(c^2+a^2)(c+a)}+\frac{c^3a(c-a)-bc^3(b-c)}{(a^2+b^2)(a+b)}\ge 0$$

$$\Leftrightarrow ab(a-b)\left(\frac{a^2}{(b^2+c^2)(b+c)}-\frac{b^2}{(c^2+a^2)(c+a)}\right)$$

$$+bc(b-c)\left(\frac{b^2}{(c^2+a^2)(c+a)}-\frac{c^2}{(a^2+b^2)(a+b)}\right)+$$

$$+ca(c-a)\left(\frac{c^2}{(a^2+b^2)(a+b)}-\frac{a^2}{(b^2+c^2)(b+c)}\right)\ge 0$$

Which is true because $a-b$ *and* $\frac{a^2}{(b^2+c^2)(b+c)}-\frac{b^2}{(c^2+a^2)(c+a)}$ *have the same sign (and analogs)*

$$\left(\therefore a\ge b\Leftrightarrow\frac{a^2}{(b^2+c^2)(b+c)}\ge\frac{b^2}{(c^2+a^2)(c+a)}\ (\text{and analogs})\right).$$

So the proof is completed. Equality holds iff ΔABC *is equilateral.*

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\frac{a^4}{b^2+c^2}+\frac{b^4}{c^2+a^2}+\frac{c^4}{a^2+b^2}=\sum_{cyc}\frac{a^6}{a^2b^2+a^2c^2}\overset{\text{Bergstrom}}{\ge}\frac{\left(\sum_{cyc}a^3\right)^2}{2\sum_{cyc}a^2b^2}$$

$$=\frac{4s^2\left(s^2-6Rr-3r^2\right)^2}{2((s^2+4Rr+r^2)^2-16Rrs^2)}\therefore\sum_{cyc}\frac{a^4}{b^2+c^2}\overset{(*)}{\ge}\frac{2s^2\left(s^2-6Rr-3r^2\right)^2}{(s^2+4Rr+r^2)^2-16Rrs^2}$$

$$\frac{a^3}{b+c}+\frac{b^3}{c+a}+\frac{c^3}{a+b}=\sum_{cyc}\frac{\sum_{cyc}a^3-\left(b^3+c^3\right)}{b+c}=\sum_{cyc}a^3.\sum_{cyc}\frac{1}{b+c}-\sum_{cyc}\frac{(b+c)\left(b^2-bc+c^2\right)}{b+c}$$

$$=2s\left(s^2-6Rr-3r^2\right).\frac{\sum_{cyc}((c+a)(a+b))}{2s(s^2+2Rr+r^2)}+\sum_{cyc}ab-2\sum_{cyc}a^2$$

$$=\frac{s^2-6Rr-3r^2}{s^2+2Rr+r^2}\cdot\left(\left(\sum_{cyc}a^2+2\sum_{cyc}ab\right)+\sum_{cyc}ab\right)+s^2+4Rr+r^2-4(s^2-4Rr-r^2)$$

$$=\frac{s^2-6Rr-3r^2}{s^2+2Rr+r^2}\cdot(4s^2+s^2+4Rr+r^2)-(3s^2-20Rr-5r^2)$$

$$=\frac{(s^2-6Rr-3r^2)(5s^2+4Rr+r^2)-(3s^2-20Rr-5r^2)(s^2+2Rr+r^2)}{s^2+2Rr+r^2}$$

$$\Rightarrow\sum_{cyc}\frac{a^3}{b+c}\overset{(**)}{=}\frac{2\left(s^4-(6Rr+6r^2)s^2+r^2(8R^2+6Rr+r^2)\right)}{s^2+2Rr+r^2}\ \therefore(*),(**)$$

$\Rightarrow$ **in order to prove :**

$$\sum_{cyc}\frac{a^4}{b^2+c^2}\geq\sum_{cyc}\frac{a^3}{b+c},\text{ it suffices to prove : }\frac{2s^2(s^2-6Rr-3r^2)^2}{(s^2+4Rr+r^2)^2-16Rrs^2}$$

$$\geq\frac{2\left(s^4-(6Rr+6r^2)s^2+r^2(8R^2+6Rr+r^2)\right)}{s^2+2Rr+r^2}$$

$$\Leftrightarrow s^2(s^2+2Rr+r^2)(s^2-6Rr-3r^2)^2$$

$$\geq\left((s^2+4Rr+r^2)^2-16Rrs^2\right)\left(s^4-(6Rr+6r^2)s^2+r^2(8R^2+6Rr+r^2)\right)$$

$$\Leftrightarrow(4R-r)s^6-(60R^2+38Rr-13r^2)rs^4+r^2s^2(232R^3+284R^2r+104Rr^2+13r^3)$$

$$-r^3(128R^4+160R^3r+72R^2r^2+14Rr^3+r^4)\overset{(\bullet)}{\geq}0\text{ and}$$

$$\because(4R-r)(s^2-16Rr+5r^2)^3\overset{\text{Gerretsen}}{\geq}0\ \therefore\text{ in order toprove }(\bullet),\text{ it suffices to prove :}$$

$$(4R-r)s^6-(60R^2+38Rr-13r^2)rs^4+r^2s^2(232R^3+284R^2r+104Rr^2+13r^3)$$

$$-r^3(128R^4+160R^3r+72R^2r^2+14Rr^3+r^4)$$

$$\geq(4R-r)(s^2-16Rr+5r^2)^3$$

$$\Leftrightarrow(132R^2-146Rr+28r^2)s^4-rs^2(2840R^3-2972R^2r+676Rr^2-88r^3)$$

$$+r^2(16256R^4-19616R^3r+8568R^2r^2-1714Rr^3+124r^4)\overset{(\bullet\bullet)}{\geq}0\text{ and}$$

$$\because(132R^2-146Rr+28r^2)(s^2-16Rr+5r^2)^2$$

$$=(59R^2+73R(R-2r)+28r^2)(s^2-16Rr+5r^2)^2\overset{\text{Euler + Gerretsen}}{\geq}0$$

$\therefore$ **in order toprove $(\bullet\bullet)$, it suffices to prove :**

$$(132R^2-146Rr+28r^2)s^4-rs^2(2840R^3-2972R^2r+676Rr^2-88r^3)$$

$$+r^2(16256R^4-19616R^3r+8568R^2r^2-1714Rr^3+124r^4)$$

$$\geq(132R^2-146Rr+28r^2)(s^2-16Rr+5r^2)^2$$

$$\Leftrightarrow(1384R^3-3020R^2r+1680Rr^2-192r^3)s^2\overset{(\bullet\bullet\bullet)}{\geq}r(17536R^4-38880R^3r$$

$$+25260R^2r^2-6416Rr^3+576r^4)$$

Now, $\because 1384R^3-3020R^2r+1680Rr^2-192r^3$

$$=(R-2r)(1258R^2+126R(R-2r)+1176r^2)+2160r^3\overset{\text{Euler}}{\geq}2160r^3>0$$

$\therefore$ **LHS of $(\bullet\bullet\bullet)$**

$$\overset{\text{Gerretsen}}{\geq}(1384R^3-3020R^2r+1680Rr^2-192r^3)(16Rr-5r^2)\overset{?}{\geq} r(17536R^4-38880R^3r+25260R^2r^2-6416Rr^3+576r^4)$$

$$\Leftrightarrow 576t^4-2045t^3+2090t^2-632t+48\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((t-2)(576t^2+259t+822)+1620\right)\overset{?}{\geq}0\to \text{true}\because t\overset{\text{Euler}}{\geq}2$$

$$\Rightarrow(\bullet\bullet\bullet)\Rightarrow(\bullet\bullet)\Rightarrow(\bullet)\text{ is true}$$

$$\therefore\sum_{cyc}\frac{a^4}{b^2+c^2}\geq\sum_{cyc}\frac{a^3}{b+c}\text{ and }\because\frac{R^4-16r^4}{r^2}\overset{\text{Euler}}{\geq}0\therefore\text{ in any }\Delta ABC,\sum_{cyc}\frac{a^4}{b^2+c^2}+\frac{R^4-16r^4}{r^2}\geq\sum_{cyc}\frac{a^3}{b+c}\ (\text{QED})$$

585. In ΔABC the following relationship holds:

$$3\leq\sum_{cyc}\frac{m_a^2}{r_br_c}\leq\frac{3R}{2r}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Adrian Popa-Romania

$$r_br_c=\frac{F}{s-b}\cdot\frac{F}{s-c}=\frac{s(s-a)(s-b)(s-c)}{(s-b)(s-c)}=s(s-a)$$

$$m_a\geq\sqrt{s(s-a)}\Rightarrow m_a^2\geq s(s-a)$$

$$\sum_{cyc}\frac{m_a^2}{r_br_c}\geq\sum_{cyc}\frac{s(s-a)}{s(s-a)}=3$$

$$m_a\leq 2R\cos^2\frac{A}{2}=2R\cdot\frac{s(s-a)}{bc}$$

$$\frac{m_a^2}{r_br_c}=\frac{4R^2s^2(s-a)^2}{b^2c^2s(s-a)}=4R^2\cdot\frac{s(s-a)}{b^2c^2}$$

$$\sum_{cyc}\frac{m_a^2}{r_br_c}=\sum_{cyc}4R^2\cdot\frac{s(s-a)}{b^2c^2}=4R^2s\cdot\frac{a^2(s-a)+b^2(s-b)+c^2(s-c)}{a^2b^2c^2}=$$

$$=4R62\,s\cdot\frac{s(a^2+b^2+c^2)-(a^3+b^3+c^3)}{16R^2r^2s^2}=$$

$$=4R^2s\cdot\frac{s(2s^2-8Rr-2r^2)-(2s^3-6sr^2-12Rrs)}{16R^2r^2s^2}=$$

$$=\frac{2s^3-8Rrs-2sr^2-2s^3+6r^2s+12sRr}{4r^2s}=$$

$$=\frac{4sr^2+4sRr}{4sr^2}=1+\frac{R}{r}\overset{(*)}{\leq}\frac{3R}{2r}$$

$$(1)\Leftrightarrow 2r+2R\leq 3R\Leftrightarrow 2r\leq r\ (Euler).$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\boxed{\sum_{cyc}\frac{m_a^2}{r_br_c}}=\sum_{cyc}\frac{\left((b-c)^2+4s(s-a)\right).4(s-b)(s-c)}{16s(s-a)(s-b)(s-c)}$$

$$=\sum_{cyc}\frac{16s(s-a)(s-b)(s-c)}{16s(s-a)(s-b)(s-c)}+\sum_{cyc}\frac{4(s-b)(s-c)(b-c)^2}{16s(s-a)(s-b)(s-c)}$$

$$=3+\frac{1}{16r^2s^2}\cdot\sum_{cyc}\left(\left(a^2-(b-c)^2\right)(b-c)^2\right)$$

$$=3+\frac{1}{16r^2s^2}\cdot\left(\sum_{cyc}a^2(b-c)^2-\sum_{cyc}(b-c)^4\right)$$

$$=3+\frac{1}{8r^2s^2}\cdot\left(-\left(\sum_{cyc}a^4+\sum_{cyc}2a^2b^2\right)+2\sum_{cyc}(a^3b+ab^3)-abc\sum_{cyc}a\right)$$

$$=3+\frac{1}{8r^2s^2}\cdot\left(-\left(\sum_{cyc}a^2\right)^2+2\sum_{cyc}\left(ab\left(\sum_{cyc}a^2-c^2\right)\right)-abc\sum_{cyc}a\right)$$

$$=3+\frac{1}{8r^2s^2}\cdot\left(-\left(\sum_{cyc}a^2\right)^2+2\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}ab\right)-3abc\sum_{cyc}a\right)$$

$$=3+\frac{1}{8r^2s^2}\cdot\left(\left(\sum_{cyc}a^2\right)\left(2\sum_{cyc}ab-\sum_{cyc}a^2\right)-24Rrs^2\right)$$

$$=3+\frac{2(s^2-4Rr-r^2)\left(2(s^2+4Rr+r^2)-2(s^2-4Rr-r^2)\right)-24Rrs^2}{8r^2s^2}$$

$$=3+\frac{(s^2-4Rr-r^2)(4R+r)-3Rs^2}{rs^2}$$

$$\boxed{\le \frac{3R}{2r}} \Leftrightarrow \frac{3(R-2r)}{2r} \ge \frac{(R+r)s^2 - r(4R+r)^2}{rs^2} \Leftrightarrow 3(R-2r)s^2 \ge 2(R+r)s^2 - 2r(4R+r)^2$$

$$\Leftrightarrow \boxed{(R-8r)s^2 + 2r(4R+r)^2 \overset{(*)}{\ge} 0}$$

$\boxed{Case\ 1}$ $R - 8r \ge 0$ *and* then, LHS of $(*) \ge 2r(4R+r)^2 > 0 \Rightarrow (*)$ is true (strict inequ*al*ity)

$\boxed{Case\ 2}$ $R - 8r < 0$ *and* then, LHS of $(*)$

$$= -(8r-R)s^2 + 2r(4R+r)^2 \overset{\text{Gerretsen}}{\ge} -(8r-R)(4R^2+4Rr+3r^2)$$

$$+ 2r(4R+r)^2 \overset{?}{\ge} 0 \Leftrightarrow 4t^3 + 4t^2 - 13t - 22 \overset{?}{\ge} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(4t^2+12t+11) \overset{?}{\ge} 0 \to \text{true} \because t \overset{\text{Euler}}{\ge} 2 \Rightarrow (*) \text{ is true, with equality iff } t$$

$= 2$, th*at* is, iff Δ ABC is equil*ateral*

$\therefore$ combining *cases* 1 *and* 2, $(*)$ is true in *any* Δ ABC

$\therefore$ in *any* Δ ABC, $\boxed{\boxed{\sum_{cyc} \frac{m_a^2}{r_b r_c} \le \frac{3R}{2r}}}$ with equ*al*ity iff Δ ABC is equil*ateral and*

$$\sum_{cyc} \frac{m_a^2}{r_b r_c} \overset{\text{Lascu + A-G}}{\ge} \sum_{cyc} \frac{s(s-a)}{s(s-a)} = 3 \therefore 3 \le \sum_{cyc} \frac{m_a^2}{r_b r_c}$$

$\le \frac{3R}{2r}$, equ*al*ities iff Δ ABC is equil*ateral* (QED)

586. If a, b, c are positive integers with $a = \frac{2bc}{b-c}$ =odd and $(a,b,c) = 1$, then prove that a and $b + c$ can be the legs of a right angles triangle and abc is perfect square.

Proposed by Neculai Stanciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $d = (b,c)$ *and* b', c' *be positive integers such that* $b = db'$ *and* $c = dc'$

$\therefore (b', c') = 1.$ *The equation becomes :* $a(b'-c') = 2db'c'$

We have a divide $2db'c'$ *and since a is odd and*

$(a,b,c) = 1$ *then we have* $(a, 2d) = 1$ *and we conclude that a divide* $b'c'$.

Also we have b' *divide* $a(b'-c')$ *and since* $(b',c') = 1$ *we have* $(b', b'-c') = 1$

$$then\ b'\ divide\ a.\ Similarly\ we\ find\ c'\ divide\ a.$$

$$And\ since\ (b',c')=1\ we\ conclude\ that\ b'c'\ divide\ a.\ Then:$$

$$a=b'c'\ and\ b'-c'=2.\ Now\ we\ have:$$

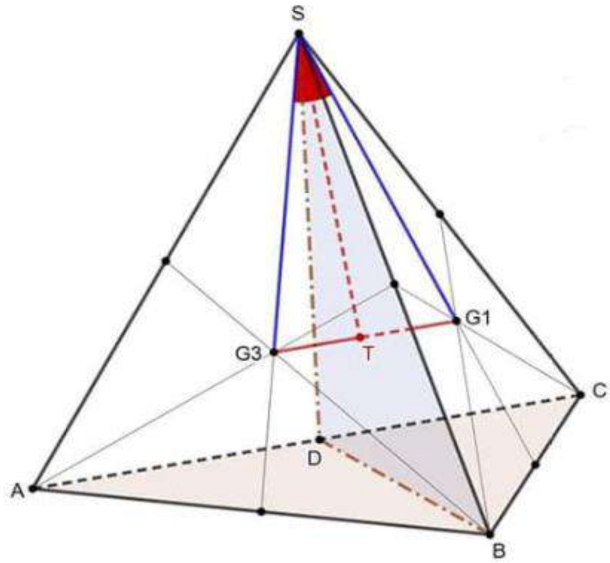
$$a^2+(b+c)^2=(b'c')^2+(db'+dc')^2=(b'c')^2+d^2[(b'-c')^2+4b'c']=$$

$$=(b'c')^2+d^2[(2d)^2+4b'c']=(b'c'+2d^2)^2.$$

$$Then\ a\ and\ b+c\ can\ be\ the\ legs\ of\ a\ right\ angles\ triangle$$

$$Also\ we\ have:\ \ abc=b'c'.db'.dc'=(db'c')^2\ \ then\ abc\ is\ perfect\ square.$$

587.



$SABC$ –regular tetrahedron, $SA=a, G_1, G_3$ –centroids of $\Delta SBC, \Delta SAB$ respectively, $\measuredangle G_1SG_3=\theta, G_1G_3\cap P=\{T\}$. Prove that:

$$(i)\ \cos\theta=\frac{5}{6},\qquad (i)\ ST=\frac{a\sqrt{11}}{6}$$

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by Jose Ferreira Queiroz-Olinda-Brazil

$$SG_1=SG_3=\frac{2}{3}\cdot\frac{a\sqrt{3}}{2}=\frac{a\sqrt{3}}{3}$$

$$MN=\frac{a}{2}, SN=SM=\frac{a\sqrt{3}}{2}$$

$$i)\ MN^2=SM^2+SN^2-2SN\cdot SM\cdot\cos\theta$$

$$\frac{a^2}{4}=\frac{3a^2}{4}+\frac{3a^2}{4}=\frac{6a^2}{4}\cos\theta\Rightarrow\cos\theta=\frac{5}{6}$$

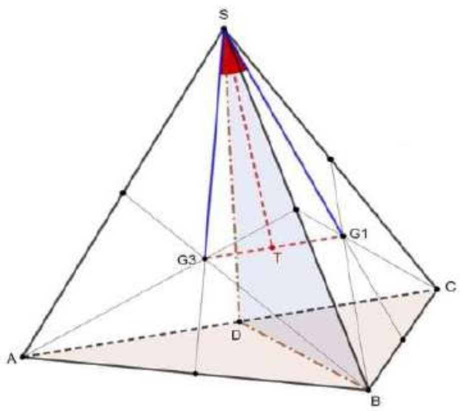
$$ii)\ \Delta SG_1G_3\sim\Delta SNM\Rightarrow G_1G_3=\frac{a}{3}$$

$$G_1T=G_3T=\frac{a}{6}$$

$$SG_1^2=G_1T^2+ST^2$$

$$ST^2=\frac{3a^2}{9}-\frac{a^2}{36}=\frac{11a^2}{36},\quad ST=\frac{a\sqrt{11}}{6}$$

588.



$SABC$ –tetrahedron, $\measuredangle BSC=\theta_1=60°,\measuredangle CSA=\theta_2=60°,\measuredangle ASB=\theta_3=60°$,

$SA=a,SB=b,SC=c,G_1,G_3$ –centroids of $\Delta SBC,\Delta SAB$ respectively,

$\measuredangle G_1SG_3=\theta,(SBD)=P,P$ –is bisector plane of dihedral

$(SBA-SBC),\ G_1G_3\cap P=\{T\}$.

$$(I)\text{ Prove: }\cos\theta=\frac{ab+bc+ca+2b^2}{2\sqrt{a^2+b^2+ab}+\sqrt{b^2+c^2+bc}}$$

$$(II)\text{ Prove: }ST^2=\frac{a^2b^2+b^2c^2+3c^2a^2+2abc(a+b+c)}{9(a+c)^2}$$

(III) If $ST\perp G_1G_3$ prove: $a=c$.

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by proposer

Plagiogonal 3D system: $SA\equiv SX,SB\equiv Sy,SC\equiv Sz$

$$S(0,0,0), A(a,0,0), B(0,b,0), C(0,0,c), G_1\left(0,\frac{b}{3},\frac{c}{3}\right), G_3\left(\frac{a}{3},\frac{b}{3},0\right)$$

$$\left\{G_1G_3: \frac{x-\frac{a}{3}}{0-\frac{a}{3}} = \frac{z-0}{\frac{c}{3}-0}, P: x = y\right\} \Rightarrow T\left(\frac{ac}{3(a+c)}, \frac{b}{3}, \frac{ac}{3(a+c)}\right)$$

$$\overrightarrow{SG_1}\left(0,\frac{b}{3},\frac{c}{3}\right), \overrightarrow{SG_2}\left(\frac{a}{3},\frac{b}{3},0\right)$$

$$\left|\overrightarrow{SG_1}\right|^2 = 0^2 + \left(\frac{b}{3}\right)^2 + \left(\frac{c}{3}\right)^2 + \frac{bc}{9} = \frac{b^2+c^2+bc}{9}$$

$$\left|\overrightarrow{SG_2}\right|^2 = \frac{a^2+b^2+ab}{9}$$

$$\overrightarrow{SG_1}\cdot\overrightarrow{SG_2} = \frac{b^2}{9} + \frac{ab+bc+ca}{18} = \frac{ab+bc+ca+2b^2}{18}$$

$$\cos\theta = \frac{\overrightarrow{SG_1}\cdot\overrightarrow{SG_2}}{\left|\overrightarrow{SG_1}\right|\cdot\left|\overrightarrow{SG_2}\right|} = \frac{ab+bc+ca+2b^2}{2\sqrt{a^2+b62+ab}\cdot\sqrt{b^2+c^2+bc}}$$

If $a = b = c \Rightarrow \cos\theta = \frac{5}{6}$

$$ST^2 = t_{162} + t_2^2 + t_3^2 + t_1t_2 + t_2t_3 + t_3t_1$$

$$ST^2 = \frac{a^2b^2 + b^2c^2 + 3c^2a^2 + 2abc(a+b+c)}{9(a+c)^2}$$

If $a = b = c \Rightarrow ST = \frac{a\sqrt{11}}{6}$

$$\overrightarrow{G_1G_3}\left(\frac{a}{3}, 0, -\frac{c}{3}\right), \overrightarrow{G_1G_3}(g_1,g_2,g_3), \overrightarrow{ST}(t_1,t_2,t_3)$$

$$ST \perp G_1G_3 \Rightarrow \overrightarrow{ST}\cdot\overrightarrow{G_1G_3} = 0$$

$$\Rightarrow g_1t_1 + g_2t_2 + g_3t_3 + (g_2t_3 + g_3t_2)\left(\frac{1}{2}\right) + (g_3t_1 + g_1t_3)\left(\frac{1}{2}\right) + (g_1t_2 + g_2t_1)\left(\frac{1}{2}\right) = 0$$

$$\frac{(a-c)(ab+bc+3ac)}{18(a+c)} = 0 \Rightarrow a - c = 0 \Rightarrow a = c.$$

589. If $t > 0$ and $x, y, z \in (0, t)$ then in ΔABC holds:

$$\frac{a^2}{(t^2-x^2)(y+z)} + \frac{b^2}{(t^2-y^2)(z+x)} + \frac{c^2}{(t^2-z^2)(z+x)} \geq \frac{9}{t^3}\cdot F$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$By\ AM-GM\ inequality\ we\ have:$$

$$2x^2(t^2-x^2)^2 \le \left(\frac{2x^2+(t^2-x^2)+(t^2-x^2)}{3}\right)^3 = \frac{8t^6}{27}\ \ then:\ t^2-x^2 \le \frac{2t^3}{3\sqrt{3}.x}$$

$$Similarly\ we\ have:\ \ t^2-y^2 \le \frac{2t^3}{3\sqrt{3}.y}\ \ \&\ t^2-z^2 \le \frac{2t^3}{3\sqrt{3}.z}$$

$$Then\ we\ have:\ \frac{a^2}{(t^2-x^2)(y+z)}+\frac{b^2}{(t^2-y^2)(z+x)}+\frac{c^2}{(t^2-z^2)(z+x)} \ge$$

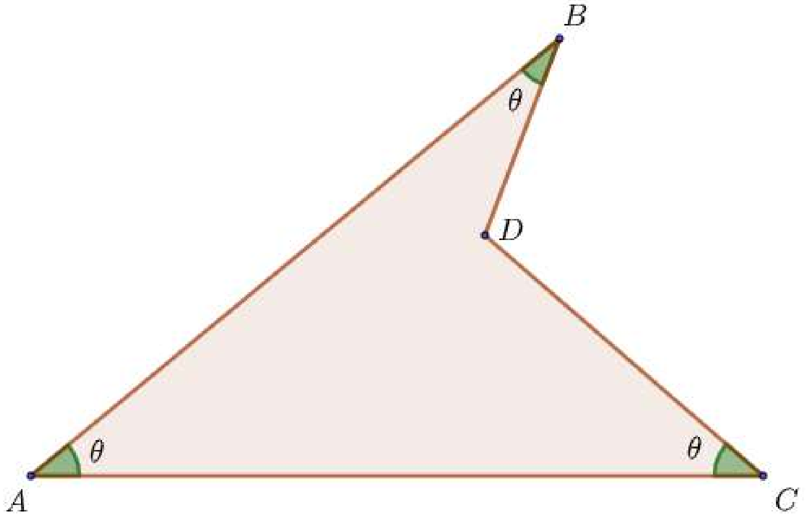
$$\frac{3\sqrt{3}.xa^2}{2t^3(y+z)}+\frac{3\sqrt{3}.yb^2}{2t^3(z+x)}+\frac{3\sqrt{3}.zc^2}{2t^3(z+x)} = \frac{3\sqrt{3}}{2t^3}\left(\frac{x}{y+z}.a^2+\frac{y}{z+x}.b^2+\frac{z}{z+x}.c^2\right).$$

$$By\ Tsinsifas\ inequality\ we\ have:\ \ \frac{x}{y+z}.a^2+\frac{y}{z+x}.b^2+\frac{z}{z+x}.c^2 \ge 2\sqrt{3}.F$$

$$Therefore,$$

$$\frac{a^2}{(t^2-x^2)(y+z)}+\frac{b^2}{(t^2-y^2)(z+x)}+\frac{c^2}{(t^2-z^2)(z+x)} \ge \frac{3\sqrt{3}}{2t^3}.2\sqrt{3}.F = \frac{9}{t^3}.F.$$

$$Equality\ holds\ iff\ \Delta ABC\ is\ equilateral\ and\ x=y=z=\frac{\sqrt{3}t}{3}$$

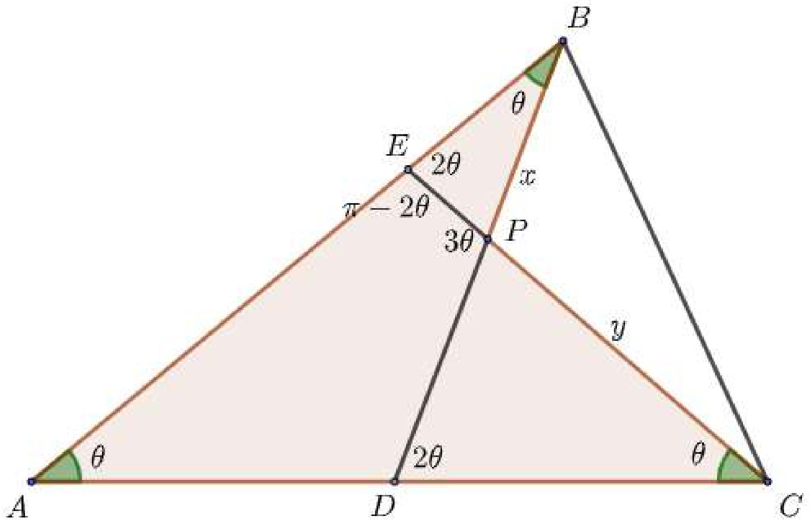


590.

$AB = b, AC = c, [ABCD]$ −area of $ABCD$. Find $[ABCD] = f(b, c, \theta)$.

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by Jose Ferreira Queiroz-Olinda-Brazil



$$\measuredangle BAD = \measuredangle DBA \Rightarrow AD = BD$$

$$\measuredangle CAE = \measuredangle ACE \Rightarrow AE = CE$$

Using Law of sines in the ΔACE and ΔABD, we have

$$AE = CE = \frac{c}{2\cos\theta} \text{ and } AD = BD = \frac{b}{2\cos\theta}$$

Now, $BP = x$ and $CP = y$, then $DP = \frac{b}{2\cos\theta} - x$ and $EP = \frac{c}{2\cos\theta} - y$

Using Law of sines in the ΔDPC and ΔPEB, we have:

$$\frac{\frac{b}{2\cos\theta} - x}{\sin\theta} = \frac{y}{\sin 2\theta} \Rightarrow y = b - 2x\cos\theta$$

$$\frac{\frac{c}{2\cos\theta} - y}{\sin\theta} = \frac{x}{\sin 2\theta} \Rightarrow x = c - 2y\cos\theta$$

Hence,

$$x = \frac{c - 2b\cos\theta}{1 - 4\cos^2\theta}, y = \frac{b - 2c\cos\theta}{1 - 4\cos^2\theta}$$

$$[ABPD] = [DBC] - [BPC]$$

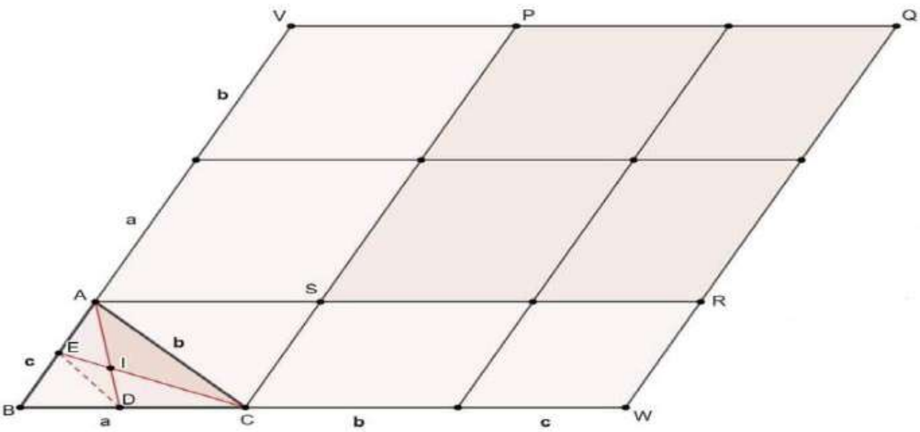
$$[ABPD] = \frac{1}{2}bc\sin\theta - \frac{1}{2}xy\sin 3\theta$$

$$[ABPD] = \frac{1}{2}bc\sin\theta - \frac{1}{2}\cdot\frac{(c - 2bc\cos\theta)(b - 2c\cos\theta)}{(1 - 4\cos^2\theta)^2}\sin 3\theta =$$

$$=\frac{1}{2}bc\sin\theta-\frac{1}{2}\cdot\frac{(c-2b\cos\theta)(b-2c\cos\theta)(3\sin\theta-4\sin^3\theta)}{(1-4\cos^2\theta)^2}=$$

$$=\frac{1}{2}\sin\theta\left[bc+\frac{(c-2b\cos\theta)(b-2c\cos\theta)}{1-4\cos^2\theta}\right]=$$

$$=\sin\theta\cdot\frac{bc-(b^2+c^2)\cos 2\theta}{1-4\cos^2\theta}$$



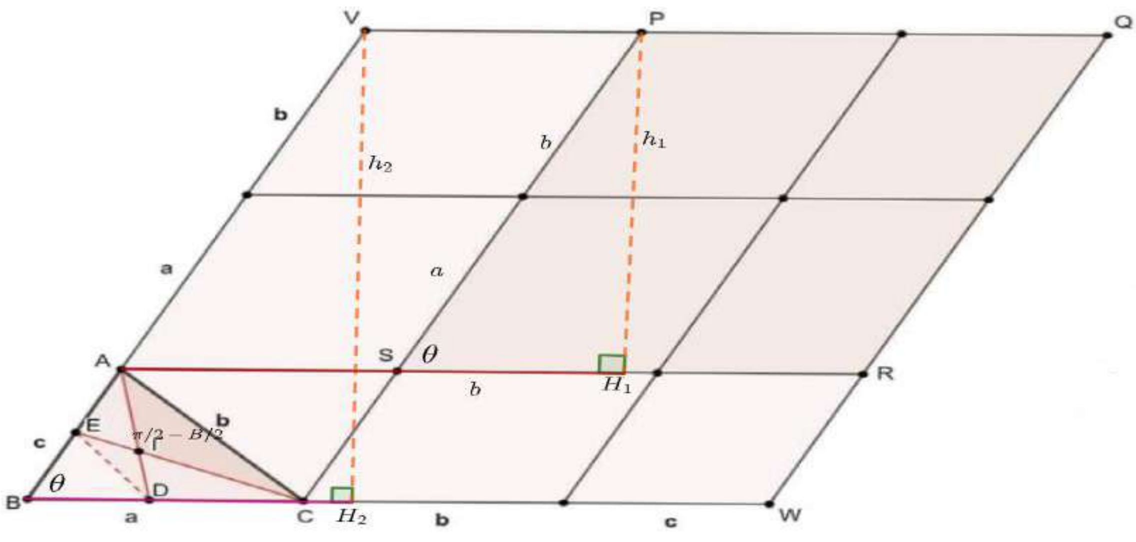
591.

I **−incenter of** ΔABC**. Prove:**

$$\frac{[ACI]}{[ACDE]}=\frac{[PQRS]}{[VQWB]}$$

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by Jose Ferreira Queiroz-Olinda-Brazil



$$2s=a+b+c, F=[ABC]=\frac{1}{2}ac\cdot\sin\theta$$

$$h_1=(a+b)\sin\theta\Rightarrow[PQRS]=\frac{1}{2}(b+c)(a+b)\sin\theta$$

$$h_2=(a+b+c)\sin\theta\Rightarrow[VQWB]=\frac{1}{2}(a+b+c)^2\sin\theta$$

$$\sin\left(\frac{\pi}{2}-\frac{\theta}{2}\right)=\cos\frac{\theta}{2}=\sqrt{\frac{s(s-b)}{ac}}$$

In the ΔAIC we have:

$$AI=\frac{\sqrt{bcs(s-a)}}{s}; CI=\frac{\sqrt{abs(s-c)}}{s}$$

$$[ACI]=\frac{1}{2}AI\cdot CI=\frac{1}{2}\cdot\frac{\sqrt{bcs(s-a)}}{s}\cdot\frac{\sqrt{abs(s-c)}}{s}\sin\left(\frac{\pi}{2}-\frac{\theta}{2}\right)$$

$$[ACI]=\frac{b}{2s}\sqrt{ac(s-a)(s-c)}\cdot\sqrt{\frac{s(s-b)}{ac}}=\frac{b}{2s}\cdot F$$

$$[ACDE]=[ABC]-[BED]$$

We know that:

$$BD=\frac{ac}{b+c}, BE=\frac{ac}{b+a}$$

$$[ACDE]=F-\frac{1}{2}BD\cdot BE\cdot\sin\theta=F-\frac{1}{2}\cdot\frac{ac}{b+c}\cdot\frac{ac}{b+a}\sin\theta$$

$$[ACDE]=F-\frac{acF}{(b+c)(b+a)}=\frac{F}{(b+c)(b+a)}(b^2+ab+bc)=\frac{2bsF}{(b+c)(a+b)}$$

$$\frac{[ACI]}{[ACDE]}=\frac{\frac{bF}{2s}}{\frac{2bsF}{(c+b)(a+b)}}=\frac{(b+c)(a+b)}{4s^2}$$

$$\frac{[PQRS]}{[VQWB]}=\frac{\frac{1}{2}(b+c)(a+b)\sin\theta}{\frac{1}{2}(2s)^2\sin\theta}=\frac{(b+c)(a+b)}{4s^2}$$

592. Let be the triangles ABC and $X_1Y_1Z_1$ with sides a,b,c and x_1,y_1,z_1 respectively,

$F=[ABC], T_1=[X_1Y_1Z_1]$ **areas. For** $P\in Int(\Delta ABC)$ **prove that:**

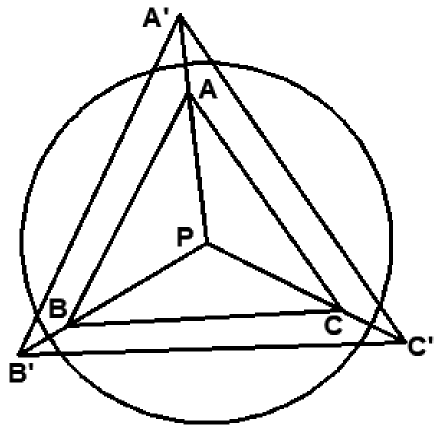
$$\sum_{cyc} x_1\cdot BP\cdot CP\geq\sqrt{\frac{1}{2}\sum_{cyc}(a\cdot AP)^2(y_1^2+z_1^2-x_1^2)-2T_1\cdot\omega_1}$$

$$\omega_1=\sqrt{\left(\sum_{cyc}a\cdot AP\right)\left(\sum_{cyc}a\cdot AP-2a\cdot AP\right)\left(\sum_{cyc}a\cdot AP-2b\cdot BP\right)\left(\sum_{cyc}a\cdot AP-2c\cdot PC\right)}$$

Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let A', B', C' *be the inverse of points* A, B, C *by means of a circle of center* P *and radius* $\rho=\sqrt{AP.BP.CP}$.



By the definition of inversion we have: $A'P.AP=\rho^2$ *then*

$$A'P=BP.CP\ \ (\text{and analogs})$$

By the Law of Cosines in $\Delta B'PC'$ *we have:*

$$B'C'^2=a'^2=B'P^2+C'P^2-2B'P.C'P.\cos\widehat{B'PC'}$$
$$=(CP.AP)^2+(AP.BP)^2-2(CP.AP).(AP.BP).\cos\widehat{B'PC'}=$$
$$=AP^2.\left(BP^2+CP^2-2BP.CP.\cos\widehat{BPC}\right)=AP^2.a^2.$$

$$\text{Then: } a'=a.AP\ \ (\text{and analogs})$$

Let F' *be the area of* $\Delta A'B'C'$. *From Heron's formula we have:*

$$F' = \frac{1}{4}\sqrt{(a'+b'+c')(-a'+b'+c')\left(a'-b'+c'\right)\left(a'+b'-c'\right)} = \frac{\omega_1}{4}.$$

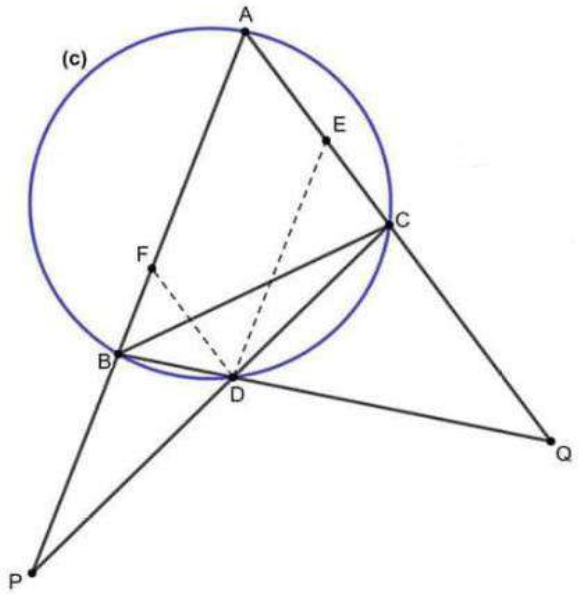
Now by Bottema's inequality for triangles $\Delta A'B'C'$ *and* $\Delta X_1Y_1Z_1$,

and for any point $P \in Int(\Delta ABC)$,

$$We\ have:\ x_1.A'P + y_1.B'P + z_1.C'P \geq \sqrt{\frac{1}{2}\sum_{cyc} a'^2(y_1^2 + z_1^2 - x_1^2) + 8F'T_1}$$

$$Therefore,\ x_1.BP.CP + y_1.CP.AP + z_1.AP.BP$$
$$\geq \sqrt{\frac{1}{2}\sum_{cyc}(a.AP)^2(y_1^2 + z_1^2 - x_1^2) + 2T_1.\omega_1}.$$

593.



$\Delta ABC, (c)$ −circumcircle, $D \in \left(\widehat{BC}\right), CD \cap AB = \{P\}, BD \cap AC = \{Q\}$,

$DE \parallel AB, DF \parallel AC$. Prove that:

$$\frac{AB}{AQ} - \frac{FB}{AE} + \frac{AC}{AP} - \frac{EC}{AF} = 0$$

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by Jose Ferreira Queiroz-Olinda-Brazil

Using new criterion for cyclic quadrilateral $NCCQ_2$ (Thanasis Gakopoulos). We have:

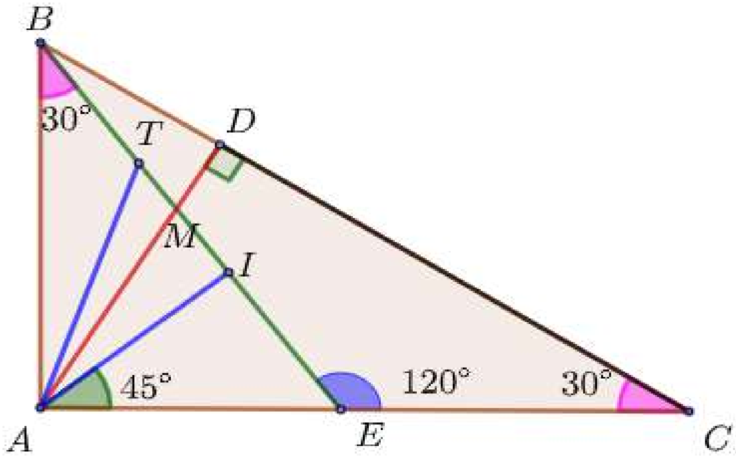
Let $\theta = \measuredangle PAQ$.

$$\cos\theta = \frac{1}{2}\left(\frac{FB}{DE} + \frac{EC}{AF}\right); (1)$$

We know that: $\cos\theta = \frac{1}{2}\left(\frac{DB}{AQ} + \frac{AC}{AP}\right); (2)$

Then: $\frac{FB}{DE} + \frac{EC}{AF} = \frac{AB}{AQ} + \frac{AC}{AP}$

Therefore $\frac{AB}{AQ} - \frac{FB}{AE} + \frac{AC}{AP} - \frac{EC}{AF} = 0$

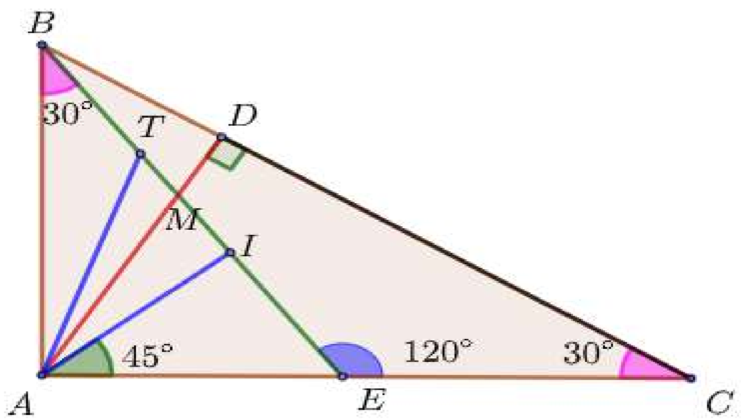


594.

If ABC is a triangle with $\mu(\widehat{A}) = \frac{\pi}{2}, \mu(\widehat{C}) = \frac{\pi}{6}, AD$ is altitude from A, I incenter and T midpoint of BI, then prove that AT is angle bisector of $\sphericalangle BAD$.

Proposed by Neculai Stanciu-Romania

Solution 1 by Adrian Popa-Romania



I −incenter$\Rightarrow \mu(\sphericalangle IAE) = 45°, \mu(\sphericalangle IBA) = \frac{B}{2} = 30°$

Let be $\{M\} = AD \cap BI$ and $T_1 \in BI$ such that $\mu(\sphericalangle T_1AM) = 15°$.

$$\sphericalangle MAE = \sphericalangle DAC = 60°, \sphericalangle IAE = 45° \Rightarrow \sphericalangle MAI = 15°$$

$\sphericalangle BAT_1 = 15° \Rightarrow$ in $\Delta ABE(\widehat{A} = 90°)$: $\sphericalangle BEA = 60°$

In ΔAIE: $\sphericalangle AIE = 75° \Rightarrow \sphericalangle T_1IA = 105°$

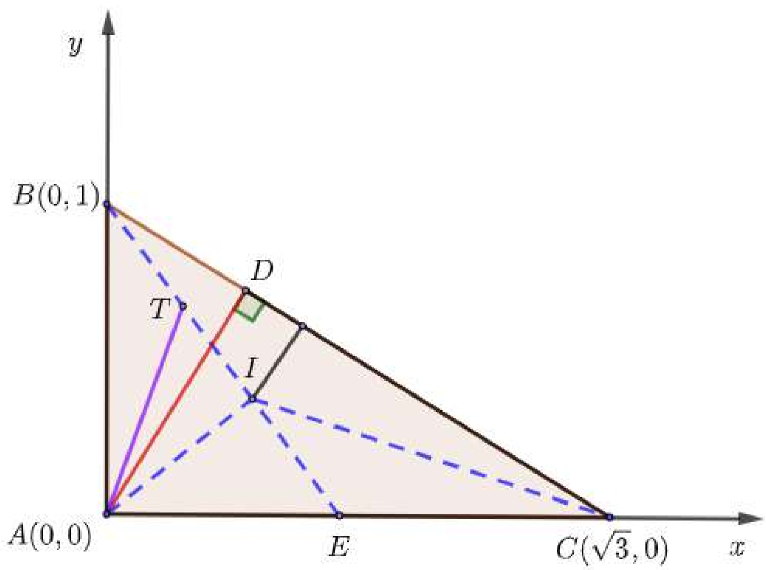
Using Law of sines in ΔAT_1I: $\frac{T_1I}{\sin 30°} = \frac{AT_1}{\sin 105°}$

$$T_1I = \frac{AT_1 \cdot \sin 30°}{\sin 105°} = \frac{AT_1 \cdot \sin 30°}{\sin 105°}$$

In ΔABT_1: $\frac{BT_1}{\sin 15°} = \frac{AT_1}{\sin 30°} \Rightarrow BT_1 = \frac{AT_1 \cdot \sin 15°}{\sin 30°} \Rightarrow T_1I = BT_1 \Rightarrow T = T_1$

So, AT −inner bisector of $\sphericalangle BAD$.

Solution 2 by Benny Le Van-Ho Chi Minh-Vietnam



$$a = BC = 2, b = CA = \sqrt{3}, c = AB = 1$$

Since $\sphericalangle BAD = \sphericalangle ACB = 30°$. We shall prove that:

$\sphericalangle BAT = 15°$. As I is the incenter of ΔABC, we get

$$a\overrightarrow{AI} + b\overrightarrow{BI} + c\overrightarrow{CI} = 0 \Leftrightarrow 2(x_I, y_I) + \sqrt{3}(x_I, y_I - 1) + (x_I - \sqrt{3}, y_I) = 0$$

$$\begin{cases} (3+\sqrt{3})x_I = \sqrt{3} \\ (3+\sqrt{3})y_I = \sqrt{3} \end{cases} \Rightarrow x_I = y_I = \frac{\sqrt{3}-1}{2}$$

T is midpoint of BI, then $I\left(\frac{\sqrt{3}-1}{4}, \frac{\sqrt{3}+1}{4}\right)$. We obtain:

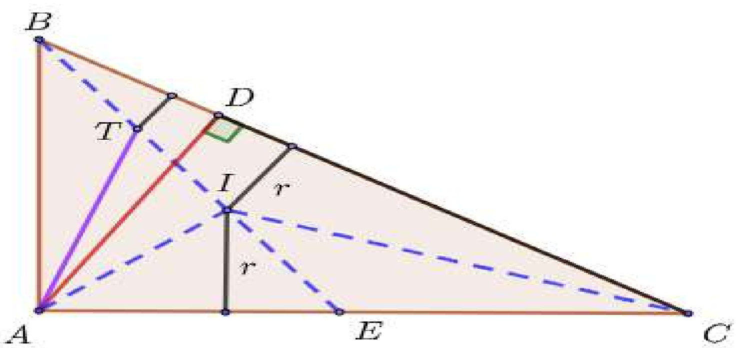
$$\overrightarrow{AB} = (0,1) \text{ and } \overrightarrow{AT} = \left(\frac{\sqrt{3}-1}{4}; \frac{\sqrt{3}+1}{4}\right)$$

$$|\overrightarrow{AB}\} = 1, |\overrightarrow{AT}| = \frac{\sqrt{2}}{2} \text{ and } \overrightarrow{AB} \cdot \overrightarrow{AT} = \frac{\sqrt{3}+1}{4}$$

$$\text{Therefore: } \cos(\widehat{BAT}) = \frac{\overrightarrow{AB} \cdot \overrightarrow{AT}}{|\overrightarrow{AB}| \cdot |\overrightarrow{AT}|} = \frac{\sqrt{6}+\sqrt{2}}{4} \Rightarrow \measuredangle BAT = 15^\circ$$

Therefore, AT is the inner bisector of $\measuredangle BAD$.

Solution 3 by Geanina Tudose-Romania



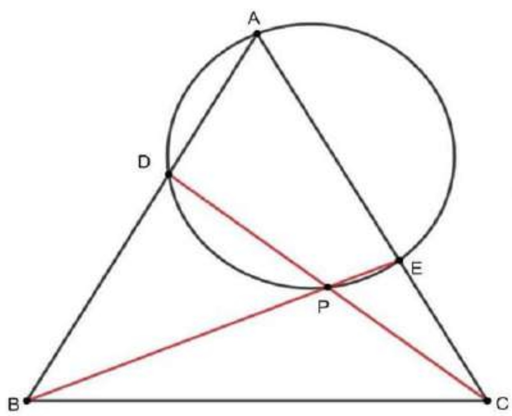
$$\text{In } \Delta BIE: |T, BC| = \frac{IE}{2} = \frac{r}{2}; (1)$$

$$\Delta ABD \sim \Delta CB \Rightarrow \frac{AB}{CB} = \frac{BD}{AB} = \frac{AD}{CA} = \frac{1}{2}, r_{ABD} = \frac{1}{2} r_{ABC} = \frac{r}{2}; (2)$$

Since $T \in Bis(\measuredangle ABD)$ and from (1),(2) we get

T −incenter in $\Delta ABD \Rightarrow AT$ is the inner bisector of $\measuredangle BAD$.

595.

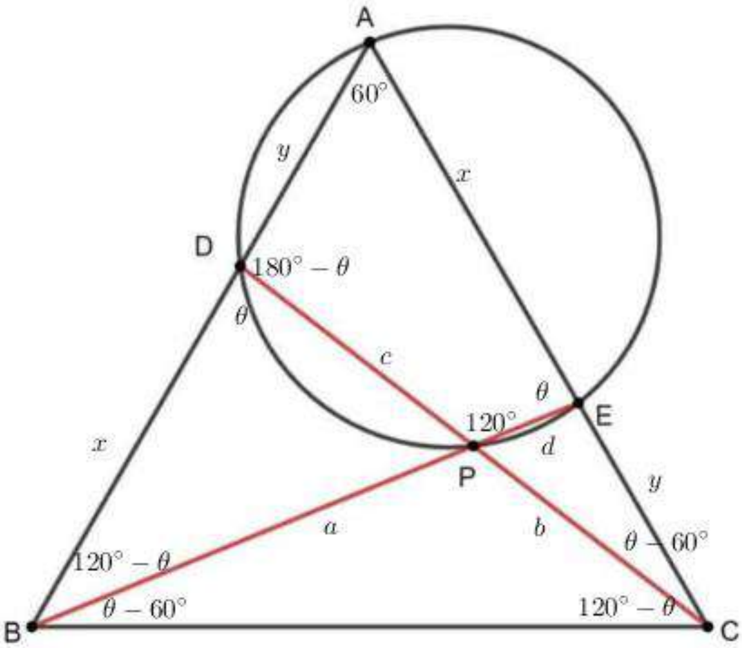


ΔABC equilateral. Prove that:

$$\frac{PB}{PC}=\sqrt{\frac{PD}{PE}}=\frac{AE}{AD}$$

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by Jose Ferreira Queiroz-Olinda-Brazil



$$\Delta BCD \equiv \Delta ABE \Rightarrow AE = BD$$

$$\Delta BEC \equiv \Delta ADC \Rightarrow AD = EC$$

$$BP = a, EP = d, DP = c, CP = b$$

$$AB = l \Rightarrow x + y = l$$

Using Law of sines, we have:

ΔBDP:

$$\frac{a}{\sin\theta}=\frac{x}{\sin 60^\circ}=\frac{c}{\sin(120^\circ-\theta)};(1)$$

ΔCEP:

$$\frac{b}{\sin(180^\circ-\theta)}=\frac{y}{\sin 60^\circ}=\frac{d}{\sin(\theta-60^\circ)};(2)$$

$$\sin(180^\circ-\theta)=\sin\theta \Rightarrow \frac{a}{x}=\frac{b}{y}\Rightarrow \frac{a}{b}=\frac{x}{y}$$

Hence,

$$\frac{PB}{PC}=\frac{AE}{AD}$$

$$\Delta ABE: \frac{l}{\sin\theta}=\frac{a+d}{\sin 60^\circ}=\frac{x}{\sin(120^\circ-\theta)};(3)$$

$$\Delta ACD: \frac{l}{\sin(180^\circ-\theta)}=\frac{c+d}{\sin 60^\circ}=\frac{y}{\sin(\theta-60^\circ)};(4)$$

Using (1) and (4):

$$\frac{a}{c}=\frac{l}{x}\Rightarrow x=\frac{cl}{a}$$

Using (2) and (3):

$$\frac{b}{d}=\frac{l}{y}\Rightarrow y=\frac{dl}{b}$$

$$\frac{a}{b}=\frac{x}{y}=\frac{\frac{cl}{a}}{\frac{dl}{b}}=\frac{bcl}{adl}\Rightarrow\frac{a^2}{b^2}=\frac{c}{d}\Rightarrow\frac{a}{b}=\sqrt{\frac{c}{d}}$$

Therefore,

$$\frac{PB}{PC}=\sqrt{\frac{PD}{PE}}=\frac{AE}{AD}$$

596. a,b,c,r,s,a',b',c',r',s' **−sides, inradii, semiperimeter in** ΔABC **and** $\Delta A'B'C'$. **If** $a'=\sqrt{a},\ \ b'=\sqrt{b},\ \ c'=\sqrt{c}$ **then :** $r'\ge\frac{3r}{\sqrt{6s}}$.

Proposed by Mehmet Şahin-Ankara-Turkiye

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let F' be the area of $\Delta A'B'C'$. We have :

$$16F'^2=2\left(a'^2b'^2+b'^2c'^2+c'^2a'^2\right)-\left(a'^4+b'^4+c'^2\right)$$
$$=2(ab+bc+ca)-(a^2+b^2+c^2)=$$

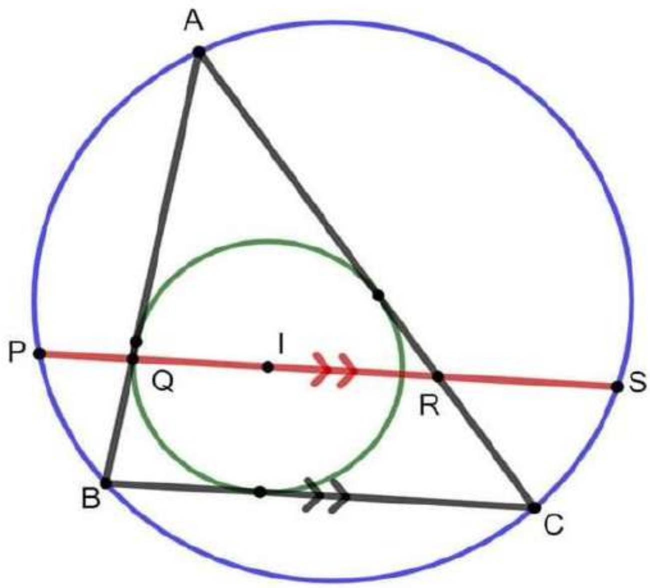
$$=2(s^2+r^2+4Rr)-2(s^2-r^2-4Rr)=4r(4R+r)\ \ then:\ \ 2F'=\sqrt{r(4R+r)}.$$

$$We\ have:\ \ r'=\frac{2F'}{2s'}=\frac{2F'}{a'+b'+c'}=\frac{\sqrt{r(4R+r)}}{\sqrt{a}+\sqrt{b}+\sqrt{c}}\overset{Euler}{\ge}\frac{\sqrt{r(4.2r+r)}}{\sqrt{a}+\sqrt{b}+\sqrt{c}}$$
$$=\frac{3r}{\sqrt{a}+\sqrt{b}+\sqrt{c}}$$

$$By\ CBS\ inequality\ we\ have:\ \sqrt{a}+\sqrt{b}+\sqrt{c}\le\sqrt{3(a+b+c)}=\sqrt{6s}.$$

$$Therefore,\ \ r'\ge\frac{3r}{\sqrt{6s}}.$$

597.

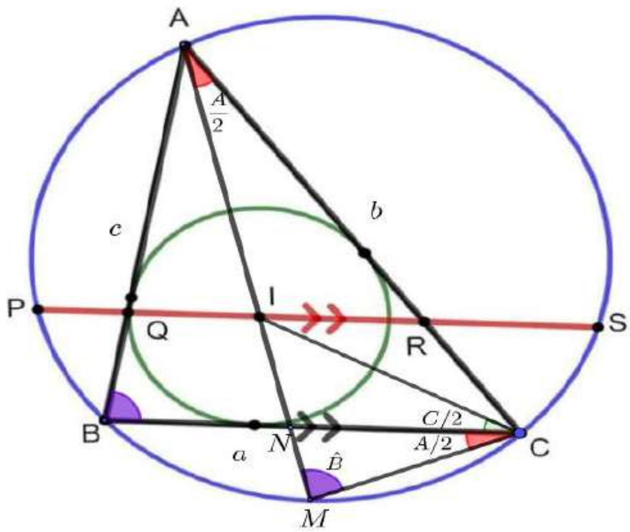


Prove that:

$$\frac{AC}{IS} - \frac{PQ}{QI} = 1$$

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by Jose Ferreira Queiroz-Olinda-Brazil



$$\frac{AC}{IS} = \frac{PQ}{QI} + 1 \Rightarrow AC - IS = \frac{PI}{QI} \Rightarrow AC \cdot QI = PI \cdot IS$$

We know that:

$$\frac{AN}{AI} = \frac{2s}{b+c}, AI = \frac{\sqrt{bcs(s-a)}}{s}, CI = \frac{\sqrt{abs(s-c)}}{s}$$

$$F=\frac{abc}{4R},\cos\frac{B}{2}=\sqrt{\frac{s(s-b)}{ac}},BN=\frac{ac}{b+c}$$

$$\Delta ABN\sim\Delta AQI:\frac{AN}{AI}=\frac{BN}{QI}\Rightarrow\frac{ac}{(b+c)QI}=\frac{2s}{b+c},\qquad QI=\frac{ac}{2s}$$

Now, $PI\cdot IS=AC\cdot QI=\frac{b\cdot ac}{2s}$

We need to prove that: $PI\cdot IS=\frac{abc}{2s}$

In ΔIMC we have:

$$\widehat{CMI}=\hat{B}\text{ and }\widehat{MCI}=\frac{A}{2}+\frac{C}{2}=\frac{\pi-B}{2}=\frac{\pi}{2}-\frac{B}{2}$$

Using Law of sines:

$$\frac{IC}{\sin B}=\frac{IM}{\sin\left(\frac{\pi}{2}-\frac{B}{2}\right)}\Rightarrow\frac{\frac{\sqrt{abs(s-c)}}{s}}{\frac{b}{2R}}=\frac{IM}{\sqrt{\frac{s(s-b)}{ac}}}$$

$$IM=\frac{2R}{bc}\sqrt{bc(s-b)(s-c)}$$

$$AI\cdot IM=PI\cdot IS$$

$$\frac{\sqrt{bcs(s-a)}}{s}\cdot\frac{2R}{bc}\sqrt{bc(s-b)(s-c)}=PI\cdot IS$$

$$PI\cdot IS=\frac{2R}{bcs}\sqrt{b^2c^2s(s-a)(s-b)(s-c)}=\frac{2R}{s}\cdot F=\frac{2R}{s}\cdot\frac{abc}{4R}$$

$$PI\cdot IS=\frac{abc}{2s}$$

Therefore,

$$\frac{AC}{IS}-\frac{PQ}{QI}=1$$

598. In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{\sqrt{b^4+c^4}}{b^2-\lambda bc+c^2}\le\frac{3\sqrt{R^2-2r^2}}{(2-\lambda)r},\ \ \lambda<2$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:\ \frac{\sqrt{b^4+c^4}}{b^2-\lambda bc+c^2}=\frac{\sqrt{\frac{b^2}{c^2}+\frac{c^2}{b^2}}}{\frac{b}{c}-\lambda+\frac{c}{b}}=\frac{\sqrt{\left(\frac{b}{c}+\frac{c}{b}\right)^2-2}}{\frac{b}{c}+\frac{c}{b}-\lambda}.$$

$$By\ Bandila's\ inequality\ we\ have:\ \frac{b}{c}+\frac{c}{b}\le\frac{R}{r}$$

$$And\ by\ AM-GM\ inequality\ we\ have:\ \frac{b}{c}+\frac{c}{b}\ge 2>\lambda$$

$$Then\ we\ get:\ \frac{\sqrt{b^4+c^4}}{b^2-\lambda bc+c^2}\le\frac{\sqrt{\left(\frac{R}{r}\right)^2-2}}{2-\lambda}=\frac{\sqrt{R^2-2r^2}}{(2-\lambda)r}.$$

$$Similarly\ we\ have:\ \frac{\sqrt{c^4+a^4}}{c^2-\lambda ca+a^2}\le\frac{\sqrt{R^2-2r^2}}{(2-\lambda)r}\ \&\ \frac{\sqrt{a^4+b^4}}{a^2-\lambda ab+b^2}\le\frac{\sqrt{R^2-2r^2}}{(2-\lambda)r}.$$

Summing up these inequalities yields the desired inequality.

Equality holds iff ΔABC is equilateral.

599. If x,y,z are the lengths of the three sides of a triangle and

$$x+y+z=1,$$

$$x<2y,y<2z,z<2x.\ \text{Prove that}:$$

$$\sqrt{\frac{x}{2y-x}}+\sqrt{\frac{y}{2z-y}}+\sqrt{\frac{z}{2x-z}}\ge\frac{1}{\sqrt{3xyz}}.$$

Proposed by Hikmat Mammadov-Azerbaijan

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$By\ H\ddot{o}lder's\ inequality\ we\ have:$$

$$\left(\sqrt{\frac{x}{2y-x}}+\sqrt{\frac{y}{2z-y}}+\sqrt{\frac{z}{2x-z}}\right)^2[x^2(2y-x)+y^2(2z-y)+z^2(2x-z)]$$

$$\ge(x+y+z)^3=1$$

$$Then:\ \sqrt{\frac{x}{2y-x}}+\sqrt{\frac{y}{2z-y}}+\sqrt{\frac{z}{2x-z}}\ge\frac{1}{\sqrt{2(x^2y+y^2z+z^2x)-(x^3+y^3+z^3)}}$$

$$So\ it\ suffices\ to\ prove\ that:\ x^3+y^3+z^3+3xyz\ge 2(x^2y+y^2z+z^2x)$$

$$Using\ Ravi's\ substitution:\ x=b+c,\ y=c+a,\ z=a+b,\ a,b,c>0.$$

$$So\ we\ need\ to\ prove:\sum_{cyc}(a+b)^3+3\prod_{cyc}(a+b)\ge 2\sum_{cyc}(a+b)^2(b+c)$$

$$\Leftrightarrow\ a^2b+b^2c+c^2a\ge 3abc\ \ which\ is\ true\ by\ AM-GM\ inequality.$$

$$So\ the\ proof\ is\ completed.\ Equality\ holds\ iff\ x=y=z=\frac{1}{3}.$$

600. a,b,c,r,R,a',b',c',r',R' −sides, inradii, circumradii in ΔABC and $\Delta A'B'C'$.

$$\text{If } a'=\sqrt{a},\ \ b'=\sqrt{b},\ \ c'=\sqrt{c}\ \text{then}:\ \frac{72r'^3}{R}\le\sum_{cyc}\frac{h_a}{h_{a'}}\le\frac{9R'^3}{R}$$

Proposed by Mehmet Şahin-Ankara-Turkiye

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Let\ F,s,F',s'\ be\ the\ area,semiperimeter\ of\ \Delta ABC\ and\ \Delta A'B'C'.\ We\ have:$$

$$\frac{h_a}{h_{a'}}=\frac{bc}{2R}\cdot\frac{2R'}{b'c'}=\frac{R'}{R}\cdot\frac{bc}{\sqrt{b}.\sqrt{c}}=\frac{R'}{R}.\sqrt{bc}=\frac{R'}{R}.b'c'\ \ (and\ analogs)$$

$$Then:\sum_{cyc}\frac{h_a}{h_{a'}}=\frac{R'}{R}\sum_{cyc}b'c'\le\frac{R'}{R}\sum_{cyc}a'^2\overset{Leibniz}{\le}\frac{R'}{R}.9R'^2=\frac{9R'^3}{R}.$$

$$And:\sum_{cyc}\frac{h_a}{h_{a'}}=\frac{R'}{R}\sum_{cyc}b'c'=\frac{R'\left(s'^2+4R'r'+r'^2\right)}{R}\ge$$

$$\overset{Euler\ \&\ Mitrinovic}{\ge}\frac{2r'\left(27r'^2+4.2r'.r'+r'^2\right)}{R}=\frac{72r'^3}{R}.$$

$$Therefore,\ \frac{72r'^3}{R}\le\sum_{cyc}\frac{h_a}{h_{a'}}\le\frac{9R'^3}{R}.$$

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru