JÓZSEF WILDT INTERNATIONAL MATHEMATICAL COMPETITION

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Problem W6.

Evaluate:

$$\sum_{n=0}^{\infty} \frac{2(5n+2)}{(n+1)(2n+1)(4n+1)}$$

Solution *by Florică Anastase and Adrian Popa, Romania*

$$\frac{2(5n+2)}{(n+1)(2n+1)(4n+1)} = \frac{10n+4}{(n+1)(2n+1)((4n+1)} = \frac{A}{n+1} + \frac{B}{2n+1} + \frac{C}{4n+1}$$

$$n = -1 \Rightarrow A = -2;\ n = -\frac{1}{2} \Rightarrow B = 2;\ n = -\frac{1}{4} \Rightarrow C = 4$$

$$\sum_{n=0}^{\infty} \frac{10n+4}{(n+1)(2n+1)(4n+1)} = -\sum_{n=0}^{\infty} \frac{2}{n+1} + \sum_{n=0}^{\infty} \frac{2}{2n+1} + \sum_{n=0}^{\infty} \frac{4}{4n+1} =$$

$$= -2\sum_{n=0}^{\infty} \frac{1}{n+1} + \sum_{n=0}^{\infty} \frac{1}{1+\frac{1}{2}} + \sum_{n=0}^{\infty} \frac{n}{n+\frac{1}{4}}$$

It is known that:

$$\psi(s+1) = -\gamma + \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+s}\right)$$

Hence,

$$\psi(2) = -\gamma + \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2}\right) \Rightarrow -\sum_{n=1}^{\infty} \frac{1}{n+2} = \psi(2) + \gamma - \sum_{n=1}^{\infty} \frac{1}{n}$$

$$-2\sum_{n=1}^{\infty} \frac{1}{n+1} = 2\psi(2) + 2\gamma - 2\sum_{n=1}^{\infty} \frac{1}{n}$$

$$\psi\left(1+\frac{1}{2}\right) = -\gamma + \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+\frac{1}{2}}\right) \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n+\frac{1}{2}} = -\gamma - \psi\left(\frac{3}{2}\right) + \sum_{n=1}^{\infty} \frac{1}{n}$$

$$\psi\left(1+\frac{1}{4}\right) = -\gamma + \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+\frac{1}{4}}\right) \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n+\frac{1}{4}} = -\gamma - \psi\left(\frac{5}{4}\right) + \sum_{n=1}^{\infty} \frac{1}{n}$$

Therefore,

$$\sum_{n=0}^{\infty} \frac{2(5n+2)}{(n+1)(2n+1)(4n+1)} = 2\psi(2) - \psi\left(\frac{3}{2}\right) - \psi\left(\frac{5}{4}\right) + 4$$

Now, we have:

$$\psi(1-x) - \psi(x) = \pi\cot(\pi x) \,\text{and}\, \psi(1+x) = \psi(x) + \frac{1}{x}$$

$$\psi\left(1+\frac{1}{2}\right) = \psi\left(\frac{3}{2}\right) = \psi\left(\frac{1}{2}\right) + 2 = -2\log 2 - \gamma + 2$$

$$\psi\left(1+\frac{1}{4}\right) = \psi\left(\frac{5}{4}\right) = \psi\left(\frac{1}{4}\right) + 4 = -\frac{\pi}{2} - 3\log 2 - \gamma$$

$$\psi(2) = \psi(1) + 1 = -\gamma + 1$$

$$\sum_{n=0}^{\infty} \frac{2(5n+2)}{(n+1)(2n+1)(4n+1)} = 5\log 2 + \frac{\pi}{2} + 4$$

Problem W13.

Let r_a, r_b and r_c be the length of the radii of excircles of a triangle ΔABC with circumradiu R and inradius r. Let a, b and c be the length of the sides of ΔABC. Prove that:

$$\frac{a^2}{r_a r_b} + \frac{b^2}{r_b r_c} + \frac{c^2}{r_a r_b} \geq 2$$

Solution *by Florică Anastase, Romania*

Using Bergstrom's Inequality, we have:

$$\frac{a^2}{r_a r_b} + \frac{b^2}{r_b r_c} + \frac{c^2}{r_a r_b} \overset{Bergstrom}{\geq} \frac{(a+b+c)^2}{r_b r_c + r_c r_a + r_a r_b} \geq$$

$$\geq \frac{4s^2}{r_b r_c + r_c r_a + r_a r_b};\ (1)$$

Now, using identities $r_a = \frac{F}{s-a}, r_c = \frac{F}{s-b}$ and $r_c = \frac{F}{s-c}$, it follows that:

$$r_b r_c + r_c r_a + r_a r_b = \frac{F^2}{(s-b)(s-c)} + \frac{F^2}{(s-c)(s-a)} + \frac{F^2}{(s-a)(s-b)} =$$

$$= F^2 \cdot \frac{(s-a+s-b+s-c)}{(s-a)(s-b)(s-c)} = s^2;\ (2)$$

From (1) and (2), we get:

$$\frac{a^2}{r_a r_b} + \frac{b^2}{r_b r_c} + \frac{c^2}{r_a r_b} \geq 2$$

Problem W 15.

Let be the sequences $(a_n)_{n\geq 1}$, $(b_n)_{n\geq}$:

$$a_n = \sum_{k=1}^{n} \arctan\frac{1}{k^2-k+1} \text{ and } \lim_{n\to\infty} \frac{b_{n+1}}{n\cdot b_n} = b \in \mathbb{R}_+$$

Compute:

$$\lim_{n\to\infty}\left(\frac{\pi}{2}-a_n\right)\sqrt[n]{b_n}$$

Solution *by Floricǎ Anastase, Romania*

Using the well-known formulae

$$\arctan(\alpha)-\arctan(\beta)=\arctan\left(\frac{\alpha-\beta}{1+\alpha\beta}\right)$$

we get:

$$\arctan\frac{1}{k^2-k+1}=\arctan\frac{\frac{1}{k-1}-\frac{1}{k}}{1+\frac{1}{k(k-1)}}=\arctan\frac{1}{k-1}-\arctan\frac{1}{k}$$

Adding from $k=1$ to n, to obtain:

$$a_n=\sum_{k=1}^{n}\arctan\frac{1}{k^2-k+1}=\sum_{k=1}^{n}\left(\arctan\frac{1}{k-1}-\arctan\frac{1}{k}\right)=\frac{\pi}{2}-\arctan\frac{1}{n}$$

$$\lim_{n\to\infty}\left(\frac{\pi}{2}-a_n\right)\sqrt[n]{b_n}=\lim_{n\to\infty}\left(\arctan\frac{1}{n}\right)\cdot\sqrt[n]{b_n}=$$

$$=\lim_{n\to\infty}\frac{\arctan\frac{1}{n}}{\frac{1}{n}}\cdot\frac{\sqrt[n]{b_n}}{n};\ (1)$$

Now, we have:

$$\lim_{n\to\infty}\frac{\sqrt[n]{b_n}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{b_n}{n^n}}\overset{C-D'A}{=}\lim_{n\to\infty}\frac{b_{n+1}}{(n+1)^{n+1)}}\cdot\frac{n^n}{b_n}=$$

$$=\lim_{n\to\infty}\frac{b_{n+1}}{n\cdot b_n}\cdot\left(\frac{n}{n+1}\right)^{n+1}=\frac{b}{e};\ (2)$$

From (1) and (2), it follows that:

$$\lim_{n\to\infty}\left(\frac{\pi}{2}-a_n\right)\sqrt[n]{b_n}=\frac{b}{e}$$

Problem W 17.

If $x,y,z>0$ and $A_1B_1C_1, A_2B_2C_2$ are two triangles with the circumradius R_1, respectively R_2 then holds the following inequality:

$$\frac{x+y}{z\sqrt{a_1a_2}}+\frac{y+z}{x\sqrt{b_1b_2}}+\frac{z+x}{y\sqrt{c_1c_2}}\geq\frac{2\sqrt{3}}{\sqrt{R_1R_2}}$$

Solution *by Floricǎ Anastase, Romania*

$$\frac{x+y}{z\sqrt{a_1a_2}}+\frac{y+z}{x\sqrt{b_1b_2}}+\frac{z+x}{y\sqrt{c_1c_2}}\overset{AGM}{\geq}\frac{2\sqrt{xy}}{z\sqrt{a_1a_2}}+\frac{2\sqrt{yz}}{x\sqrt{b_1b_2}}+\frac{2\sqrt{zx}}{y\sqrt{c_1c_2}}\overset{AGM}{\geq}$$

$$\geq 3\cdot\sqrt[3]{\frac{8xyz}{xyz\sqrt{a_1a_2a_3}\cdot\sqrt{b_1b_2b_3}}}=\frac{6}{\sqrt[3]{\sqrt{a_1a_2a_3}\cdot\sqrt{b_1b_2b_3}}}=$$

$$= \frac{6}{\sqrt[3]{\sqrt{4R_1F_1}\sqrt{4R_2F_2}}} = \frac{6}{\sqrt[3]{\sqrt{4R_1r_1s_1}\sqrt{4R_2r_2s_2}}} \overset{Euler}{\geq}$$

$$\geq \frac{6}{\sqrt[3]{\sqrt{4R_1\frac{R_1}{2}s_1}\sqrt{4R_2\frac{R_2}{2}s_2}}} = \frac{6}{\sqrt[3]{\sqrt{2R_1^2s_1}\sqrt{2R_2^2s_2}}} \overset{Mitrinovic}{\geq}$$

$$\geq \frac{6}{\sqrt[3]{\sqrt{2R_1^2\left(\frac{3\sqrt{3}R_1}{2}\right)}\sqrt{2R_2^2\left(\frac{3\sqrt{3}R_2}{2}\right)}}} =$$

$$= \frac{6}{\sqrt[6]{3\sqrt{3}R_1^3\cdot 3\sqrt{3}R_2^3}} = \frac{2\sqrt{3}}{\sqrt{R_1R_2}}$$

Problem W26.

If $a, b, c > 0$ then:

$$(a^a\cdot b^b\cdot c^c)^{\frac{1}{a+b+c}} + (a^b\cdot b^c\cdot c^a)^{\frac{1}{a+b+c}} + (a^c\cdot b^a\cdot c^b)^{\frac{1}{a+b+c}} \leq a+b+c$$

Solution *by Florică Anastase, Romania*

Let us denote: $m = a+b+c$, then:

$$(a^a\cdot b^b\cdot c^c)^{\frac{1}{a+b+c}} + (a^b\cdot b^c\cdot c^a)^{\frac{1}{a+b+c}} + (a^c\cdot b^a\cdot c^b)^{\frac{1}{a+b+c}} =$$

$$= a^{\frac{a}{m}}\cdot b^{\frac{b}{m}}\cdot c^{\frac{c}{m}} + a^{\frac{b}{m}}\cdot b^{\frac{c}{m}}\cdot c^{\frac{a}{m}} + a^{\frac{c}{m}}\cdot b^{\frac{a}{m}}\cdot c^{\frac{b}{m}} \overset{Weighted\,AGM}{\leq}$$

$$\leq \frac{a^2+b^2+c^2}{m} + \frac{ab+bc+ca}{m} + \frac{ab+bc+ca}{m} = \frac{(a+b+c)^2}{m} = a+b+c$$

Equality holds for $a = b = c$.

Problem W38.

In $\Delta ABC; a, b, c \in (0,1)$. Prove that:

$$\frac{(s-2)^2+r^2+4rR-1}{(s-1)^2+r^2+4R(r-s)} \geq \frac{3}{\sqrt[3]{(1-a)(1-b)(1-c)}}$$

Solution *by Florică Anastase-Romania*

Let $f(x) = x^3 - 2sx^2 + (s^2+r^2+4rR)x - 4srR$ then $f(x) = (x-a)(x-b)(x-c)$ and $f'(x) = 3x^2 - 4sx + s^2 + r^2 + 4rR$. Hence, we have:

$$\frac{f'(x)}{f(x)} = \frac{1}{x-a} + \frac{1}{x-b} + \frac{1}{x-c}$$

$$\frac{f'(1)}{f(1)} = \frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} \geq \frac{3}{\sqrt[3]{(1-a)(1-b)(1-c)}};\ (I)$$

$$\frac{f'(x)}{f(x)}\frac{3x^2-4sx+s^2+r^2+4rR}{x^3-2sx^2+(s^2+r^2+4rR)x-4srR}$$

$$\frac{f'(1)}{f(1)}=\frac{3-4s+s^2+r^2+4rR}{1-2s+s^2+r^2+4Rr-4Rrs}=\frac{(s-2)^2+r^2+4rR-1}{(s-1)^2+r^2+4R(r-s)};(II)$$

From (I) and (II), it follows that:

$$\frac{(s-2)^2+r^2+4rR-1}{(s-1)^2+r^2+4R(r-s)}\geq\frac{3}{\sqrt[3]{(1-a)(1-b)(1-c)}}$$

Problem W40.

In a tetrahedron $ABCD$ let be r the radius of inscribed sphere and r_A, r_B, r_C, r_D radii of exinscribed spheres. Prove that:

$$\frac{2r_A-r}{2r_A+r}+\frac{2r_B-r}{2r_B+r}+\frac{2r_C-r}{2r_C+r}+\frac{2r_D-r}{2r_D+r}\geq\frac{12}{5}$$

Solution *by Florică Anastase-Romania*

We known that:

$$\sum_{cyc}\frac{1}{r_A}=\frac{2}{R}$$

Then,

$$\sum_{cyc}\frac{1}{r_A}-\frac{1}{r_A}=\frac{2}{r}-\frac{1}{r_A}=\frac{2r_A-r}{rr_A}$$

$$\sum_{cyc}\frac{1}{r_A}+\frac{1}{r_A}=\frac{2}{r}+\frac{1}{r_A}=\frac{2r_A+r}{rr_A}$$

Hence, it follows that:

$$4+\sum_{cyc}\frac{2r_A-r}{2r_A+r}=4+\sum_{cyc}\frac{\frac{2}{r}-\frac{1}{r_A}}{\frac{2}{r}+\frac{1}{r_A}}=\sum_{cyc}\left(1+\frac{\frac{2}{r}-\frac{1}{r_A}}{\frac{2}{r}+\frac{1}{r_A}}\right)=$$

$$=\frac{4}{r}\sum_{cyc}\frac{1}{\frac{2}{r}+\frac{1}{r_A}}\overset{Bergstrom}{\geq}\frac{4}{r}\cdot\frac{(1+1+1+1)^2}{\sum_{cyc}\left(\frac{2}{r}+\frac{1}{r_A}\right)}=\frac{4\cdot 16}{r\left(\frac{8}{r}+\sum_{cyc}\frac{1}{r_a}\right)}=$$

$$=\frac{4\cdot 16}{r\left(\frac{8}{r}+\frac{2}{r}\right)}=\frac{32}{5}$$

Therefore,

$$4+\sum_{cyc}\frac{2r_A-r}{2r_A+r}\geq\frac{32}{5}$$

So, we get:

$$\frac{2r_A-r}{2r_A+r}+\frac{2r_B-r}{2r_B+r}+\frac{2r_C-r}{2r_C+r}+\frac{2r_D-r}{2r_D+r}\geq\frac{12}{5}$$