Special function and q-series

Introduction

In this article, we will form various identities related to q-series and a special function.

Prerequisites

Definition 1

If $|q| < 1$ and $a \neq 0$, then,

$$(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k)$$

where

$$(q)_\infty \coloneqq (q;q)_\infty = \prod_{k=1}^{\infty}(1-q^k)$$

Definition 2

If $m, n \in \mathbb{N}$ and $1 \leq m \leq n$, then,

$$G(m,n) \coloneqq G(m,n;q) = (e^{\pi i m/n}q)_\infty$$

Definition 3

A function $\psi(q)$ is in q-closed form if it can be written as:

$$\psi(q) = \prod_{k=1}^{n} \frac{(q^{a_k})_\infty^{b_k}}{(q^{c_k})_\infty^{d_k}}$$

where, a_k, b_k, c_k and d_k are some positive integer sequences.

Properties and Special Values

Result 1

If G is defined as above, then we have,

$$\left|G(m,n)\right|^2 = \prod_{k=1}^{\infty}\left(1-2\cos\left(\frac{\pi mk}{n}\right)q^k + q^{2k}\right) \implies \left|G(n,n)\right|^2 = \frac{(q^2)_\infty^6}{(q)_\infty^2 (q^4)_\infty^2}$$

and

$$\left|G(n-m,n;-q)\right|^2 = \left|G(m,n;q)\right|^2$$

Result 2

We have,

$$\left|G(1,2)\right|^2 = \frac{(q^4)_\infty^8}{(q^2)_\infty^3 (q^8)_\infty^3}$$

$$\left|G(2,3)\right|^2 = \frac{(q^3)_\infty^4}{(q)_\infty (q^9)_\infty}$$

and

$$\left|G(1,3)\right|^2 = \frac{(q)_\infty (q^4)_\infty (q^6)_\infty^{12} (q^9)_\infty (q^{36})_\infty}{(q^2)_\infty^3 (q^3)_\infty^4 (q^{12})_\infty^4 (q^{18})_\infty^3}$$

Proofs

Proof of Result 1

From definition 2, we have,

$$G(m,n) = \prod_{k=1}^{\infty}(1-e^{\pi imk/n}q^k) \implies \left|G(m,n)\right|^2 = \prod_{k=1}^{\infty}\left|(1-e^{\pi imk/n}q^k)\right|^2$$

writing $(1-e^{\pi imk/n}q^k)$ in the form of $a+ib$ and then taking the magnitude, completes the proof of the first part.
Substituting $m=n$ in the above product, we obtain,

$$\left|G(n,n)\right|^2 = \prod_{k=1}^{\infty}(1-(-q)^k)^2 = (-q)_\infty^2$$

From definition 1, it can be shown that,

$$(-q)_\infty = \frac{(q^2)^3_\infty}{(q)_\infty(q^4)_\infty} = \left|G(n,n)\right|$$

which completes the proof of the second part.
Replacing m by $n-m$ and q by $-q$ in the first product, completes the proof of Result 1.

Proof of Result 2

Observe that,

$$\left|G(1,2)\right|^2 = \prod_{k=1}^{\infty}\left(1 - 2\cos\left(\frac{\pi k}{2}\right)q^k + q^{2k}\right)$$

the above product can be written as,

$$\left|G(1,2)\right|^2 = \prod_{k=1}^{\infty}\left(1-2\cos(\pi k)q^{2k}+q^{4k}\right)\left(1-2\cos\left(\frac{\pi(2k-1)}{2}\right)q^{2k-1}+q^{4k-2}\right)$$

which can be simplified further as,

$$\left|G(1,2)\right|^2 = \prod_{k=1}^{\infty}\left(1 - 2(-1)^k q^{2k} + q^{4k}\right)\left(1 + q^{4k-2}\right)$$

breaking the first product into two parts, we obtain,

$$\left|G(1,2)\right|^2 = \prod_{k=1}^{\infty}\left(1 - 2q^{4k} + q^{8k}\right)\left(1 + 2q^{4k-2} + q^{8k-4}\right)\left(1 + q^{4k-2}\right)$$

using simple algebraic identities, we have,

$$\left|G(1,2)\right|^2 = \prod_{k=1}^{\infty}\left(1 - q^{4k}\right)^2\left(1 + q^{4k-2}\right)^3$$

It can be shown from definition 1, that,

$$\prod_{k=1}^{\infty}\left(1 - q^{4k}\right) = (q^4)_\infty$$

and

$$\prod_{k=1}^{\infty}\left(1+q^{4k-2}\right)=\frac{(q^4)_\infty^2}{(q^2)_\infty(q^8)_\infty}$$

which completes the proof of the first part.
Observe again that,

$$\left|G(2,3)\right|^2=\prod_{k=1}^{\infty}\left(1-2\cos\left(\frac{2\pi k}{3}\right)q^k+q^{2k}\right)$$

the above product can be written, after simplification, as,

$$\left|G(2,3)\right|^2=\prod_{k=1}^{\infty}\left(1-2q^{3k}+q^{6k}\right)\left(1+q^{3k-1}+q^{6k-2}\right)\left(1+q^{3k-2}+q^{6k-4}\right)$$

above product can be written as,

$$\left|G(2,3)\right|^2=\prod_{k=1}^{\infty}\frac{\left(1-q^{3k}\right)^3\left(1-q^{9k-6}\right)\left(1-q^{9k-3}\right)\left(1-q^{9k}\right)}{\left(1-q^{3k-2}\right)\left(1-q^{3k-1}\right)\left(1-q^{3k}\right)\left(1-q^{9k}\right)}$$

it can be shown that,

$$\prod_{k=1}^{\infty}\left(1-q^{9k-6}\right)\left(1-q^{9k-3}\right)\left(1-q^{9k}\right)=(q^3)_\infty$$

and

$$\prod_{k=1}^{\infty}\left(1-q^{3k-2}\right)\left(1-q^{3k-1}\right)\left(1-q^{3k}\right)=(q)_\infty$$

thus,

$$\left|G(2,3)\right|^2=\frac{(q^3)_\infty^3(q^3)_\infty}{(q)_\infty(q^9)_\infty}$$

which completes the proof of the second part.
The proof of the last part is also based on the same lines, that is, breaking the initial product into simpler terms and finding a closed form for $\left|G(1,3)\right|^2$ using elementary algebraic identities along with some product manipulations.

Conclusion

It can be seen that for some values of m and n, $G(m,n)$ is in q-closed form. Most of the times, $G(m,n)$ can be written, after a lot of simplification, as,

$$G(m,n) = \psi(q) \prod_{j=1}^{r} \prod_{k=1}^{\infty} (1 + \lambda q^{\nu} + q^{2\nu})$$

where, $\psi(q)$ is in q-closed form, $r \in \mathbb{N}$, λ (mostly a constant after breaking the product into parts) and ν are functions of m, n, j and k. More investigation on the products of the form,

$$f(\lambda; q) = \prod_{k=1}^{\infty} (1 + \lambda q^k + q^{2k})$$

is required, where, $\lambda \in \mathbb{Z}$ and $f(q)$ is in q-closed form when $\lambda = 0, \pm 1, \pm 2$.

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