TRẦN QUỐC ANH

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1 Topic:

In a acute triangle:

$$\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\leq\frac{R+r}{r}$$

Identity occur if and only if the triangle is equilateral triangle.

2 Lemma:

In the ΔABC:

$$am_a^2+bm_b^2+cm_c^2=\frac{p(p^2+2Rr+5r^2)}{2}$$

Proof lemma:

$$\begin{aligned}am_a^2+bm_b^2+cm_c^2&=a\left(\frac{2(b^2+c^2)-a^2}{4}\right)+b\left(\frac{2(a^2+c^2)-b^2}{4}\right)+c\left(\frac{2(a^2+b^2)-c^2}{4}\right)\\&=a\left(\frac{b^2+c^2+2bc\cos A}{4}\right)+b\left(\frac{a^2+c^2+2ac\cos B}{4}\right)+c\left(\frac{a^2+b^2+2ab\cos C}{4}\right)\\&=\frac{(a+b+c)(ab+ac+bc)}{4}-\frac{3abc}{4}+\frac{abc(\cos A+\cos B+\cos C)}{2}\\&=\frac{2p(p^2+r^2+4Rr)}{4}-\frac{3}{4}.4pRr+\frac{4pRr}{2}.\left(\frac{R+r}{R}\right)=\frac{p(p^2+2Rr+5r^2)}{2}\end{aligned}$$

Proof: We have:

$$h_a=\frac{2T}{a}\quad,\quad h_b=\frac{2T}{b}\quad,\quad h_c=\frac{2T}{c}$$

Thus:

$$\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}=\frac{am_a+bm_b+cm_c}{2T}$$

Application Bunyakovsky's inequality:

$$\left(\frac{am_a+bm_b+cm_c}{2T}\right)^2\leq(a+b+c)\left(\frac{am_a{}^2+bm_b{}^2+cm_c{}^2}{4T^2}\right)=2p\left(\frac{p(p^2+2Rr+5r^2)}{4T^2.2}\right)=\frac{p^2(p^2+2Rr+5r^2)}{4T^2}$$

We will prove that:

$$\frac{p^2(p^2+2Rr+5r^2)}{4S^2}\leq\left(\frac{R+r}{r}\right)^2\quad(1)$$

In fact:

$$\begin{aligned}(1)&\Leftrightarrow p^2+2Rr+5r^2\leq4R^2+8Rr+4r^2\\&\Leftrightarrow p^2\leq4R^2+6Rr-r^2\end{aligned}$$

Alternatively:

$$p^2 \leq 4R^2 + 4Rr + 3r^2 \text{ (Gerretsen's inequality)}$$

We will prove that:

$$4R^2 + 4Rr + 3r^2 \leq 4R^2 + 6Rr - r^2 \quad (2)$$

In fact:

$$(2) \quad \Leftrightarrow \quad 4r^2 \leq 2Rr \quad \Leftrightarrow \quad 2r \leq R \qquad (3)$$

(3) is Euler's inequality, thus (2) is true:

$$\Rightarrow \quad \left(\frac{am_a + bm_b + cm_c}{2S}\right)^2 \leq \left(\frac{R+r}{r}\right)^2$$

$$\Leftrightarrow \quad \frac{m_a}{h_a} + \frac{m_b}{h_b} + \frac{m_c}{h_c} = \frac{am_a + bm_b + cm_c}{2S} \leq \frac{R+r}{r} = 1 + \frac{R}{r}$$

Identity occur if and only if $\frac{\sqrt{a}m_a}{\sqrt{a}} = \frac{\sqrt{b}m_b}{\sqrt{b}} = \frac{\sqrt{c}m_c}{\sqrt{c}}$, $R = 2r$ and $p^2 = 4R^2 + 4Rr + 3r^2 \Leftrightarrow$ the triangle is equilateral triangle.