Dr. Said ATTAOUI/PhD. Thesis in Applied Mathematics
University of Sciences and Technology, ORAN-ALGERIA
Faculty of Mathematics and Informatics
Department of Mathematics.

Problem. Let $H_n = \sum_{k=1}^{n} \frac{1}{k}$ be the nth harmonic number and G the Catalan's constant defined as $G = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)^2}$. Prove

$$\sum_{n=1}^{\infty} \binom{2n}{n} \frac{H_{2n} - H_n}{4^n(2n-1)^2} = 2 + \frac{3\pi}{2} \log 2 - 2G - \pi$$

Proposed by Narendra Bhandari, Bajura, Nepal

Solution. Applying Theorem 3 of Hongwei Chen[1], we can write for all $|t| < 1$

$$\sum_{n=1}^{\infty} \binom{2n}{n} (H_{2n} - H_n) t^n \quad = \quad -\frac{1}{\sqrt{1-4t}} \log \left(\frac{1+\sqrt{1-4t}}{2} \right)$$

In particular, we have

$$\sum_{n=1}^{\infty} \binom{2n}{n} \frac{(H_{2n} - H_n)}{4^n} t^{2n} \quad = \quad -\frac{1}{\sqrt{1-t^2}} \log \left(\frac{1+\sqrt{1-t^2}}{2} \right) \tag{1}$$

Next, dividing both sides of (1) by t^2 and then integrating from 0 to x, we get

$$\sum_{n=1}^{\infty} \binom{2n}{n} \frac{(H_{2n} - H_n)}{4^n(2n-1)} x^{2n-1} \quad = \quad -\int_0^x \frac{1}{t^2\sqrt{1-t^2}} \log \left(\frac{1+\sqrt{1-t^2}}{2} \right) dt$$

Making the change of variable $t = \sin\theta$ in $J = \int \frac{1}{t^2\sqrt{1-t^2}} \log \left(\frac{1+\sqrt{1-t^2}}{2} \right) dt$, we find that

$$\begin{aligned} J &= \int \frac{1}{\sin^2\theta \cos\theta} \log \left(\frac{1+\cos\theta}{2} \right) \cos\theta \, d\theta \\ & \quad \left\{ \text{since } \cos\theta > 0, \ \text{ for } \ \theta \in [0, \pi/2] \right\} \\ &= \int \frac{\log\left(\cos^2(\theta/2)\right)}{\sin^2\theta} \, d\theta \\ &= 2\int \frac{\log\left(\cos(\theta/2)\right)}{\sin^2\theta} \, d\theta \end{aligned}$$

[1]Interesting Series Associated with Central Binomial Coefficients, Catalan Numbers and Harmonic Numbers. Journal of Integer Sequences, Vol. 19 (2016), Article 16.1.5

This last integral is so long to evaluate, the reader can be solved as Exercise, so we give only the final result:

$$\begin{aligned} 2\int \frac{\log\left(\cos(\theta/2)\right)}{\sin^2\theta}\,d\theta &= -2\log(\cos(\theta/2))\cot\theta+\tan(\theta/2)-\theta \\ &= -2\log(\cos(\arcsin t/2))\cot(\arcsin t)+\tan(\arcsin t/2)-\arcsin t \end{aligned}$$

Thereby

$$\begin{aligned} \sum_{n=1}^{\infty}\binom{2n}{n}\frac{(H_{2n}-H_n)}{4^n(2n-1)}x^{2n-1} &= -[J]_0^x \\ &= 2\log(\cos(\arcsin x/2))\cot(\arcsin x)-\tan(\arcsin x/2)+\arcsin x \quad (2) \end{aligned}$$

Now, dividing both sides of (2) by x and then integrating from 0 to 1, we can write

$$\begin{aligned} \sum_{n=1}^{\infty}\binom{2n}{n}\frac{(H_{2n}-H_n)}{4^n(2n-1)^2} &= 2\int_0^1 \frac{1}{x}\left(\log(\cos(\arcsin x/2))\cot(\arcsin x)\right)\,dx \\ &\quad -\int_0^1 \frac{1}{x}\tan(\arcsin x/2)\,dx+\int_0^1\frac{1}{x}\arcsin x\,dx \end{aligned}$$

Making the change of variable $x=\sin\theta$, we obtain

$$\begin{aligned} \sum_{n=1}^{\infty}\binom{2n}{n}\frac{(H_{2n}-H_n)}{4^n(2n-1)^2} &= 2\int_0^{\pi/2}\frac{1}{\sin\theta}\left(\log(\cos(\theta/2))\cot\theta\right)\cos\theta\,d\theta \\ &\quad -\int_0^{\pi/2}\frac{1}{\sin\theta}\tan(\theta/2)\,\cos\theta\,d\theta+\int_0^{\pi/2}\frac{\theta}{\sin\theta}\,\cos\theta\,d\theta \\ &= 2\int_0^{\pi/2}\cot^2\theta\log(\cos(\theta/2))\,d\theta-\int_0^{\pi/2}\tan(\theta/2)\cot\theta\,d\theta+\int_0^{\pi/2}\theta\cot\theta\,d\theta \\ &= 2\int_0^{\pi/2}\cot^2\theta\log(\cos(\theta/2))\,d\theta \\ &\quad -\left(\underbrace{[\tan(\theta/2)\log(\sin\theta)]_0^{\pi/2}}_{=0}-\underbrace{\frac{1}{2}\int_0^{\pi/2}\frac{1}{\cos^2(\theta/2)}\log(\sin\theta)\,d\theta}_{=2-\pi}\right) \\ &\quad +\left(\underbrace{[\theta\log(\sin\theta)]_0^{\pi/2}}_{=0}-\underbrace{\int_0^{\pi/2}\log(\sin\theta)\,d\theta}_{=-\frac{\pi}{2}\log 2}\right) \\ &= 2I+1-\frac{\pi}{2}+\frac{\pi}{2}\log 2. \end{aligned}$$

Let us evaluate I. Indeed

$$\begin{aligned}
I &= -\left[(\theta+\cot\theta)\log(\cos(\theta/2))\right]_0^{\pi/2} + \int_0^{\pi/2} (\theta+\cot\theta)\left(\frac{-\frac{1}{2}\sin(\theta/2)}{\cos(\theta/2)}\right)\,d\theta \\
&= -\frac{\pi}{2}\log(1/\sqrt{2}) - \frac{1}{2}\int_0^{\pi/2}\theta\tan(\theta/2)\,d\theta - \frac{1}{2}\int_0^{\pi/2}\cot\theta\tan(\theta/2)\,d\theta \\
&= \frac{\pi}{4}\log 2 - 2\underbrace{\int_0^{\pi/4}\theta\tan\theta\,d\theta}_{=\frac{1}{2}G-\frac{\pi}{8}\log 2} - \underbrace{\int_0^{\pi/4}\cot(2\theta)\tan\theta\,d\theta}_{=\frac{1}{4}(\pi-2)} \\
&= -G + \frac{\pi}{2}\log 2 - \frac{\pi}{4} + \frac{1}{2}.
\end{aligned}$$

Finally

$$\begin{aligned}
\sum_{n=1}^{\infty}\binom{2n}{n}\frac{(H_{2n}-H_n)}{4^n(2n-1)^2} &= -2G + \pi\log 2 - \frac{\pi}{2} + 1 + 1 - \frac{\pi}{2} + \frac{\pi}{2}\log 2 \\
&= 2 + \frac{3\pi}{2}\log 2 - 2G - \pi.
\end{aligned}$$