Dr. Said ATTAOUI/PhD. Thesis in Applied Mathematics
University of Sciences and Technology, ORAN-ALGERIA
Faculty of Mathematics and Informatics
Department of Mathematics.

Problem. Prove that

$$J = \int_0^1 \left(\frac{\sqrt{x}\log^2(x)}{1+x^2} \right)^2 dx = \frac{21}{80}\zeta^2(2).$$

Solution. The integral can be rewriting after making the change of variable $t = x^2$, as

$$\begin{aligned} J &= \int_0^1 \frac{x\log^4(x)}{(1+x^2)^2}\, dx \\ &= \frac{1}{32}\int_0^1 \frac{2x(2\log(x))^4}{(1+x^2)^2}\, dx \\ &= \frac{1}{32}\int_0^1 \frac{2x(\log(x^2))^4}{(1+x^2)^2}\, dx \\ &= \frac{1}{32}\int_0^1 \frac{\log^4(t)}{(1+t)^2}\, dt. \end{aligned}$$

Now, by using the expansion series $\sum_{n=1}^{\infty} n(-t)^{n-1} = \frac{1}{(1+t)^2}$, we obtain

$$\begin{aligned} J &= \frac{1}{32}\int_0^1 \log^4(t) \left(\sum_{n=1}^{\infty} n(-t)^{n-1} \right) dt \\ &= \frac{1}{32}\sum_{n=1}^{\infty}(-1)^{n-1} n \left(\int_0^1 t^{n-1}\log^4(t)\, dt \right) \\ & \quad \left\{ \text{ interchange between integral and summation } \right\} \\ &= \frac{1}{32}\sum_{n=1}^{\infty}(-1)^{n-1} n \left(\int_0^{\infty} z^4 e^{-nz}\, dz \right) \\ & \quad \left\{ t = e^{-z},\, dt = -e^{-z}dz \right\} \\ &= \frac{1}{32}\sum_{n=1}^{\infty}(-1)^{n-1} n \left(\frac{\Gamma(5)}{n^5} \right) \\ & \quad \left\{ \text{by the fact that for all } a>0,\, b>0 \,:\, \int_0^{\infty} z^a e^{-bz}\, dt = \frac{\Gamma(a+1)}{b^{a+1}} \right\} \\ &= -\frac{4!}{32}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4} = -\frac{3}{4}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}. \end{aligned}$$

Using the property $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^p} = -(1-2^{1-p})\zeta(p)$, $p>1$, we can deduce

$$\begin{aligned} J &= \frac{3}{4}\left(1-\frac{1}{8}\right)\zeta(4) = \frac{21}{32}\zeta(4) \\ &= \frac{21}{32}\left(\frac{2}{5}\zeta^2(2)\right) = \frac{21}{80}\zeta^2(2). \end{aligned}$$