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Problem. Prove

$$\frac{1}{2\sqrt{2}}\int_0^{\pi/2}\frac{\log(1+\tan y)}{\left(\cos y+\sqrt{2}\sin(y+\frac{\pi}{4})\right)\sqrt{1+\sqrt{2}\sin(2y+\frac{\pi}{4})}}\,dy \quad = \quad G$$

where G is the Catalan's constant defined as $G=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}$.

Solution. Recall that $\sin(x+\frac{\pi}{4})=\frac{1}{\sqrt{2}}(\cos x+\sin x)$ and $\cos(2x)=2\cos^2 x-1$.

Put $J=\frac{1}{2\sqrt{2}}\int_0^{\pi/2}\frac{\log(1+\tan x)}{\left(\cos x+\sqrt{2}\sin(x+\frac{\pi}{4})\right)\sqrt{1+\sqrt{2}\sin(2x+\frac{\pi}{4})}}\,dx$, so we have

$$\begin{aligned}
J &= \frac{1}{2\sqrt{2}}\int_0^{\pi/2}\frac{\log(1+\tan y)}{(2\cos y+\sin y)\sqrt{1+\cos(2y)+\sin(2y)}}\,dy\\
&= \frac{1}{2\sqrt{2}}\int_0^{\pi/2}\frac{\log(1+\tan y)}{(2\cos y+\sin y)\sqrt{2\cos^2 y+2\sin y\cos y}}\,dy\\
&= \frac{1}{4}\int_0^{\pi/2}\frac{\log(1+\tan y)}{(2+\tan y)\sqrt{1+\tan y}}\,\frac{dy}{\cos^2 y}
\end{aligned}$$

Now, substitute $x=\tan y$, we have $dx=\frac{dy}{\cos^2 y}$, and so

$$\begin{aligned}
J &= \frac{1}{4}\int_0^{\infty}\frac{\log(1+x)}{(2+x)\sqrt{1+x}}\,dx\\
&= \frac{1}{4}\int_1^{\infty}\frac{\log(x)}{(1+x)\sqrt{x}}\,dx;\ \text{ Substitute } x \text{ by } x-1\\
&= \frac{1}{4}\int_1^{0}\frac{\log(\frac{1}{x})}{(1+\frac{1}{x})\sqrt{\frac{1}{x}}}\left(-\frac{dx}{x^2}\right);\ \text{ Substitute } x \text{ by } \frac{1}{x}
\end{aligned}$$

Thus

$$\begin{aligned}
J &= -\frac{1}{4}\int_0^1 \frac{\log(x)}{\sqrt{x}\,(1+x)}\,dx \\
&= -\frac{1}{4}\int_0^1 \log(x)\left(\sum_{n=0}^{\infty}(-1)^n x^{n-\frac{1}{2}}\right)\,dx \\
&= -\frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\left(\int_0^1 x^{n-\frac{1}{2}}\log(x)\,dx\right) \\
&= -\frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\left(\left[\frac{1}{(n+\frac{1}{2})}x^{n+\frac{1}{2}}\log(x)\right]_0^1 - \frac{1}{(n+\frac{1}{2})}\int_0^1 x^{n-\frac{1}{2}}\,dx\right) \\
&= -\frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\left(-\frac{1}{(n+\frac{1}{2})^2}\left[x^{n+\frac{1}{2}}\right]_0^1\right) \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2} \\
&= G.
\end{aligned}$$

Finally

$$\frac{1}{2\sqrt{2}}\int_0^{\pi/2}\frac{\log(1+\tan x)}{\left(\cos x+\sqrt{2}\sin(x+\frac{\pi}{4})\right)\sqrt{1+\sqrt{2}\sin(2x+\frac{\pi}{4})}}\,dx = G$$

Remark. We have

$$\int_0^1 \frac{\log(x)}{1+x^2}\,dx = -G.$$

Proof. Based on the previous result, we get

$$\begin{aligned}
-G &= \frac{1}{4}\int_0^1 \frac{\log(x)}{\sqrt{x}\,(1+x)}\,dx \\
&= \frac{1}{4}\int_0^1 \frac{2\log(x)}{x\,(1+x^2)}\,(2xdx);\ \ \text{Substitute } x \text{ by } x^2 \\
&= \int_0^1 \frac{\log(x)}{1+x^2}\,dx
\end{aligned}$$