RMM - Geometry Marathon 601 - 700

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601.

$(OABC)$ −tetrahedron

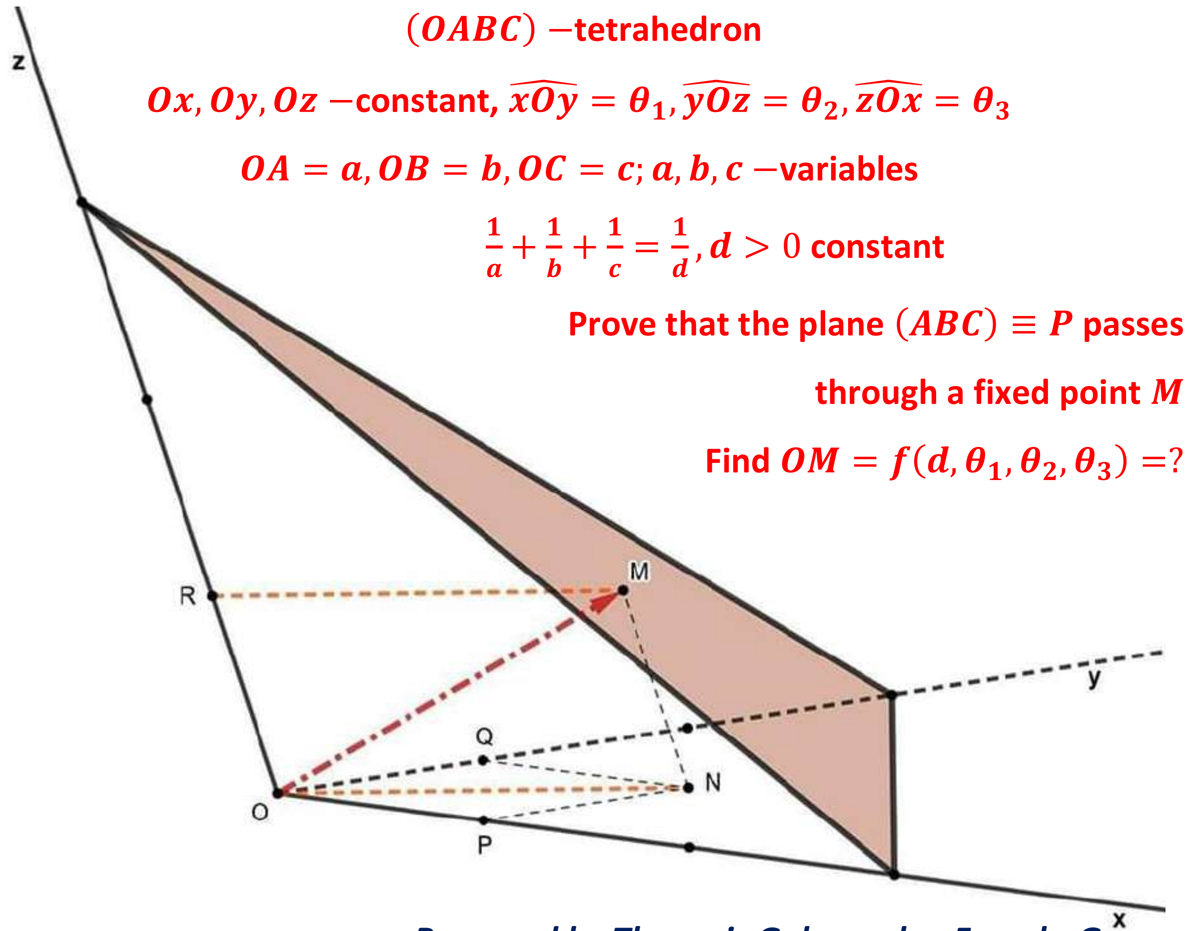
Ox, Oy, Oz −constant, $\widehat{xOy} = \theta_1, \widehat{yOz} = \theta_2, \widehat{zOx} = \theta_3$

$OA = a, OB = b, OC = c; a, b, c$ −variables

$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{d}, d > 0$ constant

Prove that the plane $(ABC) \equiv P$ passes through a fixed point M

Find $OM = f(d, \theta_1, \theta_2, \theta_3) = ?$



Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by proposer

(I) Plagiogonal 3rd system: $(Ox, Oy, Oz, \theta_1, \theta_2, \theta_3)$

$$P: \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \text{ and let } M(d, d, d)$$

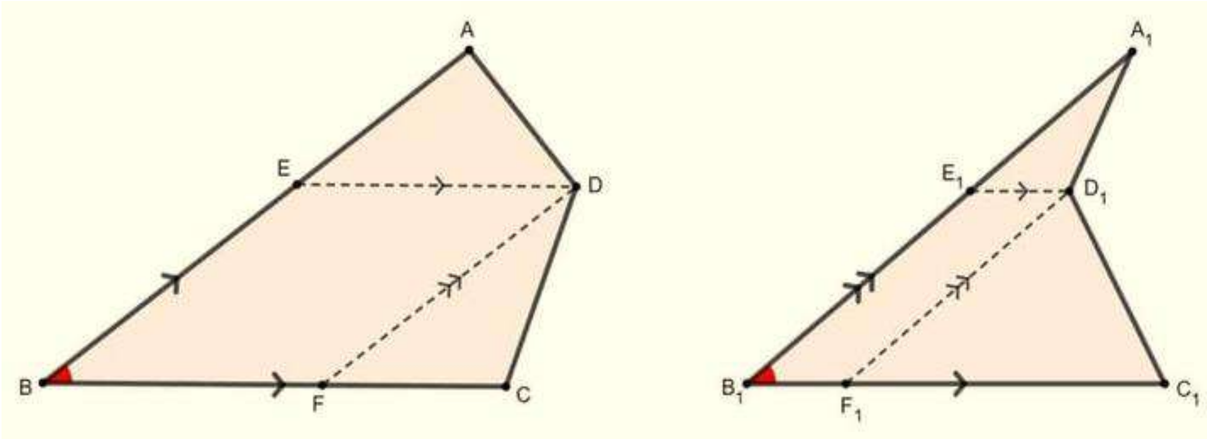
$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{d} \Rightarrow \frac{d}{a} + \frac{d}{b} + \frac{d}{c} = 1$$

Therefore, $M \in P$, then M passes through fixed point $M(d, d, d)$.

(II) $\overrightarrow{OM}(d, d, d), \left|\overrightarrow{OM}\right|^2 = d^2 + d^2 + d^2 + 2d^2(\cos\theta_1 + \cos\theta_2 + \cos\theta_3)$

$$\overrightarrow{OM} = d\sqrt{3 + 2(\cos\theta_1 + \cos\theta_2 + \cos\theta_3)}$$

602.

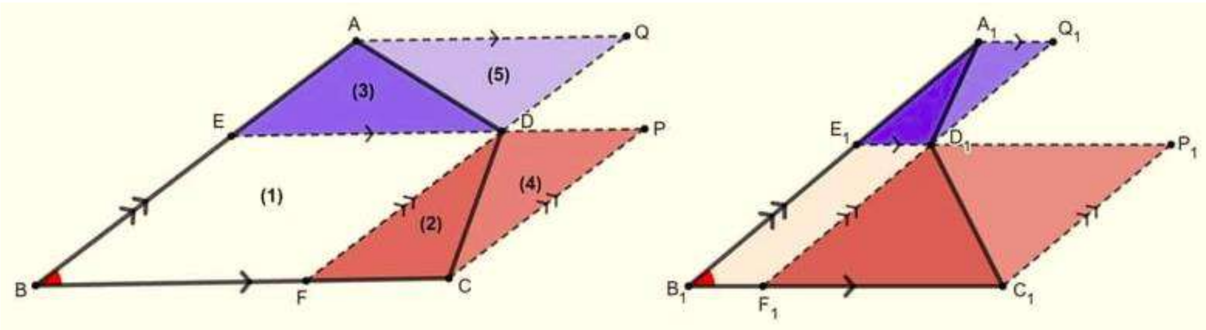


Prove:

$$[ABCD] = \frac{\sin B}{2}(BC \cdot BE + BA \cdot BF)$$

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by proposer



$$2[ABCD] = (1) + (1) + (2) + (2) + (3) + (3) = (1) + (2) + (4) + (1) + (3) + (5) =$$

$$= [BCPE] + [BFQA] \Rightarrow [ABCD] = \frac{\sin B}{2}(BC \cdot BE + BA \cdot BF)$$

$$2[ABCD] = 2([BFDE] + [DFC] + [DAE]) =$$

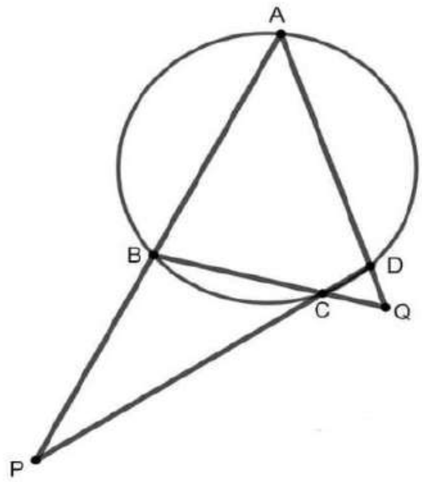
$$= ([BFDE] + [DFC] + [DCF]) + ([BFDE] + [DAE] + [DQA]) =$$

$$= [BCPE] + [BFQA] = BC \cdot BE \cdot \sin B + BA \cdot BF \cdot \sin B$$

Hence,

$$[ABCD] = \frac{\sin B}{2}(BC \cdot BE + BA \cdot BF)$$

603.



$ABCD$ −cyclic if and only if

$$\cos A = \frac{1}{2} \cdot \frac{AB \cdot AP + AD \cdot AQ}{AP \cdot AQ}$$

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by proposer

Let $AB = b, AP = p, AD = d, AQ = q$. Plaiogonal system:

$$AB \equiv Ax, AD \equiv Ay, A(0,0), B(b,0), P(p,0), D(0,d), Q(0,q), C(c_1, c_2)$$

$$BQ: \frac{x}{b} + \frac{y}{q} = 1, BD: \frac{x}{p} + \frac{y}{d} = 1 \Rightarrow c_1 = \frac{bp(q-d)}{pq - bd}, c_2 = \frac{qd(p-b)}{pq - bd}$$

From $NCCQ1$ we have: $ABCD$ −cyclic if and only if

$$AC^2 = c_1 \cdot AB + c_2 \cdot AD \Leftrightarrow c_1^2 + c_2^2 + 2c_1c_2 \cos A = bc_1 + dc_2 \Leftrightarrow$$

$$\cos A = \frac{1}{2} \frac{bp + dq}{pq} \Leftrightarrow \cos A = \frac{1}{2} \cdot \frac{AB \cdot AP + AD \cdot AQ}{AP \cdot AQ}$$

604. In ΔABC the following relationship holds:

$$\sum \frac{b + c - a}{4a\sqrt{bc}} \geq \frac{25r^2 - 4R^2}{3\sqrt{3}R^3}$$

Proposed by Mehmet Șahin-Ankara-Turkiye

Solution by Nguyen Van Canh-BenTre-Vietnam

Let $s = \frac{a+b+c}{2}$. By AM − GM Inequality, we have:

$$\sum\frac{b+c-a}{4a\sqrt{bc}}\ge 3\sqrt[3]{\frac{(b+c-a)(a+b-c)(a+c-b)}{64a\sqrt{bc}.b\sqrt{ac}.c\sqrt{ab}}}=3\sqrt[3]{\frac{(s-a)(s-b)(s-c)}{8a^2b^2c^2}}$$

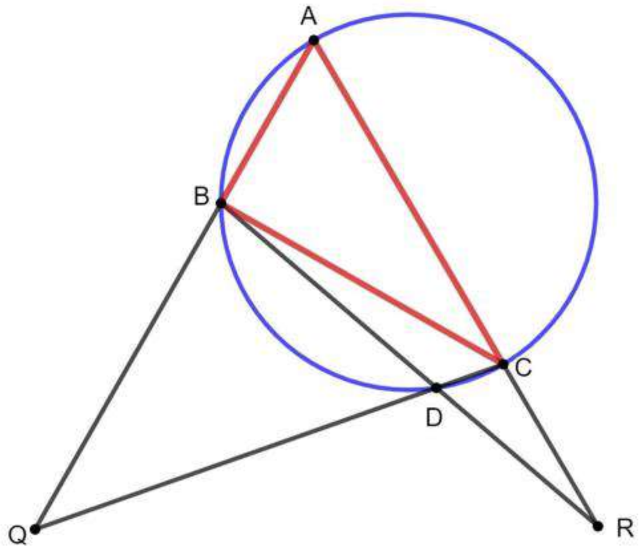
$$=\frac{3}{2}\sqrt[3]{\frac{sr^2}{(4Rrs)^2}}=\frac{3}{2\sqrt[3]{16sR^2}}\overset{s\le\frac{3\sqrt{3}}{2}R}{\underset{(3)}{\ge}}\frac{3}{2\sqrt[3]{8.3\sqrt{3}.R^3}}$$

$$=\frac{3}{4\sqrt{3}R}\overset{(*)}{\underset{(3)}{\ge}}\frac{25r^2-4R^2}{3\sqrt{3}R^3};$$

$$(*)\Leftrightarrow 9R^2\ge 4(25r^2-4R^2);$$

$$\Leftrightarrow 25R^2\ge 4.25r^2\Leftrightarrow R^2\ge 4r^2\Leftrightarrow R\ge 2r\ (\text{Euler})\Rightarrow (*)\ \text{true. Proved.}$$

605.



$$\frac{BC^2}{AB\cdot AC}=k,\frac{AB}{AC}=l,\frac{AB}{AR}=m,\frac{AC}{AQ}=n$$

Find: $k=f(l,m,n)=?$

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by Jose Ferreira Queiroz-Olinda-Brazil

We know that if $ABDC$ is cyclic, then

$$\cos\theta=\frac{AB\cdot AQ+AC\cdot AR}{2AQ\cdot AR},\text{where }\widehat{\theta}=\widehat{QAR}$$

$$\Delta ABC: BC^2 = AB^2 + AC^2 - 2AB\cdot AC\cdot\cos\theta$$

$$BC^2 = AB^2 + AC^2 - 2AB\cdot AC\cdot\frac{AB\cdot AQ + AC\cdot AR}{2AQ\cdot AR}$$

$$BC^2 = \frac{CR}{AR}\cdot AB^2 + \frac{BQ}{AQ}\cdot AC^2$$

$$\frac{BC^2}{AB\cdot AC} = \frac{CR}{AR}\cdot\frac{AB^2}{AB\cdot AC} + \frac{BQ}{AQ}\cdot\frac{AC^2}{AB\cdot AC}$$

$$\frac{BC^2}{AB\cdot AC} = \frac{CR}{AR}\cdot\frac{AB}{AC} + \frac{BQ}{AQ}\cdot\frac{AC}{AB}$$

$$\frac{BC^2}{AB\cdot AC} = \frac{AR-AC}{AR}\cdot\frac{AB}{AC} + \frac{AQ-AB}{AQ}\cdot\frac{AC}{AB}$$

$$k = \frac{AB}{AC} - \frac{AB}{AR} + \frac{AC}{AB} - \frac{AC}{AQ}$$

$$k = l - m + \frac{1}{l} - n = \left(l + \frac{1}{l}\right) - (m+n)$$

606. In ΔABC the following relationship holds:

$$216r^3 \le (h_a+h_b)(h_b+h_c)(h_c+h_a) \le \frac{8r}{3}(4R+r)^2$$

Proposed by Marin Chirciu-Romania

Solution 1 by Marian Ursărescu-Romania

$$\because h_a = \frac{2F}{a} \Rightarrow (h_a+h_b)(h_b+h_c)(h_c+h_a) = \frac{s^2r(s^2+r^2+2Rr)}{R^2}$$

$$\frac{s^2r(s^2+r^2+2Rr)}{R^2} \le \frac{8r}{3}(4R+r)^2 \Leftrightarrow 3s^2(s^2+r^2+2Rr) \le 8R^2(4R+r)^2; (1)$$

But $3s^2 \le (4R+r)^2$ $(Doucet); (2)$. From (1) and (2) we must show:

$$s^2+r^2+2Rr \le 8R^2 \Leftrightarrow s^2 \le 8R^2-2Rr-r^2; (3)$$

$$s^2 \le 4R^2+4Rr+3r^2\ (Gerretsen); (4)$$

From (3) and (4) we must show:

$$4R^2+4Rr+3r^2 \le 8R^2-2Rr-r^2 \Leftrightarrow 4R^2-6Rr-4r^2 \ge 0$$

$2R^2-3Rr-2r^2 \ge 0 \Leftrightarrow (R-2r)(2R-r) \ge 0$, true from $R \ge 2r(Euler)$.

$$\frac{s^2r(s^2+r^2+2Rr)}{R^2} \ge 216r^3 \Leftrightarrow s^2(s^2+r^2+2Rr) \ge 216R^2r^2; (5)$$

$$s^2 \ge \frac{27Rr}{2}(Cosnita-Turtoiu); (6)$$

From (5) and (6) we must show:

$$\frac{27Rr}{2}(s^2+r^2+2Rr)\geq 216R^2r^2\Leftrightarrow s^2+r^2+2Rr\geq 16Rr\Leftrightarrow$$
$$s^2\geq 14Rr-r^2;(7)$$

But $s^2\geq 16Rr-5r^2\ (Gerretsen);(8)$. From (7) and (8) we must show:

$$16Rr-5r^2\geq 14Rr-r^2\Leftrightarrow R\geq 2r\ (Euler).$$

Solution 2 by Avishek Mitra-West Bengal-India

$$\prod_{cyc}(h_a+h_b)\overset{AGM}{\geq}\prod_{cyc}2\sqrt{h_ah_b}=8\prod_{cyc}h_a\overset{GHM}{\geq}8\cdot\left(\frac{3}{\sum\frac{1}{h_a}}\right)^3 8\cdot\frac{27}{\left(\frac{1}{r}\right)^3}=216r^3$$

$$\prod_{cyc}(h_a+h_b)=8F^3\prod_{cyc}\left(\frac{1}{a}+\frac{1}{b}\right)=8F^3\cdot\frac{\prod(a+b)}{(abc)^2}=8F^3\cdot\frac{2abc+\sum ab(a+b)}{16R^2F^2}=$$

$$=\frac{F}{2R^2}\left(2abc+\sum_{cyc}ab(2s-c)\right)=\frac{F}{2R^2}\left(2s\sum_{cyc}ab-abc\right)=$$

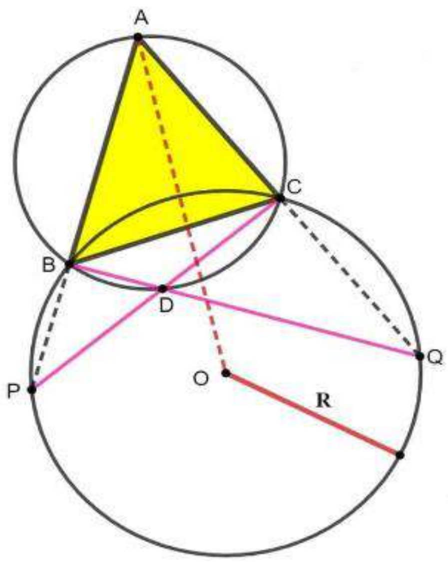
$$=\frac{F}{2R^2}(2s(s^2+r^2+4Rr)-4Rrs)=\frac{F}{2R^2}\cdot 2s(s^2+r^2+2Rr)$$

$\because 4R+r\geq\sqrt{3}s(Trucht)$.Need to show:

$$3s^2\cdot\frac{8r}{3}\geq\frac{27r}{4}(s^2+r^2+2Rr),\qquad 32s^2\geq 27s^2+27r^2+54Rr$$
$$5s^2\geq 54Rr+27r^2,\ \ s^2\geq 16Rr-5r^2(Gerretsen)$$
$$80Rr-25r^2\geq 54Rr+27r^2\Leftrightarrow 26Rr-52r^2\geq 0$$

$26r(R-2r)\geq 0$ true from $R\geq 2r\ (Euler)$.

607.



Prove:

$$R^2=\frac{(AP^2-AO^2)+(AQ^2-AO^2)}{2}$$

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by Jose Ferreira Queiroz-Olinda-Brazil

By Stewart's theorem:

$$AP^2\cdot R+AQ^2\cdot R-AO^2\cdot 2R=R\cdot R\cdot 2R^2$$

$$AP^2+AQ^2-2AO^2=2R^2$$

Therefore,

$$R^2=\frac{(AP^2-AO^2)+(AQ^2-AO^2)}{2}$$

608. If $m>0$ then in ΔABC holds:

$$\left(\frac{a^{\frac{m}{2}}+b^{\frac{m}{2}}+c^{\frac{m}{2}}}{3}\right)^{\frac{2}{m}}\geq\frac{2}{\sqrt[4]{3}}\cdot\sqrt{F}$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

Solution by Tapas Das-India

$$\frac{a^{\frac{m}{2}}+b^{\frac{m}{2}}+c^{\frac{m}{2}}}{3}\geq\sqrt[3]{a^{\frac{m}{2}}\cdot b^{\frac{m}{2}}\cdot c^{\frac{m}{2}}}=(abc)^{\frac{m}{6}}$$

$$\left(\frac{a^{\frac{m}{2}}+b^{\frac{m}{2}}+c^{\frac{m}{2}}}{3}\right)^{\frac{2}{m}}\geq(abc)^{\frac{m}{6}\cdot\frac{2}{m}}\geq(abc)^{\frac{1}{3}}\geq\left(\frac{4F}{\sqrt{3}}\right)^{\frac{3}{2}\cdot\frac{1}{3}}=\frac{2\sqrt{F}}{\sqrt[4]{3}}$$

$$Carlitz:\ (abc)^2\geq\left(\frac{4F}{\sqrt{3}}\right)^3\Rightarrow abc\geq\left(\frac{4F}{\sqrt{3}}\right)^{\frac{3}{2}}$$

609. If $m,x,y,z>0$ and $xyz=1$ then in ΔABC holds:

$$\frac{(x+y)^m}{z}\cdot a^2+\frac{(y+z)^m}{x}\cdot b^2+\frac{(z+x)^m}{y}\cdot c^2\geq 2^{m+2}\sqrt{3}\cdot F$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

Solution 1 by Avishek Mitra-West Bengal-India

$$\sum_{cyc}\frac{(x+y)^m}{z}\cdot a^2\overset{AGM}{\geq}3\sqrt[3]{\frac{1}{xyz}\prod_{cyc}(x+y)^m a^2}\overset{AGM}{\geq}3\sqrt[3]{\frac{1}{xyz}\prod_{cyc}\left(2\sqrt{xy}\right)^m(4Rrs)^2}=$$

$$= 3\sqrt[3]{(8xyz)^m 16R^2r^2s^2} \overset{\substack{Euler\\Mitrinovic}}{\geq} 3\sqrt[3]{8^m\cdot 16\cdot 2r\cdot \frac{2s}{3\sqrt{3}}r^2s^2} =$$

$$= \frac{3}{\sqrt{3}}\cdot 4rs\sqrt[3]{8^m} = 2^{m+2}\sqrt{3}F$$

Solution 2 by Tapas Das-India

$$(x+y)^m \geq \left(2\sqrt{xy}\right)^m = 2^m(xy)^{\frac{m}{2}}; (\text{and analogs})$$

$$\sum_{cyc}\frac{(x+y)^m}{z}\cdot a^2 \overset{AGM}{\geq} 3\sqrt[3]{\frac{1}{xyz}\prod_{cyc}(x+y)^m a^2} \overset{AGM}{\geq} 3\sqrt[3]{\frac{1}{xyz}\prod_{cyc}\left(2\sqrt{xy}\right)^m a^2} =$$

$$= 2^m\cdot 3\sqrt[3]{(xyz)^m(abc)^2} = 2^m\cdot 3(abc)^{\frac{2}{3}}; (\because abc = 1)$$

$$\geq 2^m\cdot 3\left(\frac{4F}{\sqrt{3}}\right)^{3\cdot\frac{1}{3}} = 2^{m+2}F\sqrt{3}$$

Solution 3 by Debopriyo Dawn-India

$$\sum_{cyc}\frac{(x+y)^m}{z}\cdot a^2 \overset{AGM}{\geq} 3\sqrt[3]{\frac{1}{xyz}\prod_{cyc}(x+y)^m a^2} \overset{AGM}{\geq} 3\sqrt[3]{\frac{1}{xyz}\prod_{cyc}\left(2\sqrt{xy}\right)^m a^2} =$$

$$= 2^m\cdot 3\sqrt[3]{(xyz)^m(abc)^2} = 2^m\cdot 3(abc)^{\frac{2}{3}}; (\because abc = 1)$$

$$\geq 2^m\cdot 3\left(\frac{4F}{\sqrt{3}}\right)^{3\cdot\frac{1}{3}} = 2^{m+2}F\sqrt{3}$$

$$(abc)^2 \geq \left(\frac{4F}{\sqrt{3}}\right)^{\frac{1}{3}} (Carlitz),\qquad a+b+c \leq 3\sqrt{3}R\ (Mitrinovic),\qquad F = \frac{abc}{4R}$$

$$\frac{4F}{\sqrt{3}} = \frac{abc}{R\sqrt{3}} \leq \frac{2abc}{2s} \Rightarrow \frac{3abc}{a+b+c} = \frac{abc}{\frac{a+b+c}{3}} \leq \frac{abc}{(abc)^{\frac{1}{3}}} \Rightarrow (abc)^2 \geq \left(\frac{4F}{\sqrt{3}}\right)^{\frac{1}{3}}$$

610. In ΔABC the following relationship holds:

$$a^2+b^2+c^2 \geq 4F\sqrt{\frac{abc}{(a+b)(b+c)(c+a)}}\cdot\sum_{cyc}\csc C\cdot\sqrt{\frac{a+b}{c}}$$

Proposed by Daniel Sitaru-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:$$

$$4F\sqrt{\frac{abc}{(a+b)(b+c)(c+a)}}\cdot\sum_{cyc}\csc C\cdot\sqrt{\frac{a+b}{c}}=4F\sum_{cyc}\frac{2R}{c}\cdot\sqrt{\frac{ab}{(b+c)(c+a)}}=$$

$$=\sum_{cyc}\frac{2abc}{c}\cdot\sqrt{\frac{ab}{(b+c)(c+a)}}=\sum_{cyc}ab.2\sqrt{\frac{ab}{(b+c)(c+a)}}\overset{AM-GM}{\leq}\sum_{cyc}ab\left(\frac{a}{b+c}+\frac{b}{c+a}\right)$$

$$=\sum_{cyc}\frac{a^2b}{b+c}+\sum_{cyc}\frac{ab^2}{c+a}=\sum_{cyc}\frac{a^2b}{b+c}+\sum_{cyc}\frac{ca^2}{b+c}=\sum_{cyc}a^2\left(\frac{b}{b+c}+\frac{c}{b+c}\right)=\sum_{cyc}a^2.$$

$$Therefore,\ a^2+b^2+c^2\geq 4F\sqrt{\frac{abc}{(a+b)(b+c)(c+a)}}\cdot\sum_{cyc}\csc C\cdot\sqrt{\frac{a+b}{c}}.$$

611. In ΔABC the following relationship holds:

$$\frac{3}{2}\sqrt{\frac{2R}{r}}\geq\frac{\sec\frac{A}{2}}{\sec\frac{B}{2}}+\frac{\sec\frac{B}{2}}{\sec\frac{C}{2}}+\frac{\sec\frac{C}{2}}{\sec\frac{A}{2}}$$

Proposed by Marin Chirciu-Romania

Solution by Marian Ursărescu-Romania

We must show:

$$\left(\frac{\cos\frac{B}{2}}{\cos\frac{A}{2}}+\frac{\cos\frac{C}{2}}{\cos\frac{B}{2}}+\frac{\cos\frac{A}{2}}{\cos\frac{C}{2}}\right)^2\leq\frac{9}{2}\cdot\frac{R}{r};(1)$$

$$\left(\frac{\cos\frac{B}{2}}{\cos\frac{A}{2}}+\frac{\cos\frac{C}{2}}{\cos\frac{B}{2}}+\frac{\cos\frac{A}{2}}{\cos\frac{C}{2}}\right)^2\overset{C-S}{\leq}\sum_{cyc}\cos^2\frac{A}{2}\cdot\sum_{cyc}\frac{1}{\cos^2\frac{A}{2}};(2)$$

$$\sum_{cyc}\cos^2\frac{A}{2}=\frac{4R+r}{2R}\text{ and }\sum_{cyc}\frac{1}{\cos^2\frac{A}{2}}=1+\frac{(4R+r)^2}{s^2};(3)$$

From (1) and (2) we must show:

$$\frac{4R+r}{2R}\left(1+\frac{(4R+r)^2}{s^2}\right)\le\frac{9}{2}\cdot\frac{R}{r}$$

$$(4R+r)\left(1+\frac{(4R+r)^2}{s^2}\right)\le\frac{9R^2}{r};(4)$$

$$s^2\ge 3r(4R+r)(Doucet)\Rightarrow\frac{1}{s^2}\le\frac{1}{3r(4R+r)}\Rightarrow\frac{(4R+r)^2}{s^2}\le\frac{4R+r}{3r};(5)$$

From (4) and (5) we must show:

$$(4R+r)\left(1+\frac{4R+r}{3r}\right)\le\frac{9R^2}{r}\Leftrightarrow 4(4R+r)(R+r)\le 27R^2$$

which is true, because $4R+r\le 4R+\frac{R}{2}=\frac{9R}{2}$ **and** $R+r\le R+\frac{R}{2}=\frac{3R}{2}$

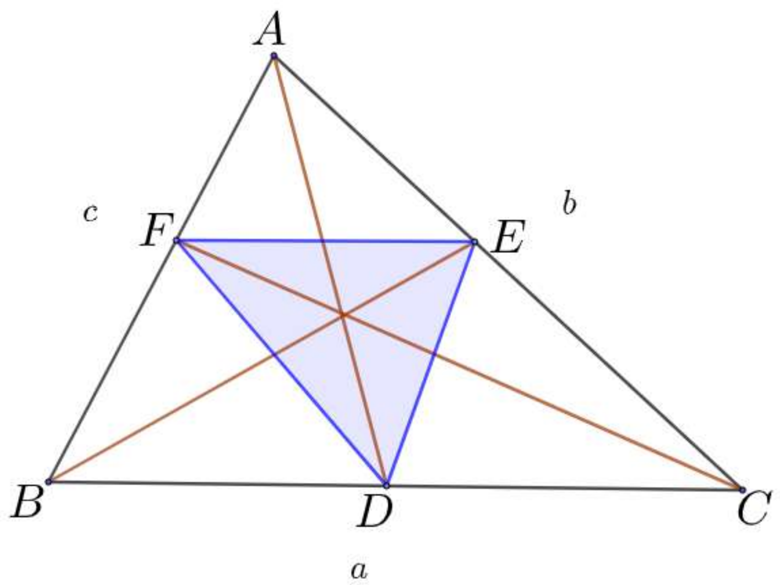
$$4(4R+r)(R+r)\le 4\cdot\frac{4R}{2}\cdot\frac{3R}{2}=27R^2$$

612. If in $\Delta ABC, AD, BE, CF$ –internal bisectors, R_A, R_B, R_C –circumradii of $\Delta AFE, \Delta BDF, \Delta CED, AA_1, BB_1, CC_1$ –altitudes in $\Delta AFE, \Delta BDF, \Delta CED$ then:

$$\frac{h_a^2}{AA_1\cdot R_A}+\frac{h_b^2}{BB_1\cdot R_B}+\frac{h_c^2}{CC_1\cdot R_C}\le 18$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution by Marian Ursărescu-Romania



$$AA_1=\frac{2S_{AFE}}{EF}\text{ and }R_A=\frac{AF\cdot AE\cdot EF}{4S_{AFE}}\Rightarrow AA_1\cdot R_A=\frac{AE\cdot AF}{2};(1)$$

From bisector theorem:

$$\frac{AF}{BF}=\frac{b}{a}\Rightarrow\frac{AF}{AB}=\frac{b}{a+b}\Rightarrow AF=\frac{bc}{a+b};(2)$$

$$\frac{AE}{EC}=\frac{c}{a}\Rightarrow\frac{AE}{AC}=\frac{c}{a+c}\Rightarrow AF=\frac{bc}{a+c};(3)$$

From (1),(2) and (3) it follows that:

$$AA_1\cdot R_A=\frac{b^2c^2}{2(a+b)(a+c)}\text{ and }h_a^2=\left(\frac{2F}{a}\right)^2=\frac{4F^2}{a^2}$$

$$\frac{h_a^2}{AA_1\cdot R_A}=\frac{8F^2(a+b)(a+c)}{a^2b^2c^2}$$

We must show that:

$$\frac{4F^2}{a^2b^2c^2}\sum_{cyc}(a+b)(a+c)\le 9;(4)$$

$$\frac{4F^2}{a^2b^2c^2}=\frac{4s^2r^2}{16s^2r^2R^2}=\frac{1}{4R^2};(5)$$

From (4) and (5), we must show that:

$$a^2+b^2+c^2+3(ab+bc+ca)\le 36R^2;(6)$$

But $ab+bc+ca\le a^2+b^2+c^2;(7)$. From (6) and (7), we must show:

$$4(a^2+b^2+c^2)\le 36R^2\Leftrightarrow a^2+b^2+c^2\le 9R^2(Lebniz)$$

613. Prove that for any acute triangle ABC :

$$\sqrt[4]{2}<\frac{\sqrt{\sin\frac{A}{2}}+\sqrt{\sin\frac{B}{2}}+\sqrt{\sin\frac{C}{2}}}{\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}}\le\sqrt{2}$$

Proposed by Vasile Mircea Popa-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:\sqrt{\sin\frac{A}{2}}+\sqrt{\sin\frac{B}{2}}+\sqrt{\sin\frac{C}{2}}=\sqrt{\frac{\sin A}{2\cos\frac{A}{2}}}+\sqrt{\frac{\sin B}{2\cos\frac{B}{2}}}+\sqrt{\frac{\sin C}{2\cos\frac{C}{2}}}\le$$

$$\overset{CBS}{\leq}\sqrt{\frac{1}{2}(\sin A+\sin B+\sin C)\left(\frac{1}{\cos\frac{A}{2}}+\frac{1}{\cos\frac{B}{2}}+\frac{1}{\cos\frac{C}{2}}\right)}=$$

$$=\sqrt{\frac{1}{2}.4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.\left(\frac{1}{\cos\frac{A}{2}}+\frac{1}{\cos\frac{B}{2}}+\frac{1}{\cos\frac{C}{2}}\right)}=$$

$$=\sqrt{2\left(\cos\frac{A}{2}\cos\frac{B}{2}+\cos\frac{B}{2}\cos\frac{C}{2}+\cos\frac{C}{2}\cos\frac{A}{2}\right)}\overset{?}{\leq}\sqrt{2}\left(\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\right)$$

Now let us make the change $\{A,B,C\}$

$\rightarrow\{\pi-2A,\pi-2B,\pi-2C\}$ *so we need to prove*

$$\sqrt{\sin A\sin B+\sin B\sin C+\sin C\sin A}\leq\cos A+\cos B+\cos C,\forall\ \text{acute}\ \Delta ABC.$$

Using sRr notations we have :

$$\sin A\sin B+\sin B\sin C+\sin C\sin A=\frac{s^2+r^2+4Rr}{4R^2}\quad \text{and}$$

$$\cos A+\cos B+\cos C=\frac{R+r}{R}$$

So the inequality is equivalent to :

$$\frac{s^2+r^2+4Rr}{4R^2}\leq\left(\frac{R+r}{R}\right)^2\Leftrightarrow s^2\leq 4R^2+4Rr+3r^2$$

Which is Gerretsen's inequality. Then : $\dfrac{\sqrt{\sin\frac{A}{2}}+\sqrt{\sin\frac{B}{2}}+\sqrt{\sin\frac{C}{2}}}{\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}}\leq\sqrt{2}.$

Since $x\rightarrow\sqrt{\sin x}$ *is increasing on* $\left(0,\frac{\pi}{2}\right)$ *then we have :*

$$\sqrt{\sin\frac{A}{2}}\overset{A<\frac{\pi}{2}}{<}\sqrt{\sin\frac{\pi}{4}}=\frac{1}{\sqrt[4]{2}}\ \Rightarrow\ \sqrt[4]{2}.\sin\frac{A}{2}<\sqrt{\sin\frac{A}{2}}.$$

Similarly we have : $\sqrt[4]{2}.\sin\frac{B}{2}<\sqrt{\sin\frac{B}{2}}$ *and* $\sqrt[4]{2}.\sin\frac{C}{2}<\sqrt{\sin\frac{C}{2}}.$

$$\textit{Summing these inequalities we get}:\ \sqrt[4]{2} < \frac{\sqrt{\sin\frac{A}{2}} + \sqrt{\sin\frac{B}{2}} + \sqrt{\sin\frac{C}{2}}}{\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}}.$$

$$\textit{Therefore},\ \sqrt[4]{2} < \frac{\sqrt{\sin\frac{A}{2}} + \sqrt{\sin\frac{B}{2}} + \sqrt{\sin\frac{C}{2}}}{\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}} \le \sqrt{2}.$$

614. If $m \in [0,\infty)$ and $M \in Int(\Delta ABC), x = MA, y = MB, z = MC$ then :

$$(x.m_a)^{2m+1} + (y.w_b)^{2m+1} + (z.h_c)^{2m+1} \ge 2^{2m+1}.F^{2m+1}.\left(\sqrt{3}\right)^{1-2m}$$

Proposed by D.M.Bătineţu-Giurgiu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\textit{Since}\ m_a \ge h_a\ \textit{and}\ w_b \ge h_b\ \textit{then we have}:$$

$$(x.m_a)^{2m+1} + (y.w_b)^{2m+1} + (z.h_c)^{2m+1} \ge (x.h_a)^{2m+1} + (y.h_b)^{2m+1} + (z.h_c)^{2m+1} =$$

$$= \left(2F.\frac{x}{a}\right)^{2m+1} + \left(2F.\frac{y}{b}\right)^{2m+1} + \left(2F.\frac{z}{c}\right)^{2m+1} \overset{\text{Hölder}}{\ge} (2F)^{2m+1}.\frac{\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)^{2m+1}}{3^{2m}} \ge$$

$$\ge \frac{(2F)^{2m+1}}{3^{2m}}.\left(\sqrt{3\left(\frac{xy}{ab}+\frac{yz}{bc}+\frac{zx}{ca}\right)}\right)^{2m+1} \overset{Hayashi}{\ge} \frac{(2F)^{2m+1}}{3^{2m}}.\left(\sqrt{3.1}\right)^{2m+1}.$$

$$\textit{Therefore},\ (x.m_a)^{2m+1} + (y.w_b)^{2m+1} + (z.h_c)^{2m+1} \ge 2^{2m+1}.F^{2m+1}.\left(\sqrt{3}\right)^{1-2m}.$$

615. In ΔABC the following relationship holds:

$$\frac{R}{r} \ge \sqrt{\frac{(r_a+r_b+r_c)^3}{(m_a+m_b+m_c)^2(h_a+h_b+h_c)}} \cdot max\left\{\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}, \sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}, \sqrt{\frac{c}{a}}+\sqrt{\frac{a}{c}}\right\}$$

Proposed by Bogdan Fuştei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\textit{Lemma}:\ \textit{In}\ \Delta ABC:\ \frac{b}{c}+\frac{c}{b} \le \frac{\sqrt{a^2b^2+b^2c^2+c^2a^2}}{2F}\quad (*)$$

$$We\ have:\ (*)\ \Leftrightarrow (b^2+c^2)\sqrt{2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)}$$
$$\le 2bc\sqrt{a^2b^2+b^2c^2+c^2a^2}$$
$$\overset{squaring}{\Leftrightarrow} \left(2b^2c^2+b^4+c^4\right)\left[2\left(a^2b^2+b^2c^2+c^2a^2\right)-\left(a^4+b^4+c^4\right)\right]$$
$$\le 4b^2c^2\left(a^2b^2+b^2c^2+c^2a^2\right)$$
$$\Leftrightarrow -a^4(b^2+c^2)^2-2b^2c^2(b^4+c^4)+2(b^4+c^4)(a^2b^2+b^2c^2+c^2a^2)-(b^4+c^4)^2\le 0$$
$$\Leftrightarrow -a^4(b^2+c^2)^2+2(b^4+c^4)(a^2b^2+c^2a^2)-(b^4+c^4)^2$$
$$=-[a^2(b^2+c^2)-(b^4+c^4)]^2\le 0$$

Which is true and the proof of the lemma is completed.

Now, $\sqrt{a},\sqrt{b},\sqrt{c}$ *can be the sides of a triangle* Δ' *and let* F' *be the area of* Δ'.

We have:

$$4F'=\sqrt{2(ab+bc+ca)-(a^2+b^2+c^2)}=\sqrt{4r(4R+r)}=2\sqrt{r(r_a+r_b+r_c)}$$

Using the lemma in Δ' *we obtain:* $\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\le\frac{\sqrt{ab+bc+ca}}{2F'}$

$$=\frac{\sqrt{2R(h_a+h_b+h_c)}}{\sqrt{r(r_a+r_b+r_c)}}\ (and\ analogs)$$

$$Then:\ \omega\le\sqrt{\frac{2R}{r}\cdot\frac{h_a+h_b+h_c}{r_a+r_b+r_c}}\ \ or\ \ \sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\cdot\omega\le\sqrt{\frac{2R}{r}}\ \ (1)$$

$$Now\ we\ have:\ m_a\overset{Lascu}{\ge}\frac{b+c}{2}\cdot\cos\frac{A}{2}$$

$$=\frac{a+2(s-a)}{2}\cdot\cos\frac{A}{2}\overset{AM-GM}{\ge}\sqrt{a.2(s-a)}\cdot\sqrt{\frac{s(s-a)}{bc}}=\frac{a(s-a)}{\sqrt{2Rr}}.$$

$$Similarly\ we\ have:\ m_b\ge\frac{b(s-b)}{\sqrt{2Rr}}\ \ and\ \ m_c\ge\frac{c(s-c)}{\sqrt{2Rr}}$$

Summing up these inequalities we get:

$$m_a+m_b+m_c\ge\frac{a(s-a)+b(s-b)+c(s-c)}{\sqrt{2Rr}}=\frac{2r(4R+r)}{\sqrt{2Rr}}=\sqrt{\frac{2r}{R}}\cdot(r_a+r_b+r_c)$$

$$Then: \frac{r_a+r_b+r_c}{m_a+m_b+m_c} \le \sqrt{\frac{R}{2r}} \quad (2)$$

$$Multiplying\ (1)\ and\ (2)\ we\ get: \sqrt{\frac{(r_a+r_b+r_c)^3}{(m_a+m_b+m_c)^2(h_a+h_b+h_c)}}\cdot\omega \le \frac{R}{r}.$$

616. In ΔABC the following relationship holds:

$$\sum_{cyc}\left(\frac{1}{m_b}+\frac{\lambda}{m_c}\right)^2 \ge \frac{4(\lambda+1)^2}{3R^2}, \lambda \ge 0$$

Proposed by Marin Chirciu-Romania

Solution 1 by Avishek Mitra-West Bengal-India

$$\sum_{cyc}\left(\frac{1}{m_b}+\frac{\lambda}{m_c}\right)^2 = \sum_{cyc}\frac{1}{m_b^2}+2\lambda\sum_{cyc}\frac{1}{m_bm_c}+\lambda^2\sum_{cyc}\frac{1}{m_c^2} \overset{Bergstrom}{\ge}$$

$$\ge (1+\lambda^2)\cdot\frac{(1+1+1)^2}{\sum m_b^2}+2\lambda\cdot\frac{(1+1+1)^2}{\sum m_bm_c} \ge$$

$$\ge \frac{9(1+\lambda^2)}{\sum m_b^2}+\frac{18\lambda}{\sum m_b^2} = \frac{9(1+\lambda^2)}{\frac{3}{4}\sum b^2}+\frac{18\lambda}{\frac{3}{4}\sum b^2} \overset{Lebniz}{\ge}$$

$$\ge \frac{12(1+\lambda^2)}{9R^2}+\frac{24\lambda}{9R^2} = \frac{12(1+2\lambda+\lambda^2)}{9R^2} = \frac{4(1+\lambda)^2}{3R^2}$$

Equality holds for $a=b=c$.

Solution 2 by Aggeliki Papaspyropoulou-Greece

$$m_a \le \frac{RF}{ar}, m_b \le \frac{FR}{br}, m_c \le \frac{FR}{cr}$$

$$\frac{1}{m_b}+\frac{\lambda}{m_c} \ge \frac{br}{FR}+\frac{\lambda cr}{FR} = \frac{r}{FR}(b+\lambda c) = \frac{r(b+\lambda c)}{Rrs} = \frac{b+\lambda c}{Rs}$$

So, we have to prove:

$$\sum_{cyc}\frac{(b+\lambda c)^2}{R^2s^2} \ge \frac{4(\lambda+1)^2}{3R^2} \Leftrightarrow 3\sum_{cyc}(b+\lambda c)^2 \ge (\lambda+1)^2(a+b+c)^2; (1)$$

But $3(x^2+y^2+z^2) \ge (x+y+z)^2$

$$3\sum_{cyc}(b+\lambda c)^2 \ge (b+\lambda c+c+\lambda a+a+\lambda b)^2 = [(\lambda+1)(a+b+c)]^2 =$$

$$=(\lambda+1)(a+b+c)^2$$

Equality holds for $a=b=c$.

617. In ΔABC the following relationship holds:

$$\sum_{cyc}\sqrt[3]{\frac{\csc^4\frac{A}{2}}{\csc^2\frac{B}{2}+\csc\frac{C}{2}\left(\csc\frac{A}{2}+\csc\frac{B}{2}\right)}} \ge \sqrt[3]{36}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\sin A+\sin B+\sin C \le \frac{3\sqrt{3}}{2} = 3\sin\frac{\pi}{3}$$

$$\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2} \le \frac{3}{2} = 3\sin\frac{\pi}{6}$$

$$\csc\frac{A}{2}+\csc\frac{B}{2}+\csc\frac{C}{2} = \frac{1}{\sin\frac{A}{2}}+\frac{1}{\sin\frac{B}{2}}+\frac{1}{\sin\frac{C}{2}} \ge 6$$

$$\sum_{cyc}\sqrt[3]{\frac{\csc^4\frac{A}{2}}{\csc^2\frac{B}{2}+\csc\frac{C}{2}\left(\csc\frac{A}{2}+\csc\frac{B}{2}\right)}} \ge \frac{\left(\sum\csc\frac{A}{2}\right)^{\frac{4}{3}}}{\left(\sum\csc^2\frac{A}{2}+\sum\csc\frac{B}{2}\csc\frac{C}{2}\right)^{\frac{1}{3}}} \ge \sqrt[3]{36}$$

$$\frac{\left(\sum\csc\frac{A}{2}\right)^4}{\left(\sum\csc\frac{A}{2}\right)^2} \ge 36 \Leftrightarrow \left(\sum\csc\frac{A}{2}\right)^2 \ge 36 \Leftrightarrow \sum\csc\frac{A}{2} \ge 6(true!)$$

Solution 2 by Tapas Das-India

$$\sum_{cyc}\sqrt[3]{\frac{\csc^4\frac{A}{2}}{\csc^2\frac{B}{2}+\csc\frac{C}{2}\left(\csc\frac{A}{2}+\csc\frac{B}{2}\right)}} \overset{Radon}{\ge} \frac{\left(\sum\csc\frac{A}{2}\right)^{\frac{4}{3}}}{\left(\sum\csc^2\frac{A}{2}+\sum\csc\frac{B}{2}\csc\frac{C}{2}\right)^{\frac{1}{3}}} =$$

$$=\frac{\left(\sum\csc\frac{A}{2}\right)^{\frac{4}{3}}}{\left(\sum\csc\frac{A}{2}\right)^{\frac{2}{3}}}=\left(\sum\csc\frac{A}{2}\right)^{\frac{2}{3}}\ge 3^{\frac{2}{3}}\left[\left(\sum\csc\frac{A}{2}\right)^{\frac{2}{3}}\right]^{\frac{1}{3}}=\frac{3^{\frac{2}{3}}}{\left(\sum\sin\frac{A}{2}\right)^{\frac{2}{9}}}=3^{\frac{2}{3}}\left(\frac{4R}{r}\right)^{\frac{2}{9}}\ge$$

$$\ge 3^{\frac{2}{3}}\left(\frac{4\cdot 2r}{r}\right)^{\frac{2}{9}}=3^{\frac{2}{3}}\cdot 8^{\frac{2}{9}}=\sqrt[3]{36}$$

618. If $m\ge 0; u,v>0, M\in Int(\Delta ABC), x=MA, y=MB, z=MC$, then:

$$\sum_{cyc}\left(\frac{x}{a}\left(\frac{uy}{b}+\frac{vz}{c}\right)\right)^{m+1}\ge\frac{(u+v)^{m+1}}{3^m}$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

Solution by Tapas Das-India

$$\sum_{cyc}\left(\frac{x}{a}\left(\frac{uy}{b}+\frac{vz}{c}\right)\right)^{m+1}\ge\frac{1}{3^m}\left[u\sum_{cyc}\frac{xy}{ab}+v\sum_{cyc}\frac{xy}{ab}\right]^{m+1}=$$

$$=\frac{1}{3^m}\left[(u+v)\left(\sum_{cyc}\frac{xy}{ab}\right)\right]^{m+1}\overset{Hayashi}{\ge}\frac{1}{3^m}(u+v)^{m+1}$$

Lemma (Hayashi). For any triangle ABC and for any arbitrary point P, it holds:

$$a\cdot PB\cdot PC+b\cdot PC\cdot PA+c\cdot PA\cdot PB\ge abc$$

619. In ΔABC the following relationship holds:

$$\left(\frac{3\sin^2 A}{\cos A}+\frac{2\sin^2 B}{\cos B}+\frac{\sin^2 C}{\cos C}\right)\cos\left(\frac{\pi+2A+B}{6}\right)\ge 6\sin^2\left(\frac{\pi+2A+B}{6}\right)$$

Proposed by Daniel Sitaru-Romania

Solution by Tapas Das-India

$$\text{Let } f(x)=\frac{\sin^2 x}{\cos x}=\frac{1-\cos^2 x}{\cos x}=\sec x-\cos x$$

$$f'(x)=\sec x\tan x+\sin x$$

$$f''(x)=\sec x+\tan^2 x+\sec^3 x+\cos x>0\Rightarrow f\text{ −convex function on }\left(0,\frac{\pi}{2}\right)$$

$$\frac{3\sin^2 A}{\cos A}+\frac{2\sin^2 B}{\cos B}+\frac{\sin^2 C}{\cos C}=3f(A)+2f(b)+f(C)\geq$$

$$\geq(3+2+1)f\left(\frac{3A+2B+C}{3+2+1}\right)=6f\left(\frac{3A+2B+C}{6}\right)=$$

$$=6\cdot\frac{\sin^2\left(\frac{3A+2B+C}{6}\right)}{\cos\left(\frac{3A+2B+C}{6}\right)}=6\cdot\frac{\sin^2\left(\frac{\pi+2A+B}{6}\right)}{\cos\left(\frac{\pi+2A+B}{6}\right)}$$

Therefore,

$$\left(\frac{3\sin^2 A}{\cos A}+\frac{2\sin^2 B}{\cos B}+\frac{\sin^2 C}{\cos C}\right)\cos\left(\frac{\pi+2A+B}{6}\right)\geq 6\sin^2\left(\frac{\pi+2A+B}{6}\right)$$

620. In ΔABC the following relationship holds:

$$\sum_{cyc}\cos\left(\frac{A+\pi}{8}\right)\geq\frac{3\sqrt{3}r}{R}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\textit{Let us prove that}:\ \cos\left(\frac{x+\pi}{8}\right)\geq\frac{1}{4}\cos\frac{x}{2}+\frac{3\sqrt{3}}{8},\forall x\in(0,\pi).$$

$$\textit{Let } f(x)=\cos\left(\frac{x+\pi}{8}\right)-\frac{1}{4}\cos\frac{x}{2},\ \ x\in(0,\pi).$$

$$\textit{We have}:\ f'(x)=-\frac{1}{8}\left[\sin\left(\frac{x+\pi}{8}\right)-\sin\frac{x}{2}\right]=-\frac{1}{4}\sin\left(\frac{\pi-3x}{16}\right)\cos\left(\frac{\pi+5x}{16}\right),\forall x\in(0,\pi)$$

$$\textit{Since } \frac{\pi+5x}{16}\in\left(\frac{\pi}{16},\frac{3\pi}{8}\right)\textit{ and } \frac{\pi-3x}{16}\in\left(-\frac{\pi}{8},\frac{\pi}{16}\right)\textit{ then we have}:$$

$$\cos\left(\frac{\pi+5x}{16}\right)>0,\forall x\in(0,\pi)$$

$$\textit{And } f \textit{ is decreasing on } \left(0,\frac{\pi}{3}\right)\textit{ and increasing on }\left(\frac{\pi}{3},\pi\right)\Rightarrow$$

$$\min_{x\in(0,\pi)}\{f(x)\}=f\left(\frac{\pi}{3}\right)=\frac{3\sqrt{3}}{8}.$$

$$\textit{Then}:\ \cos\left(\frac{A+\pi}{8}\right)\geq\frac{1}{4}\cos\frac{A}{2}+\frac{3\sqrt{3}}{8}\ \ (\textit{and analogs})$$

$$Thus,\ \sum_{cyc}\cos\left(\frac{A+\pi}{8}\right)\ge\frac{1}{4}\sum_{cyc}\cos\frac{A}{2}+3.\frac{3\sqrt{3}}{8}\overset{AM-GM}{\ge}\frac{1}{4}.3\sqrt[3]{\prod_{cyc}\cos\frac{A}{2}}+\frac{9\sqrt{3}}{8}$$

$$=\frac{3}{4}.\sqrt[3]{\frac{s}{4R}}+\frac{9\sqrt{3}}{8}\ge$$

$$\overset{Mitrinovic}{\ge}\frac{3}{4}.\sqrt[3]{\frac{3\sqrt{3}r}{4R}}+\frac{9\sqrt{3}}{8}\overset{Euler}{\ge}\frac{3}{4}.\sqrt[3]{\frac{3\sqrt{3}r}{4R}.\left(\frac{2r}{R}\right)^2}+\frac{9\sqrt{3}}{8}.\frac{2r}{R}=\frac{3}{4}.\frac{\sqrt{3}r}{R}+\frac{9\sqrt{3}r}{4R}=\frac{3\sqrt{3}r}{R}.$$

So the proof is completed. Equality holds iff ΔABC is equilateral.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We will use Popoviciu's inequality :

If $f: I\to R$ is a concave function, then for any $x,y,z\in I$ we have :

$$f\left(\frac{x+y}{2}\right)+f\left(\frac{y+z}{2}\right)+f\left(\frac{z+x}{2}\right)\ge\frac{1}{2}\left(f(x)+f(y)+f(z)+3f\left(\frac{x+y+z}{3}\right)\right).$$

Let $f_1(t)=\cos\left(\frac{t}{4}\right), t\in(0,\pi)$ and $x_1=A+B,\ y_1=B+C,\ z_1=C+A$.

Since f_1 is concave then by Popoviciu's inequality for (x_1,y_1,z_1) we have :

$$\sum_{cyc}\cos\left(\frac{(C+A)+(A+B)}{8}\right)\ge\frac{1}{2}\left(\sum_{cyc}\cos\left(\frac{B+C}{4}\right)+3\cos\frac{\pi}{6}\right)$$

$$Then:\ \sum_{cyc}\cos\left(\frac{A+\pi}{8}\right)\ge\frac{1}{2}\left(\sum_{cyc}\cos\left(\frac{B+C}{4}\right)+\frac{3\sqrt{3}}{2}\right)\quad(1)$$

Let $f_2(t)=\cos\left(\frac{t}{2}\right), t\in(0,\pi)$ and $x_2=A,\ y_2=B,\ z_2=C$.

Since f_2 is concave then by Popoviciu's inequality for (x_2,y_2,z_2) we have :

$$\sum_{cyc}\cos\left(\frac{B+C}{4}\right)\ge\frac{1}{2}\left(\sum_{cyc}\cos\frac{A}{2}+3\cos\frac{\pi}{6}\right)=\frac{1}{2}\left(\sum_{cyc}\cos\frac{A}{2}+\frac{3\sqrt{3}}{2}\right)\quad(2)$$

Let $f_3(t)=\sin t, t\in(0,\pi)$ and $x_3=A,\ y_3=B,\ z_3=C$.

Since f_3 is concave then by Popoviciu's inequality for (x_3,y_3,z_3) we have :

$$\sum_{cyc}\sin\left(\frac{B+C}{2}\right)\ge\frac{1}{2}\left(\sum_{cyc}\sin A+3\sin\frac{\pi}{3}\right)\Leftrightarrow\sum_{cyc}\cos\frac{A}{2}\ge\frac{1}{2}\left(\frac{s}{R}+\frac{3\sqrt{3}}{2}\right)\quad(3)$$

From (1), (2) and (3) we get :

$$\sum_{cyc}\cos\left(\frac{A+\pi}{8}\right)\ge\frac{s}{8R}+\frac{21\sqrt{3}}{16}\overset{Mitrinovic}{\ge}\frac{3\sqrt{3}r}{8R}+\frac{21\sqrt{3}}{16}\overset{Euler}{\ge}\frac{3\sqrt{3}r}{8R}+\frac{21\sqrt{3}}{16}.\frac{2r}{R}=\frac{3\sqrt{3}r}{R}.$$

So the proof is completed. Equality holds iff ΔABC is equilateral.

621. a,b,c −sides in ΔABC, $\frac{1}{a+b},\frac{1}{b+c},\frac{1}{c+a}$ −sides in $\Delta A'B'C'$. Prove that:

$$\frac{9}{4s}\le 2s'\le\frac{\sqrt{3}}{4r}$$

Proposed by Mehmet Şahin-Ankara-Turkiye

Solution by Daniel Sitaru-Romania

$$2s'=\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\overset{Bergstrom}{\geq}\frac{(1+1+1)^2}{a+b+b+c+c+a}=\frac{9}{4s}$$

$$f:(0,2s)\to\mathbb{R}, f(x)=\frac{1}{2s-x}, f'(x)=\frac{-1}{(2s-x)^2},$$

$$f''(x)=\frac{2}{(2s-x)^3}>0, f-convex$$

$$2s'=\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}=\frac{1}{2s-a}+\frac{1}{2s-b}+\frac{1}{2s-c}=$$

$$=f(a)+f(b)+f(c)\overset{Jensen}{\geq}3f\left(\frac{a+b+c}{3}\right)=3f\left(\frac{2s}{3}\right)=$$

$$=3\cdot\frac{1}{2s-\frac{2s}{3}}=\frac{9}{4s}\overset{Mitrinovic}{\geq}\frac{9}{4\cdot3\sqrt{3}r}=\frac{\sqrt{3}}{4r}$$

622. In ΔABC the following relationship holds:

$$(\sin A+2\sin B)^4+(\sin B+2\sin C)^4+(\sin C+2\sin A)^4\le\frac{2187}{8}\left(1-\frac{r}{R}\right).$$

Proposed by Marian Ursărescu-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$b^4+c^4-b^3c-bc^3=b^3(b-c)-c^3(b-c)=(b-c)(b^3-c^3)$$
$$=(b-c)(b-c)(b^2+bc+c^2)=(b^2+bc+c^2)(b-c)^2\ge0$$
$$\Rightarrow b^3c+bc^3\overset{(i)}{\le}b^4+c^4$$

Now, $(b+2c)^4$

$$= b^4+8b^3c+8bc^3+24b^2c^2+24bc^3+16c^4 \overset{via\ (i)}{\leq} b^4+8b^4+8c^4+24b^2c^2$$
$$+12.2.bc.c^2+16c^4 \overset{A-G}{\leq} 9b^4+24c^4+24b^2c^2+12(b^2c^2+c^4)$$
$$\Rightarrow (b+2c)^4 \overset{(*)}{\leq} 9b^4+36c^4+36b^2c^2$$
$$\therefore (\sin A+2\sin B)^4+(\sin B+2\sin C)^4+(\sin C+2\sin A)^4$$
$$=\frac{1}{16R^4}\sum_{cyc}(b+2c)^4 \overset{via\ (*)\ and\ analogs}{\leq} \frac{1}{16R^4}\left(9\sum_{cyc}b^4+36\sum_{cyc}c^4+36\sum_{cyc}b^2c^2\right)$$
$$\leq \frac{1}{16R^4}\left(9\sum_{cyc}a^4+36\sum_{cyc}a^4+36\sum_{cyc}a^4\right)=\frac{81}{16R^4}\sum_{cyc}a^4 \overset{?}{\leq} \frac{2187}{8}\left(1-\frac{r}{R}\right)$$
$$\Leftrightarrow \sum_{cyc}a^4 \overset{?}{\leq} 54R^3(R-r) \Leftrightarrow \sum_{cyc}a^2b^2-8r^2s^2 \overset{?}{\leq} 27R^3(R-r)$$
$$\Leftrightarrow \left(s^2+4Rr+r^2\right)^2-16Rrs^2-8r^2s^2 \overset{?}{\leq} 27R^3(R-r)$$
$$\Leftrightarrow s^4-(8Rr+6r^2)s^2+(4Rr+r^2)^2 \underset{(*)}{\overset{?}{\leq}} 27R^3(R-r)$$
$$\text{Now, LHS of } (*) \overset{Gerretsen}{\leq} (4R^2-4Rr-3r^2)s^2$$
$$+(4Rr+r^2)^2 \overset{Gerretsen}{\leq} (4R^2-4Rr-3r^2)(4R^2+4Rr+3r^2)$$
$$+(4Rr+r^2)^2 \left(\because 4R^2-4Rr-3r^2=(R-2r)(4R+4r)+5r^2 \overset{Euler}{>} 0\right)$$
$$\overset{?}{\leq} 27R^3(R-r) \Leftrightarrow 11t^4-27t^3+16t+8 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right)$$
$$\Leftrightarrow (t-2)\left((t-2)(11t^2+17+24)+44\right) \overset{?}{\geq} 0 \to \text{true} \because t \overset{Euler}{\geq} 2 \Rightarrow (*) \text{ is true}$$
$$\therefore \text{in any } \Delta\, ABC, (\sin A+2\sin B)^4+(\sin B+2\sin C)^4+(\sin C+2\sin A)^4$$
$$\leq \frac{2187}{8}\left(1-\frac{r}{R}\right), \text{with equality iff } \Delta\, ABC \text{ is equilateral (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Since\ \ (x+2y)^4 \overset{Hölder}{\leq} (x^4+2y^4)(1+2)^3=27(x^4+2y^4),\ \ \forall x,y > 0,\ \ then\ we\ have:$$
$$(\sin A+2\sin B)^4 \leq 27(\sin^4 A+2\sin^4 B)$$
$$(\sin B+2\sin C)^4 \leq 27(\sin^4 B+2\sin^4 C)$$
$$And:\ \ (\sin C+2\sin A)^4 \leq 27(\sin^4 C+2\sin^4 A)$$
$$Summing\ up\ these\ inequalities\ we\ get:$$
$$(\sin A+2\sin B)^4+(\sin B+2\sin C)^4+(\sin C+2\sin A)^4$$
$$\leq 81(\sin^4 A+\sin^4 B+\sin^4 C)$$
$$So\ it\ suffice\ to\ prove\ that:$$
$$\sin^4 A+\sin^4 B+\sin^4 C \leq \frac{27}{8}\left(1-\frac{r}{R}\right)\ \ or\ \ a^4+b^4+c^4 \leq 54R^3(R-r)$$
$$We\ have:$$

$$a^4+b^4+c^4=2[(s^2-4Rr-3r^2)^2-8r^3(2R+r)]\overset{Gerretsen}{\leq}2[(4R^2)^2-8r^3(2R+r)]=$$

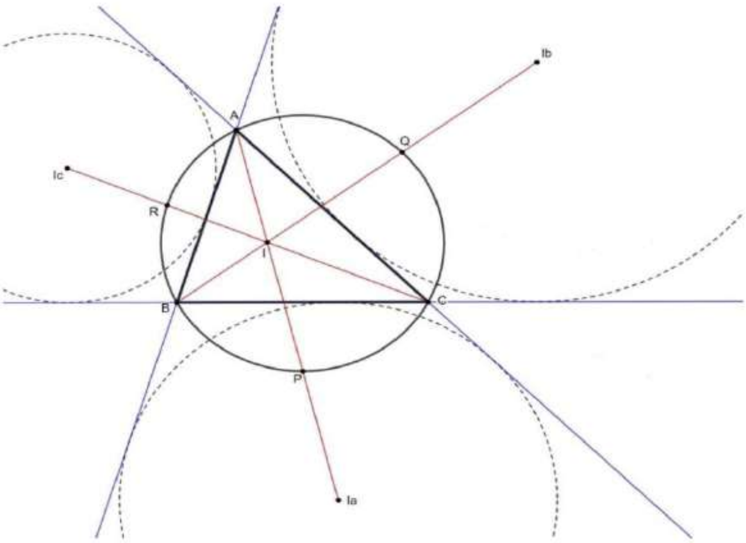
$$=16(2R^4-2Rr^3-r^4)\overset{?}{\leq}54R^3(R-r)\Leftrightarrow 11R^4-27R^3r+16Rr^3+8r^4\geq 0$$

$$\Leftrightarrow (R-2r)[(R-2r)(11R^2+17Rr+24r^2)+44r^3]\geq 0 \text{ which is true by Euler's inequality.}$$

Therefore,

$$(\sin A+2\sin B)^4+(\sin B+2\sin C)^4+(\sin C+2\sin A)^4\leq\frac{2187}{8}\left(1-\frac{r}{R}\right).$$

623.



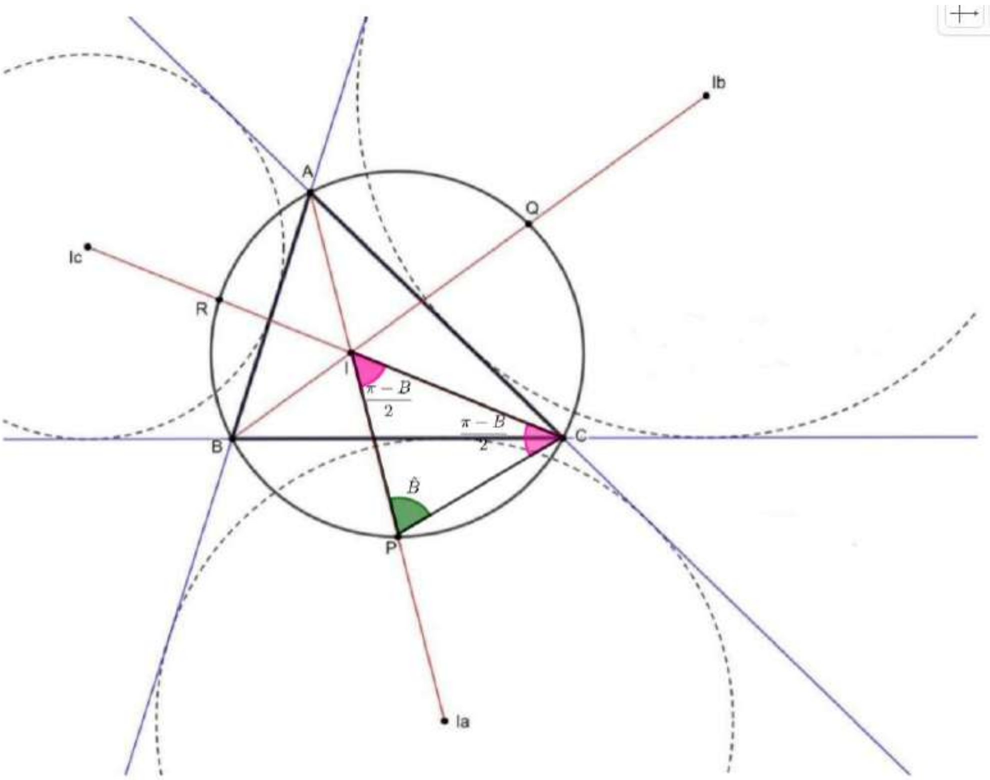
Prove that:

$$\sum_{cyc}\frac{II_a}{IP}\cdot\sum_{cyc}\left(\frac{II_a}{IP}\right)^2\geq 72$$

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by Jose Ferreira Queiroz-Olinda-Brazil

$$\begin{cases} II_A=4R\cdot\sin\frac{A}{2}\\ II_B=4R\cdot\sin\frac{B}{2},\\ II_C=4R\cdot\sin\frac{C}{2}\end{cases}\begin{cases} IA=\frac{\sqrt{bcs(s-a)}}{s}\\ IB=\frac{\sqrt{acs(s-b)}}{s}\\ IC=\frac{\sqrt{abs(s-c)}}{s}\end{cases}$$



$$\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}},\sin\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{ca}},\sin\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{ab}}$$

$$\Delta IPC:\ \frac{IC}{\sin B}=\frac{IP}{\sin\left(\frac{\pi}{2}-\frac{B}{2}\right)}\Rightarrow IP=\frac{IC}{2\sin\frac{B}{2}}$$

Now, we have:

$$\frac{II_A}{IP}=\frac{4R\sin\frac{A}{2}}{\frac{IC}{2\sin\frac{B}{2}}}=\frac{8R}{IC}\cdot\sin\frac{A}{2}\sin\frac{B}{2}$$

$$\frac{II_A}{IP}=\frac{8R}{\frac{\sqrt{abs(s-c)}}{s}}\cdot\sqrt{\frac{(s-b)(s-c)}{bc}}\cdot\sqrt{\frac{(s-c)(s-a)}{ac}}$$

$$\frac{II_A}{IP}=\frac{8R}{abc}\cdot F=2$$

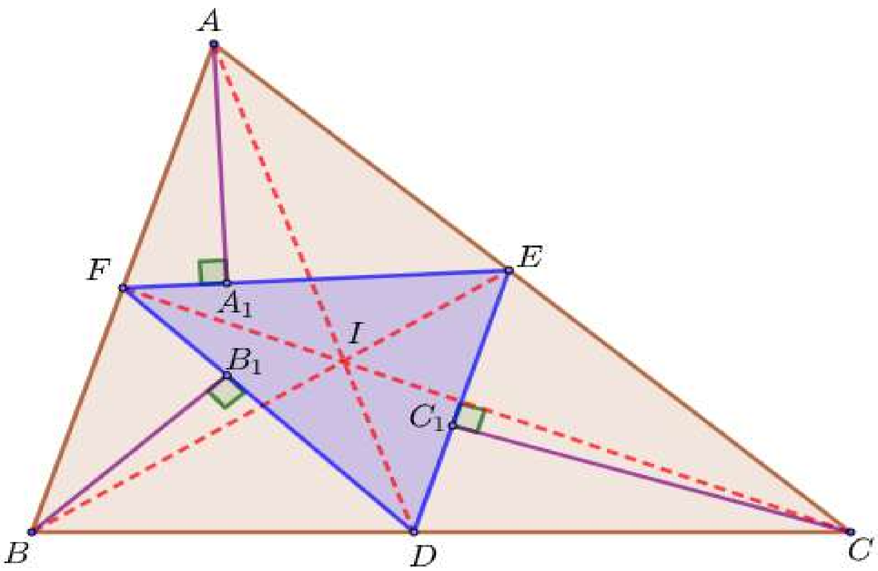
Similarly, $\frac{II_B}{IQ}=2$ **and** $\frac{II_C}{IR}=2$**. So,**

$$\sum_{cyc} \frac{II_a}{IP} = 6 \text{ and } \sum_{cyc} \left(\frac{II_a}{IP}\right)^2 = 12$$

Therefore,

$$\sum_{cyc} \frac{II_a}{IP} \cdot \sum_{cyc} \left(\frac{II_a}{IP}\right)^2 = 72$$

624. In $\Delta ABC, I$ –incenter, R_A, R_B, R_C –circumradius of $\Delta AFE, \Delta BDF, \Delta CED$ respectively. Prove that:



$$\frac{h_a^2}{AA_1 \cdot R_A} + \frac{h_b^2}{BB_1 \cdot R_B} + \frac{h_c^2}{CC_1 \cdot R_C} \leq 18$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let F_A *be the area of* ΔAEF. *We have* : $2F_A = AA_1.EF = \frac{AE.EF.FA}{2R_A}$ *then* : $AA_1.R_A = \frac{AE.FA}{2}$.

Since : $AE = \frac{bc}{c+a}$ *and* $AF = \frac{bc}{a+b}$ *then we get* : $AA_1.R_A = \frac{(bc)^2}{2(c+a)(a+b)}$

Also we have : $h_a = \frac{bc}{2R}$ *then* : $\frac{{h_a}^2}{AA_1.R_A} = \frac{(c+a)(a+b)}{2R^2}$.

$$\textit{Similarly we have}: \frac{h_b{}^2}{BB_1.R_B} = \frac{(a+b)(b+c)}{2R^2} \ \textit{and} \ \frac{h_c{}^2}{CC_1.R_C} = \frac{(b+c)(c+a)}{2R^2}$$

$$\textit{Then}: \frac{h_a{}^2}{AA_1.R_A} + \frac{h_b{}^2}{BB_1.R_B} + \frac{h_c{}^2}{CC_1.R_C} = \frac{(a^2+b^2+c^2)+3(ab+bc+ca)}{2R^2} \le$$

$$\le \frac{4(a^2+b^2+c^2)}{2R^2} \overset{Leibniz}{\le} \frac{2.9R^2}{R^2} = 18.$$

Equality holds iff $\Delta\Delta ABC$ *is equilateral.*

625. If $x, y, z \in \left(0, \frac{\pi}{2}\right)$ then in ΔABC holds:

$$\frac{\tan y + \tan z}{\sin x} \cdot a + \frac{\tan z + \tan x}{\sin y} \cdot b + \frac{\tan x + \tan y}{\sin z} \cdot c > 4 \cdot \sqrt[4]{27} \cdot \sqrt{F}$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

Solution 1 by Tapas Das-India

$$\frac{\tan y + \tan z}{\sin x} \ge \frac{2\sqrt{\tan y \tan z}}{\sin x} \ (\textit{and analogs})$$

$$\frac{\tan y + \tan z}{\sin x} \cdot a + \frac{\tan z + \tan x}{\sin y} \cdot b + \frac{\tan x + \tan y}{\sin z} \cdot c \ge$$

$$\ge \frac{2\sqrt{\tan y \tan z}}{\sin x} \cdot a + \frac{2\sqrt{\tan z \tan x}}{\sin y} \cdot b + \frac{2\sqrt{\tan x \tan y}}{\sin z} \cdot c >$$

$$> 6 \cdot \sqrt[3]{abc \cdot \frac{\tan x \tan y \tan z}{\sin x \sin y \sin z}} = 6 \cdot \sqrt[3]{\sec x \sec y \sec z} \cdot \sqrt[3]{abc} >$$

$$> 6 \cdot \sqrt[3]{abc} = 6 \cdot \sqrt[3]{\sqrt{\left(\frac{4F}{\sqrt{3}}\right)^3}} = 4\sqrt{F} \cdot \sqrt[4]{27}$$

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

Let $\tan x = n, \tan y = m, \tan z = p$, then:

$$\frac{m+p}{n} \cdot a + \frac{p+n}{m} \cdot b + \frac{n+m}{p} \cdot c > 4\sqrt{F} \cdot \sqrt[4]{27}$$

$$\sum_{cyc} \left(\frac{m+n+p}{n} \cdot a\right) - 2p = \sum_{cyc} n \cdot \sum_{cyc} \frac{a}{n} - 2p \overset{CBS}{\ge}$$

$$\geq\left(\sum_{cyc}\sqrt{a}\right)^2-2p=2p+2\sum_{cyc}\sqrt{ab}-2p=2\sum_{cyc}\sqrt{ab}=2\sum_{cyc}\sqrt{\frac{2F}{\sin C}}=$$

$$=2\sqrt{2}\cdot 3\sqrt{3}\cdot\sqrt{F}\sum_{cyc}\frac{(1)^{\frac{3}{2}}}{(\sin C)^{\frac{1}{2}}}\overset{Radon}{\geq}2\sqrt{2}\cdot 3\sqrt{3}\cdot\sqrt{F}\cdot\frac{1}{(\sum\sin A)^{\frac{1}{2}}}=$$

$$=2\sqrt{2}\cdot 3\sqrt{3}\cdot\sqrt{F}>4\sqrt{F}\cdot\sqrt[4]{27}$$

Solution 3 by Debopriyo Dawn-India

$$\frac{\tan y+\tan z}{\sin x}\geq\frac{2\sqrt{\tan y\tan z}}{\sin x}\ (and\ analogs)$$

$$\frac{\tan y+\tan z}{\sin x}\cdot a+\frac{\tan z+\tan x}{\sin y}\cdot b+\frac{\tan x+\tan y}{\sin z}\cdot c\geq$$

$$\geq\frac{2\sqrt{\tan y\tan z}}{\sin x}\cdot a+\frac{2\sqrt{\tan z\tan x}}{\sin y}\cdot b+\frac{2\sqrt{\tan x\tan y}}{\sin z}\cdot c>$$

$$>6\cdot\sqrt[3]{abc\cdot\frac{\tan x\tan y\tan z}{\sin x\sin y\sin z}}=6\cdot\sqrt[3]{\sec x\sec y\sec z}\cdot\sqrt[3]{abc}>$$

$$>6\cdot\sqrt[3]{abc}=6\cdot\sqrt[3]{\sqrt{\left(\frac{4F}{\sqrt{3}}\right)^3}}=4\sqrt{F}\cdot\sqrt[4]{27}$$

626. In ΔABC the following relationship holds:

$$\frac{\sin\frac{3A}{2}}{\cos\frac{A}{2}}+\frac{\sin\frac{3B}{2}}{\cos\frac{B}{2}}+\frac{\sin\frac{3C}{2}}{\cos\frac{C}{2}}\leq\frac{9R}{s}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Tapas Das-India

$$\text{Let } f(x)=\frac{\sin 3x}{\cos x}\Rightarrow f'(x)=\frac{\cos 4x+2\cos 2x}{\cos^2 x}$$

$$f''(x)=\frac{-4(\cos^2 x\sin 4x+\cos^2 x\cos 2x)+(\cos 4x\sin 2x+\sin 4x)}{\cos^4 x}\leq 0$$

Because: $-4(\cos^2 x\sin 4x+\cos^2 x\cos 2x)+(\cos 4x\sin 2x+\sin 4x)=$

$$= \sin 4x\,(1 - 4\cos^2 x) + \cos 4x \sin 2x - 4\sin^2 x \cos 2x =$$

$$= -\sin 4x\,(1 + \cos 2x) - \cos 4x\,(1 - \sin 2x) - 2\cos 2x < 0$$

$\Rightarrow f$ —concave on $\left(0, \frac{\pi}{2}\right)$. Using Jensen's inequality, we get:

$$f\left(\frac{A}{2}\right) + f\left(\frac{B}{2}\right) + f\left(\frac{C}{2}\right) \le 3f\left(\frac{A+B+C}{6}\right) = 3f\left(\frac{\pi}{6}\right)$$

$$\frac{\sin\frac{3A}{2}}{\cos\frac{A}{2}} + \frac{\sin\frac{3B}{2}}{\cos\frac{B}{2}} + \frac{\sin\frac{3C}{2}}{\cos\frac{C}{2}} \le 2\sqrt{3}$$

We need to prove that: $2\sqrt{3} \le \frac{9R}{s} \Leftrightarrow 2s \le 3\sqrt{3}R$; $(Mitrinovic)$

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\sum_{cyc} \cot A = \frac{\sum a^2}{4F};(1), \qquad 2\sum_{cyc} \frac{1}{\sin A} = \frac{\sum ab}{2F};(2), \qquad 2\sum_{cyc} \sin A = \frac{2s}{R};(3)$$

$$\sum_{cyc} \frac{\sin\frac{3A}{2}}{\cos\frac{A}{2}} = \sum_{cyc} \frac{2\sin\frac{A}{2}\sin\frac{3A}{2}}{\sin A} = \sum_{cyc} \frac{\cos A - \cos 2A}{\cos\frac{A}{2}} =$$

$$= \sum_{cyc} \cot A + 2\sum_{cyc} \sin A - \sum_{cyc} \frac{1}{\sin A} \overset{(1)-(3)}{=} \frac{\sum a^2 - 2\sum ab}{4F} + \frac{2s}{R} =$$

$$= -\frac{4R+r}{s} + \frac{2s}{R} \overset{(*)}{\le} \frac{9R}{s}, \qquad (*) \Leftrightarrow 13R^2 + Rr \ge 2s^2$$

But $s^2 \le 4R^2 + 4Rr + 3r^2$ $(Gerretsen)$. So, $(*) \Leftrightarrow$

$$2s^2 \le 8R^2 + 8Rr + 6r^2 \le 13R^2 + Rr \Leftrightarrow$$

$$5R62 - 7Rr - 6r^2 \ge 0 \Leftrightarrow (R - 2r)(5R + 3r) \ge 0(Euler)$$

627. If $x, y, z > 0$ such that $xy + yz + zx = 3$ then in acute ΔABC holds:

$$\sum_{cyc} \frac{\tan^{n+1} A \tan^{n+1} B}{x^n y^n} \ge 3^{n+2}, n \in \mathbb{N}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Marian Ursărescu-Romania

$$\sum_{cyc} \frac{\tan^{n+1} A \tan^{n+1} B}{x^n y^n} \overset{Holder}{\ge} \frac{(\sum \tan A \tan B)^{n+1}}{(\sum xy)^n} = \frac{(\sum \tan A \tan B)^{n+1}}{3^n};(1)$$

In acute ΔABC: $\tan A, \tan B, \tan C > 0 \Rightarrow \sum \tan A \tan B \geq 3\sqrt[3]{\tan A \tan B \tan C}$; (2)

$$\text{But } \sum \tan A = \prod \tan A \Rightarrow \sum \tan A \geq 3\sqrt[3]{\prod \tan A}$$

$$\Rightarrow \sum \tan A = \prod \tan A \geq 3\sqrt{3}; (3)$$

From (2) and (3), we get:

$$\sum_{cyc} \tan A \tan B \geq 3^2; (4)$$

From (1) and (4), we get:

$$\sum_{cyc} \frac{\tan^{n+1} A \tan^{n+1} B}{x^n y^n} \geq \frac{3^{2n+2}}{3^n} = 3^{n+2}, n \in \mathbb{N}$$

Solution 2 by Tapas Das-India

$$\sum_{cyc} \frac{\tan^{n+1} A \tan^{n+1} B}{x^n y^n} = \sum_{cyc} \frac{(\tan A \tan B)^{n+1}}{(xy)^n} \geq 3 \cdot \sqrt[3]{\frac{(\tan^2 A \tan^2 B \tan^2 C)^{n+1}}{(x^2y^2z^2)^n}} =$$

$$= 3 \cdot \frac{(\tan A \tan B \tan C)^{\frac{2n+2}{3}}}{(xyz)^{\frac{2n}{3}}} \geq 3 \cdot (\tan A \tan B \tan C)^{\frac{2n+2}{3}} \geq ; \left(\because (xyz)^{\frac{2}{3}} \leq 1\right)$$

$$\geq 3\left(3\sqrt{3}\right)^{\frac{2n+2}{3}} = 3^{n+2}$$

$$xy + yz + zx \geq 3(xyz)^{\frac{2}{3}} \Rightarrow 1 \geq (xyz)^{\frac{2}{3}}$$

Let $f(x) = \tan x$ is convex on $\left(0, \frac{\pi}{2}\right)$ and using Jensen's inequality, we have:

$$\tan A + \tan B + \tan C \geq 3 \tan \frac{A+B+C}{3} = 3\sqrt{3}$$

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C$$

628. If $x, y > 0$ then in ΔABC holds:

$$(x^2 + y^2)(a^2 + b^2 + x^2) \geq 8xy\sqrt{3} \cdot F + (xa - yb)^2 + (xb - yc)^2 + (xc - ya)^2$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

Solution by Tapas Das-India

$$(x^2 + y^2)(a^2 + b^2 + x^2) - (xa - yb)^2 - (xb - yc)^2 - (xc - ya)^2 =$$

$$= 2xy(ab + bc + ca) \geq 2xy \cdot 3 \cdot \sqrt[3]{(abc)^2} \overset{Carlitz}{\geq} 6xy\sqrt[3]{\left(\frac{4F}{\sqrt{3}}\right)^3} =$$

$$= 6xy \cdot \frac{4F}{\sqrt{3}}$$

Hence,

$$(x^2+y^2)(a^2+b^2+x^2)-(xa-yb)^2-(xb-yc)^2-(xc-ya)^2 \geq 8xy\sqrt{3}\cdot F$$

Therefore,

$$(x^2+y^2)(a^2+b^2+x^2) \geq 8xy\sqrt{3}\cdot F+(xa-yb)^2+(xb-yc)^2+(xc-ya)^2$$

629. If $t \in (-\infty, 0] \cup [1, \infty)$, then prove that in any ΔABC holds:

$$\left(\frac{r_ar_b}{h_c}\right)^t+\left(\frac{r_br_c}{h_a}\right)^t+\left(\frac{r_cr_a}{h_b}\right)^t \geq \frac{(4R+r)^t}{3^{t-1}}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

Solution by Soumitra Mandal-Chandar Nagore-India

We know that:

$$\frac{1}{x^t}+\frac{1}{y^t}+\frac{1}{z^t} \geq \frac{1}{3^{t-1}}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^t, \text{where } t \in (-\infty, 0] \cup [1, \infty)$$

$$h_a=\frac{2F}{a}, h_b=\frac{2F}{b}, h_c=\frac{2F}{c}, r_a=\frac{F}{s-a}, r_b=\frac{F}{s-b}, r_c=\frac{F}{s-c}$$

$$\sum_{cyc}\left(\frac{r_ar_b}{h_c}\right)^t=(r_ar_br_c)^t\sum_{cyc}\frac{1}{(r_ah_a)^t}=\frac{(r_ar_br_c)^t}{3^{t-1}}\left(\sum_{cyc}\frac{1}{r_ah_a}\right)^t=$$

$$=\frac{(r_ar_br_c)^t}{3^{t-1}}\left(\sum_{cyc}\frac{1}{\frac{2F}{a}\cdot\frac{F}{s-a}}\right)^t=$$

$$=\frac{1}{3^{t-1}}\left(\frac{F^3}{(s-a)(s-b)(s-c)}\right)^t\left(\frac{as+bs+cs-a^2-b^2-c^2}{2F^2}\right)^t=$$

$$=\frac{1}{3^{t-1}}\left(\frac{\frac{F}{2}}{s-a)(s-b)(s-c)}\right)^t[2s^2-2(s^2-4Rr-3r^2)]^t=$$

$$=\frac{1}{3^{t-1}}\cdot\left(\frac{\frac{F}{2}}{\frac{F^2}{s}}\right)^t(8Rr+6r^2)^t=\frac{1}{3^{t-1}}\left(\frac{s}{2F}\right)^t(8Rr+6r^2)^t=\frac{(4R+3r)^t}{3^{t-1}}$$

630. If $m,t\geq 0$ and $x,y,z>0$ then in ΔABC holds:

$$\sum_{cyc}\frac{x+y}{z}\cdot\frac{a^{m+1}b^{t+1}}{c^{m+t}}\geq 8\sqrt{3}\cdot F$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

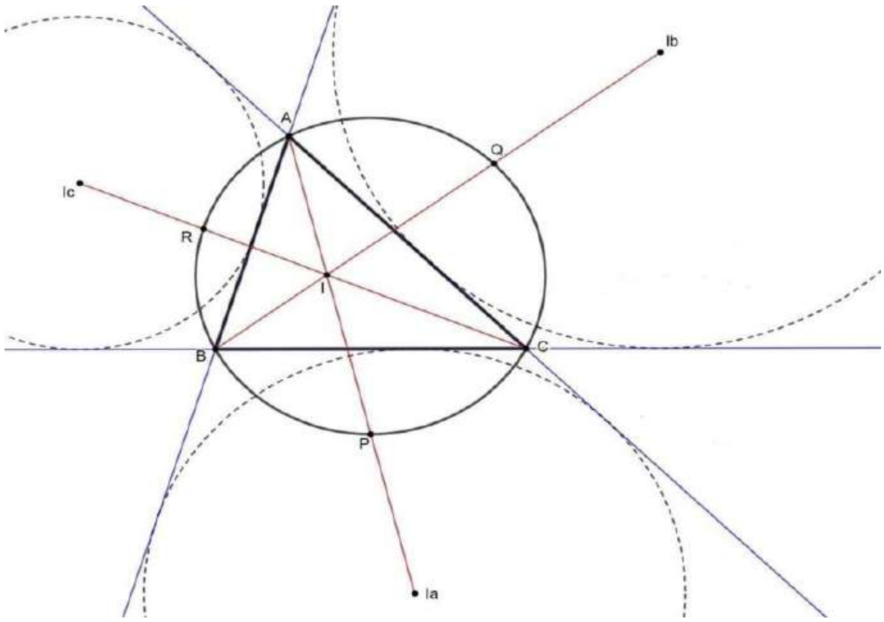
Solution by Tapas Das-India

$$\sum_{cyc}\frac{x+y}{z}\cdot\frac{a^{m+1}b^{t+1}}{c^{m+t}}\overset{AGM}{\geq}\sum_{cyc}\frac{2\sqrt{xy}}{z}\cdot\frac{a^{m+1}b^{t+1}}{c^{m+t}}\overset{AGM}{\geq}$$

$$\geq 3\cdot\sqrt[3]{8\cdot\frac{xyz}{xyz}\cdot\frac{a^{m+t}b^{m+t}c^{m+t}}{a^{m+t}b^{m+t}c^{m+t}}(abc)^2}=6\cdot\sqrt[3]{(abc)^2}\overset{CARLITZ}{\geq}6\cdot\sqrt[3]{\left(\frac{4F}{\sqrt{3}}\right)^{\frac{3}{2}\cdot 2}}=8\sqrt{3}\cdot F$$

631. Prove:

$$\sum_{cyc}\frac{II_a}{IQ}\cdot\sum_{cyc}\left(\frac{II_a}{IQ}\right)^2\geq 72$$



Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by proposer

$$(II_a)^2=\frac{abc\cdot a}{s(s-a)},\qquad IQ^2=\frac{abc\cdot b}{4s(s-a)},\quad\left(\frac{II_a}{IQ}\right)^2=\frac{4a(s-b)}{b(s-a)}$$

$$\sum_{cyc}\frac{II_a}{IQ}\ge 3\sqrt[3]{2^3\sqrt{\frac{abc(s-a)(s-b)(s-c)}{abc(s-a)(s-b)(s-c)}}}=3\cdot 2=6$$

$$\sum_{cyc}\left(\frac{II_a}{IQ}\right)^2=3\sqrt[3]{4^3\cdot\frac{abc(s-a)(s-b)(s-c)}{abc(s-a)(s-b)(s-c)}}=3\cdot 4=12$$

Therefore,

$$\sum_{cyc}\frac{II_a}{IQ}\cdot\sum_{cyc}\left(\frac{II_a}{IQ}\right)^2\ge 72$$

632. If $t\in(-\infty,0]\cup[1,\infty)$ then prove that in any triangle ABC holds:

$$\left(\frac{r_ar_b}{h_c}\right)^t+\left(\frac{r_br_c}{h_a}\right)^t+\left(\frac{r_cr_a}{h_b}\right)^t\ge\frac{(4R+r)^t}{3^{t-1}}$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu-Romania

Solution 1 by Soumitra Mandal-Chandar-Nagore-India

We know that:

$$\frac{1}{x^t}+\frac{1}{y^t}+\frac{1}{z^t}\ge\frac{1}{3^{t-1}}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^t,\text{where } t\in(-\infty,0]\cup[1,\infty)$$

$$h_a=\frac{2F}{a},h_b=\frac{2F}{b},h_c=\frac{2F}{c}$$

$$r_a=\frac{F}{s-a},r_b=\frac{F}{s-b},r_c=\frac{F}{s-c}$$

$$\sum_{cyc}\left(\frac{r_ar_b}{h_c}\right)^t=(r_ar_br_c)^t\sum_{cyc}\frac{1}{(r_ah_a)^t}=\frac{(r_ar_br_c)^t}{3^{t-1}}\left(\sum_{cyc}\frac{1}{r_ah_a}\right)^t=$$

$$=\frac{(r_ar_br_c)^t}{3^{t-1}}\left(\sum_{cyc}\frac{1}{\frac{2F}{a}\cdot\frac{F}{s-a}}\right)^t=\frac{\left(\frac{F^3}{(s-a)(s-b)(s-c)}\right)^t}{3^{t-1}}\cdot\left(\frac{as+bs+cs-a^2-b^2-c^2}{2F^2}\right)^t=$$

$$=\frac{\left(\frac{\frac{F}{2}}{(s-a)(s-b)(s-c)}\right)^t}{3^{t-1}}\{2s^2-2(s^2-4Rr-3r^2)\}^t=$$

$$=\frac{\left(\frac{\frac{F}{2}}{\frac{F^2}{s}}\right)^t}{3^{t-1}}(8Rr+6r^2)^t=\frac{\left(\frac{s}{2F}\right)^t}{3^{t-1}}(8Rr+6r^2)^t=\frac{(4R+3r)^t}{3^{t-1}}$$

Solution 2 b y Tapas Das-India

$$F^2=s(s-a)(s-b)(s-c);(1)$$

$$\frac{F^2}{s}=(s-a)(s-b)(s-c);(2)$$

$$F=sr;(3)$$

$$\frac{x_1^m+x_2^m+\cdots+x_n^m}{n}\ge\left(\frac{x_1+x_2+\cdots+x_n}{n}\right)^m;(4)$$

$$c=2s-(a+b),b=2s-(c+a),a=2s-(b+c);(5)$$

$$\frac{r_ar_b}{h_c}=\frac{F}{s-a}\cdot\frac{F}{s-b}\cdot\frac{c}{2F}=\frac{cF}{2(s-a)(s-b)}\ (and\ analogs)$$

$$\sum_{cyc}\frac{r_ar_b}{h_c}=\frac{F}{2}\sum_{cyc}\frac{c}{(s-a)(s-b)}=\frac{F}{2}\sum_{cyc}\frac{2s-(a+b)}{(s-a)(s-b)}=$$

$$=\frac{F}{2}\cdot2\sum_{cyc}\frac{1}{s-a}=F\cdot\frac{3s^2-4s^2+(ab+bc+ca)}{\frac{F^2}{s}}=$$

$$=\frac{s}{F}[(ab+bc+ca)-s^2]=\frac{s}{rs}(s^2+r^2+4Rr-s^2)=\frac{s}{rs}(r^2+4Rr)$$

$$\sum_{cyc}\left(\frac{r_ar_b}{h_c}\right)^t\overset{Holder}{\ge}\frac{1}{3^{t-1}}\left(\sum_{cyc}\frac{r_ar_b}{h_c}\right)^t=\frac{1}{3^{t-1}}(4R+r)^t$$

633. In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{w_a^2}{w_c}\le9r\left(\frac{R}{2r}\right)^8$$

Proposed by Marin Chirciu-Romania

Solution by Avishek Mitra-West Bengal-India

$$\sum_{cyc}\frac{w_a^2}{w_c}\overset{w_a\le\sqrt{s(s-a)};w_c\ge h_c}{\le}\sum_{cyc}\frac{s(s-a)}{h_c}=\frac{s}{2F}\sum_{cyc}c(s-a)=$$

$$=\frac{1}{2r}\left(s\cdot 2s-\sum_{cyc}ac\right)=\frac{1}{2r}(2s^2-s^2-r^2-4Rr)=$$

$$=\frac{s^2-4Rr-r^2}{2r}=\frac{1}{4r}\sum_{cyc}a^2\overset{Leibniz}{\leq}\frac{9R^2}{4r}$$

We need to show:

$$\frac{9R^2}{4r}\leq 9r\left(\frac{R}{2r}\right)^8\Leftrightarrow\frac{1}{(2r)^2}\leq\frac{R^6}{(2r)^8}\Leftrightarrow R^6\geq(2r)^6\Leftrightarrow R\geq 2r\ (Euler).$$

634. In acute ΔABC the following relationship holds:

$$\prod_{cyc}(1+\tan A\cot B)\geq 2+32F^2\prod_{cyc}\sqrt[3]{\frac{1}{(b^2+c^2-a^2)^2}}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Tapas Das-India

$$32F^2\prod_{cyc}\sqrt[3]{\frac{1}{(b^2+c^2-a^2)^2}}=32F^2\prod_{cyc}\sqrt[3]{\frac{4b^2c^2}{(b^2+c^2-a^2)^2}}\cdot\frac{1}{4\sqrt[3]{(abc)^4}}=$$

$$=32F^2\cdot\sqrt[3]{\frac{1}{(\cos A\cos B\cos C)^2}}\leq\frac{8F^2}{\left(\frac{4F}{\sqrt{3}}\right)^{\frac{3}{2}\cdot\frac{4}{3}}}\cdot\sqrt[3]{\frac{1}{(\cos A\cos B\cos C)^2}}=$$

$$=\frac{3}{2}\cdot\frac{1}{\sqrt[3]{(\cos A\cos B\cos C)^2}}\geq\left(\frac{4F}{\sqrt{3}}\right)^3$$

Let $\cos A\cos B\cos C=x$.

$$\prod_{cyc}(1+\tan A\cot B)=\prod_{cyc}\frac{\sin(A+B)}{\cos A\cos B}=\frac{\sin A\sin B\sin C}{\cos A\cos B\cos C\sin A\sin B\sin C}=$$

$$=\frac{1}{\cos A\cos B\cos C}$$

$$\because A+B+C=\pi\Rightarrow\sin(A+B)=\sin(\pi-C)=\sin C$$

$$\prod_{cyc}(1+\tan A\cot B)-2=\frac{1}{\cos A\cos B\cos C}-2=$$

$$=\frac{1-2\cos A\cos B\cos C}{\cos A\cos B\cos C}=\frac{\cos^2 A+\cos^2 B+\cos^2 C}{\cos A\cos B\cos C}\geq$$

$$\geq\frac{3}{4}\cdot\frac{1}{\cos A\cos B\cos C}=\frac{3}{4x}$$

$$\because \cos^2 A+\cos^2 B+\cos^2 C\geq\frac{3}{4}$$

We need to show:

$$\frac{3}{4x}\geq\frac{3}{2}\cdot\frac{2}{x^{\frac{2}{3}}}\Leftrightarrow x^{\frac{2}{3}}\geq 2x\Leftrightarrow 2x^{\frac{1}{3}}\leq 1\Leftrightarrow x\leq\frac{1}{8}\Leftrightarrow \cos A\cos B\cos C\leq\frac{1}{8}\text{true, because:}$$

$$\sin^2 A+\sin^2 B+\sin^2 C=2+2\cos A\cos B\cos C\leq 2+2\cdot\frac{1}{8}=\frac{9}{4}$$

$$\cos^2 A+\cos^2 B+\cos^2 C=3-(\sin^2 A+\sin^2 B+\sin^2 C)\geq\frac{3}{4}$$

$$\cos^2 A+\cos^2 B+\cos^2 C=\frac{1}{2}(1+\cos 2A+1+\cos 2B)+\cos^2 C=$$

$$=\frac{1}{2}(2+\cos 2A+\cos 2B)+\cos^2 C=1-\frac{1}{2}\cdot 2\cos(A+B)\cos(A-B)+\cos^2 C=$$

$$=1-\cos C\cos(A-B)+\cos^2 C=1-2\cos A\cos B\cos C$$

$$\prod_{cyc}(1+\tan A\cot B)-2\geq 32F^2\prod_{cyc}\sqrt[3]{\frac{1}{(b^2+c^2-a^2)^2}}\Leftrightarrow$$

$$\prod_{cyc}(1+\tan A\cot B)\geq 2+32F^2\prod_{cyc}\sqrt[3]{\frac{1}{(b^2+c^2-a^2)^2}}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\prod_{cyc}(1+\tan A\cot B)=\prod_{cyc}\left(1+\frac{\sin A.\cos B}{\cos A.\sin B}\right)=\prod_{cyc}\frac{\sin A.\cos B+\cos A.\sin B}{\cos A.\sin B}$$

$$=\frac{\prod_{cyc}\sin(A+B)}{(\prod_{cyc}\cos A)(\prod_{cyc}\sin B)}=\frac{\prod_{cyc}\sin C}{(\prod_{cyc}\cos A)(\prod_{cyc}\sin B)}=\frac{1}{\prod_{cyc}\cos A}$$

$$\boxed{\prod_{cyc}(1+\tan A\cot B)\overset{(i)}{=}\frac{1}{t}\left(t=\prod_{cyc}\cos A\right)}$$

$$\prod_{cyc}\sqrt[3]{\frac{1}{(b^2+c^2-a^2)^2}}=\frac{1}{\sqrt[3]{\prod_{cyc}(4b^2c^2\cos^2 A)}}=\frac{1}{\sqrt[3]{64(abc)^4.\prod_{cyc}\cos^2 A}}$$

$$=\frac{1}{4(\prod_{cyc}\cos A)^{\frac{2}{3}}.\sqrt[3]{(16R^2r^2s^2)^2}}=\frac{1}{4(\prod_{cyc}\cos A)^{\frac{2}{3}}.\sqrt[3]{(16R.R.r^2s^2)^2}}$$

$$\overset{\text{Euler + Mitrinovic}}{\leq} \frac{1}{4\left(\prod_{cyc}\cos A\right)^{\frac{2}{3}}.\sqrt[3]{\left(16.2r.\frac{2s}{3\sqrt{3}}r^2s^2\right)^2}} = \frac{1}{4\left(\prod_{cyc}\cos A\right)^{\frac{2}{3}}.\sqrt[3]{\left(\frac{64r^3s^3}{3\sqrt{3}}\right)^2}}$$

$$= \frac{3}{4\left(\prod_{cyc}\cos A\right)^{\frac{2}{3}}.16r^2s^2} \Rightarrow 2+32F^2\prod_{cyc}\sqrt[3]{\frac{1}{(b^2+c^2-a^2)^2}}$$

$$\leq 2+\frac{96r^2s^2}{64\left(\prod_{cyc}\cos A\right)^{\frac{2}{3}}.r^2s^2}$$

$$\Rightarrow \boxed{2+32F^2\prod_{cyc}\sqrt[3]{\frac{1}{(b^2+c^2-a^2)^2}} \overset{(ii)}{\leq} 2+\frac{3}{2t^{\frac{2}{3}}}} \therefore (i),(ii) \Rightarrow \text{it suffices to prove}: \frac{1}{t} \geq 2+\frac{3}{2t^{\frac{2}{3}}}$$

$$\Leftrightarrow \frac{1}{m^3} \geq 2+\frac{3}{2m^2}\left(t^{\frac{1}{3}}=m\right) \Leftrightarrow \frac{1}{m^3} \geq \frac{3+4m^2}{2m^2} \Leftrightarrow 2 \geq 3m+4m^3$$

$$\Leftrightarrow 4m^3+3m-2 \leq 0 \Leftrightarrow 4m^3-2m^2+2m^2-m+4m-2 \leq 0 \Leftrightarrow (2m-1)(2m^2+m+2) \leq 0 \Leftrightarrow 2m-1 \leq 0$$

$$\left(\because m = t^{\frac{1}{3}} = \sqrt[3]{\prod_{cyc}\cos A} > 0 \text{ for acute triangles} \Rightarrow 2m^2+m+2>0\right) \Leftrightarrow m = t^{\frac{1}{3}}$$

$$= \sqrt[3]{\prod_{cyc}\cos A} \leq \frac{1}{2} \to \text{true} \because \sqrt[3]{\prod_{cyc}\cos A} \overset{A-G}{\leq} \frac{\sum_{cyc}\cos A}{3} = \frac{1+\frac{r}{R}}{3} \overset{\text{Euler}}{\leq} \frac{1+\frac{1}{2}}{3} = \frac{1}{2}$$

$$\therefore \text{in any acute } \Delta\,ABC, \prod_{cyc}(1+\tan A\cot B) \geq 2+32F^2\prod_{cyc}\sqrt[3]{\frac{1}{(b^2+c^2-a^2)^2}} \text{ (QED)}$$

Solution 3 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\textit{We have}: \prod_{cyc}(1+\tan A\cot B) = 2+\sum_{cyc}\tan A\cot B+\sum_{cyc}\tan B\cot C.\tan C\cot A$$

$$= 2+\sum_{cyc}\frac{\tan A}{\tan B}+\sum_{cyc}\frac{\tan B}{\tan A}.$$

By AM − GM inequality we have :

$$\sum_{cyc}\frac{\tan A}{\tan B} = \sum_{cyc}\frac{1}{3}\left(\frac{\tan A}{\tan B}+\frac{\tan A}{\tan B}+\frac{\tan B}{\tan C}\right) \geq \sum_{cyc}\sqrt[3]{\frac{\tan^2 A}{\tan B.\tan C}}$$

$$= \frac{\tan A+\tan B+\tan C}{\sqrt[3]{\tan A.\tan B.\tan C}}$$

And since $\tan A+\tan B+\tan C = \tan A.\tan B.\tan C$ *then we get :*

$$\sum_{cyc}\frac{\tan A}{\tan B}\geq\sqrt[3]{(\tan A.\tan B.\tan C)^2}.\ \textit{Similarly we have}:\ \sum_{cyc}\frac{\tan B}{\tan A}\geq\sqrt[3]{(\tan A.\tan B.\tan C)^2}$$

$$\textit{Then}:\ \prod_{cyc}(1+\tan A\cot B)\geq 2+2\prod_{cyc}\sqrt[3]{\tan^2 A}=2+2\prod_{cyc}\sqrt[3]{\left(\frac{4F}{b^2+c^2-a^2}\right)^2}$$

$$=2+32F^2\prod_{cyc}\sqrt[3]{\frac{1}{(b^2+c^2-a^2)^2}}.$$

Equality holds iff ΔABC is equilateral.

635. In ΔABC the following relationship holds:

$$\frac{9R}{4r}-\frac{3}{2}\leq\sum_{cyc}\frac{m_a{}^2}{h_bh_c}\leq\frac{R^2}{r^2}+\frac{r}{2R}-\frac{5}{4}$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\textit{We have}:\ \sum_{cyc}\frac{m_a{}^2}{h_bh_c}=\sum_{cyc}\frac{(2b^2+2c^2-a^2)bc}{4.(2F)^2}=\frac{2\sum_{cyc}bc(b^2+c^2)-abc\sum_{cyc}a}{16F^2}=$$

$$=\frac{2(\sum_{cyc}ab)(\sum_{cyc}a^2)-3abc\sum_{cyc}a}{16F^2}$$

$$=\frac{2\left(s^2+r(4R+r)\right).2\left(s^2-r(4R+r)\right)-3.4Rsr.2s}{16F^2}$$

$$\textit{Then}:\ \sum_{cyc}\frac{m_a{}^2}{h_bh_c}=\frac{s^4-r^2(4R+r)^2-6s^2Rr}{4s^2r^2}=\frac{s^2-6Rr}{4r^2}-\frac{(4R+r)^2}{4s^2}$$

$$\textit{Now we have}:\ \sum_{cyc}\frac{m_a{}^2}{h_bh_c}\overset{Gerretsen}{\geq}\frac{(16Rr-5r^2)-6Rr}{4r^2}-\frac{(4R+r)^2}{4(16Rr-5r^2)}$$

$$=\frac{10R}{4r}-\frac{5}{4}-\frac{(4R+r)^2}{4r(16R-5r)}=$$

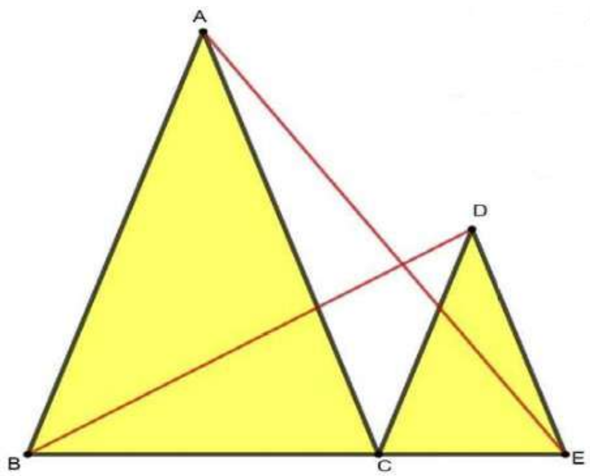
$$=\frac{9R}{4r}-\frac{3}{2}+\left(\frac{R}{4r}+\frac{1}{4}-\frac{(4R+r)^2}{4r(16R-5r)}\right)=\frac{9R}{4r}-\frac{3}{2}+\frac{3(R-2r)}{4(16R-5r)}\overset{Euler}{\geq}\frac{9R}{4r}-\frac{3}{2}.$$

$$\textit{Also we have}:\ \sum_{cyc}\frac{m_a{}^2}{h_bh_c}\overset{Gerretsen\ \&\ Blundon\ Gerretsen}{\leq}\frac{(4R^2+4Rr+3r^2)-6Rr}{4r^2}-\frac{1}{4}.\frac{2(2R-r)}{R}=$$

$$=\frac{R^2}{r^2}+\frac{r}{2R}-\frac{R}{2r}-\frac{1}{4}\overset{Euler}{\lesssim}\frac{R^2}{r^2}+\frac{r}{2R}-1-\frac{1}{4}=\frac{R^2}{r^2}+\frac{r}{2R}-\frac{5}{4}.$$

$$Therefore,\ \frac{9R}{4r}-\frac{3}{2}\le\sum_{cyc}\frac{m_a^{\ 2}}{h_bh_c}\le\frac{R^2}{r^2}+\frac{r}{2R}-\frac{5}{4}.$$

636.

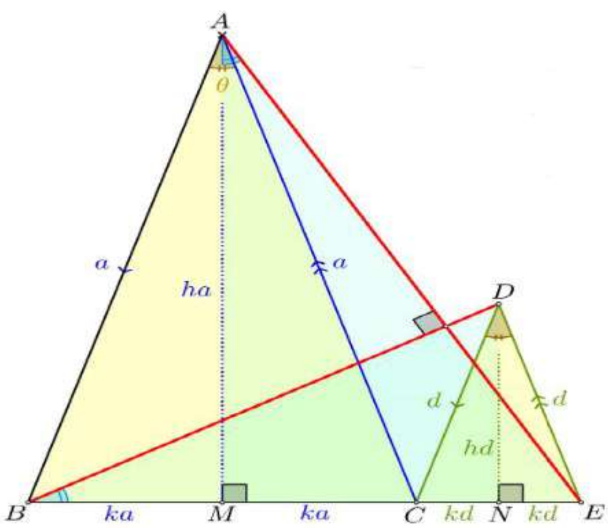


$AB=AC=a, DC=DE=d, AB\parallel DC, AC\parallel DE, AE\perp BD,$

$\sphericalangle BAC=\theta, S=[ABC]+[CDE](\text{ area}),$ a,d,θ –variables, a,d,S –positive integers. If $\theta=\theta_{max.}$, find $S=?$

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by Amirul Faiz-Japan



$$\Delta AME\sim\Delta BND\Rightarrow\frac{hd}{k(2a+d)}=\frac{k(a+2d)}{ha}$$

$$\tan^2\frac{\theta}{2}=\frac{k^2}{h^2}=\frac{ad}{2(a^2+d^2)+5ad}\le\frac{ad}{4ad+5ad}=\frac{1}{9}$$

Equality holds when $a=d\Rightarrow h^2=9k^2=9(1-h^2)$

$$h=\frac{3}{\sqrt{10}},k=\frac{1}{\sqrt{10}},S=2hka^2=\frac{3}{5}a^2$$

Since $a,d,S\in\mathbb{Z}_+\Rightarrow a=5n,S=15n^2$, where $n=1,2,3,\dots$

637. In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{\csc^4 A+(\csc B+\csc C)^4}{\csc B\csc C}\ge\frac{34R}{r}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $x,y,z>0$ *we have* :

$$\sum_{cyc}\frac{x^4+(y+z)^4}{yz}=\sum_{cyc}\frac{x^4}{yz}+\sum_{cyc}\frac{(y+z)^4}{yz}\overset{CBS\,\&\,AM-GM}{\ge}\frac{(x^2+y^2+z^2)^2}{yz+zx+xy}+\sum_{cyc}\frac{\left(2\sqrt{yz}\right)^4}{yz}\ge$$

$$\ge\frac{(xy+yz+zx)^2}{xy+yz+zx}+16\sum_{cyc}yz=17\sum_{cyc}yz.$$

$$Then:\ \sum_{cyc}\frac{x^4+(y+z)^4}{yz}\ge17\sum_{cyc}yz,\quad\forall x,y,z>0.$$

For $x=\csc A,y=\csc B,z=\csc C$ *we obtain* :

$$\sum_{cyc}\frac{\csc^4 A+(\csc B+\csc C)^4}{\csc B\csc C}\ge17\sum_{cyc}\csc B\csc C=17\sum_{cyc}\frac{4R^2}{bc}=17.\frac{4R^2.2s}{4Rsr}=\frac{34R}{r}.$$

Equality holds iff ΔABC is equilateral.

Solution 2 by Tapas Das-India

$$\frac{\csc^5 A+\csc^5 B+\csc^5 C}{3}\ge\frac{\csc^2 A+\csc^2 B+\csc^2 C}{3}\cdot\frac{\csc^3 A+\csc^3 B+\csc^3 C}{3}$$

$$\csc^5 A+\csc^5 B+\csc^5 C\ge(\csc^2 A+\csc^2 B+\csc^2 C)\sqrt[3]{\csc^3 A\csc^3 B\csc^3 C}$$

$$\csc^2 A+\csc^2 B+\csc^2 C\ge\csc^2 A\csc^2 B+\csc^2 B\csc^2 C+\csc^2 C\csc^2 A$$

$$\sum_{cyc}\frac{\csc^4 A+(\csc B+\csc C)^4}{\csc B\csc C}=\sum_{cyc}\frac{\csc^4 A}{\csc B\csc C}+\sum_{cyc}\frac{(\csc B+\csc C)^4}{\csc B\csc C}=$$

$$=\frac{1}{\prod\csc A}\sum_{cyc}\csc^5 A+\sum_{cyc}\frac{(\csc B+\csc C)^4}{\csc B\csc C}\geq$$

$$\geq\sum_{cyc}\csc A\csc B+\sum_{cyc}\frac{\left(2\sqrt{\csc A\csc B}\right)^4}{\csc B\csc C}=$$

$$=\sum_{cyc}\csc A\csc B+16\sum_{cyc}\frac{\csc^2 A\csc^2 B}{\csc A\csc B}=17\sum_{cyc}\csc A\csc B=$$

$$=17\sum_{cyc}\frac{1}{\sin A\sin B}=17\sum_{cyc}\frac{4R^2}{ab}=68R^2\cdot\frac{a+b+c}{abc}=68R^2\cdot\frac{2s}{abc}=34\cdot\frac{R}{r}$$

638. In any Δ ABC, the following relationship holds:

$$2\left(3-\frac{2r}{R}\right)\leq\sum_{cyc}\frac{a^2}{r_a^2}\leq\frac{2R}{r}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{a^2}{r_a^2}=\sum_{cyc}\frac{a^2(s-a)^2}{r^2s^2}=\frac{1}{r^2s^2}\sum_{cyc}a^2(s^2-2sa+a^2)$$

$$=\frac{1}{r^2s^2}\left(s^2\sum_{cyc}a^2-2s\sum_{cyc}a^3+\sum_{cyc}a^4\right)$$

$$=\frac{1}{r^2s^2}\left(2s^2(s^2-4Rr-r^2)-4s^2(s^2-6Rr-3r^2)+2\sum_{cyc}a^2b^2-16r^2s^2\right)$$

$$=\frac{1}{r^2s^2}\Big(2s^2(s^2-4Rr-r^2)-4s^2(s^2-6Rr-3r^2)+2\left(s^2+4Rr+r^2\right)^2-32Rrs^2$$

$$-16r^2s^2\Big)=2.\frac{(4R+r)^2-s^2}{s^2}\Rightarrow\sum_{cyc}\frac{a^2}{r_a^2}\overset{(*)}{=}2.\frac{(4R+r)^2-s^2}{s^2}$$

$$(*)\Rightarrow\sum_{cyc}\frac{a^2}{r_a^2}\leq\frac{2R}{r}\Leftrightarrow\frac{(4R+r)^2-s^2}{s^2}\leq\frac{R}{r}\Leftrightarrow(R+r)s^2\overset{(**)}{\geq}r(4R+r)^2$$

$$\text{Now}, (R+r)s^2 \overset{\text{Gerretsen}}{\geq} (R+r)(16Rr-5r^2) \overset{?}{\geq} r(4R+r)^2 \Leftrightarrow 3r(R-2r) \overset{?}{\geq} 0$$

$$\to \text{true via Euler} \Rightarrow (**) \text{ is true} \Rightarrow \sum_{cyc} \frac{a^2}{r_a^2} \leq \frac{2R}{r}$$

$$\text{Again}, 2\left(3-\frac{2r}{R}\right) \leq \sum_{cyc} \frac{a^2}{r_a^2} \Leftrightarrow 2.\frac{(4R+r)^2-s^2}{s^2} \geq 2\left(3-\frac{2r}{R}\right) \Leftrightarrow \frac{(4R+r)^2-s^2}{s^2} \geq \frac{3R-2r}{R}$$

$$\Leftrightarrow R(4R+r)^2 \overset{(***)}{\geq} (4R-2r)s^2$$

$$\text{Now, RHS of } (4R-2r)s^2 \overset{\text{Rouche}}{\leq} (4R-2r)\left(2R^2+10Rr-r^2+2(R-2r)\sqrt{R^2-2Rr}\right) \overset{?}{\leq} R(4R+r)^2$$

$$\Leftrightarrow R(4R+r)^2-(2R^2+10Rr-r^2)(4R-2r) \overset{?}{\underset{\sim}{\geq}} 2(4R-2r)(R-2r)\sqrt{R^2-2Rr}$$

$$\Leftrightarrow (R-2r)(8R^2-12Rr+r^2) \underset{(i)}{\overset{?}{\geq}} 2(4R-2r)(R-2r)\sqrt{R^2-2Rr}$$

$$\because R-2r \overset{\text{Euler}}{\geq} 0 \therefore \text{in order to prove (i), it suffices to prove}: 8R^2-12Rr+r^2 > 2(4R-2r)\sqrt{R^2-2Rr} \Leftrightarrow (8R^2-12Rr+r^2)^2-4(R^2-2Rr)(4R-2r)^2 > 0 \Leftrightarrow r^2(4R+r)^2 > 0$$

$$\to \text{true} \Rightarrow \text{(i) is true} \therefore (4R-2r)s^2 \leq R(4R+r)^2 \Rightarrow (***) \text{ is true} \Rightarrow 2\left(3-\frac{2r}{R}\right) \leq \sum_{cyc} \frac{a^2}{r_a^2}$$

$$\therefore 2\left(3-\frac{2r}{R}\right) \leq \sum_{cyc} \frac{a^2}{r_a^2} \leq \frac{2R}{r} \text{ (QED)}$$

639. Let be G −centroid of $\Delta ABC, x, y, z$ altitudes from G to BC, CA, AB, respectively. Prove that:

$$\frac{54}{F} \leq \frac{b+c-a}{x^3}+\frac{c+a-b}{y^3}+\frac{a+b-c}{z^3} \leq \frac{27}{F}\cdot\frac{R}{r}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have: \ x = \frac{2F(\Delta BGC)}{a} = \frac{2F}{3a} \ (and\ analogs).\ Then:$$

$$\sum_{cyc} \frac{b+c-a}{x^3} = \left(\frac{3}{2F}\right)^3 . \sum_{cyc} a^3(b+c-a).$$

$$We\ have: \sum_{cyc} a^3(b+c-a) = \sum_{cyc} ab(a^2+b^2) - \sum_{cyc} a^4 \overset{AM-GM}{\geq} 2\sum_{cyc} a^2b^2 - \sum_{cyc} a^4$$
$$= 16F^2.$$

$$Then: \sum_{cyc} \frac{b+c-a}{x^3} \geq \left(\frac{3}{2F}\right)^3 . 16F^2 = \frac{54}{F} \quad (1)$$

$$Now\ we\ have: \sum_{cyc} a^3(b+c-a) = \sum_{cyc} a^2.[bc-(a-b)(a-c)]$$
$$= abc\sum_{cyc} a - \sum_{cyc} a^2(a-b)(a-c) \leq$$

$$\overset{Schur}{\leq} 4FR.2s - 0 = 8FRs. \quad Then: \sum_{cyc} \frac{b+c-a}{x^3} \leq \left(\frac{3}{2F}\right)^3 .8FRs = \frac{27}{F}.\frac{R}{r} \quad (2)$$

From (1) and (2) yields the desired inequality.

Equality holds iff ΔABC is equilateral.

$$w_a \leq \sqrt{r_b r_c} = \sqrt{s(s-a)} \leq m_a$$

First, we prove that $a^5+b^5 \geq a^2b^2(a+b),\ \forall a,b>0.$

$$a^5+b^5-a^2b^2(a+b) = a^3(a^2-b^2)-b^3(a^2-b^2) = (a^3-b^3)(a^2-b^2) =$$
$$= (a-b)^2(a^2+ab+b^2)(a+b) \geq 0.\ alors: a^5+b^5 \geq a^2b^2(a+b),\ \forall a,b>0.$$

$$So: \frac{ab}{a^5+b^5+ab} \leq \frac{ab}{a^2b^2(a+b)+ab} = \frac{1}{ab(a+b)+1} = \frac{c}{a+b+c}$$

$$Analogously: \frac{bc}{b^5+c^5+bc} \leq \frac{a}{a+b+c}\ \ et\ \ \frac{ca}{c^5+a^5+ca} \leq \frac{b}{a+b+c}.$$

$$D'où, \frac{ab}{a^5+b^5+ab} + \frac{bc}{b^5+c^5+bc} + \frac{ca}{c^5+a^5+ca}$$
$$\leq \frac{c}{a+b+c} + \frac{a}{a+b+c} + \frac{b}{a+b+c} = \frac{a+b+c}{a+b+c} = 1.$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\text{Now}, \frac{1}{2}.x.\text{BC} = \frac{\text{F}}{3} \Rightarrow \frac{1}{x} = \frac{3a}{2\text{rs}}\ and\ analogs \Rightarrow \frac{\text{b}+\text{c}-a}{x^3} + \frac{\text{c}+a-\text{b}}{\text{y}^3} + \frac{a+\text{b}-\text{c}}{\text{z}^3}$$

$$= \sum_{\text{cyc}} \frac{54a^3(\text{s}-a)}{8\text{r}^3\text{s}^3} = \frac{27}{4\text{r}^3\text{s}^3}\left(\text{s}\sum_{\text{cyc}} a^3 - \sum_{\text{cyc}} a^4\right)$$

$$=\frac{27}{4r^3s^3}\left(2s^2(s^2-6Rr-3r^2)+16r^2s^2-2\sum_{cyc}a^2b^2\right)$$

$$=\frac{27}{4r^3s^3}\left(2s^2(s^2-6Rr-3r^2)+16r^2s^2-2(s^2+4Rr+r^2)^2+32Rrs^2\right)$$

$$=\frac{27r\left((2R+3r)s^2-r(4R+r)^2\right)}{2r^3s^3}$$

$$\therefore\frac{b+c-a}{x^3}+\frac{c+a-b}{y^3}+\frac{a+b-c}{z^3}\overset{(*)}{=}\frac{27\left((2R+3r)s^2-r(4R+r)^2\right)}{2r^2s^3}\ \therefore (*)$$

$$\Rightarrow\frac{b+c-a}{x^3}+\frac{c+a-b}{y^3}+\frac{a+b-c}{z^3}\le\frac{27}{F}\cdot\frac{R}{r}$$

$$\Leftrightarrow\frac{27\left((2R+3r)s^2-r(4R+r)^2\right)}{2r^2s^3}\le\frac{27R}{r^2s}$$

$$\Leftrightarrow 2Rs^2\ge(2R+3r)s^2-r(4R+r)^2\Leftrightarrow(4R+r)^2\ge 3s^2\to \text{true via Trucht}$$

$$\therefore\boxed{\frac{b+c-a}{x^3}+\frac{c+a-b}{y^3}+\frac{a+b-c}{z^3}\le\frac{27}{F}\cdot\frac{R}{r}}$$

$$\text{Again, } (*)\Rightarrow\frac{54}{F}\le\frac{b+c-a}{x^3}+\frac{c+a-b}{y^3}+\frac{a+b-c}{z^3}\Leftrightarrow\frac{27r\left((2R+3r)s^2-r(4R+r)^2\right)}{2r^3s^3}$$

$$\ge\frac{54}{rs}\Leftrightarrow(2R-r)s^2\overset{(i)}{\ge}r(4R+r)^2$$

$$\text{Now, }(2R-r)s^2\overset{Gerretsen}{\ge}(2R-r)(16Rr-5r^2)\overset{?}{\ge}r(4R+r)^2\Leftrightarrow 8R^2-17Rr+2r^2\overset{?}{\ge}0$$

$$\Leftrightarrow(R-2r)(8R-r)\overset{?}{\ge}0\to\text{true}\because R\overset{Euler}{\ge}2r\Rightarrow(i)\text{ is true}$$

$$\therefore\boxed{\frac{54}{F}\le\frac{b+c-a}{x^3}+\frac{c+a-b}{y^3}+\frac{a+b-c}{z^3}}\ (QED)$$

640. In $\Delta ABC, R_a, R_b$ and R_c are the circumradius of $\Delta BIC, \Delta CIA$ and ΔAIB respectively. Prove that:

$$a\cdot\frac{r_a}{R_a}+b\cdot\frac{r_b}{R_b}+c\cdot\frac{r_c}{R_c}\le\frac{9\sqrt{3}}{2}R$$

Proposed Ertan Yildirim-Izmir-Turkiye

Solution by Aggeliki Papaspyropoulou-Greece

$$AI = \frac{r}{\sin\frac{A}{2}};(1)$$

$$r_a = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2};(2)$$

$$[IBC] = \frac{ar}{2} = \frac{a\cdot IB\cdot IC}{4R_a} \Rightarrow R_a = \frac{IB\cdot IC}{2r};(3)$$

From (1), (2) and (3), it follows:

$$a\cdot\frac{r_a}{R_a} = a\cdot r_a\cdot\frac{2r}{IB\cdot IC} = 2r\cdot\frac{a\cdot r_a}{IB\cdot IC} = \left(2r\cdot a\cdot 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\right)\cdot\frac{1}{\frac{r}{\sin\frac{B}{2}}\cdot\frac{r}{\sin\frac{C}{2}}} =$$

$$= \frac{2R}{r}\cdot a\cdot\sin\frac{A}{2}\sin B\sin C = \frac{4Rr\cdot s}{2Rr}\cdot\sin\frac{A}{2} = 2s\cdot\sin\frac{A}{2};(4)$$

So, we have to prove:

$$2s\left(\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\right) \le \frac{9\sqrt{3}}{2}R$$

But $2s \le 3\sqrt{3}R\ (Mitrinovic)$**, so we need to prove:**

$$\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2} \le \frac{3}{2}$$

Let $f(x) = \sin x, x \in \left(0,\frac{\pi}{2}\right), f'(x) = \cos x, f''(x) = -\sin x < 0 \Rightarrow f$ **–convace function,**

hence by Jensen's inequality, we have:

$$\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2} \le 3\sin\frac{\pi}{6} = \frac{3}{2}$$

Equality holds for $a = b = c$.

641. In $\Delta ABC, AE$ **–internal bisector,** d **–antiparallel through the incenter,**

$d \cap [AB] = \{K\}, d \cap [AC] = \{L\}$**. Prove that:**

$$\frac{KI}{AE}+\frac{LI}{CE} \ge \frac{b+c}{s}$$

Proposed by Mehmet Şahin-Ankara-Turkiye

Solution by Marian Ursărescu-Romania

From bisector theorem:

$$\frac{BE}{EC}=\frac{AB}{AC}=\frac{c}{b}\Rightarrow\frac{BE}{a}=\frac{c}{b+c}\Rightarrow BE=\frac{ac}{b+c};(1)$$

$$\text{Again:}\frac{AI}{IE}=\frac{AB}{BE}=\frac{c}{\frac{ac}{b+c}}=\frac{b+c}{a}\Rightarrow\frac{AI}{AA'}=\frac{b+c}{2s}\Rightarrow AI=\frac{b+c}{2s}\cdot w_a;(2)$$

$$w_a=\frac{2bc}{b+c}\cdot\cos\frac{A}{2};(3)\ \text{From }(2),(3): AI=\frac{bc\cos\frac{A}{2}}{s};(4)$$

In ΔAKI from sines theorem:

$$\frac{AI}{\sin C}=2R_{\Delta KAI}\overset{(3)}{\Rightarrow}2R_{\Delta KAI}=\frac{bc\cos\frac{A}{2}}{s\sin C}=\frac{2R\cdot b\cdot\cos\frac{A}{2}}{s}\Rightarrow R_{\Delta KAI}=\frac{Rb\cos\frac{A}{2}}{s};(5)$$

$$\text{Again in }\Delta AKI:\frac{KI}{\sin\frac{A}{2}}=2R_{\Delta AKI}\Rightarrow KI=\frac{2Rb\cos\frac{A}{2}\sin\frac{A}{2}}{s}=\frac{Rb\sin A}{s}$$

$$\text{But:}\frac{a}{\sin A}=2R\Rightarrow\sin A=\frac{a}{2R}.\text{Hence}, KI=\frac{ab}{2s};(6)$$

From (1), (6) we get:

$$\frac{KI}{BE}=\frac{ab}{2s}\cdot\frac{b+c}{ac}=\frac{b(b+c)}{2sc};(7)$$

Similarly:

$$\frac{LI}{CE}=\frac{c(c+b)}{2sb};(8)$$

From (7) and (8) we have:

$$\frac{b(b+c)}{2sc}+\frac{c(c+b)}{2sb}\geq\frac{b+c}{s}\Leftrightarrow\frac{b^2+bc}{c}+\frac{c^2+bc}{b}\geq 2(b+c)\Leftrightarrow$$

$$\frac{b^2}{c}+\frac{c^2}{b}\geq b+c\ \text{ true from CBS inequality.}$$

642. If $x,y>0$ then in ΔABC holds:

$$\left(\frac{x}{a}+\frac{y}{b}\right)^2w_c^2+\left(\frac{x}{b}+\frac{y}{c}\right)^2w_a^2+\left(\frac{x}{c}+\frac{y}{a}\right)^2w_b^2\geq\frac{18xy\cdot r}{R}$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

Solution 1 by Tapas Das-India

$$w_a \geq h_a \geq \frac{2F}{a} (and\ analogs)$$

$$F = rs; abc = 4RF; s = \frac{a+b+c}{2} \geq \frac{3}{2}\sqrt[3]{abc}$$

$$\sum_{cyc}\left(\frac{x}{a}+\frac{y}{b}\right)^2 w_c^2 \overset{AGM}{\geq} \sum_{cyc}\frac{4xy}{ab}w_c^2 = 4xy\sum_{cyc}\frac{w_c^2}{ab} \overset{AGM}{\geq}$$

$$\geq 4xy \cdot 3\sqrt[3]{\prod_{cyc}\frac{w_c^2}{ab}} = 12xy\sqrt[3]{\left(\frac{w_a w_b w_c}{abc}\right)^2} =$$

$$= 12xy\sqrt[3]{\left(\frac{2F}{a}\cdot\frac{2F}{b}\cdot\frac{2F}{c}\cdot\frac{1}{abc}\right)^2} = 12xy \cdot \frac{4F^2}{abc}\cdot\frac{1}{\sqrt[3]{abc}} =$$

$$= 12xy \cdot \frac{rs}{F}\cdot\frac{1}{\sqrt[3]{abc}} \geq 12xy \cdot \frac{r}{R}\cdot\frac{3}{2}\cdot\frac{\sqrt[3]{abc}}{\sqrt[3]{abc}} = \frac{18xy \cdot r}{R}$$

Solution 2 by Alex Szoros-Romania

$$\left(\frac{x}{a}+\frac{y}{b}\right)^2 \geq \frac{4xy}{ab};\ \ w_c^2 \geq h_c^2 \Rightarrow$$

$$\left(\frac{x}{a}+\frac{y}{b}\right)^2 w_c^2 \geq \frac{4xy}{ab}h_c^2 = \frac{4xy}{ab}\left(\frac{2F}{c}\right)^2 = \frac{16xyF^2}{abc^2}$$

Hence,

$$\sum_{cyc}\left(\frac{x}{a}+\frac{y}{b}\right)^2 w_c^2 \geq \frac{16xyF^2}{abc}\sum_{cyc}\frac{1}{c} = \frac{16xy \cdot s^2r^2}{4Rrs}\sum_{cyc}\frac{1}{a} =$$

$$= 4xy \cdot \frac{sr}{R}\sum_{cyc}\frac{1}{a} = 2xy \cdot \frac{r}{R}\sum_{cyc}a \cdot \sum_{cyc}\frac{1}{a} \geq 2xy \cdot \frac{r}{R}\cdot 9 = \frac{18xy \cdot r}{R}$$

643. In ΔABC the following relationship holds:

$$36r^2s \leq \sum_{cyc}(m_b^2+m_c^2)a \leq \frac{9R^3s}{2r}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Marian Ursărescu-Romania

We must show:

$$\sum_{cyc} a \cdot m_b m_c \geq 18r^2 s; (1)$$

But: $\sum_{cyc} a\cdot m_b m_c \geq 3\sqrt[3]{abc(m_a m_b m_c)^2};\ (2)$

From (1) and (2) we must show:

$$abc(m_a m_b m_c)^2 \geq 6^3 r^6 s^3;\ (3)$$

$$abc = 4Rrs \text{ and } m_a \geq \sqrt{s(s-a)} \Rightarrow m_a m_b m_c \geq sF = s^2 r;\ (4)$$

From (3) and (4) we must show:

$4s^5 Rr^3 \geq 6^3 r^6 s^3 \Leftrightarrow 4s^2 R \geq 6^3 r^3$ **true, because** $s^3 \geq 27r^2(Mitrinovic)$ **and** $4R \geq 8r(Euler)$. **For the right sides, we have:**

$$\sum_{cyc}(m_a^2+m_b^2+m_c^2-m_a^2)\,a \leq \frac{9R^3 s}{2r} \Leftrightarrow$$

$$(m_a^2+m_b^2+m_c^2)(a+b+c)-\sum_{cyc} am_a^2 \leq \frac{9R^3 s}{2r} \Leftrightarrow$$

$$\frac{3}{4}(a^2+b^2+c^2)\cdot 2s-\sum_{cyc} am_a^2 \leq \frac{9R^3 s}{2r};(5)$$

$$a^2+b^2+c^2 \leq 9R^2;\ (6)$$

From (5) and (6) we must show:

$$\frac{27R^2 s}{2}-\sum_{cyc} am_a^2 \leq \frac{9R^3 s}{2r} \Leftrightarrow \sum_{cyc} am_a^2 \geq \frac{s(27R^2 r-9R^3)}{2r};(7)$$

$$m_a \geq \sqrt{s(s-a)} \Rightarrow m_a^2 \geq s(s-a);\ (8)$$

From (7) and (8) we must show:

$$s\sum_{cyc} a(s-a) \geq \frac{s(27R^2 r-9R^3)}{2r} \Leftrightarrow \sum_{cyc} a(s-a) \geq \frac{27R^2 r-9R^3}{2r};(9)$$

But: $\sum_{cyc} a(s-a) = 2r(4R+r);(10)$

From (9) and (10) we must show:

$$2r(4R+r) \geq \frac{27R^2 r-9r^3}{2r} \Leftrightarrow 16Rr^2+4r^3 \geq 27R^2 r-9R^3 \Leftrightarrow$$

$9R^3-27R^2 r+16Rr^2+4r^3 \geq 0 \Leftrightarrow (R-2r)(9R^2-9Rr-2r^2) \geq 0$ **true, because**

$R \geq 2r$ $(Euler)$ and $9R^2 - 9Rr - 2r^2 > 0.$

Solution 2 by Alex Szoros-Romania

$$m_b^2 + m_c^2 \geq s(s-b) + s(s-c) = as$$

$$(m_b^2 + m_c^2)a \geq a^2 s \Rightarrow \sum_{cyc}(m_b^2 + m_c^2)a \geq \sum_{cyc} a^2 s = s\sum_{cyc} a^2 \overset{Leibniz}{\geq}$$

$$\geq s \cdot 9R^2 \overset{Euler}{\geq} 9s \cdot (2r)^2 = 36r^2 s; \ (1)$$

$$m_b^2 + m_c^2 = \frac{4a^2 + b^2 + c^2}{4} \Rightarrow \sum_{cyc}(m_b^2 + m_c^2)a = \sum_{cyc} a\left(a^2 + \frac{b^2c^2}{4}\right) =$$

$$= \sum_{cyc} a^3 + \frac{1}{4}\sum_{cyc} ab(a+b) \leq \sum_{cyc} a^3 + \frac{1}{4}\sum_{cyc}(a^3 + b^3) = \frac{3}{2}\sum_{cyc} a^3 =$$

$$= \frac{3}{2} \cdot 2s(s^2 - 3r^2 - 6Rr) \overset{Gerretsen}{\leq} 3s(4R^2 + 4Rr + 3r^2 - 3r^2 - 6Rr)$$

$$\sum_{cyc}(m_b^2 + m_c^2)a \leq 3s(4R^2 - 2Rr)$$

It is enough to prove that:

$$2r(4R - 2r) \leq 3R^2 \Leftrightarrow 3R62 - 8Rr + 4r^2 \geq 0$$

$(R - 2r)(3R - 2r) \geq 0$ which is true.

From (2) and (3), it follows that:

$$\sum_{cyc}(m_b^2 + m_c^2)a \leq \frac{9R^3 s}{2r}$$

Solution 3 by Tapas Das-India

$$m_b^2 + m_c^2 = \frac{1}{4}(2a^2 + 2c^2 - b^2) + \frac{1}{4}(2a^2 + 2b^2 - c^2) = \frac{1}{4}(4a^2 + b^2 + c^2)$$

$$a(m_b^2 + m_c^2) = \frac{a}{4}(4a^2 + b^2 + c^2) = \frac{1}{4}(4a^3 + ab^2 + ac^2)$$

$$\sum_{cyc} a(m_b^2 + m_c^2) = \frac{1}{4}\sum_{cyc}(4a^3 + ab^2 + ac^2) =$$

$$= \sum_{cyc} a^3 + \frac{1}{4}\sum_{cyc} a(b^2 + c^2) \geq 3abc + \frac{1}{4}(a \cdot 2bc + b \cdot 2ca + c \cdot 2ab) =$$

$$=\frac{9}{2}abc=\frac{9}{2}\cdot 4RF=18RF\geq 18\cdot 2r\cdot rs=36r^2s$$

$$\sum_{cyc}a(m_b^2+m_c^2)=\frac{1}{4}\sum_{cyc}(4a^3+ab^2+ac^2)=$$

$$=\sum_{cyc}a^3+\frac{1}{4}\sum_{cyc}a(b^2+c^2)\leq\sum_{cyc}a^3+\frac{1}{4}\sum_{cyc}(a^3+b^3)\leq$$

$$\leq\sum_{cyc}a^3+\frac{1}{2}\sum_{cyc}a^3=\frac{3}{2}\sum_{cyc}a^3=\frac{3}{2}\cdot 2(s^3-3r^2s-6Rrs)=$$

$$=3s(s^2-3r^2-6Rr)\leq 3s(4R^2+4Rr+3r^2-3r^2-6Rr)=$$

$$=3s(4R^2+4Rr-6Rr)=3s(4R^2-2Rr)$$

We need to show:

$$\frac{9R^3s}{2r}\geq 3s(4R^2-2Rr)\Leftrightarrow\frac{3R^3}{2r}\geq 4R^2-2Rr$$

$\Leftrightarrow 3R^2\geq 2r(4R-2r)\Leftrightarrow(R-2r)(3R-2r)\geq 0$ true from $R\geq 2r$ $(Euler)$.

644. In cyclic $ABCD$ the following relationship holds:

$$\sum_{cyc}\csc\frac{A}{2}\leq 8\sqrt{2}\frac{R^2}{F}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution by Aggeliki Papaspyropoulou-Greece

$$\sum_{cyc}\csc\frac{A}{2}=\sum_{cyc}\frac{1}{\sin\frac{A}{2}}$$

$$2R\sin A=2R\sin C=BD;2R\sin B=2R\sin D=AC$$

$$\frac{1}{\sin\frac{A}{2}}=\frac{4R\cos\frac{A}{2}}{BD},\frac{1}{\sin\frac{B}{2}}=\frac{4R\cos\frac{B}{2}}{AC},\frac{1}{\sin\frac{C}{2}}=\frac{4R\cos\frac{C}{2}}{BD},\frac{1}{\sin\frac{D}{2}}=\frac{4R\cos\frac{D}{2}}{AC}$$

$$\sum_{cyc}\frac{1}{\sin\frac{A}{2}}=\frac{4R}{BD}\left(\cos\frac{A}{2}+\cos\frac{C}{2}\right)+\frac{4R}{AC}\left(\cos\frac{B}{2}+\cos\frac{D}{2}\right)=$$

$$=\frac{4R}{BD}\cdot 2\cos\frac{A+C}{2}\cos\frac{A-C}{2}+\frac{4R}{AC}\cdot 2\cos\frac{B+D}{2}\cos\frac{B-D}{2}=$$

$$=\frac{8R}{BD}\cdot\frac{\sqrt{2}}{2}\cos\frac{A-C}{2}+\frac{8R}{AC}\cdot\frac{\sqrt{2}}{2}\cos\frac{B-D}{2}\le$$

$$\le\frac{4R}{BD}\sqrt{2}+\frac{4R}{AC}\sqrt{2}$$

So, we have to prove:

$$4R\sqrt{2}\left(\frac{1}{BD}+\frac{1}{AC}\right)\le 8\sqrt{2}\frac{R^2}{F};(*)$$

$$\frac{AC+BD}{AC\cdot BD}\le\frac{2R}{F};(**)$$

$$\varphi=\mu(\widehat{AKD}),\text{where }\{K\}=AC\cap BD$$

$$F=[ABCCD]=\frac{1}{2}KD\cdot KA\cdot\sin\varphi+\frac{1}{2}\cdot KD\cdot KC\cdot\sin\varphi+$$

$$+\frac{1}{2}\cdot KC\cdot KB\cdot\sin\varphi+\frac{1}{2}\cdot KB\cdot KA\cdot\sin\varphi=$$

$$=\frac{KD}{2}(KA+KC)\sin\varphi+\frac{KB}{2}(KC+KA)\sin\varphi=$$

$$=\frac{(KD+KB)(KA+KC)}{2}\sin\varphi=\frac{1}{2}BD\cdot AC\cdot\sin\varphi$$

So, from $(**)$, we have to prove:

$$\frac{BD\cdot AC}{2}\left(\frac{AC+BD}{BD\cdot AC}\right)\sin\varphi\le 2R\Leftrightarrow(AC+BD)\sin\varphi\le 4R$$

$$(\text{since: }\sin\varphi\le 1)\Rightarrow AC+BD\le 4R\text{ which is true.}$$

Equality holds if and only if $ABCD$ is a square.

645. If $x,y,z>0$ then in ΔABC the following relationship holds:

$$\frac{x\cdot a\sqrt{a}}{(y+z)\sqrt{h_a}}+\frac{y\cdot b\sqrt{b}}{(z+x)\sqrt{h_b}}+\frac{z\cdot c\sqrt{c}}{(x+y)\sqrt{h_c}}\ge\sqrt{6F}$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

Solution by Tapas Das-India

Using Tsintsifas' inequality:

$$\frac{x}{y+z}\cdot a^2+\frac{y}{z+x}\cdot b^2+\frac{z}{x+y}\cdot c^2\ge 2\sqrt{3}F;\quad(1)$$

$$h_a=\frac{2F}{a}\Rightarrow ah_a=2F,\text{similarly}: bh_b=2F; ch_c=2F$$

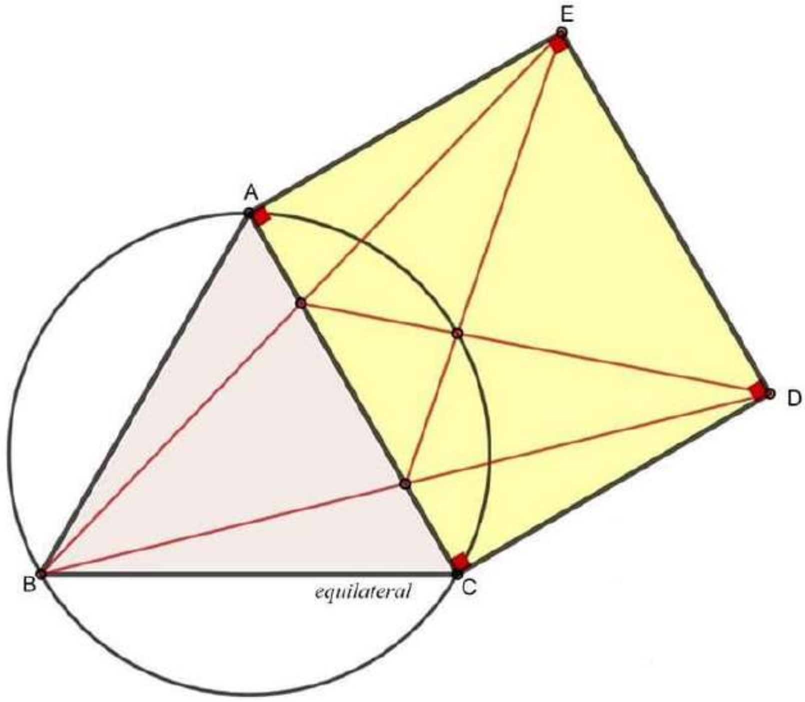
$$\frac{x\cdot a\sqrt{a}}{(y+z)\sqrt{h_a}}+\frac{y\cdot b\sqrt{b}}{(z+x)\sqrt{h_b}}+\frac{z\cdot c\sqrt{c}}{(x+y)\sqrt{h_c}}=$$

$$=\frac{x\cdot a\cdot a}{(y+z)\sqrt{ah_a}}+\frac{y\cdot b\cdot b}{(z+x)\sqrt{bh_b}}+\frac{z\cdot c\cdot c}{(x+y)\sqrt{ch_c}}=$$

$$=\frac{x\cdot a^2}{(y+z)\sqrt{ah_a}}+\frac{y\cdot b^2}{(z+x)\sqrt{bh_b}}+\frac{z\cdot c^2}{(x+y)\sqrt{ch_c}}=$$

$$=\frac{1}{\sqrt{2F}}\left(\frac{x}{y+z}\cdot a^2+\frac{y}{z+x}\cdot b^2+\frac{z}{x+y}\cdot c^2\right)\overset{(1)}{\geq}\frac{1}{\sqrt{2F}}\cdot 2\sqrt{3}F=\sqrt{6F}$$

646.



$$\frac{[ACDE]}{[ABC]}=?\,(\text{area})$$

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by Jose Ferreira Queiroz-Olinda-Brazil

$$DC=a, AB=l, MP=PN=x, PF=\frac{l\sqrt{3}}{6}$$

$$BP=\frac{l\sqrt{3}}{2}$$

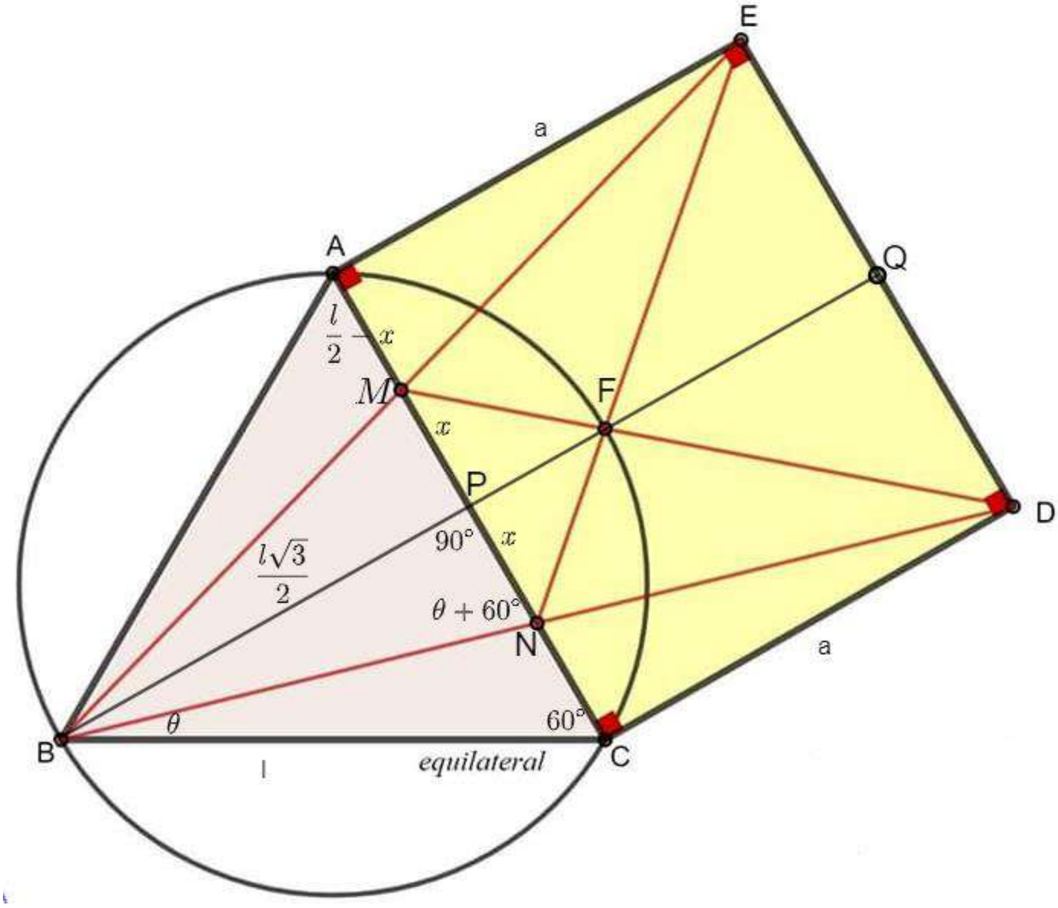
$$\Delta BPN\sim\Delta DCN\Rightarrow\frac{a}{\frac{l\sqrt{3}}{2}}=\frac{\frac{l}{2}-x}{x}\Rightarrow ax=\frac{l^2\sqrt{3}}{4}-\frac{xl\sqrt{3}}{2};(I)$$

$$\Delta FPN\sim\Delta EAN\Rightarrow\frac{a}{\frac{l\sqrt{3}}{6}}=\frac{\frac{l}{2}+x}{x}\Rightarrow ax=\frac{l^2\sqrt{3}}{12}+\frac{xl\sqrt{3}}{6};(II)$$

From(I) and (II) it follows $x=\frac{l}{4}$ and $a=\frac{l\sqrt{3}}{2}$

So,

$$[ACDE]=al=\frac{l^2\sqrt{3}}{2}\text{ and }[ABC]=\frac{l^2\sqrt{3}}{2}\Rightarrow\frac{[ACDE]}{[ABC]}=2$$



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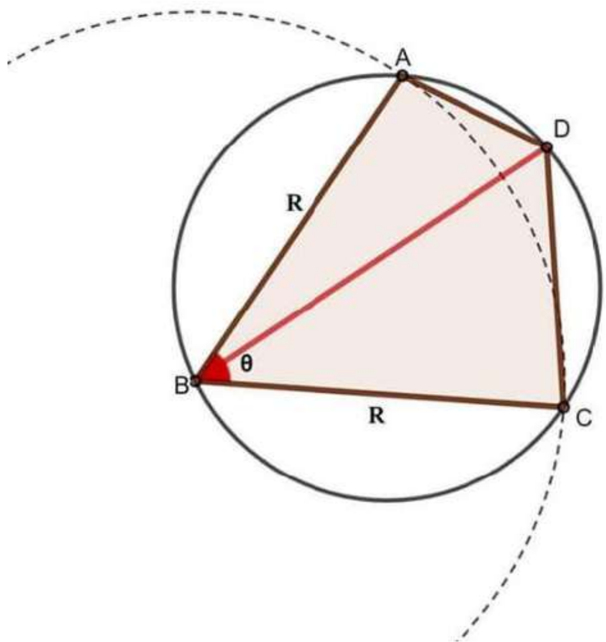


$$\text{Prove: } [ABCD] = \frac{BD^2}{2} \cdot \sin\theta$$

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution 1 by Ahmet Cetin-Turkiye

From Law of sines in ΔABC:

$$\frac{AC}{AB} = \frac{\sin\theta}{\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right)}$$

From Ptolemy's theorem in $ABCD$:

$$AB \cdot CD + BC \cdot AD = AC \cdot BD$$

$$AB(AD + DC) = AC \cdot BD; (AB = BC)$$

$$AD + DC = \frac{AC \cdot BD}{AB}$$

$$[ABCD] = \frac{BD \cdot AD \cdot \sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right)}{2} + \frac{BD \cdot DC \cdot \sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right)}{2} =$$

$$= BD \cdot \sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right) \cdot \frac{AC \cdot BD}{2}$$

$$[ABCD] = \frac{BD^2}{2} \cdot \sin\theta$$

Solution 2 by Jose Ferreira Queiroz-Olinda-Brazil

$$[BCD] + [ABD] = \frac{R \cdot DC \cdot BD}{4r} + \frac{R \cdot AD \cdot BD}{4r} = \frac{R \cdot BD}{4r}(DC + AD); (1)$$

$ABCD$ –is cyclic: $AB \cdot DC + BC \cdot AD = BD \cdot AC$

$$R \cdot DC + R \cdot AD = BD \cdot AC, \quad \frac{BD \cdot AC}{R} = DC + AD; (2)$$

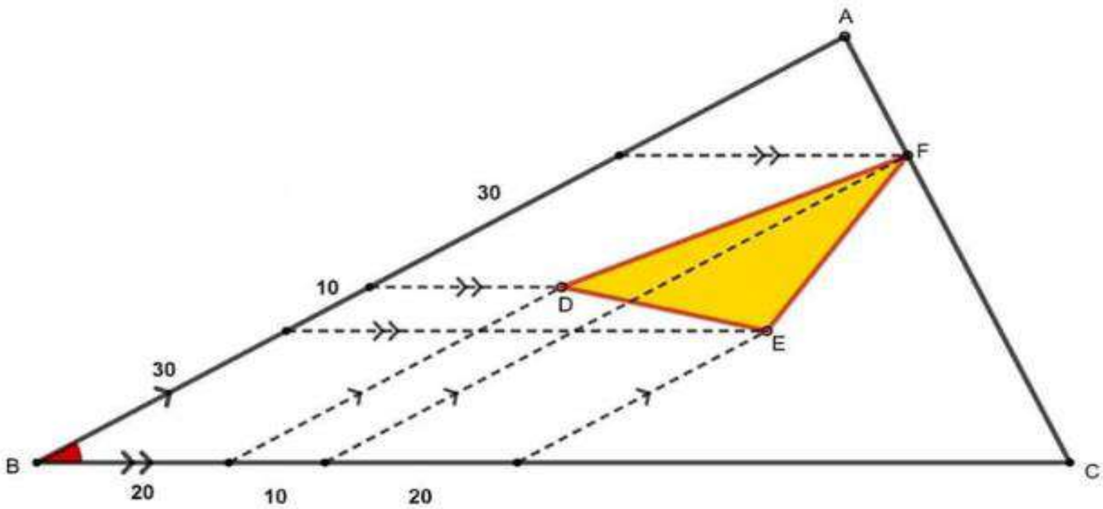
$\frac{AC}{\sin\theta} = 2r \Rightarrow AC = 2r \cdot \sin\theta$. From (1), (2) and (3) we have:

$$[ABCD] = \frac{R \cdot BD}{4r} \cdot \frac{BD \cdot AC}{R} = \frac{BD^2 \cdot AC}{4r} = \frac{BD^2 \cdot 2r \cdot \sin\theta}{4r}$$

Therefore,

$$[ABCD] = \frac{BD^2}{2} \cdot \sin\theta$$

648.



$[DEF] = 250$ (area). Find: $\measuredangle ABC = ?$

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by Remedy Ogswubaba-Nigeria

Taking B as the origin $O(0,0)$ we obtain the points:

D as $(20 + 40\cos B, 40\sin B)$, E as $(50 + 30\cos B, 30\sin B)$

F as $(30 + 70\cos B, 70\sin B)$

Area of Δ passing through, $(x_1,y_1),(x_2,y_2)$ and (x_3,y_3) is given by:

$$F=\frac{1}{2}\begin{vmatrix}1 & 1 & 1\\ y_1 & y_2 & y_3\\ x_1 & x_2 & x_3\end{vmatrix}$$

Hence,

$$[DEF]=\frac{1}{2}\begin{vmatrix}1 & 1 & 1\\ 40\sin B & 30\sin B & 70\sin B\\ 20+40\cos B & 50+80\cos B & 30+70\cos B\end{vmatrix}\overset{\substack{R_1-R_2\\R_3-R_2}}{\Rightarrow}$$

$$250=\frac{1}{2}\begin{vmatrix}0 & 1 & 0\\ 10\sin B & 30\sin B & 40\sin B\\ -30+10\cos B & 50+80\cos B & -20+40\cos B\end{vmatrix}$$

$$500=\begin{vmatrix}10\sin B & 40\sin B\\ -30+10\cos B & -20+40\cos B\end{vmatrix}$$

$$500=-200\sin B+400\sin B\cos B+1200\sin B-40\sin B\cos B$$

$$\sin B=\frac{1}{2}\Rightarrow \measuredangle ABC=30^\circ$$

649. In ΔABC the following relationship holds:

$$a^2(s-a)+b^2(s-b)+c^2(s-c)\ge 4\sqrt[4]{3}\cdot\sqrt{F^3}$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

Solution 1 by Tapas Das-India

$$r_a=\frac{F}{s-a}\Rightarrow s-a=\frac{F}{r_a},\text{similarly: } s-b=\frac{F}{s_b}\text{ and } s-c=\frac{F}{r_c};(1)$$

We have: $2s\le 3R\sqrt{3}\Leftrightarrow 6s\le 9R\sqrt{3}$

$$2rs^2\sqrt{3}\le 9Rrs\Leftrightarrow 8rs^2\le 36Rrs\Leftrightarrow 8rs^2\le 3\sqrt{3}(4R\cdot rs)\Leftrightarrow 8rs^2\le 3\sqrt{3}\cdot 4RF\Leftrightarrow$$

$$\because 8r_ar_br_c\le 3abc\sqrt{3};\ (2),\quad (abc)^2\ge\left(\frac{4F}{\sqrt{3}}\right)^3;\ (3)$$

$$a^2(s-a)+b^2(s-b)+c^2(s-c)=a^2\cdot\frac{F}{r_a}+b^2\cdot\frac{F}{r_b}+c^2\cdot\frac{F}{r_c}=$$

$$=F\left(\frac{a^2}{r_a}+\frac{b^2}{r_b}+\frac{c^2}{r_c}\right)\ge 3F\cdot\sqrt[3]{\frac{(abc)^2}{r_ar_br_c}}=3F\cdot\sqrt[3]{\frac{(abc)^2}{\frac{3abc\sqrt{3}}{8}}}=$$

$$=\frac{3F\cdot 2\sqrt[3]{abc}}{\sqrt{3}}=2\sqrt{3}F\cdot\sqrt[3]{abc}\geq 2\sqrt{3}F\cdot\frac{\sqrt{4F}}{\sqrt[4]{3}}=4\sqrt[4]{3}\cdot\sqrt{F^3}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}a^2(s-a)=s\sum_{cyc}a^2-\sum_{cyc}a^3=2s(s^2-4Rr-r^2)-2s(s^2-6Rr-3r^2)$$

$$=4rs(R+r)\geq 4\sqrt[4]{3}.\sqrt{F^3}\Leftrightarrow r^4s^4(R+r)^4\geq 3r^6s^6\Leftrightarrow (R+r)^4\overset{(*)}{\geq}3r^2s^2$$

$$\text{Now, }(R+r)^4\overset{\text{Euler}}{\geq}3r(R+r)^3\overset{?}{\geq}3r^2s^2\Leftrightarrow rs^2\underbrace{\overset{?}{\leq}}_{(**)}(R+r)^3\ and$$

$$\because rs^2\overset{\text{Gerretsen}}{\leq}r(4R^2+4Rr+3r^2)$$

$\therefore$ in order to prove $(**)$, it suffices to prove : $(R+r)^3$
$\geq r(4R^2+4Rr+3r^2)$

$$\Leftrightarrow R^3-R^2r-Rr^2-2r^3\geq 0\Leftrightarrow (R-2r)(R^2+Rr+r^2)\geq 0\to \text{true}\because R\overset{\text{Euler}}{\geq}2r\Rightarrow (**)$$

$$\Rightarrow (*)\text{ is true}\Rightarrow a^2(s-a)+b^2(s-b)+c^2(s-c)\geq 4\sqrt[4]{3}.\sqrt{F^3}\ (QED)$$

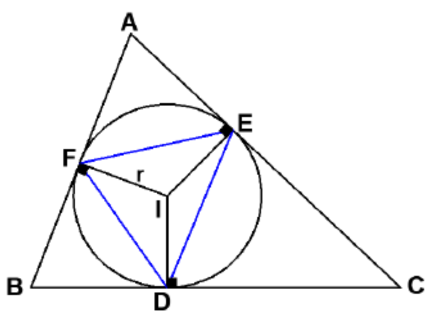
650. In ΔABC the following relationship holds:

$$\frac{x}{{h_a}^2}+\frac{y}{{h_b}^2}+\frac{z}{{h_c}^2}\geq\frac{1}{2F}\sqrt{\frac{xy}{\sin^2\frac{C}{2}}+\frac{yz}{\sin^2\frac{A}{2}}+\frac{zx}{\sin^2\frac{B}{2}}}$$

Proposed by D.M.Bătineţu-Giurgiu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let DEF be the intouch triangle of ΔABC and S be the area of ΔDEF.



We have :

$$a'=EF=2R_{\Delta DEF}.\sin(\angle EDF)=2r\sin\left(\frac{B+C}{2}\right)=2r\cos\frac{A}{2}\ \ (and\ analogs)$$

$$Also\ we\ have:\ S=\frac{a'b'c'}{4R_{\Delta DEF}}=\frac{2r\cos\frac{A}{2}.2r\cos\frac{B}{2}.2r\cos\frac{C}{2}}{4r}=2r^2.\frac{s}{4R}=\frac{sr^2}{2R}.$$

Using now Oppenheim's inequality in ΔDEF we have:

$$u.a'^2+v.b'^2+w.c'^2\geq 4S\sqrt{uv+vw+wu},\ \forall u,v,w>0.$$

$$Let\ u=x\sin^2\frac{A}{2},\ v=y\sin^2\frac{B}{2},\ w=z\sin^2\frac{C}{2},\ then\ we\ have:$$

$$\sum_{cyc}x\sin^2\frac{A}{2}.\left(2r\cos\frac{A}{2}\right)^2\geq 4.\frac{sr^2}{2R}\sqrt{\sum_{cyc}x\sin^2\frac{A}{2}.y\sin^2\frac{B}{2}}\Leftrightarrow$$

$$\sum_{cyc}x.\left(\frac{a}{4R}\right)^2\geq\frac{s}{2R}\sqrt{\left(\frac{r}{4R}\right)^2\sum_{cyc}\frac{xy}{sin^2\frac{C}{2}}}$$

$$\Leftrightarrow\sum_{cyc}\frac{x}{{h_a}^2}\geq\frac{(4R)^2}{(2F)^2}.\frac{sr}{8R^2}\sqrt{\sum_{cyc}\frac{xy}{sin^2\frac{C}{2}}}=\frac{1}{2F}\sqrt{\sum_{cyc}\frac{xy}{sin^2\frac{C}{2}}}$$

651. If $A_kB_kC_k,k=\overline{1,3}$ are three triangles with circumradii $R_k,k=\overline{1,3}$, then:

$$\frac{1}{a_1a_2a_3}+\frac{1}{b_1b_2b_3}+\frac{1}{c_1c_2c_3}\geq\frac{9\sqrt{3}}{R_1+R_2+R_3}$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sqrt[3]{a_1a_2a_3}\overset{A-G}{\leq}\frac{a_1+a_2+a_3}{3}\Rightarrow a_1a_2a_3\leq\frac{(a_1+a_2+a_3)^3}{27}\Rightarrow\frac{1}{a_1a_2a_3}$$
$$\geq\frac{27}{(a_1+a_2+a_3)^3}\ and\ similarly,\frac{1}{b_1b_2b_3}\geq\frac{27}{(b_1+b_2+b_3)^3}\ and\ \frac{1}{c_1c_2c_3}$$
$$\geq\frac{27}{(c_1+c_2+c_3)^3}$$

$$\Rightarrow\frac{1}{a_1a_2a_3}+\frac{1}{b_1b_2b_3}+\frac{1}{c_1c_2c_3}\geq 27\left(\frac{1}{(a_1+a_2+a_3)^3}+\frac{1}{(b_1+b_2+b_3)^3}+\frac{1}{(c_1+c_2+c_3)^3}\right)$$
$$=27\left(\frac{1^4}{(a_1+a_2+a_3)^3}+\frac{1^4}{(b_1+b_2+b_3)^3}+\frac{1^4}{(c_1+c_2+c_3)^3}\right)$$

$$\overset{Radon}{\geq}\frac{(1+1+1)^4}{(a_1+a_2+a_3+b_1+b_2+b_3+c_1+c_2+c_3)^3}$$

$$=\frac{81}{\left((a_1+b_1+c_1)+(a_2+b_2+c_2)+(a_3+b_3+c_3)\right)^3}$$

$$=\frac{81}{8(s_1+s_2+s_3)^3}\overset{Mitrinovic}{\geq}\frac{81}{8\left(\frac{3\sqrt{3}}{2}.R_1+\frac{3\sqrt{3}}{2}.R_2+\frac{3\sqrt{3}}{2}.R_3\right)^3}$$

$$=\frac{81}{27.3\sqrt{3}.(R_1+R_2+R_3)^3}=\frac{9\sqrt{3}}{(R_1+R_2+R_3)^3}\text{ (QED)}$$

Solution 2 by Tapas Das-India

$$\frac{1}{a_1a_2a_3}+\frac{1}{b_1b_2b_3}+\frac{1}{c_1c_2c_3}\geq 3\sqrt[3]{\frac{1}{a_1a_2a_3}\cdot\frac{1}{b_1b_2b_3}\cdot\frac{1}{c_1c_2c_3}}\geq$$

$$\geq 3\sqrt[3]{\left(\frac{1}{3\sqrt{3}}\right)^3\cdot\frac{1}{(R_1R_2R_3)^3}}=\frac{1}{\sqrt{3}}\cdot\frac{1}{R_1R_2R_3}\geq$$

$$\geq\frac{27}{\sqrt{3}(R_1+R_2+R_3)^3}=\frac{9\sqrt{3}}{(R_1+R_2+R_3)^3}$$

Now, $a_1b_1c_1=2R_1\sin A_1\,2R_1\sin B_1\,2R_1\sin C_1=$

$$=8R_1^3(\sin A_1\sin B_1\sin C_1)\leq 8R_1^3\cdot\frac{3\sqrt{3}}{8}=3\sqrt{3}R_1^3$$

Similarly,

$a_2b_2c_3\leq 3\sqrt{3}R_2^3$ **and** $a_3b_3c_3\leq 3\sqrt{3}R_3^3$

Hence,

$$R_1R_2R_3\leq\frac{(R_1+R_2+R_3)^3}{27}$$

$$\frac{\sin A+\sin B+\sin C}{3}\leq\sin\left(\frac{A+B+C}{3}\right)=\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$$

$$\sin A+\sin B+\sin C\leq\frac{3\sqrt{3}}{2}$$

Now,

$$\sin A\sin B\sin C\le\left(\frac{\sin A+\sin B+\sin C}{3}\right)^3=\frac{3\sqrt{3}}{8}$$

652. Find the minimum value of $\lambda\ge 0$ so that the inequality :

$$\frac{3R}{2r}\ge\sum_{cyc}\frac{(a+b)^4}{a^4+\lambda a^2b^2+b^4}\text{ is true in any }\Delta ABC.$$

Proposed by Alex Szoros-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\text{In equilateral }\Delta ABC\text{ we have : }a=b=c\text{ and }R=2r\text{ then}$$

$$3\ge 3.\frac{2^4}{2+\lambda}\Leftrightarrow\lambda\ge 14.$$

$$\text{Next, we will prove that the desired inequality is true for }\lambda=14.$$

$$\text{We have :}$$

$$\frac{(a+b)^4}{a^4+14a^2b^2+b^4}=\frac{(a+b)^4}{[(a^2+b^2)^2+4a^2b^2]+8a^2b^2}\overset{AM-GM}{\le}\frac{(a+b)^4}{4ab(a^2+b^2)+8a^2b^2}=$$

$$=\frac{(a+b)^4}{4ab(a+b)^2}=\frac{(a+b)^2}{4ab}=\frac{1}{4}\left(\frac{a}{b}+\frac{b}{a}\right)+\frac{1}{2}\overset{Bandila\ \&\ Euler}{\le}\frac{1}{4}\cdot\frac{R}{r}+\frac{1}{2}\cdot\frac{R}{2r}=\frac{R}{2r}$$

$$\text{Then : }\frac{(a+b)^4}{a^4+14a^2b^2+b^4}\le\frac{R}{2r}\ (\text{and analogs})$$

$$\text{Therefore, }\sum_{cyc}\frac{(a+b)^4}{a^4+14a^2b^2+b^4}\le\frac{3R}{2r}\text{ and }\lambda_{min}=14.$$

653. In ΔABC the following relationship holds:

$$\frac{a^2}{(b+\sqrt{bc}+\sqrt{ca})h_a}+\frac{b^2}{(c+\sqrt{ca}+\sqrt{ab})h_b}+\frac{c^2}{(a+\sqrt{ab}+\sqrt{bc})h_c}\ge\frac{2\sqrt{3}}{3}$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

Solution 1 by Tapas Das-India

$$h_a=\frac{2F}{a},h_b=\frac{2F}{b},h_c=\frac{2F}{c}$$

$$\sum_{cyc}\frac{a^2}{(b+\sqrt{bc}+\sqrt{ca})h_a}=\sum_{cyc}\frac{a^3}{2F(b+\sqrt{bc}+\sqrt{ca})}=$$

$$=\frac{1}{2F}\sum_{cyc}\frac{a^3}{b+\sqrt{bc}+\sqrt{ca}}=\frac{1}{2F}\sum_{cyc}\frac{(\sqrt{a})^6}{b+\sqrt{bc}+\sqrt{ca}}\ge$$

$$\overset{Holder}{\ge}\frac{1}{2F}\cdot\frac{(\sqrt{a}+\sqrt{b}+\sqrt{c})^6}{3^4[a+b+c+2(\sqrt{ab}+\sqrt{bc}+\sqrt{ca})]}=$$

$$=\frac{1}{2F}\cdot\frac{(\sqrt{a}+\sqrt{b}+\sqrt{c})^4}{81}\ge\frac{1}{2F}\cdot\frac{1}{81}\left[3(abc)^{\frac{1}{6}}\right]^4=\frac{1}{81\cdot 2F}\cdot 81[(abc)^2]^{\frac{1}{3}}\ge$$

$$\ge\frac{1}{81\cdot 2F}\cdot 81\left(\frac{4F}{\sqrt{3}}\right)^{3\cdot\frac{1}{3}}=\frac{2\sqrt{3}}{3}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{a^2}{(\mathrm{b}+\sqrt{\mathrm{bc}}+\sqrt{ca})\mathrm{h}_a}$$

$$=\frac{1}{2\mathrm{rs}}\sum_{cyc}\frac{a^3}{(\mathrm{b}+\sqrt{\mathrm{bc}}+\sqrt{ca})}\overset{\text{Holder}}{\ge}\frac{\left(\sum_{cyc}a\right)^3}{6\mathrm{rs}\left(\sum_{cyc}a+2\sum_{cyc}\sqrt{\mathrm{bc}}\right)}\overset{\text{CBS}}{\ge}\frac{\left(\sum_{cyc}a\right)^3}{6\mathrm{rs}\left(\sum_{cyc}a+2\sqrt{\sum_{cyc}a}.\sqrt{\sum_{cyc}a}\right)}$$

$$=\frac{\left(\sum_{cyc}a\right)^2}{18\mathrm{rs}}=\frac{\sum_{cyc}a^2+2\sum_{cyc}ab}{18\mathrm{rs}}$$

$$\overset{\substack{\text{Ionescu-Weitzenbock}\\+\\\text{Gordon}}}{\ge}\frac{(4\sqrt{3}+8\sqrt{3})\mathrm{rs}}{18\mathrm{rs}}=\frac{2\sqrt{3}}{3}\ (\text{QED})$$

654. If $x, y\in\mathbb{R}$, then in ΔABC the following relationship holds:

$$\frac{x^2m_a^8+y^2m_b^8}{c^4}+\frac{x^2m_b^8+y^2m_c^8}{a^4}+\frac{x^2m_c^8+y^2m_a^8}{b^4}\ge\frac{81(x+y)^2}{32}\cdot F^2$$

Proposed by Alex Szoros-Romania

Solution 1 by Adrian Popa-Romania

$$\sum_{cyc}\frac{x^2m_a^8+y^2m_b^8}{c^4}=x^2\sum_{cyc}\frac{(m_a^2)^4}{(c^2)^2}+y^2\sum_{cyc}\frac{(m_b^2)^4}{(c^2)^2}\ge$$

$$\ge x^2\cdot\frac{(\sum m_a^2)^4}{3(\sum a^2)^2}+y^2\cdot\frac{(\sum m_a^2)^4}{3(\sum a^2)^2}=x^2\cdot\frac{\left[\frac{3}{4}(\sum a^2)\right]^4}{3(\sum a^2)^2}+y^2\cdot\frac{\left[\frac{3}{4}(\sum a^2)\right]^4}{3(\sum a^2)^2}=$$

$$=\frac{27}{256}(x^2+y^2)(a^2+b^2+c^2)^2\overset{Ionescu-W}{\geq}$$

$$\geq\frac{27}{256}\left(4\sqrt{3}F\right)^2(x^2+y^2)=\frac{81}{16}(x^2+y^2)\cdot F^2\overset{Bergstrom}{\geq}$$

$$\geq\frac{81}{16}\cdot\frac{(x+y)^2}{2}\cdot F^2=\frac{81(x+y)^2}{32}\cdot F^2$$

Solution 2 by Tapas Das-India

$$\left(\frac{m_a^8}{c^4}+\frac{m_b^8}{a^4}+\frac{m_c^8}{b^4}\right)^{\frac{1}{4}}(c^2+a^2+b^2)^{\frac{2}{4}}(1+1+1)^{\frac{1}{4}}\overset{Holder}{\geq}$$

$$\geq\left(\frac{m_a^8}{c^4}\cdot c^2\cdot c^2\cdot 1\right)^{\frac{1}{4}}+\left(\frac{m_b^8}{a^4}\cdot a^2\cdot a^2\cdot 1\right)^{\frac{1}{4}}+\left(\frac{m_c^8}{b^4}\cdot b^2\cdot b^2\cdot 1\right)^{\frac{1}{4}}=$$

$$=m_a^2+m_b^2+m_c^2$$

$$\Rightarrow\frac{m_a^8}{c^4}+\frac{m_b^8}{a^4}+\frac{m_c^8}{b^4}\geq\frac{1}{3}\frac{\left(m_a^2+m_b^2+m_c^2\right)^4}{(a^2+b^2+c^2)^2}$$

Similarly,

$$\frac{m_a^8}{c^4}+\frac{m_b^8}{a^4}+\frac{m_c^8}{b^4}\geq\frac{1}{3}\frac{\left(m_a^2+m_b^2+m_c^2\right)^4}{(a^2+b^2+c^2)^2}$$

Therefore,

$$\frac{x^2m_a^8+y^2m_b^8}{c^4}+\frac{x^2m_b^8+y^2m_c^8}{a^4}+\frac{x^2m_c^8+y^2m_a^8}{b^4}=$$

$$=x^2\sum_{cyc}\frac{(m_a^2)^4}{(c^2)^2}+y^2\sum_{cyc}\frac{\left(m_b^2\right)^4}{(c^2)^2}\geq$$

$$\geq x^2\cdot\frac{(\sum m_a^2)^4}{3(\sum a^2)^2}+y^2\cdot\frac{(\sum m_a^2)^4}{3(\sum a^2)^2}=x^2\cdot\frac{\left[\frac{3}{4}(\sum a^2)\right]^4}{3(\sum a^2)^2}+y^2\cdot\frac{\left[\frac{3}{4}(\sum a^2)\right]^4}{3(\sum a^2)^2}=$$

$$=\frac{27}{256}(x^2+y^2)(a^2+b^2+c^2)^2\geq\frac{1}{3}\cdot\frac{81}{256}\left[3(abc)^{\frac{2}{3}}\right]^2\cdot\frac{(x+y)^2}{2}\geq$$

$$\geq\frac{1}{3}\cdot\frac{81}{256}\cdot 9\cdot\frac{16F^2}{3}\cdot\frac{(x+y)^2}{2}=\frac{81(x+y)^2}{32}\cdot F^2$$

655. In ΔABC the following relationship holds:

$$\sqrt[3]{\prod_{cyc}\left(\sin^4\frac{A}{2}+\sin^4\frac{B}{2}\right)}\geq\frac{1}{6}\left(\frac{2r}{R}-\left(\frac{r}{R}\right)^2\right)$$

Proposed by Marian Ursărescu-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Lemma:\quad If\ x,y,z\geq 0\ then\ \sqrt[3]{(x^4+y^4)(y^4+z^4)(z^4+x^4)}\geq\frac{2(x^2y^2+y^2z^2+z^2x^2)}{3}.$$

$$We\ have:9\prod_{cyc}(x^4+y^4)$$

$$=9\left(\sum_{cyc}x^4\right)\left(\sum_{cyc}x^4y^4\right)-9x^4y^4z^4\overset{AM-GM}{\geq}8\left(\sum_{cyc}x^4\right)\left(\sum_{cyc}x^4y^4\right)$$

$$\overset{CBS}{\geq}8\left(\sum_{cyc}x^2y^2\right).\frac{\left(\sum_{cyc}x^2y^2\right)^2}{3}=\frac{\left(2\sum_{cyc}x^2y^2\right)^3}{3}.$$

$$Then:\ \sqrt[3]{(x^4+y^4)(y^4+z^4)(z^4+x^4)}\geq\frac{2(x^2y^2+y^2z^2+z^2x^2)}{3},\quad\forall x,y,z\geq 0.$$

$$For\ x=\sin\frac{A}{2},\quad y=\sin\frac{B}{2},\quad z=\sin\frac{C}{2}\ we\ get:$$

$$\sqrt[3]{\prod_{cyc}\left(\sin^4\frac{A}{2}+\sin^4\frac{B}{2}\right)}\geq\frac{2}{3}\sum_{cyc}\sin^2\frac{B}{2}\sin^2\frac{C}{2}=\frac{2}{3}\sum_{cyc}\frac{(s-c)(s-a)}{ca}.\frac{(s-a)(s-b)}{ab}$$

$$=\frac{2sr^2}{3.4Rsr}\sum_{cyc}\frac{s-a}{a}=$$

$$=\frac{r}{6R}\left(\frac{s(ab+bc+ca)}{abc}-3\right)=\frac{r}{6R}\left(\frac{s^2+r^2+4Rr}{4Rr}-3\right)$$

$$=\frac{s^2+r^2-8Rr}{24R^2}\overset{Gerretsen}{\geq}\frac{8Rr-4r^2}{24R^2}.$$

$$Therefore,\ \sqrt[3]{\prod_{cyc}\left(\sin^4\frac{A}{2}+\sin^4\frac{B}{2}\right)}\geq\frac{1}{6}\left(\frac{2r}{R}-\left(\frac{r}{R}\right)^2\right).$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sqrt[3]{\prod_{cyc}\left(\sin^4\frac{A}{2}+\sin^4\frac{B}{2}\right)} \geq \sqrt[3]{\prod_{cyc}\frac{\left(\sin^2\frac{A}{2}+\sin^2\frac{B}{2}\right)^2}{2}} = \frac{1}{2}\left(\prod_{cyc}\left(\sin^2\frac{A}{2}+\sin^2\frac{B}{2}\right)\right)^{\frac{2}{3}}$$

$$= \frac{1}{2}\left(\left(\sum_{cyc}\sin^2\frac{A}{2}\right)\left(\sum_{cyc}\sin^2\frac{A}{2}\sin^2\frac{B}{2}\right)-\prod_{cyc}\sin^2\frac{A}{2}\right)^{\frac{2}{3}}$$

$$= \frac{1}{2}\left(\left(\prod_{cyc}\sin^2\frac{A}{2}\right)\left(\sum_{cyc}\sin^2\frac{A}{2}\right)\left(\sum_{cyc}\text{cosec}^2\frac{A}{2}\right)-\prod_{cyc}\sin^2\frac{A}{2}\right)^{\frac{2}{3}}$$

$$= \frac{1}{2}\left(\left(\frac{r^2}{16R^2}\right)\left(\frac{2R-r}{2R}\right)\left(\frac{s^2-8Rr+r^2}{r^2}\right)-\frac{r^2}{16R^2}\right)^{\frac{2}{3}}$$

$$= \frac{1}{2}\left(\frac{(2R-r)(s^2-8Rr+r^2)-2Rr^2}{32R^3}\right)^{\frac{2}{3}} \overset{?}{\geq} \frac{1}{6}\left(\frac{2r}{R}-\left(\frac{r}{R}\right)^2\right)$$

$$\Leftrightarrow \frac{\left((2R-r)(s^2-8Rr+r^2)-2Rr^2\right)^2}{1024R^6} \overset{?}{\geq} \frac{r^3(2R-r)^3}{27R^6}$$

$$\Leftrightarrow 27\left((2R-r)(s^2-8Rr+r^2)-2Rr^2\right)^2 \underset{(*)}{\overset{?}{\geq}} 1024r^3(2R-r)^3$$

Now, LHS of $(*) \overset{\text{Gerretsen}}{\geq} 27\left((2R-r)(8Rr-4r^2)-2Rr^2\right)^2 \overset{?}{\geq} 1024r^3(2R-r)^3$

$$\Leftrightarrow 27(8R^2-9Rr+2r^2)^2 \overset{?}{\geq} 256r(2R-r)^3$$

$$\Leftrightarrow 1728t^4-5936t^3+6123t^2-2508t+364 \overset{?}{\geq} 0 \quad \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((t-2)(1728t^2+976t+3115)+6048\right) \overset{?}{\geq} 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$\Rightarrow (*)$ is true

$$\therefore \text{ in any } \Delta\, ABC, \sqrt[3]{\prod_{cyc}\left(\sin^4\frac{A}{2}+\sin^4\frac{B}{2}\right)} \geq \frac{1}{6}\left(\frac{2r}{R}-\left(\frac{r}{R}\right)^2\right)$$

656. If $x, y, z > 0$ and in ΔABC, g_a –Gergonne's cevian, then holds:

$$a^4(a^2g_b^2+y^2g_c^2)+b^4(x^2g_c^2+y^2g_a^2)+c^4(x^2g_a^2+y^2g_b^2)$$
$$\geq 8\sqrt{3}(x+y)^2\cdot F^3$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

Solution 1 by Tapas Das-India

$$g_a \geq h_a = \frac{2F}{a} (and\ analogs)$$

$$a^4(a^2 g_b^2 + y^2 g_c^2) + b^4(x^2 g_c^2 + y^2 g_a^2) + c^4(x^2 g_a^2 + y^2 g_b^2) \geq$$

$$\geq a^4\left(x^2 \frac{4F^2}{b^2} + y^2 \frac{4F^2}{c^2}\right) + b^4\left(x^2 \frac{4F^2}{c^2} + y^2 \frac{4F^2}{a^2}\right) + c^4\left(x^2 \frac{4F^2}{a^2} + y^2 \frac{4F^2}{b^2}\right) =$$

$$= 4F^2\left(\frac{a^4}{b^2 x^2} + \frac{a^4}{c^2} y^2\right) + 4F^2\left(\frac{b^4}{c^2} x^2 + \frac{b^4}{a^2} y^2\right) + 4F^2\left(\frac{c^4}{a^2} x^2 + \frac{c^4}{b^2} y^2\right) =$$

$$= 4F^2\left[x^2\left(\frac{a^4}{b^2} + \frac{b^4}{c^2} + \frac{c^4}{a^2}\right) + y^2\left(\frac{a^4}{c^2} + \frac{b^4}{a^2} + \frac{c^4}{b^2}\right)\right] =$$

$$= 4F^2 \cdot (x^2 + y^2)\left(\frac{a^4}{b^2} + \frac{b^4}{c^2} + \frac{c^4}{a^2}\right) \geq 4F^2 \cdot \frac{(x+y)^2}{2} \cdot 3\left(\frac{a^4}{b^2} \cdot \frac{b^4}{c^2} \cdot \frac{c^4}{a^2}\right)^{\frac{1}{3}} \overset{AGM}{\geq}$$

$$\geq 4F^2 \cdot \frac{(x+y)^2}{2} \cdot 3(a^2 b^2 c^2)^{\frac{1}{3}} = 2F^2(x+y)^2 3[(abc)^2]^{\frac{1}{3}} \geq$$

$$\geq 2F^2(x+y)^2 \cdot 3\left(\frac{4F}{\sqrt{3}}\right)^{3 \cdot \frac{1}{3}} = 2F^2(x+y)^2 \cdot 3 \cdot \frac{4F}{\sqrt{3}} = 8F^3\sqrt{3}(x+y)^2$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$a^4(x^2 g_b^2 + y^2 g_c^2) + b^4(x^2 g_c^2 + y^2 g_a^2) + c^4(x^2 g_a^2 + y^2 g_b^2)$$
$$= x^2(a^4 g_b^2 + b^4 g_c^2 + c^4 g_a^2) + y^2(a^4 g_c^2 + b^4 g_a^2 + c^4 g_b^2)$$
$$= x^2(a^2.a^2 g_b^2 + b^2.b^2 g_c^2 + c^2.c^2 g_a^2) + y^2(a^2.a^2 g_c^2 + b^2.b^2 g_a^2 + c^2.c^2 g_b^2)$$
$$\overset{\text{Oppenheim}}{\geq} x^2.4F.\sqrt{a^2 g_b^2.b^2 g_c^2 + b^2 g_c^2.c^2 g_a^2 + c^2 g_a^2.a^2 g_b^2}$$
$$+ y^2.4F.\sqrt{a^2 g_c^2.b^2 g_a^2 + b^2 g_a^2.c^2 g_b^2 + c^2 g_b^2.a^2 g_c^2}$$

$$\overset{g_a\geq h_a \text{ and analogs}}{\geq} x^2.4F.\sqrt{a^2h_b^2.b^2h_c^2+b^2h_c^2.c^2h_a^2+c^2h_a^2.a^2h_b^2}$$

$$+y^2.4F.\sqrt{a^2h_c^2.b^2h_a^2+b^2h_a^2.c^2h_b^2+c^2h_b^2.a^2h_c^2}$$

$$=x^2.4F.4F^2.\sqrt{\frac{a^2b^2}{b^2c^2}+\frac{b^2c^2}{c^2a^2}+\frac{c^2a^2}{a^2b^2}}$$

$$+y^2.4F.4F^2.\sqrt{\frac{a^2b^2}{c^2a^2}+\frac{b^2c^2}{a^2b^2}+\frac{c^2a^2}{b^2c^2}}\overset{A-G}{\geq}x^2.16F^3.\sqrt{3.\sqrt[3]{\frac{a^2}{c^2}.\frac{b^2}{a^2}.\frac{c^2}{b^2}}}$$

$$+y^2.16F^3.\sqrt{3.\sqrt[3]{\frac{b^2}{c^2}.\frac{c^2}{a^2}.\frac{a^2}{b^2}}}=16\sqrt{3}F^3(x^2+y^2)\geq\frac{16\sqrt{3}F^3}{2}(x+y)^2$$

$$\Rightarrow a^4(x^2g_b^2+y^2g_c^2)+b^4(x^2g_c^2+y^2g_a^2)+c^4(x^2g_a^2+y^2g_b^2)$$
$$\geq 8\sqrt{3}(x+y)^2.F^3 \text{ in any } \Delta\, ABC\ \forall\, x,y>0\ (QED)$$

Solution 3 by Debopriyo Dawn-India

$$g_a\geq h_a=\frac{2F}{a}(\text{and analogs})$$

$$a^4(a^2g_b^2+y^2g_c^2)+b^4(x^2g_c^2+y^2g_a^2)+c^4(x^2g_a^2+y^2g_b^2)\geq$$

$$\geq a^4\left(x^2\frac{4F^2}{b^2}+y^2\frac{4F^2}{c^2}\right)+b^4\left(x^2\frac{4F^2}{c^2}+y^2\frac{4F^2}{a^2}\right)+c^4\left(x^2\frac{4F^2}{a^2}+y^2\frac{4F^2}{b^2}\right)=$$

$$=4F^2\left(\frac{a^4}{b^2x^2}+\frac{a^4}{c^2}y^2\right)+4F^2\left(\frac{b^4}{c^2}x^2+\frac{b^4}{a^2}y^2\right)+4F^2\left(\frac{c^4}{a^2}x^2+\frac{c^4}{b^2}y^2\right)=$$

$$=4F^2\left[x^2\left(\frac{a^4}{b^2}+\frac{b^4}{c^2}+\frac{c^4}{a^2}\right)+y^2\left(\frac{a^4}{c^2}+\frac{b^4}{a^2}+\frac{c^4}{b^2}\right)\right]\geq$$

$$\geq 4F^2\left[x^2\cdot 3\sqrt[3]{a^2b^2c^2}+y^2\cdot 3\sqrt[3]{a^2b^2c^2}\right]\geq$$

$$\geq 8F^2\cdot\frac{(x+y)^2}{4}\cdot 3\left[\left(\frac{4F}{\sqrt{3}}\right)^{\frac{3}{2}}\right]^{\frac{2}{3}}\geq 8F^2\cdot\frac{(x+y)^2}{4}\cdot 3\cdot\frac{4F}{\sqrt{3}}=8\sqrt{3}(x+y)^2\cdot F^3$$

657. In ΔABC the following relationship holds:

$$\frac{(2R-r)^2}{4r^2}\geq\left(\sum_{cyc}\frac{a}{b+c}\right)\left(\sum_{cyc}\frac{h_a}{h_b+h_c}\right)$$

Proposed by Alex Szoros-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{h_a}{h_b+h_c}=\sum_{cyc}\left(\frac{1}{a}\cdot\frac{bc}{b+c}\right)\overset{H\le G}{\le}\frac{1}{2}\sum_{cyc}\frac{\sqrt{bc}}{a}=\frac{1}{8Rrs}\sum_{cyc}\left(b\sqrt{b}.c\sqrt{c}\right)\overset{CBS}{\le}\frac{1}{8Rrs}\cdot\sqrt{\sum_{cyc}b^3}\cdot\sqrt{\sum_{cyc}c^3}$$

$$=\frac{1}{8Rrs}\sum_{cyc}a^3=\frac{s^2-6Rr-3r^2}{4Rr}$$

$$\therefore\left(\sum_{cyc}\frac{a}{b+c}\right)\left(\sum_{cyc}\frac{h_a}{h_b+h_c}\right)\le\frac{s^2-6Rr-3r^2}{4Rr}\left(\sum_{cyc}\frac{a}{b+c}\right)$$

$$=\frac{s^2-6Rr-3r^2}{4Rr}\cdot\frac{1}{\prod_{cyc}(b+c)}\cdot\sum_{cyc}a(c+a)(a+b)$$

$$=\frac{s^2-6Rr-3r^2}{4Rr.2s(s^2+2Rr+r^2)}\sum_{cyc}\left(a\left(\sum_{cyc}ab+a^2\right)\right)$$

$$=\frac{s^2-6Rr-3r^2}{4Rr.2s(s^2+2Rr+r^2)}\cdot\left(2s(s^2+4Rr+r^2)+2s(s^2-6Rr-3r^2)\right)$$

$$=\frac{(s^2-6Rr-3r^2)(s^2-Rr-r^2)}{2Rr(s^2+2Rr+r^2)}\overset{CBS}{\le}\frac{(4R^2+4Rr+3r^2-6Rr-3r^2)(s^2-Rr-r^2)}{2Rr(s^2+2Rr+r^2)}$$

$$=\frac{(2R-r)(s^2-Rr-r^2)}{r(s^2+2Rr+r^2)}$$

$$\overset{?}{\le}\frac{(2R-r)^2}{4r^2}\Leftrightarrow(2R-r)(s^2+2Rr+r^2)\overset{?}{\ge}4r(s^2-Rr-r^2)$$

$$\Leftrightarrow(2R-5r)s^2+r(4R^2+4Rr+3r^2)\underset{(*)}{\overset{?}{\ge}}0$$

Now, LHS of $(*)\overset{Gerretsen}{\ge}(2R-5r)s^2+rs^2=(2R-4r)s^2\overset{Euler}{\ge}0\Rightarrow(*)$ is true

$$\therefore\left(\sum_{cyc}\frac{a}{b+c}\right)\left(\sum_{cyc}\frac{h_a}{h_b+h_c}\right)\le\frac{(2R-r)^2}{4r^2}\ \text{(QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have : $$\sum_{cyc}\frac{a}{b+c}\overset{CBS}{\le}\sum_{cyc}\frac{a}{4}\left(\frac{1}{b}+\frac{1}{c}\right)=\frac{1}{4}\sum_{cyc}\left(\frac{a}{b}+\frac{b}{a}\right)\overset{Bandila}{\le}\frac{1}{4}\sum_{cyc}\frac{R}{r}$$

$$=\frac{3R}{4r}\overset{Euler}{\le}\frac{2R-r}{2r}.$$

Similarly we have : $$\sum_{cyc}\frac{h_a}{h_b+h_c}\overset{CBS}{\le}\sum_{cyc}\frac{h_a}{4}\left(\frac{1}{h_b}+\frac{1}{h_c}\right)=\frac{1}{4}\sum_{cyc}\left(\frac{b}{a}+\frac{a}{b}\right)\overset{Bandila}{\le}\frac{1}{4}\sum_{cyc}\frac{R}{r}$$

$$=\frac{3R}{4r}\overset{Euler}{\le}\frac{2R-r}{2r}.$$

Multiplying these inequalities yields the desired inequality.

Equality holds iff ΔABC is equilateral.

658. In ΔABC the following relationship holds:

$$\sqrt[3]{\prod_{cyc}\left(\cos^4\frac{A}{2}+\cos^4\frac{B}{2}\right)}\geq\frac{1}{6}\left(4+\frac{6r}{R}-\left(\frac{r}{R}\right)^2\right)$$

Proposed by Marian Ursărescu-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Lemma : $If\ x,y,z\geq 0\ then\ \sqrt[3]{(x^4+y^4)(y^4+z^4)(z^4+x^4)}\geq\frac{2(x^2y^2+y^2z^2+z^2x^2)}{3}.$

We have :

$$9\prod_{cyc}(x^4+y^4)=9\left(\sum_{cyc}x^4\right)\left(\sum_{cyc}x^4y^4\right)-9x^4y^4z^4\overset{AM-GM}{\geq}8\left(\sum_{cyc}x^4\right)\left(\sum_{cyc}x^4y^4\right)\geq$$

$$\overset{AM-GM\ \&\ CBS}{\geq}8\left(\sum_{cyc}x^2y^2\right).\frac{\left(\sum_{cyc}x^2y^2\right)^2}{3}=\frac{\left(2\sum_{cyc}x^2y^2\right)^3}{3}.$$

$$Then:\ \sqrt[3]{(x^4+y^4)(y^4+z^4)(z^4+x^4)}\geq\frac{2(x^2y^2+y^2z^2+z^2x^2)}{3},\quad\forall x,y,z\geq 0.$$

$$For\ x=\cos\frac{A}{2},\ y=\cos\frac{B}{2},\ z=\cos\frac{C}{2}\ we\ get:$$

$$\sqrt[3]{\prod_{cyc}\left(\cos^4\frac{A}{2}+\cos^4\frac{B}{2}\right)}\geq\frac{2}{3}\sum_{cyc}\cos^2\frac{B}{2}\cos^2\frac{C}{2}=\frac{2}{3}\sum_{cyc}\frac{s(s-b)}{ca}.\frac{s(s-c)}{ab}$$

$$=\frac{2s.sr^2}{3.4Rsr}\sum_{cyc}\frac{s}{a(s-a)}=$$

$$=\frac{sr}{6R}\sum_{cyc}\left(\frac{1}{a}+\frac{1}{s-a}\right)=\frac{sr}{6R}\left(\frac{s^2+r^2+4Rr}{4Rsr}+\frac{4R+r}{sr}\right)$$

$$=\frac{s^2+(4R+r)^2}{24R^2}\overset{Gerretsen}{\geq}\frac{16R^2+24Rr-4r^2}{24R^2}.$$

$$Therefore,\ \sqrt[3]{\prod_{cyc}\left(\cos^4\frac{A}{2}+\cos^4\frac{B}{2}\right)}\geq\frac{1}{6}\left(4+\frac{6r}{R}-\left(\frac{r}{R}\right)^2\right).$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sqrt[3]{\prod_{cyc}\left(\cos^4\frac{A}{2}+\cos^4\frac{B}{2}\right)}\geq\sqrt[3]{\prod_{cyc}\frac{\left(\cos^2\frac{A}{2}+\cos^2\frac{B}{2}\right)^2}{2}}=\frac{1}{2}\left(\prod_{cyc}\left(\cos^2\frac{A}{2}+\cos^2\frac{B}{2}\right)\right)^{\frac{2}{3}}$$

$$=\frac{1}{2}\left(\left(\sum_{cyc}\cos^2\frac{A}{2}\right)\left(\sum_{cyc}\cos^2\frac{A}{2}\cos^2\frac{B}{2}\right)-\prod_{cyc}\cos^2\frac{A}{2}\right)^{\frac{2}{3}}$$

$$=\frac{1}{2}\left(\left(\prod_{cyc}\cos^2\frac{A}{2}\right)\left(\sum_{cyc}\cos^2\frac{A}{2}\right)\left(\sum_{cyc}\sec^2\frac{A}{2}\right)-\prod_{cyc}\cos^2\frac{A}{2}\right)^{\frac{2}{3}}$$

$$=\frac{1}{2}\left(\left(\frac{s^2}{16R^2}\right)\left(\frac{4R+r}{2R}\right)\left(\frac{s^2+(4R+r)^2}{s^2}\right)-\frac{s^2}{16R^2}\right)^{\frac{2}{3}}$$

$$=\frac{1}{2}\left(\frac{(2R+r)s^2+(4R+r)^3}{32R^3}\right)^{\frac{2}{3}}\overset{?}{\geq}\frac{1}{6}\left(4+\frac{6r}{R}-\left(\frac{r}{R}\right)^2\right)$$

$$\Leftrightarrow\left(\frac{(2R+r)s^2+(4R+r)^3}{32R^3}\right)^2\overset{?}{\geq}\frac{\left(4R^2+6Rr-r^2\right)^3}{27R^6}$$

$$\Leftrightarrow\boxed{27\left((2R+r)s^2+(4R+r)^3\right)^2\underbrace{\overset{?}{\geq}}_{(*)}1024\left(4R^2+6Rr-r^2\right)^3}$$

$$\text{Now, LHS of }(*)\overset{\text{Gerretsen}}{\geq}27\left((2R+r)\left(16Rr-5r^2\right)+(4R+r)^3\right)^2\overset{?}{\geq}1024\left(4R^2+6Rr-r^2\right)^3$$

$$\Leftrightarrow 11264t^6-4608t^5-39552t^4-2448t^3+22443t^2-5580t$$

$$+364\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)\left((t-2)\left(11264t^4+40448t^3+77184t^2+144496t+291691\right)+583200\right)\overset{?}{\geq}0$$

$$\to\text{true}\because t\overset{\text{Euler}}{\geq}2\Rightarrow(*)\text{ is true}$$

$$\therefore\text{ in any }\Delta\,ABC,\ \sqrt[3]{\prod_{cyc}\left(\cos^4\frac{A}{2}+\cos^4\frac{B}{2}\right)}\geq\frac{1}{6}\left(4+\frac{6r}{R}-\left(\frac{r}{R}\right)^2\right)\ \text{(QED)}$$

659. In ΔABC the following relationship holds:

$$12R^2\leq\sum_{cyc}\frac{(b^2+c^2-a^2)^2}{bc.\sin^2 A}\leq\frac{16R^4-208r^4}{r^2}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:\ \sum_{cyc}\frac{(b^2+c^2-a^2)^2}{bc.\sin^2 A}=\sum_{cyc}\frac{4bc.\cos^2 A}{\sin^2 A}=4\sum_{cyc}bc\left(\frac{1}{\sin^2 A}-1\right)$$

$$=16R^2\sum_{cyc}\frac{bc}{a^2}-4\sum_{cyc}bc\geq$$

$$\underbrace{\geq}^{AM-GM}16R^2.3-4\sum_{cyc}a^2\underbrace{\geq}^{Leibniz}48R^2-4.9R^2=12R^2.$$

$$\text{Now we have}: \sum_{cyc}\frac{(b^2+c^2-a^2)^2}{bc.\sin^2 A}=16R^2\sum_{cyc}\frac{bc}{a^2}-4\sum_{cyc}bc$$

$$=16R^2\sum_{cyc}\frac{bc}{[(s-b)+(s-c)]^2}-4abc\sum_{cyc}\frac{1}{a}\le$$

$$\overset{AM-GM\ \&\ CBS}{\le}16R^2\sum_{cyc}\frac{bc}{4(s-b)(s-c)}-\frac{4abc.9}{a+b+c}=4R^2.\frac{4Rsr}{sr^2}.\sum_{cyc}\frac{s-a}{a}-\frac{36.4Rsr}{2s}=$$

$$=\frac{16R^3}{r}.\frac{s^2+r^2-8Rr}{4Rr}-72Rr\overset{Gerretsen}{\le}\frac{4R^2(4R^2-4Rr+4r^2)}{r^2}-72Rr\le$$

$$\overset{Euler}{\le}\frac{4R^2(4R^2-4.2r.r+4r^2)}{r^2}-72.2r.r\overset{Euler}{\le}\frac{16R^4-4(2r)^2.4r^2}{r^2}-144r^2$$

$$=\frac{16R^4-208r^4}{r^2}.$$

$$\text{Therefore},\ 12R^2\le\sum_{cyc}\frac{(b^2+c^2-a^2)^2}{bc.\sin^2 A}\le\frac{16R^4-208r^4}{r^2}.$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{(b^2+c^2-a^2)^2}{bc.\sin^2A}=4R^2\sum_{cyc}\frac{\left(\sum_{cyc}a^2-2a^2\right)^2}{bca^2}$$

$$=\frac{4R^2}{4Rrs}\left(\left(\sum_{cyc}a^2\right)^2.\frac{\sum_{cyc}ab}{4Rrs}-4\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a\right)+4\sum_{cyc}a^3\right)$$

$$=\frac{R}{rs}\left(\frac{(s^2+4Rr+r^2)(s^2-4Rr-r^2)^2}{Rrs}-16s(s^2-4Rr-r^2)+8s(s^2-6Rr-3r^2)\right)$$

$$=\frac{R}{rs}\left(\frac{(s^2+4Rr+r^2)(s^2-4Rr-r^2)^2}{Rrs}-8s(s^2-2Rr+r^2)\right)$$

$$=\frac{s^6-(12Rr+r^2)s^4-r^2s^2(16Rr+r^2)+r^3(4R+r)^3}{r^2s^2}$$

$$\therefore\sum_{cyc}\frac{(b^2+c^2-a^2)^2}{bc.\sin^2A}\overset{(*)}{=}\frac{s^6-(12Rr+r^2)s^4-r^2s^2(16Rr+r^2)+r^3(4R+r)^3}{r^2s^2}\ \therefore(*)$$

$$\Rightarrow 12R^2\le\sum_{cyc}\frac{(b^2+c^2-a^2)^2}{bc.\sin^2A}$$

$$\Leftrightarrow s^6-(12Rr+r^2)s^4-r^2s^2(16Rr+r^2)+r^3(4R+r)^3\ge 12R^2r^2s^2$$

$$\Leftrightarrow s^6-(12Rr+r^2)s^4-r^2s^2(12R^2+16Rr+r^2)+r^3(4R+r)^3\overset{(\bullet)}{\ge}0$$

$$\text{Now, LHS of }(\bullet)\overset{\text{Gerretsen}}{\ge}(4Rr-6r^2)s^4-r^2s^2(12R^2+16Rr+r^2)$$

$$+r^3(4R+r)^3\overset{\text{Gerretsen}}{\ge}r^2s^2\Big((4R-6r)(16R-5r)$$

$$-(12R^2+16Rr+r^2)\Big)+r^3(4R+r)^3\overset{?}{\ge}0$$

$$\Leftrightarrow(52R^2-132Rr+29r^2)s^2+r^3(4R+r)^3\underset{(\bullet\bullet)}{\overset{?}{\ge}}0$$

Case 1 $52R^2-132Rr+29r^2\ge 0$ and then, LHS of $(\bullet\bullet)\ge r^3(4R+r)^3>0$

$\Rightarrow(\bullet\bullet)$ is true (strict inequality)

Case 2 $52R^2-132Rr+29r^2<0$ and then, LHS of $(\bullet\bullet)$

$$=-\Big(-(52R^2-132Rr+29r^2)\Big)s^2+r^3(4R+r)^3$$

$$\overset{\text{Gerretsen}}{\ge}-\Big(-(52R^2-132Rr+29r^2)\Big)(4R^2+4Rr+3r^2)+r^3(4R+r)^3\overset{?}{\ge}0$$

$$\Leftrightarrow 52t^4-64t^3-52t^2-67t+22\overset{?}{\ge}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)(52t^3+40t^2+22t+6(t-2)+1)\overset{?}{\ge}0\to\text{true}\because t\overset{\text{Euler}}{\ge}2\Rightarrow(\bullet\bullet)\text{ is true}$$

$\therefore$ combining both cases, $(\bullet\bullet)$ is true in any triangle

$$\Rightarrow\text{in any }\Delta\,ABC,\ \boxed{12R^2\le\sum_{cyc}\frac{(b^2+c^2-a^2)^2}{bc.\sin^2A}}$$

$$\text{Again, }\sum_{cyc}\frac{(b^2+c^2-a^2)^2}{bc.\sin^2A}$$

$$\le\frac{16R^4-208r^4}{r^2}\overset{\text{via }(*)}{\Leftrightarrow}\frac{s^6-(12Rr+r^2)s^4-r^2s^2(16Rr+r^2)+r^3(4R+r)^3}{r^2s^2}$$

$$\le\frac{16R^4-208r^4}{r^2}$$

$$\Leftrightarrow s^6-(12Rr+r^2)s^4-\Big(16R^4-208r^4+r^2(16Rr+r^2)\Big)s^2+r^3(4R+r)^3\overset{(\bullet\bullet\bullet)}{\le}0$$

$$\text{Now, LHS of } (\bullet\bullet\bullet) \overset{\text{Gerretsen}}{\le} (4R^2-8Rr+2r^2)s^4$$

$$-\left(16R^4-208r^4+r^2(16Rr+r^2)\right)s^2+r^3(4R+r)^3$$

$$\overset{\text{Gerretsen}}{\le} (4R^2-8Rr+2r^2)(4R^2+4Rr+3r^2)s^2$$

$$-\left(16R^4-208r^4+r^2(16Rr+r^2)\right)s^2+r^3(4R+r)^3 \overset{?}{\le} 0$$

$$\Leftrightarrow (16R^3+12R^2r+32Rr^2-213r^3)s^2 \overset{(\bullet\bullet\bullet\bullet)}{\ge} r^2(4R+r)^3$$

$$\because 16R^3+12R^2r+32Rr^2-213r^3$$

$$=(R-2r)(16R^2+44Rr+120r^2)+27r^3 \overset{\text{Euler}}{\ge} 27r^3>0$$

$$\therefore \text{LHS of } (\bullet\bullet\bullet\bullet) \overset{\text{Gerretsen}}{\ge} (16R^3+12R^2r+32Rr^2-213r^3)(16Rr-5r^2)$$

$$\overset{?}{\ge} r^2(4R+r)^3 \Leftrightarrow 64t^4+12t^3+101t^2-895t+266 \overset{?}{\ge} 0$$

$$\Leftrightarrow (t-2)(64t^3+140t^2+314t+67(t-2)+1) \overset{?}{\ge} 0 \to \text{true} \because t \overset{\text{Euler}}{\ge} 2$$

$$\Rightarrow (\bullet\bullet\bullet\bullet) \Rightarrow (\bullet\bullet\bullet) \text{ is true}$$

$$\therefore \text{in } any\ \Delta\ ABC, \boxed{\sum_{cyc}\frac{(b^2+c^2-a^2)^2}{bc.\sin^2 A} \le \frac{16R^4-208r^4}{r^2}}\ (QED)$$

660. If $f:\mathbb{R}\to\mathbb{R}, 1+f(x+y)\le f(x)+f(y)\le x+y+2, \forall x,y\in\mathbb{R}$, then in ΔABC holds:

$$\frac{f(x^2)}{y+z}\cdot a^2+\frac{f(y^2)}{z+x}\cdot b^2+\frac{f(z^2)}{x+y}\cdot c^2 \ge 4\sqrt{3}\cdot F; \forall x,y,z\in(0,\infty)$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

Solution by Alex Szoros-Romania

For $x=y=0 \Rightarrow 1+f(0)\le 2f(0)\le 2 \Rightarrow 1\le f(0)\le 1 \Rightarrow f(0)=1$

For $y=x \Rightarrow f(x)+f(x)\le x+x+2 \Rightarrow$

$$f(x)\le x+1, \forall x\in\mathbb{R}; (1)$$

For $y=-x \Rightarrow 1+f(0)\le f(x)+f(-x)\le 2 \Rightarrow$

$$f(x)+f(-x)=2, \forall x\in\mathbb{R}; (2)$$

From $f(x)+f(y)\le x+y+2, \forall x,y\in\mathbb{R} \Rightarrow f(-x)+f(-x)\le -x-x+2$

$$2f(x)\le 2-2x\Rightarrow f(-x)\le 1-x\overset{(2)}{\Rightarrow}2-f(x)\le 1-x,\forall x\in\mathbb{R}$$

$$\Rightarrow x+1\le f(x),\forall x\in\mathbb{R};(3)$$

From (1) and (3): $f(x)=x+1,\forall x\in\mathbb{R}\Rightarrow f(x^2)\ge 2x,\forall x\in\mathbb{R}$

For $x,y.z>0$ we have:

$$\sum_{cyc}\frac{f(x^2)}{y+z}\cdot a^2\ge\sum_{cyc}\frac{2x}{y+z}\cdot a^2=2\sum_{cyc}\frac{x}{y+z}\cdot a^2;(4)$$

$$\sum_{cyc}\frac{x}{y+z}\cdot a^2\overset{Tsintsifas}{\ge}2\sqrt{3}\cdot F;(5)$$

From (4) and (5), it follows:

$$\sum_{cyc}\frac{f(x^2)}{y+z}\cdot a^2\ge 4\sqrt{3}\cdot F;\forall x,y,z>0$$

661. In ΔABC the following relationship holds:

$$3\cdot\frac{2r}{R}\le\frac{\csc\frac{A}{2}}{\csc\frac{B}{2}}+\frac{\csc\frac{B}{2}}{\csc\frac{C}{2}}+\frac{\csc\frac{C}{2}}{\csc\frac{A}{2}}\le 3\left(\frac{R}{2r}\right)^{\frac{3}{2}}$$

Proposed by Marin Chirciu-Romania

Solution by Marian Ursărescu-Romania

For the left side:

$$\frac{\csc\frac{A}{2}}{\csc\frac{B}{2}}+\frac{\csc\frac{B}{2}}{\csc\frac{C}{2}}+\frac{\csc\frac{C}{2}}{\csc\frac{A}{2}}\overset{AGM}{\ge}3$$

We must show: $3\ge 3\cdot\frac{2r}{R}\Leftrightarrow R\ge 2r\ (Euler)$

For the right side we must show:

$$\left(\frac{\csc\frac{A}{2}}{\csc\frac{B}{2}}+\frac{\csc\frac{B}{2}}{\csc\frac{C}{2}}+\frac{\csc\frac{C}{2}}{\csc\frac{A}{2}}\right)^2\le\frac{9}{8}\left(\frac{R}{r}\right)^3;(1)$$

From Cauchy's inequality:

$$\left(\frac{\csc\frac{A}{2}}{\csc\frac{B}{2}}+\frac{\csc\frac{B}{2}}{\csc\frac{C}{2}}+\frac{\csc\frac{C}{2}}{\csc\frac{A}{2}}\right)^2\le\left(\sum_{cyc}\csc^2\frac{A}{2}\right)\left(\sum_{cyc}\frac{1}{\csc^2\frac{A}{2}}\right);(2)$$

From (1) and (2) we must show:

$$\left(\sum_{cyc}\csc^2\frac{A}{2}\right)\left(\sum_{cyc}\frac{1}{\csc^2\frac{A}{2}}\right)\le\frac{9}{8}\left(\frac{R}{r}\right)^3;(3)$$

$$\sum_{cyc}\frac{1}{\csc^2\frac{A}{2}}=1-\frac{r}{2R};(4)$$

$$\sum_{cyc}\csc^2\frac{A}{2}=\frac{s^2+r^2-8Rr}{r^2}\overset{Gerretsen}{\le}\frac{R^2-4Rr+4r^2}{r^2}=4\left(\left(\frac{R}{r}\right)^2-\left(\frac{R}{r}\right)+1\right);(5)$$

From (3), (4) and (5) we must show:

$$4\left(\left(\frac{R}{r}\right)^2-\left(\frac{R}{r}\right)+1\right)\left(1-\frac{r}{2R}\right)\le\frac{9}{8}\left(\frac{R}{r}\right)^3;(6)$$

Let $\frac{R}{r}=x, x\ge 0(Euler);(7)$ **From (6) and (7) we must show:**

$$4(x^2-x+1)\left(1-\frac{1}{2x}\right)\le\frac{9}{8}x^3\Leftrightarrow 16(x^2-x+1)(2x-1)\le 9x^4\Leftrightarrow$$

$$4\sqrt{(x^2-x+1)(2x-1)}\le 3x^2;(8)$$

But $\sqrt{(x^2-x+1)(2x-1)}\le\frac{x^2+x}{2};(9)$

From (8) and (9) we must show:

$2x^2+2x\le 3x^2\Leftrightarrow 2x\le x^2\Leftrightarrow 2\le x$ **true!**

662. In ΔABC the following relationship holds:

$$s_a^n s_b+s_b^n s_c+s_c^n s_a\ge\frac{162r^4}{R}\left(\frac{6r^2}{R}\right)^{n-2}, n\in\mathbb{N}, n\ge 2$$

Proposed by Marin Chirciu-Romania

Solution 1 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

From Holder's inequality:

$$\sum_{cyc}\frac{x^n}{a}\ge\frac{(\sum x)^n}{3^{n-2}\sum a};\ \ (1)$$

$$s_a\ge h_a;\ \ (2)$$

$$\sum_{cyc}s_a^n s_b=s_as_bs_c\sum_{cyc}\frac{s_a^{n-1}}{s_c}\overset{(1)}{\ge}s_as_bs_c\cdot\frac{(\sum s_a)^{n-2}}{3^{n-3}}\overset{(2)}{\ge}h_ah_bh_c\cdot\frac{(\sum h_a)^{n-2}}{3^{n-3}}=$$

$$=\frac{(abc)^2}{8R^3}\cdot 3\left(\frac{\sum ab}{2R^3}\right)^{n-2}=\frac{16R^2r^2s^2}{8R^3}\cdot 3\left(\frac{s^2+4Rr+r^2}{6R}\right)^{n-2}\overset{\substack{Euler\\ Mitrinovic}}{\ge}$$

$$\ge\frac{54r^4}{R}\cdot 3\left(\frac{27r^2+8r^2+r^2}{6R}\right)^{n-2}=\frac{162r^4}{R}\left(\frac{36r^2}{6R}\right)^{n-2}=\frac{162r^4}{R}\left(\frac{6r^2}{R}\right)^{n-2}$$

Solution 2 by Tapas Das-India

$$s_a\ge h_a;\ \frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}=\frac{1}{r}$$

The product $\frac{1}{h_ah_bh_c}$ has the maximum value if $\frac{1}{h_a}=\frac{1}{h_b}=\frac{1}{h_c}=\frac{1}{3r}$ where it follows that the product $h_ah_bh_c$ attains the minimum value $27r^3$ if $h_a=h_b=h_c$. Hence,

$$h_ah_bh_c\ge 27r^3$$

$$\sum_{cyc}s_a^n s_b=\sum_{cyc}s_a^{n-2}(s_a^2s_b)\overset{AGM}{\ge}3(s_as_bs_x)^{\frac{n-2}{3}}(s_a^3s_b^3s_c^3)^{\frac{1}{3}}=$$

$$=3\left(\frac{2bcm_a}{b^2+c^2}\cdot\frac{2cam_b}{c^2+a^2}\cdot\frac{2abm_c}{a^2+b^2}\right)^{\frac{n-2}{3}}\cdot s_as_bs_c\overset{Bandila}{\ge}$$

$$\ge 3\left[8\left(\frac{r}{R}\right)^3\cdot 27r^3\right]^{\frac{n-2}{3}}\cdot h_ah_bh_c\ge 3\cdot\left(2\cdot\frac{r^2}{R}\cdot 3\right)^{n-2}\cdot 27r^3=$$

$$=\left(\frac{6r^2}{R}\right)^{n-2}\cdot\frac{2\cdot 27r^4}{r}=\left(\frac{6R^2}{R}\right)^{n-2}\cdot\frac{162r^4}{R}$$

663. In ΔABC the following relationship holds:

$$\frac{R}{r}\ge\frac{(b+c)^4+2bc(b^2-bc+c^2)}{bc(b+2c)(2b+c)}\ge 2$$

Proposed by Alex Szoros-Romania

Solution 1 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$(b+c)^4+2bc(b^2-bc+c^2)\overset{AGM}{\geq}4bc(b+c)^2+2bc(b^2-bc+c^2)\geq$$

$$\geq 2bc(2b^2+2c^2+4bc+b^2+c^2-bc)\overset{AGM}{\geq}2bc(b+2c)(c+2b)$$

$$\frac{(b+c)^4+2bc(b^2-bc+c^2)}{bc(b+2c)(2b+c)}\geq\frac{2bc(b+2c)(c+2b)}{bc(b+2c)(c+2b)}\geq 2$$

$$\frac{R}{r}\overset{Bandila}{\geq}\frac{b}{c}+\frac{c}{b}\overset{(?)}{\geq}\frac{(b+c)^4+2bc(b^2-bc+c^2)}{bc(b+2c)(2b+c)}$$

$$2\left(\frac{b}{c}+\frac{c}{b}\right)bc(b+2c)(c+2b)=(b^2+c^2)(2b^2+2c^2+5bc)$$

$$2(b^2+c^2)+5bc(b^2+c^2)\overset{(?)}{\geq}(b+c)^4+2bc(b^2-bc+c^2)$$

$$2b^4+2c^4+4b^2c^2+5bc(b^2+c^2)\geq b^4+c^4+6b^2c+4b^2c^2+6bc^3$$

$$b^4+c^4\geq b^3c+bc^3\Leftrightarrow(b-c)^2(b^2+bc+c^2)\geq 0$$

Solution 2 by Tapas Das-India

$$(b+c)^4+2bc(b^2-bc+c^2)=(b+c)^4+2bc[(b+c)^2-2bc-bc]=$$

$$=(b+c)^4+2bc(b+c)^2-6b^2c^2=(b+c)^2[(b+c)^2+2bc]-6b^2c^2\geq$$

$$\geq\left(2\sqrt{bc}\right)^2[(b+c)^2+2bc]-6b^2c^2=4bc(b^2+2bc+c^2+2bc)-6b^2c^2=$$

$$=2bc(2b^2+2c^2+8bc-3bc)=2bc(2b^2+5bc+2c^2)=2bc(b+2c)(c+2b)$$

Hence,

$$(b+c)^4+2bc(b^2-bc+c^2)\geq 2bc(b+2c)(c+2b)\text{ or}$$

$$\frac{(b+c)^4+2bc(b^2-bc+c^2)}{bc(b+2c)(2b+c)}\geq 2$$

$$\frac{(b+c)^4+2bc(b^2-bc+c^2)}{bc(b+2c)(2b+c)}=\frac{(b^4+c^4)+4(b^3c+bc^3)+2(b^3c+bc^3)+4b^2c^2}{bc(2b^2+5bc+2c^2)}=$$

$$=\frac{\left(\frac{b^2}{c^2}+\frac{c^2}{b^2}\right)+4\left(\frac{b}{c}+\frac{c}{b}\right)+2\left(\frac{b}{c}+\frac{c}{b}\right)+4}{2\left(\frac{b}{c}+\frac{c}{b}\right)+5}=\frac{\left(\frac{b}{c}+\frac{c}{b}\right)^2+6\left(\frac{b}{c}+\frac{c}{b}\right)+2}{2\left(\frac{b}{c}+\frac{c}{b}\right)+5}$$

Now,

$$\left(\frac{b}{c}+\frac{c}{b}\right)\left[2\left(\frac{b}{c}+\frac{c}{b}\right)+5\right]-\left[\left(\frac{b}{c}+\frac{c}{b}\right)^2+6\left(\frac{b}{c}+\frac{c}{b}\right)+2\right]\overset{\left(\frac{b}{c}+\frac{c}{b}=x\right)}{=}$$

$$2x^2+5x-x^2-6x-2=x^2-x-2=$$

$$=\left(\frac{b}{c}+\frac{c}{b}\right)^2-\left(\frac{b}{c}+\frac{c}{b}\right)-2=\frac{(b-c)^2}{b^2c^2}(b^2+bc+c^2)\geq 0$$

$$\left(\frac{b}{c}+\frac{c}{b}\right)\left[2\left(\frac{b}{c}+\frac{c}{b}\right)+5\right]\geq\left[\left(\frac{b}{c}+\frac{c}{b}\right)^2+6\left(\frac{b}{c}+\frac{c}{b}\right)+2\right]$$

$$\Rightarrow\frac{b}{c}+\frac{c}{b}\geq\frac{\left(\frac{b}{c}+\frac{c}{b}\right)^2+6\left(\frac{b}{c}+\frac{c}{b}\right)+2}{2\left(\frac{b}{c}+\frac{c}{b}\right)+5}$$

$$\frac{R}{r}\overset{Bandila}{\geq}\frac{(b+c)^4+2bc(b^2-bc+c^2)}{bc(b+2c)(2b+c)}$$

664. a,b,c **–sides in** $\Delta ABC,\sqrt{a},\sqrt{b},\sqrt{c}$ **–sides in** $\Delta A'B'C'$**. Prove that:**

$$\sqrt{a}\cos A'+\sqrt{b}\cos B'+\sqrt{c}\cos C'\leq\frac{r_a+r_b+r_c}{\sqrt{6\sqrt{3}r}}$$

Proposed by Mehmet Şahin-Ankara-Turkiye

Solution by Marian Ursărescu-Romania

In any ΔABC **holds:**

$$a\cos A=\frac{a(b^2+c^2-a^2)}{2bc}$$

$$\sum_{cyc}a\cos A=\sum_{cyc}\frac{a(b^2+c^2-a^2)}{2bc}=\frac{1}{2abc}\sum_{cyc}a^2(b^2+c^2-a^2)$$

In our case, we have:

$$\sum_{cyc}r_a\cos A'=\frac{1}{2\sqrt{abc}}\sum_{cyc}a(b+c-a)=\frac{1}{2\sqrt{abc}}\sum_{cyc}(ab+ac-a^2)=$$

$$=\frac{1}{2\sqrt{abc}}[2(ab+bc+ca)-(a^2+b^2+c^2)];\ \ (1)$$

But $ab+bc+ca=s^2+r^2+4Rr$ **and** $a^2+b^2+c^2=2(s^2-r^2-4Rr);\ \ (2)$

From (1) and (2), it follows

$$\sum_{cyc}\sqrt{a}\cos A'=\frac{1}{2\sqrt{abc}}2(s^2+r^2+4Rr-s^2+r^2+4Rr)=$$

$$=\frac{8Rr+2r^2}{\sqrt{abc}}=\frac{2r(4R+r)}{\sqrt{abc}}$$

We must show:

$$\frac{2r(4R+r)}{\sqrt{abc}} \le \frac{r_a+r_b+r_c}{\sqrt{6\sqrt{3}r}}; (3)$$

But $abc = 4Rs$ and $r_a + r_b + r_c = 4R + r; (4)$. From (3) and (4), we must show:

$$\frac{2r(4R+r)}{\sqrt{4sRr}} \le \frac{4R+r}{\sqrt{6\sqrt{3}r}} \Leftrightarrow \sqrt{6\sqrt{3}r \cdot r} \le \sqrt{sRr} \Leftrightarrow 6\sqrt{3}r^2 \le Rs$$

which is true, because $s \ge 3\sqrt{3}r$ and $R \ge 2r$ $(Euler)$.

665. In ΔABC the following relationship holds:

$$\prod_{cyc}[2(m_bh_c + m_ch_b) - h_a(h_b+h_c)] \ge h_ah_bh_c\prod_{cyc}(h_b+h_c)$$

Proposed by Bogdan Fuştei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$By\ Tereshin's\ inequality\ we\ have:$$

$$2m_a \ge \frac{b^2+c^2}{2R} = \frac{h_ch_a}{h_b} + \frac{h_ah_b}{h_c}\ (and\ analogs).\ \left(\because\ \frac{h_ch_a}{h_b} = \frac{ab}{2R}\cdot\frac{bc}{2R}\cdot\frac{2R}{ca} = \frac{b^2}{2R}\right)$$

$$Then:\ \ 2(m_bh_c + m_ch_b) \ge \left(\frac{h_ah_b}{h_c} + \frac{h_bh_c}{h_a}\right)h_c + \left(\frac{h_bh_c}{h_a} + \frac{h_ch_a}{h_b}\right)h_b$$

$$= \left(h_a + \frac{h_bh_c}{h_a}\right)(h_b+h_c)$$

$$Thus:\ \ 2(m_bh_c + m_ch_b) - h_a(h_b+h_c) \ge \frac{h_bh_c}{h_a}.(h_b+h_c)\ \ (and\ analogs)$$

$$Multiplying\ this\ inequality\ with\ similar\ ones\ we\ get:$$

$$\prod_{cyc}[2(m_bh_c+m_ch_b) - h_a(h_b+h_c)] \ge \prod_{cyc}\left(\frac{h_bh_c}{h_a}.(h_b+h_c)\right) = h_ah_bh_c\prod_{cyc}(h_b+h_c),$$

$$as\ desired.\ \ Equality\ holds\ iff\ \Delta ABC\ is\ equilateral.$$

666. Find $x \in R$ so that the double inequality

$$\frac{R^2}{r} \geq \frac{a^x + b^x + c^x}{h_a + h_b + h_c} \geq 2R \text{ holds in any } \Delta ABC$$

Proposed by Alex Szoros-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\textit{Because the double inequality is true in any } \Delta ABC,$$

$$\textit{then it is true for equilateral } \Delta ABC.$$

$$\textit{For } a = b = c \textit{ we have}: \ 2Rh_a = bc = a^2 \text{ and } R = 2r.$$

$$\textit{Replacing on the problem we get}:$$

$$1 \geq \frac{3a^x}{3a^2} \geq 1 \Leftrightarrow a^{x-2} = 1 \Rightarrow x = 2.$$

$$\textit{So we need to prove the double inequality for } x = 2:$$

$$\textit{We have}: \frac{a^2 + b^2 + c^2}{h_a + h_b + h_c} = 2R.\frac{a^2 + b^2 + c^2}{ab + bc + ca} \geq 2R.1 = 2R.$$

$$\textit{Also, we have}: \frac{a^2 + b^2 + c^2}{h_a + h_b + h_c} \overset{\textit{Leibniz \& CBS}}{\leq} 9R^2.\frac{\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}}{9} = \frac{R^2}{r}.$$

$$\textit{Therefore}, \ x = 2.$$

667. Let $x, y > 0, \Delta A_i B_i C_i, i = 1, 2$ triangles with areas $F_i, i = 1, 2$ and altitudes $h_{a_i}, h_{b_i}, h_{c_i}, i = 1, 2$, then holds:

$$\frac{a_1^{1-x} a_2^{1-x}}{h_{a_1}^x h_{a_2}^y} + \frac{b_1^{1-x} b_1^{1-y}}{h_{b_1}^x h_{b_2}^x} + \frac{c_1^{1-x} c_2^{1-y}}{h_{c_1}^x h_{c_2}^x} \geq 2^{2-x-y}\sqrt{3} \cdot F_1^{\frac{1}{2}-x} F_2^{\frac{1}{2}-y}$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

Solution by Tapas Das-India

$$h_{a_i} h_{b_i} h_{c_i} = \frac{(2F_i)^3}{a_i b_i c_i}, i = 1, 2$$

$$\frac{a_1^{1-x} a_2^{1-x}}{h_{a_1}^x h_{a_2}^y} + \frac{b_1^{1-x} b_1^{1-y}}{h_{b_1}^x h_{b_2}^x} + \frac{c_1^{1-x} c_2^{1-y}}{h_{c_1}^x h_{c_2}^x} \geq 3\sqrt[3]{\frac{(a_1 b_1 c_1)^{1-x}}{\left(h_{a_1} h_{b_1} h_{c_1}\right)^x} \cdot \frac{(a_2 b_2 c_2)^{1-y}}{\left(h_{a_2} h_{b_2} h_{c_2}\right)^y}} =$$

$$= 3\sqrt[3]{\frac{(a_1 b_1 c_1)^{1-x} (a_1 b_1 c_1)^x}{(2F_1)^{3x}} \cdot \frac{(a_2 b_2 c_2)^{1-y} (a_2 b_2 c_2)^y}{(2F_2)^{3y}}} =$$

$$= 3\sqrt[3]{\frac{a_1b_1c_1}{(2F_1)^{3x}}\cdot\frac{a_2b_2c_2}{(2F_2)^{3y}}} \ge 3\sqrt[3]{\left(\frac{4F_1}{\sqrt{3}}\right)^{\frac{3}{2}}\cdot\frac{1}{(2F_1)^{3x}}\cdot\left(\frac{4F_2}{\sqrt{3}}\right)^{\frac{3}{2}}\cdot\frac{1}{(2F_2)^{3y}}} =$$

$$= 3\left[\frac{2\sqrt{F_1}}{\sqrt[4]{3}}\cdot\frac{1}{(2F_1)^x}\cdot\frac{2\sqrt{F_2}}{\sqrt[4]{3}}\cdot\frac{1}{(2F_2)^y}\right] = 2^{2-x-y}\sqrt{3}\cdot F_1^{\frac{1}{2}-x}F_2^{\frac{1}{2}-y}$$

668. In ΔABC the following relationship holds:

$$\frac{3R}{2r} \ge 3 + \frac{(a-b)^2}{al_a+bl_b} + \frac{(b-c)^2}{bl_b+cl_c} + \frac{(c-a)^2}{cl_c+al_a}$$

Proposed by Alex Szoros-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:\ al_a + bl_b \ge ah_a + bh_b = 2F + 2F = 4F\ \ (and\ analogs)$$

$$Then:\ 3 + \frac{(a-b)^2}{al_a+bl_b} + \frac{(b-c)^2}{bl_b+cl_c} + \frac{(c-a)^2}{cl_c+al_a} \le 3 + \frac{(a-b)^2}{4F} + \frac{(b-c)^2}{4F} + \frac{(c-a)^2}{4F} =$$

$$= 3 + \frac{(a^2+b^2+c^2)-(ab+bc+ca)}{2F} = 3 + \frac{2(s^2-r^2-4Rr)-(s^2+r^2+4Rr)}{2F} =$$

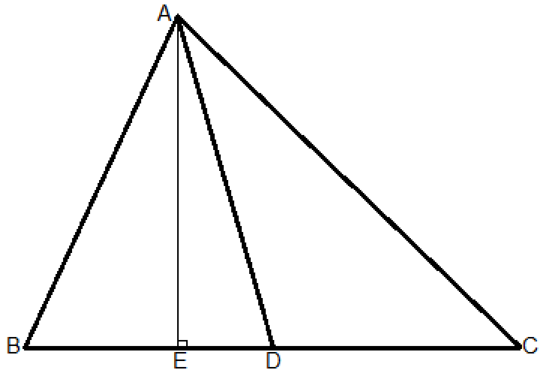
$$= 3 + \frac{s^2-3r(4R+r)}{2F} = 3 + \frac{s}{2r} - \frac{3(4R+r)}{2s} \overset{Mitrinovic}{\le} 3 + \frac{3\sqrt{3}R}{4r} - \frac{3(4R+r)}{3\sqrt{3}R} =$$

$$= 3 - \frac{4\sqrt{3}}{3} + \frac{3\sqrt{3}R}{4r} - \frac{\sqrt{3}r}{3R} = \frac{3R}{2r} - \left(\frac{R}{2r}-1\right)\left(3-\frac{3\sqrt{3}}{2}-\frac{\sqrt{3}r}{3R}\right) \overset{Euler}{\le} \frac{3R}{2r}.$$

$$Because\ \ 3 - \frac{3\sqrt{3}}{2} \ge \frac{\sqrt{3}}{6} \overset{Euler}{\ge} \frac{\sqrt{3}r}{3R}.\ \ So\ the\ proof\ is\ completed.$$

Equality holds iff ΔABC is equilateral.

Solution 2 by Soumava Chakraborty-Kolkata-India



$$AD = m_a \text{ and } AE = h_a \therefore BE = c\cos B \Rightarrow DE = \frac{a}{2} - c\cos B \text{ and } CE = b\cos C \Rightarrow DE = b\cos C - \frac{a}{2}$$

$$\therefore 2DE = b\cos C - c\cos B \text{ and similarly when } c > b, 2DE = c\cos B - b\cos C$$

$$\therefore 2DE = |b\cos C - c\cos B| = \left|b\left(\frac{a^2+b^2-c^2}{2ab}\right) - c\left(\frac{c^2+a^2-b^2}{2ca}\right)\right| = \frac{2|b^2-c^2|}{2a} \Rightarrow 4DE^2$$

$$= \frac{(b^2-c^2)^2}{a^2} \Rightarrow AD^2 - AE^2 = \frac{(b^2-c^2)^2}{4a^2} \Rightarrow m_a^2 - h_a^2 = \frac{(b^2-c^2)^2}{4a^2}$$

$$\Rightarrow \frac{m_a^2 - h_a^2}{h_a^2} = \frac{(b^2-c^2)^2}{4a^2\left(\frac{4F^2}{a^2}\right)} \Rightarrow \frac{m_a^2-h_a^2}{h_a^2} \overset{(*)}{=} \frac{(b^2-c^2)^2}{16F^2}$$

$$\text{Now, } \frac{am_a}{F} \ge 2 + \frac{(b-c)^2}{2F} \Leftrightarrow \frac{2m_a}{h_a} - 2 \ge \frac{(b-c)^2}{2F} \Leftrightarrow \left(\frac{m_a}{h_a} - 1\right)\left(\frac{m_a}{h_a} + 1\right) \ge \frac{(b-c)^2}{4F}\left(\frac{m_a}{h_a} + 1\right)$$

$$\Leftrightarrow \frac{m_a^2 - h_a^2}{h_a^2} \ge \frac{(b-c)^2}{4F}\left(\frac{m_a}{h_a}+1\right) \overset{\text{via } (*)}{\Leftrightarrow} \frac{(b^2-c^2)^2}{16F^2} \ge \frac{(b-c)^2}{4F}\left(\frac{m_a}{h_a}+1\right)$$

$$\overset{\because (b-c)^2 \ge 0}{\Leftrightarrow} \frac{(b+c)^2}{4F} \ge \frac{m_a}{h_a} + 1 = \frac{am_a}{2F} + 1 = \frac{2am_a + 4F}{4F} \Leftrightarrow (b+c)^2 - 2am_a \ge 4F$$

$$\Leftrightarrow \left((b+c)^2 - 2am_a\right)^2$$

$$\ge 16F^2 \left(\because 2am_a < 2(b+c)\left(\frac{b+c}{2}\right) \Rightarrow (b+c)^2 - 2am_a > 0\right)$$

$$\Leftrightarrow (b+c)^4 + 4a^2m_a^2 - 4am_a(b+c)^2 \ge 2\sum a^2b^2 - \sum a^4$$

$$\Leftrightarrow (b+c)^4 + a^2(2b^2+2c^2-a^2) + \sum a^4 - 2\sum a^2b^2 \ge 4am_a(b+c)^2$$

$$\Leftrightarrow b^4 + c^4 + 2b^2c^2 + 2b^3c + 2bc^3 \ge 2am_a(b+c)^2 \Leftrightarrow (b^2+c^2)^2 + 2bc(b^2+c^2)$$

$$\ge 2am_a(b+c)^2 \Leftrightarrow (b^2+c^2)(b+c)^2 \ge 2am_a(b+c)^2 \Leftrightarrow b^2+c^2 \ge 2am_a$$

$$\Leftrightarrow (b^2+c^2)^2 \ge 4a^2m_a^2 \Leftrightarrow b^4 + c^4 + 2b^2c^2 \ge a^2(2b^2+2c^2-a^2)$$

$$\Leftrightarrow b^4 + c^4 + a^4 + 2b^2c^2 - 2a^2b^2 - 2c^2a^2 \ge 0 \Leftrightarrow (b^2+c^2-a^2)^2 \ge 0$$

$$\Leftrightarrow \cos^2 A \ge 0 \rightarrow \text{true}$$

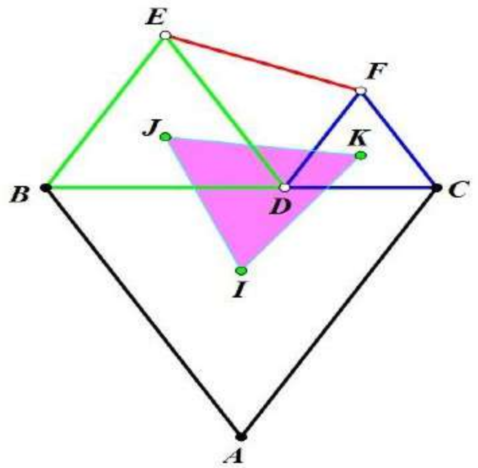
$$\Rightarrow \frac{am_a}{2F} \geq 1+\frac{(b-c)^2}{4F} \Rightarrow 1+\frac{(b-c)^2}{4F} \leq \frac{m_a}{h_a} \overset{\text{Panaitopol}}{\leq} \frac{R}{2r} \text{ and analogs}$$

$$\Rightarrow \boxed{3+\sum_{cyc}\frac{(b-c)^2}{4F} \overset{(**)}{\leq} \frac{3R}{2r}}$$

$$\text{Now, } 3+\frac{(a-b)^2}{al_a+bl_b}+\frac{(b-c)^2}{bl_b+cl_c}+\frac{(c-a)^2}{cl_c+al_a} \leq 3+\frac{(a-b)^2}{ah_a+bh_b}+\frac{(b-c)^2}{bh_b+ch_c}+\frac{(c-a)^2}{ch_c+ah_a}$$

$$= 3+\sum_{cyc}\frac{(b-c)^2}{4F} \overset{\text{via }(**)}{\leq} \frac{3R}{2r} \Rightarrow \frac{3R}{2r} \geq 3+\frac{(a-b)^2}{al_a+bl_b}+\frac{(b-c)^2}{bl_b+cl_c}+\frac{(c-a)^2}{cl_c+al_a} \text{ (QED)}$$

669.



Three equilateral triangles with centers I, J, K.

Prove $[BEFC] = 3 \cdot [IJK]$.

Proposed by Binh Luc-Vietnam

Solution by Eldeniz Hesenov-Georgia

$$\begin{cases} BJ = a \\ DK = b \end{cases} \Rightarrow \begin{cases} BD = a\sqrt{3} \\ DC = b\sqrt{3} \end{cases} \Rightarrow BI = a+b$$

$$IJ^2 = BJ^2 + BI^2 - 2BI \cdot BJ \cdot \cos 60^\circ$$

$$JI = \sqrt{a^2+b^2+ab} \text{ (and analogs)}$$

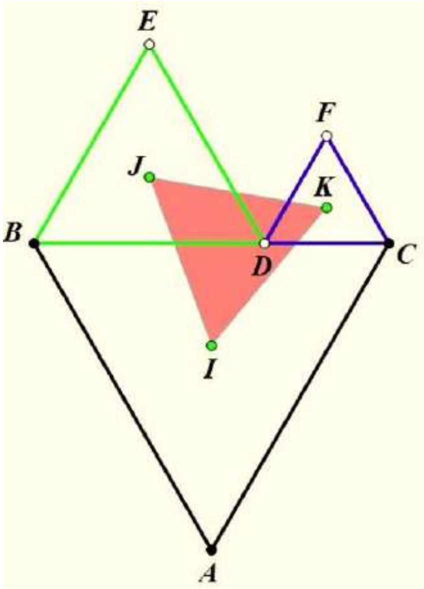
$$\Rightarrow \Delta JIK - \text{equilateral} \Rightarrow [JIK] = \frac{(a^2+b^2+ab)\sqrt{3}}{4}; (1)$$

$$[BDE] = \frac{a^2 3\sqrt{3}}{4}; [DFC] = \frac{b^2 3\sqrt{3}}{4}$$

$$[DEF] = \frac{1}{2} a\sqrt{3} \cdot b\sqrt{3} \cdot \sin 60^\circ = \frac{3\sqrt{3}ab}{4}$$

$$[BEFC] = \frac{3\sqrt{3}a^2}{4} + \frac{3\sqrt{3}b^2}{4} + \frac{3\sqrt{3}ab}{4} = \frac{3\sqrt{3}}{4}(a^2 + b^2 + ab) \overset{(1)}{=} [JIK]$$

670.

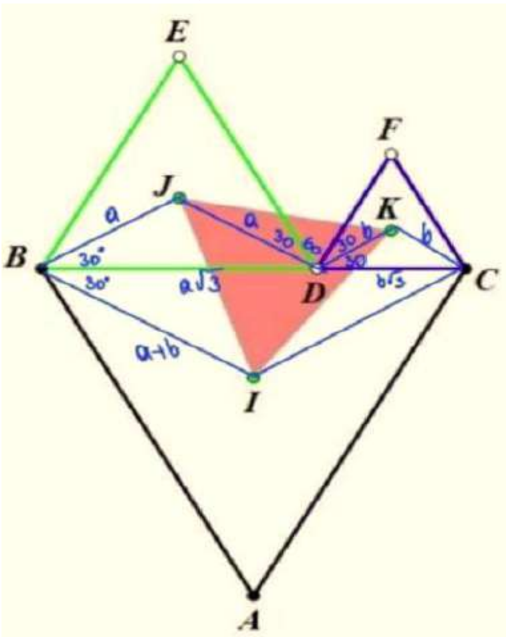


Three equilateral triangles with centers I, J, K. Prove that:

$$[ABC] + [BDE] + [CDF] = 6[IJK]$$

Proposed by Binh Luc-Vietnam

Solution 1 by Eldeniz Hesenov-Georgia



$$BJ = a \Rightarrow BD = a\sqrt{3}, CK = b \Rightarrow DC = b\sqrt{3}$$

$$BC = (a+b)\sqrt{3} \Rightarrow BI = IC = a+b$$

$$\sphericalangle JBI = \sphericalangle KCI = 60^\circ \Rightarrow IJ^2 = BJ^2 + BI^2 - 2BJ \cdot BI \cdot \cos 60^\circ$$

$$\Rightarrow IJ = \sqrt{a^2+b^2+ab}; (1)$$

$$IK^2 = IC^2 + KC^2 - 2IC \cdot KC \cdot \cos 60^\circ \Rightarrow IK = \sqrt{a^2+b^2+ab}; (2)$$

$$KJ^2 = KD^2 + JD^2 - 2JD \cdot DK \cdot \cos 120^\circ \Rightarrow KJ = \sqrt{a^2+b^2+ab}; (3)$$

From (1), (2) and (3) it follows:

$$JI = IK = KJ \Rightarrow S_{JKI} = \frac{(a^2+b^2+c^2)\sqrt{3}}{4}$$

$$S_{BED} + S_{DFC} + S_{ABC} = \frac{3\sqrt{3}a^2}{4} + \frac{3\sqrt{3}b^2}{4} + \frac{3\sqrt{3}(a+b)^2}{4} =$$

$$= \frac{3\sqrt{3}}{2}(a^2+b^2+ab) = 6S_{KJI}$$

Solution 2 by Hikmat Mammadov-Azerbaijan

Say the three equilateral triangles have side lengths a, b, c

$$a = b + c$$

$$\Delta BCL \equiv \Delta BCA$$

$$[BCL] = [ABC] = \frac{a^2\sqrt{3}}{4}$$

$$LE = a - b, LF = a - c$$

$$[LEF] = \frac{(a-b)(a-c)\sin\frac{\pi}{3}}{2} = \frac{(a-b)(a-c)\sqrt{3}}{4}$$

$$[BEFC] = [BCL] - [LEF] = \frac{\sqrt{3}}{4}[a^2 - (a-b)(a-c)] =$$

$$= \frac{\sqrt{3}}{4}[(b+c)^2 - bc] = \frac{\sqrt{3}}{4}(b^2+c^2+bc)$$

$$DJ = \frac{2}{3} \cdot \frac{\sqrt{3}b}{2} = \frac{b\sqrt{3}}{3} \text{ and } DK = \frac{2}{3} \cdot \frac{\sqrt{3}c}{2} = \frac{\sqrt{3}c}{3}$$

$$\angle JDK = 120^\circ. \text{Side length of } \Delta IJK = s$$

$$s^2=\left(\frac{\sqrt{3}b}{3}\right)^2+\left(\frac{\sqrt{3}c}{3}\right)^2+2\left(\frac{\sqrt{3}b}{3}\right)\left(\frac{\sqrt{3}c}{3}\right)\cdot\frac{1}{2}=\frac{a^2+b^2+bc}{3}$$

$$[IJK]=\frac{\sqrt{3}s^2}{4}=\frac{1}{3}\cdot\frac{\sqrt{3}(b^2+c^2+bc)}{4}=\frac{[BEFC]}{3},\qquad [BEFC]=3[IJK].$$

671. In ΔABC the following relationship holds:

$$\frac{m_bh_c}{a}+\frac{m_ch_a}{b}+\frac{m_ah_b}{c}\le\frac{9\sqrt{3}}{4}R$$

Proposed by Marian Ursărescu-Romania

Solution 1 by Alex Szoros-Romania

$$F=\frac{ch_c}{2}\Rightarrow h_c=\frac{2F}{a}\Rightarrow\frac{m_bh_c}{a}=\frac{m_b\cdot 2F}{ac}$$

$$\sum_{cyc}\frac{m_bh_c}{a}=2F\sum_{cyc}\frac{m_b}{ac}=\frac{2F}{abc}\sum_{cyc}am_a=\frac{1}{2R}\sum_{cyc}am_a$$

$$\text{But }\left(\sum_{cyc}am_a\right)^2\overset{CBS}{\le}\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}m_a^2\right)=\frac{3}{4}\left(\sum_{cyc}a^2\right)^2\le\frac{3}{4}(9R^2)^2$$

$$\sum_{cyc}am_a\le\frac{9^2\sqrt{3}}{2};(1)$$

$$\sum_{cyc}\frac{m_bh_c}{a}=\frac{1}{2R}\sum_{cyc}am_a\overset{(1)}{\le}\frac{9R^2\sqrt{3}}{4R}=\frac{9R\sqrt{3}}{4}$$

Solution 2 by Ertan Yildirim-Izmir-Turkiye

$$\sum_{cyc}\frac{m_bh_c}{a}\overset{Panaitopol}{\le}\sum_{cyc}h_b\cdot\frac{R}{2r}\cdot\frac{h_c}{a}=\frac{R}{2r}\cdot\sum_{cyc}\frac{h_bh_c}{a}=$$

$$=\frac{R}{2r}\cdot\sum_{cyc}\frac{ac}{2R}\cdot\frac{ab}{2R}\cdot\frac{1}{a}=\frac{R}{8R^2r}\cdot\sum_{cyc}abc=\frac{1}{8Rr}\cdot 3abc=$$

$$=\frac{3}{8Rr}\cdot 4Rrs=\frac{3s}{2}\overset{Mitrinovic}{\le}\frac{3}{2}\cdot\frac{3\sqrt{3}}{2}R=\frac{9\sqrt{3}}{4}R$$

Solution 3 by Eldeniz Hesenov-Georgia

$$\sum_{cyc}\frac{m_bh_c}{a}=\sum_{cyc}\frac{2F\cdot m_b}{ac}=\frac{1}{2R}\sum_{cyc}am_a\overset{CBS}{\le}\frac{1}{2R}\sqrt{\frac{3}{4}\left(\sum_{cyc}a^2\right)^2}=$$

$$=\frac{\sqrt{3}}{4R}\sum_{cyc}a^2\overset{Leibniz}{\le}\frac{\sqrt{3}}{4R}\cdot 9R^2=\frac{9\sqrt{3}R}{4}$$

672. In ΔABC the following relationship holds:

$$\sum_{cyc}\left(\frac{w_a}{w_b+w_c-w_a}\right)^n\sqrt{w_a(w_b+w_c-w_a)}\ge 9r, n\in\mathbb{N}$$

Proposed by Marin Chirciu-Romania

Solution by Marian Ursărescu-Romania

We must show that:

$$\sum_{cyc}\frac{w_a^{n+\frac{1}{2}}}{(w_b+w_c-w_a)^{n-\frac{1}{2}}}\ge 9r;(1)$$

$$\sum_{cyc}\frac{w_a^{n+\frac{1}{2}}}{(w_b+w_c-w_a)^{n-\frac{1}{2}}}\overset{Holder}{\ge}\frac{(w_a+w_b+w_c)^{n+\frac{1}{2}}}{(w_a+w_b+w_c)^{n-\frac{1}{2}}}=w_a+w_b+w_c;(2)$$

From (1) and (2) we must show:

$$w_a+w_b+w_c\ge 9r;(3)$$

But $w_a\ge h_a;(4)$. From (3) and (4) we must show:

$h_a+h_b+h_c\ge 9r$, which is true because

$$(h_a+h_b+h_c)\left(\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}\right)\overset{CBS}{\ge}9$$

$$\text{But }\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}=\frac{1}{r}\Rightarrow h_a+h_b+h_c\ge 9r$$

673. In ΔABC the following relationship holds:

$$\frac{5}{2}+\frac{1}{2}\sum_{cyc}\frac{r_a}{h_a}=2R\sum_{cyc}\frac{1}{w_a}\cos\left(\frac{B-C}{2}\right)$$

Proposed by Eldeniz Hesenov, Rahide Yusubova-Georgia

Solution 1 by Alex Szoros-Romania

$$\frac{r_a}{h_a}=\frac{F}{s-a}\cdot\frac{a}{2F}=\frac{a}{2(s-a)}$$

$$\frac{5}{2}+\frac{1}{2}\sum_{cyc}\frac{r_a}{h_a}=\frac{5}{2}+\frac{1}{2}\sum_{cyc}\frac{a}{2(s-a)}=\frac{5}{2}+\frac{1}{4}\sum_{cyc}\frac{a}{s-a}=$$

$$=\frac{5}{2}+\frac{1}{4}\sum_{cyc}\frac{a-s+s}{s-a}=\frac{5}{2}+\frac{1}{4}\left(-3+\sum_{cyc}\frac{s}{s-a}\right)=\frac{7}{4}+\frac{s}{4}\sum_{cyc}\frac{1}{s-a};(1)$$

$$\cos\left(\frac{B-C}{2}\right)=\frac{h_a}{w_a}$$

$$2R\sum_{cyc}\frac{1}{w_a}\cos\left(\frac{B-C}{2}\right)=2R\sum_{cyc}\frac{1}{w_a}\cos\left(\frac{B-C}{2}\right)=2R\sum_{cyc}\frac{h_a}{w_a^2}=$$

$$=2R\sum_{cyc}\frac{2F}{aw_a^2}=4RF\sum_{cyc}\frac{1}{aw_a^2}=abc\sum_{cyc}\frac{(b+c)^2}{4abcs(s-a)}=\frac{1}{4s}\sum_{cyc}\frac{(b+c)^2}{s-a}=$$

$$=\frac{1}{4s}\sum_{cyc}\frac{(2s-a)^2}{s-a}=\frac{1}{4s}\sum_{cyc}\frac{4s^2-4as+a^2}{s-a}=\frac{1}{4s}\sum_{cyc}\left(4s+\frac{a^2}{s-a}\right)=$$

$$=3+\frac{1}{4s}\sum_{cyc}\frac{a^2}{s-a}=3-\frac{1}{4s}\sum_{cyc}\frac{s^2-a^2-s^2}{s-a}=3-\frac{1}{4s}\sum_{cyc}\left(s+a-\frac{s^2}{s-a}\right)=$$

$$=3-\frac{1}{4s}\left(5s-\sum_{cyc}\frac{s^2}{s-a}\right)=3-\frac{5}{4}+\frac{1}{4s}\sum_{cyc}\frac{s^2}{s-a}=\frac{7}{4}+\frac{s}{4}\sum_{cyc}\frac{1}{s-a};\quad(2)$$

From(1) and (2), it follows that:

$$\frac{5}{2}+\frac{1}{2}\sum_{cyc}\frac{r_a}{h_a}=2R\sum_{cyc}\frac{1}{w_a}\cos\left(\frac{B-C}{2}\right)$$

Solution 2 by Tapas Das-India

$$\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}},\cos\left(\frac{A-B}{2}\right)=\frac{a+b}{2}\sin\frac{C}{2}$$

$$w_a^2=\frac{4bcs(s-a)}{(b+c)^2},2R=\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

$$\frac{2R}{w_a}\cos\left(\frac{B-C}{2}\right)=\frac{a}{\sin A}\cdot\frac{b+c}{2\sqrt{bcs}}\cdot\frac{1}{\sqrt{s-a}}\cdot\frac{b+c}{a}\cdot\sin\frac{A}{2}=$$

$$=\frac{a}{2\sin\frac{A}{2}\cos\frac{A}{2}}\cdot\frac{b+c}{2\sqrt{bcs}}\cdot\frac{1}{\sqrt{s-a}}\cdot\frac{b+c}{a}\cdot\sin\frac{A}{2}=$$

$$=\frac{1}{4}\cdot\frac{(b+c)^2}{\cos\frac{A}{2}\sqrt{bcs(s-a)}}=\frac{1}{4}\cdot\frac{(b+c)^2}{s(s-a)}=\frac{1}{4}\cdot\frac{(2s-a)^2}{s(s-a)}=\frac{1}{4}\cdot\frac{[s+(s-a)]^2}{s(s-a)}=$$

$$=\frac{1}{4}\cdot\frac{s^2+(s-a)^2+2s(s-a)}{s(s-a)}=\frac{1}{4}\left(\frac{s}{s-a}+\frac{s-a}{s}+2\right)$$

$$2R\sum_{cyc}\frac{1}{w_a}\cos\left(\frac{B-C}{2}\right)=\frac{1}{4}\sum_{cyc}\left(\frac{s}{s-a}+\frac{s-a}{s}+2\right)=$$

$$=\frac{1}{4}\sum_{cyc}\frac{s}{s-a}+\frac{7}{4};(1)$$

$$\frac{5}{2}+\frac{1}{2}\sum_{cyc}\frac{r_a}{h_a}=\frac{5}{2}+\frac{1}{2}\sum_{cyc}\frac{F}{s-a}\cdot\frac{a}{2F}=\frac{5}{2}+\frac{1}{4}\sum_{cyc}\frac{a}{s-a}=$$

$$=\frac{5}{2}+\frac{1}{4}\sum_{cyc}\frac{s-(s-a)}{s-a}=\frac{7}{4}+\frac{1}{4}\sum_{cyc}\frac{s}{s-a};(2)$$

From (1) and (2), it follows that:

$$\frac{5}{2}+\frac{1}{2}\sum_{cyc}\frac{r_a}{h_a}=2R\sum_{cyc}\frac{1}{w_a}\cos\left(\frac{B-C}{2}\right)$$

674. a,b,c **−sides in** $\Delta ABC,\sqrt{a},\sqrt{b},\sqrt{c}$ **−sides in** $\Delta A'B'C'$**. Prove that:**

$$a\sin A'+b\sin B'+c\sin C'\le 3F'\sqrt{\frac{2R}{r}}$$

Proposed by Mehmet Şahin-Ankara-Turkiye

Solution 1 by Marian Ursărescu-Romania

From Law of sines:

$$\sin A=\frac{a}{2R}\Rightarrow \sin A'=\frac{a'}{2R'}=\frac{\sqrt{a}}{2R'};\ \ (1)$$

$$F'=\frac{a'b'c'}{4R'}=\frac{\sqrt{abc}}{4R'};(2)$$

From (1) and (2) we must show:

$$\frac{a\sqrt{a}+b\sqrt{b}+c\sqrt{c}}{2R'} \le \frac{3\sqrt{abc}}{4R'}\sqrt{\frac{2R}{r}} \Leftrightarrow 2\left(a\sqrt{a}+b\sqrt{b}+c\sqrt{c}\right) \ge 3\sqrt{abc}\sqrt{\frac{2R}{r}}$$

$$2\left(a\sqrt{a}+b\sqrt{b}+c\sqrt{c}\right)^2 \le 9abc\cdot\frac{R}{r};\ \ (3)$$

But $abc = 4Rrs$; (4). From (3) and (4) we must show:

$$2\left(a\sqrt{a}+b\sqrt{b}+c\sqrt{c}\right)^2 \le 9\cdot 4Rrs\cdot\frac{R}{r} \Leftrightarrow \left(a\sqrt{a}+b\sqrt{b}+c\sqrt{c}\right)^2 \le 18R^2s;(5)$$

From CBS inequality, we have:

$$\left(a\sqrt{a}+b\sqrt{b}+c\sqrt{c}\right)^2 \le (a^2+b^2+c^2)(a+b+c) = 2s(a^2+b^2+c^2);(6)$$

From (5) and (6) we must show:

$$2s(a^2+b^2+c^2) \le 18R^2s \Leftrightarrow a^2+b^2+c^2 \le 9R^2\ (Leibniz)$$

Solution 2 by Tapas Das-India

Let $a' = \sqrt{a}, b' = \sqrt{b}, c' = \sqrt{c}, F' = \frac{1}{2}b'c'\sin A' \Rightarrow \frac{2F'}{b'c'} = \sin A'$ $(and\ analogs)$

$$a\sin A' + b\sin B' + c\sin C' = a\cdot\frac{2F'}{b'c'} + b\cdot\frac{2F'}{a'c'} + c\cdot\frac{2F'}{a'b'} =$$

$$= 2F'\left[\frac{a}{b'c'}+\frac{b}{c'a'}+\frac{c}{a'b'}\right] = 2F'\left[\frac{a}{\sqrt{bc}}+\frac{b}{\sqrt{ca}}+\frac{c}{\sqrt{ab}}\right] =$$

$$= 2F'\left[\frac{a^{\frac{3}{2}}}{\sqrt{abc}}+\frac{b^{\frac{3}{2}}}{\sqrt{abc}}+\frac{c^{\frac{3}{2}}}{\sqrt{abc}}\right] = \frac{2F'}{\sqrt{abc}}\left(a^{\frac{3}{2}}+b^{\frac{3}{2}}+c^{\frac{3}{2}}\right) \le$$

$$\le \frac{2F'}{\sqrt{abc}}\sqrt{(a^2+b^2+c^2)(a+b+c)} \le \frac{2F'}{\sqrt{abc}}\sqrt{9R^2\cdot 2s} =$$

$$= \frac{2F'\cdot 3R\cdot\sqrt{2s}}{\sqrt{4Rrs}} = 3F'\sqrt{\frac{2R}{r}}$$

675. In ΔABC holds:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{R}{2r} \ge 1+\frac{b}{a}+\frac{c}{b}+\frac{a}{c}$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We will prove that the problem is true.

The inequality in the statement can be written as follows :

$$\frac{R}{2r} \geq 1 + \frac{(b-a)(a-c)(c-b)}{abc}.$$

Using Ravi's substitution : $a = y+z,\ b = z+x,\ c = x+y$

The problem becomes,

$$\frac{(x+y)(y+z)(z+x)}{8xyz} \geq 1 + \frac{(x-y)(y-z)(z-x)}{(x+y)(y+z)(z+x)}$$

If $(x-y)(y-z)(z-x) \leq 0$ *the inequality is true by Cesaro's inequality.*

Assume now that : $(x-y)(y-z)(z-x) \geq 0.$

By Cesaro's inequality we have :

$(x+y)(y+z)(z+x) \geq 8xyz$ *so it suffices to prove*

$$\frac{(x+y)(y+z)(z+x)}{8xyz} \geq 1 + \frac{(x-y)(y-z)(z-x)}{8xyz} \Leftrightarrow \frac{x^2y + y^2z + z^2x - 3xyz}{4xyz} \geq 0$$

Which is true by AM − GM inequality.

So the proof is completed. Equality holds iff ΔABC is equilateral.

Solution 2 by Soumava Chakraborty-Kolkata-India

Let $s - a = x, s - b = y$ and $s - c = z \therefore s = x + y + z \Rightarrow a = y + z, b = z + x$ and $c = x + y$

$$\text{Now, } \frac{s^2}{r^2} = \frac{s^4}{\Delta^2} = \frac{s^4}{s(s-a)(s-b)(s-c)} \overset{(\bullet)}{=} \frac{\left(\sum_{cyc} x\right)^3}{xyz} \text{ and } 1 + \frac{4R}{r}$$

$$= 1 + \frac{4sabc}{4s(s-a)(s-b)(s-c)} = 1 + \frac{\prod_{cyc}(y+z)}{xyz}$$

$$\Rightarrow 1 + \frac{4R}{r} \overset{(\bullet\bullet)}{=} \frac{xyz + \prod_{cyc}(y+z)}{xyz}$$

$$\text{Now, } \sum_{cyc} \frac{b}{a} = \sum \frac{z+x}{y+z} \Rightarrow \sum_{cyc} \frac{b}{a} \overset{(\bullet\bullet\bullet)}{=} \frac{\sum_{cyc}(x+y)^2(y+z)}{\prod_{cyc}(y+z)} \therefore (\bullet), (\bullet\bullet), (\bullet\bullet\bullet) \Rightarrow \frac{s^2}{r^2}$$

$$\geq \left(\sum_{cyc} \frac{b}{a}\right)\left(1 + \frac{4R}{r}\right) \Leftrightarrow \frac{\left(\sum_{cyc} x\right)^3}{xyz} \geq \left(\frac{xyz + \prod_{cyc}(y+z)}{xyz}\right)\left(\frac{\sum_{cyc}(x+y)^2(y+z)}{\prod_{cyc}(y+z)}\right)$$

$$\Leftrightarrow \left(\prod_{cyc}(y+z)\right)\left(\sum_{cyc} x\right)^3 \geq \left(xyz + \prod_{cyc}(y+z)\right)\left(\sum_{cyc}(x+y)^2(y+z)\right)$$

$$\Leftrightarrow \sum_{cyc} x^4y^2 + \sum_{cyc} x^3y^3 \overset{(i)}{\geq} xyz\left(\sum_{cyc} xy^2\right) + 3x^2y^2z^2$$

$$\text{Now}, \forall\, u,v,w>0, u^3+u^3+v^3 \overset{A-G}{\geq} 3u^2v, v^3+v^3+w^3 \overset{A-G}{\geq} 3v^2w \text{ and } w^3+w^3+u^3 \overset{A-G}{\geq} 3w^2u \text{ and summing up}: \sum_{cyc} u^3 \geq \sum_{cyc} u^2v \text{ and choosing } u=xy, v=yz \text{ and } w=zx,$$

$$\sum_{cyc} x^3y^3 \overset{(*)}{\geq} xyz\left(\sum_{cyc} xy^2\right) \text{ and } \sum_{cyc} x^4y^2 \underbrace{\overset{A-G}{\geq}}_{(**)} 3x^2y^2z^2 \therefore (*)+(**) \Rightarrow \text{(i) is true} \Rightarrow \frac{s^2}{r^2} \geq \left(\sum_{cyc}\frac{b}{a}\right)\left(1+\frac{4R}{r}\right) \Rightarrow \boxed{\sum_{cyc}\frac{b}{a} \overset{(\bullet\bullet\bullet\bullet)}{\leq} \frac{s^2}{r(4R+r)}}$$

$$\text{Now}, \frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{R}{2r} \overset{?}{\geq} 1+\frac{b}{a}+\frac{c}{b}+\frac{a}{c} \Leftrightarrow \sum_{cyc}\frac{a}{b}+\sum_{cyc}\frac{b}{a}+\frac{R-2r}{2r} \overset{?}{\geq} 2\sum_{cyc}\frac{b}{a}$$

$$\Leftrightarrow \frac{\sum_{cyc}\left(ab\left(\sum_{cyc}a-c\right)\right)}{4Rrs}+\frac{R-2r}{2r} \overset{?}{\geq} 2\sum_{cyc}\frac{b}{a}$$

$$\Leftrightarrow \frac{2s(s^2+4Rr+r^2)-12Rrs}{4Rrs}+\frac{R-2r}{2r} \overset{?}{\geq} 2\sum_{cyc}\frac{b}{a}$$

$$\Leftrightarrow \boxed{\frac{s^2-2Rr+r^2+R(R-2r)}{2Rr} \underbrace{\overset{?}{\geq}}_{(***)} 2\sum_{cyc}\frac{b}{a}} \text{ and } \because 2\sum_{cyc}\frac{b}{a} \overset{\text{via }(\bullet\bullet\bullet\bullet)}{\leq} \frac{2s^2}{r(4R+r)}$$

$$\therefore \text{in order to prove } (***), \text{ it suffices to prove}: \frac{s^2-2Rr+r^2+R(R-2r)}{2Rr} \geq \frac{2s^2}{r(4R+r)}$$

$$\Leftrightarrow \boxed{rs^2+R(R-2r)(4R+r) \overset{(****)}{\geq} r(2R-r)(4R+r)}$$

$$\text{Now, LHS of } (****) \overset{\text{Gerretsen}}{\geq} r(16Rr-5r^2)+R(R-2r)(4R+r) \overset{?}{\geq} r(2R-r)(4R+r)$$

$$\Leftrightarrow 4t^3-15t^2+16t-4 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right) \Leftrightarrow (t-2)(4t(t-2)+t+2) \overset{?}{\geq} 0$$

$$\rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (****) \Rightarrow (***) \text{ is true} \Rightarrow \frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{R}{2r} \geq 1+\frac{b}{a}+\frac{c}{b}+\frac{a}{c} \text{ is true } \forall \text{ triangles (QED)}$$

Solution 3 by Alex Szoros-Romania

Le us denote: $M=\max\left\{\frac{a}{b}+\frac{b}{c}+\frac{c}{a};\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right\}$ and $m=\min\left\{\frac{a}{b}+\frac{b}{c}+\frac{c}{a};\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right\}$

Case I) Suppose that: $M=\frac{a}{b}+\frac{b}{c}+\frac{c}{a}$. From $M\geq m$ and $\frac{R}{2r}\geq 1$ we get:

$$M+\frac{R}{2r}\geq m+1 \Rightarrow \frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{R}{2r}\geq 1+\frac{b}{a}+\frac{c}{b}+\frac{a}{c}$$

Case II) Suppose that: $M=\frac{b}{a}+\frac{c}{b}+\frac{a}{c}$.

We show that in any ΔABC holds: $2m \geq M+3$; $(*)$

$$\text{From } \sum(a+c-b)(a-b)^2 \geq 0 \Rightarrow \sum(a+c-b)\left(a^2-2ab+b^2\right) \geq 0 \Leftrightarrow$$

$$\sum\left(a^3-2a^2b+ab^2+a^2c-2abc+b^2c-a^2b+2ab^2-b^3\right) \geq 0 \Leftrightarrow$$

$$4\sum ab^2-2\sum a^2b-6abc \geq 0 \,\Big|:2abc \Rightarrow 2\left(\frac{b}{c}+\frac{c}{a}+\frac{a}{b}\right) \geq \frac{a}{c}+\frac{b}{a}+\frac{c}{b}+3 \Rightarrow$$

$$2m \geq M+3 \Rightarrow 2m+\frac{R}{r} \overset{(*)}{\geq} M+3+\frac{R}{r} \overset{Shan\ He\ Wu}{\geq} 2+2M \Rightarrow$$

$$2m+\frac{R}{r} \geq 2(1+M) \Rightarrow m+\frac{R}{2r} \geq 1+M$$

676. In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{{s_a}^2}{s_c} \leq \frac{9R}{2}\left(\frac{R}{2r}\right)^6$$

Proposed by Marin Chirciu-Romania

Solution 1 by Marian Ursărescu-Romania

We must show that:

$$\frac{1}{s_as_bs_c}\left(s_a^3s_b+s_b^3s_c+s_c^3s_a\right) \leq \frac{9R^7}{2^7r^6}; (1)$$

Now, we use:

$$(2): 3(x^3y+y^3z+z^3x) \leq (x^2+y^2+z^2)^2 \Leftrightarrow$$

$$(m+n+p)^2 \geq 3(mn+np+pm) \text{ for } m=x^2+yz-xy, n=y^2+zx-yz$$

$$\text{and } p=z^2+xy-zx$$

$$\text{From } (1) \Rightarrow s_a^3s_b+s_b^3s_c+s_c^3s_a \leq \frac{1}{3}\left(s_a^2+s_b^2+s_c^2\right)^2$$

We must show:

$$\frac{\left(s_a^2+s_b^2+s_c^2\right)^2}{3s_as_bs_c} \leq \frac{9R^7}{2^7r^6}; (2)$$

$$\text{But: } s_a \leq m_a \Rightarrow s_a^2+s_b^2+s_c^2 \leq m_a^2+m_b^2+m_c^2 = \frac{3}{4}(a^2+b^2+c^2) \leq \frac{3}{4}\cdot 9R^2; (3)$$

From (2) and (3) we must show:

$$\frac{9}{16}\cdot\frac{81R^4}{3s_as_bs_c} \leq \frac{9R^7}{2^7r^6} \Leftrightarrow \frac{27}{s_as_bs_c} \leq \frac{R^3}{8r^6} \Leftrightarrow s_as_bs_c \geq 27\cdot 8\cdot\frac{r^6}{R^3}; (4)$$

$$\text{But: } s_a = \frac{2bc}{b^2+c^2}m_a \Rightarrow s_a s_b s_c = \frac{8(abc)^2}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}m_a m_b m_c; (5)$$

From (4) and (5) we must show:

$$\frac{8(abc)^2}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}m_a m_b m_c \ge 27\cdot 8\cdot\frac{r^6}{R^3} \Leftrightarrow$$

$$\frac{(abc)^2}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}m_a m_b m_c \ge 27\cdot\frac{r^6}{R^3}$$

But: $abc = 4Rrs; (8)$. **From (7) and (8) we must show:**

$$\frac{16R^2r^2s^2}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}m_a m_b m_c \ge \frac{27r^6}{R^3} \Leftrightarrow$$

$$\frac{16R^2s^2}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}m_a m_b m_c \ge \frac{27R^4}{R^3}; (9)$$

$$\text{But: } \sqrt[3]{(a^2+b^2)(b^2+c^2)(c^2+a^2)} \le \frac{2(a^2+b^2+c^2)}{3} \Rightarrow$$

$$\frac{m_a m_b m_c}{(a^2+b^2)(b^2+c^2)(c^2+a^2)} \ge \frac{27}{8(a^2+b^2+c^2)}; (10)$$

From (9) and (10) we must show:

$$\frac{16\cdot 27R^2s^2m_am_bm_c}{8(a^2+b^2+c^2)^3} \ge \frac{27r^4}{R^3} \Leftrightarrow \frac{2R^2s^2m_am_bm_c}{(a^2+b^2+c^2)^3} \ge \frac{r^4}{R^3}; (11)$$

$$\text{But: } a^2+b^2+c^2 \le 9R^2 \Rightarrow \frac{1}{(a^2+b^2+c^2)^3} \ge \frac{1}{9^3R^6} = \frac{1}{3^6R^6}; (12)$$

From (11) and (12) we must show:

$$\frac{2R^2s^2m_am_bm_c}{3^6R^6} \ge \frac{r^4}{R^3} \Leftrightarrow 2s^2m_am_bm_c \ge 3^6r^4R; (13)$$

$$\text{But } m_a \ge \sqrt{s(s-a)} \Rightarrow m_am_bm_c \ge s^2r; (14)$$

From (13) and (14) we must show:

$$2s^4r \ge 3^6r^4R \Leftrightarrow 2s^4 \ge 3^6Rr^3; (15)$$

$$\text{But } 2s^2 \ge 27Rr\ (Cosnita-Turtoiu); (16).$$

From (15) and (16) we must show: $s^2 - 27Rr \ge 3^6Rr^3 \Leftrightarrow$

$$s^2 \ge 27R^2(Mitrinovic).$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:\ \ h_a \le s_a = \frac{2bc}{b^2+c^2}\cdot m_a \overset{AM-GM}{\le} m_a\ \ (and\ analogs)$$

$$Then:$$

$$\sum_{cyc}\frac{{s_a}^2}{s_c} \le \sum_{cyc}\frac{{m_a}^2}{h_c} = \frac{1}{8F}\sum_{cyc}c(2b^2+2c^2-a^2)$$

$$= \frac{1}{8F}\left(2\sum_{cyc}b^2c + 2\sum_{cyc}a^3 - \sum_{cyc}ca^2\right) \overset{AM-GM}{\le}$$

$$\frac{1}{8F}\left(2\sum_{cyc}a^3 + 2\sum_{cyc}a^3 - 3abc\right) = \frac{4.2s(s^2-3r^2-6Rr)-3.4Rsr}{8sr}$$

$$= \frac{2s^2-15Rr-6r^2}{2r} \le$$

$$\overset{Gerretsen}{\le} \frac{8R^2-7Rr}{2r} = \frac{32R^2r-7R(4r^2+R^2)+7R^3}{4r.2r} \overset{AM-GM}{\le} \frac{32R^2r-7R.4Rr+7R^3}{8r^2}$$

$$= \frac{7R^3+4R^2r}{8r^2} \overset{Euler}{\le} \frac{9R^3}{8r^2} \overset{Euler}{\le} \frac{9R^3}{8r^2}\cdot\left(\frac{R}{2r}\right)^4 = \frac{9R}{2}\left(\frac{R}{2r}\right)^6,\ \ as\ desired.$$

677. In $\Delta ABC, I$ −incenter, R_a, R_b, R_c −circumradius of $\Delta BIC, \Delta CIA, \Delta AIB$.

Prove that:

$$3s\left(\frac{2r}{R}\right)^{\frac{1}{3}} \le \frac{ar_a}{R_a} + \frac{br_b}{R_b} + \frac{cr_c}{R_c} \le 3s$$

Proposed by Marin Chirciu-Romania

Solution by Marian Ursărescu-Romania

$$\widehat{BIC} = \pi - \frac{B+C}{2} = \frac{\pi}{2} + \frac{A}{2}$$

From Law of sines, we have:

$$\frac{a}{\sin(BIC)} = 2R_a \Rightarrow R_a = \frac{a}{2\cos\frac{A}{2}}$$

We must show that:

$$3s\sqrt[3]{\frac{2r}{R}}\le 2\sum_{cyc} r_a\cos\frac{A}{2}\le 3s;\ \ (1)$$

$$\text{But: } r_a=s\tan\frac{A}{2}\Rightarrow r_a\cos\frac{A}{2}=s\sin\frac{A}{2};(2)$$

From (1) and (2) we must show:

$$\frac{3}{2}\sqrt[3]{\frac{2r}{R}}\le\sum_{cyc}\sin\frac{A}{2}\le\frac{3}{2};(3)$$

For the left side, we have:

$$\sum_{cyc}\sin\frac{A}{2}\ge 3\sqrt[3]{\prod_{cyc}\sin\frac{A}{2}};(4)$$

$$\text{But: }\prod_{cyc}\sin\frac{A}{2}\ge 3\sqrt[3]{\frac{r}{4R}}=\frac{3}{2}\sqrt[3]{\frac{2r}{R}}$$

For the right side, we have:

$$\sum_{cyc}\sin\frac{A}{2}\le\frac{3}{2}\text{ true, becuse by}$$

$$\sum_{cyc}\cos\frac{A}{2}\le\frac{3}{2}\text{ put }A\to\frac{A}{2},B\to\frac{B}{2}\text{ and }C\to\frac{C}{2}.$$

678. In any ΔABC holds:

$$\frac{m_a}{m_b}+\frac{m_b}{m_c}+\frac{m_c}{m_a}+\frac{R^2}{4r^2}\ge 1+\frac{m_b}{m_a}+\frac{m_c}{m_b}+\frac{m_a}{m_c}$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\textit{Lemma}:\ \ \textit{In any }\Delta ABC,\ 2\left(\frac{m_a}{m_b}+\frac{m_b}{m_c}+\frac{m_c}{m_a}\right)\ge\frac{m_b}{m_a}+\frac{m_c}{m_b}+\frac{m_a}{m_c}+3.$$

$$We\ have:\ 2\left(\frac{m_a}{m_b}+\frac{m_b}{m_c}+\frac{m_c}{m_a}\right)-\left(\frac{m_b}{m_a}+\frac{m_c}{m_b}+\frac{m_a}{m_c}+3\right)$$

$$=\sum_{cyc}\frac{(m_a+m_b-m_c)(m_b-m_c)^2}{2m_am_bm_c}\geq 0$$

Because m_a, m_b, m_c *can be the side lengths of a triangle.*

So the proof of lemma is completed. Back to main problem.

$$From\ the\ lemma\ we\ have:\ 2\left(\frac{m_a}{m_b}+\frac{m_b}{m_c}+\frac{m_c}{m_a}+\frac{R^2}{4r^2}\right)\geq\frac{m_b}{m_a}+\frac{m_c}{m_b}+\frac{m_a}{m_c}+3+\frac{R^2}{2r^2}$$

$$So\ it\ suffices\ to\ prove:\ \frac{R^2}{2r^2}+1\geq\frac{m_b}{m_a}+\frac{m_c}{m_b}+\frac{m_a}{m_c}$$

$$We\ have:\ \sum_{cyc}\frac{m_b}{m_a}\overset{AM-GM}{\leq}\sum_{cyc}\frac{1}{2}\left(\frac{{m_b}^2}{9r^2}+\frac{9r^2}{{m_a}^2}\right)\overset{m_a\geq\sqrt{s(s-a)}}{\leq}$$

$$\frac{1}{18r^2}\cdot\frac{3}{4}\sum_{cyc}a^2+\frac{9r^2}{2}\sum_{cyc}\frac{1}{s(s-a)}\leq$$

$$\overset{Leibniz}{\leq}\frac{1}{24r^2}\cdot 9R^2+\frac{9r^2}{2s}\cdot\frac{4R+r}{sr}\overset{Doucet}{\leq}\frac{3R^2}{8r^2}+\frac{3}{2}\overset{Euler}{\leq}\frac{R^2}{2r^2}+1.$$

So the proof is completed and

$$\frac{m_a}{m_b}+\frac{m_b}{m_c}+\frac{m_c}{m_a}+\frac{R^2}{4r^2}\geq 1+\frac{m_b}{m_a}+\frac{m_c}{m_b}+\frac{m_a}{m_c}\ is\ true\ in\ any\ \Delta ABC.$$

Solution 2 by Soumava Chakraborty-Kolkata-India

Let $s-a=x, s-b=y$ and $s-c=z\ \therefore\ s=x+y+z\Rightarrow a=y+z, b=z+x$ and $c=x+y$

$$\text{Now, }\frac{s^2}{r^2}=\frac{s^4}{\Delta^2}=\frac{s^4}{s(s-a)(s-b)(s-c)}\overset{(\bullet)}{=}\frac{\left(\sum_{cyc}x\right)^3}{xyz}\ \text{and}\ 1+\frac{4R}{r}$$

$$=1+\frac{4sabc}{4s(s-a)(s-b)(s-c)}=1+\frac{\prod_{cyc}(y+z)}{xyz}$$

$$\Rightarrow 1+\frac{4R}{r}\overset{(\bullet\bullet)}{=}\frac{xyz+\prod_{cyc}(y+z)}{xyz}$$

$$\text{Now, }\sum_{cyc}\frac{b}{a}=\sum\frac{z+x}{y+z}\Rightarrow\sum_{cyc}\frac{b}{a}\overset{(\bullet\bullet\bullet)}{=}\frac{\sum_{cyc}(x+y)^2(y+z)}{\prod_{cyc}(y+z)}\ \therefore\ (\bullet),(\bullet\bullet),(\bullet\bullet\bullet)\Rightarrow\frac{s^2}{r^2}$$

$$\geq\left(\sum_{cyc}\frac{b}{a}\right)\left(1+\frac{4R}{r}\right)\Leftrightarrow\frac{\left(\sum_{cyc}x\right)^3}{xyz}\geq\left(\frac{xyz+\prod_{cyc}(y+z)}{xyz}\right)\left(\frac{\sum_{cyc}(x+y)^2(y+z)}{\prod_{cyc}(y+z)}\right)$$

$$\Leftrightarrow\left(\prod_{cyc}(y+z)\right)\left(\sum_{cyc}x\right)^3\geq\left(xyz+\prod_{cyc}(y+z)\right)\left(\sum_{cyc}(x+y)^2(y+z)\right)$$

$$\Leftrightarrow\sum_{cyc}x^4y^2+\sum_{cyc}x^3y^3\overset{(i)}{\geq}xyz\left(\sum_{cyc}xy^2\right)+3x^2y^2z^2$$

Now, $\forall\, u,v,w>0, u^3+u^3+v^3\overset{A-G}{\geq}3u^2v, v^3+v^3+w^3\overset{A-G}{\geq}3v^2w$ *and* $w^3+w^3+u^3\overset{A-G}{\geq}3w^2u$ *and* summing up : $\sum_{cyc}u^3\geq\sum_{cyc}u^2v$ *and* choosing $u=xy, v=yz$ *and* $w=zx$,

$$\sum_{cyc}x^3y^3\overset{(*)}{\geq}xyz\left(\sum_{cyc}xy^2\right)\text{ and }\sum_{cyc}x^4y^2\underset{(**)}{\overset{A-G}{\geq}}3x^2y^2z^2\ \therefore (*)+(**)\Rightarrow (i)\text{ is true}\Rightarrow\frac{s^2}{r^2}\geq\left(\sum_{cyc}\frac{b}{a}\right)\left(1+\frac{4R}{r}\right)\Rightarrow\boxed{\sum_{cyc}\frac{b}{a}\overset{(\bullet\bullet\bullet\bullet)}{\leq}\frac{s^2}{r(4R+r)}}$$

Now, $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{R}{2r}\overset{?}{\geq}1+\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\Leftrightarrow\sum_{cyc}\frac{a}{b}+\sum_{cyc}\frac{b}{a}+\frac{R-2r}{2r}\overset{?}{\geq}2\sum_{cyc}\frac{b}{a}$

$$\Leftrightarrow\frac{\sum_{cyc}\left(ab(\sum_{cyc}a-c)\right)}{4Rrs}+\frac{R-2r}{2r}\overset{?}{\geq}2\sum_{cyc}\frac{b}{a}$$

$$\Leftrightarrow\frac{2s(s^2+4Rr+r^2)-12Rrs}{4Rrs}+\frac{R-2r}{2r}\overset{?}{\geq}2\sum_{cyc}\frac{b}{a}$$

$$\Leftrightarrow\boxed{\frac{s^2-2Rr+r^2+R(R-2r)}{2Rr}\underset{(***)}{\overset{?}{\geq}}2\sum_{cyc}\frac{b}{a}}\text{ and }\because 2\sum_{cyc}\frac{b}{a}\overset{via\,(\bullet\bullet\bullet\bullet)}{\leq}\frac{2s^2}{r(4R+r)}$$

$\therefore$ in order to prove $(***)$, it suffices to prove : $\frac{s^2-2Rr+r^2+R(R-2r)}{2Rr}\geq\frac{2s^2}{r(4R+r)}$

$$\Leftrightarrow\boxed{rs^2+R(R-2r)(4R+r)\overset{(****)}{\geq}r(2R-r)(4R+r)}$$

Now, LHS of $(****)\overset{Gerretsen}{\geq}r(16Rr-5r^2)+R(R-2r)(4R+r)\overset{?}{\geq}r(2R-r)(4R+r)$

$$\Leftrightarrow 4t^3-15t^2+16t-4\overset{?}{\geq}0\left(t=\frac{R}{r}\right)\Leftrightarrow(t-2)(4t(t-2)+t+2)\overset{?}{\geq}0$$

$\rightarrow$ true $\because t\overset{Euler}{\geq}2\Rightarrow(****)\Rightarrow(***)$ is true $\Rightarrow\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{R}{2r}\geq1+\frac{b}{a}+\frac{c}{b}+\frac{a}{c}$

$\Rightarrow\boxed{\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{abc(a+b+c)}{16F^2}\geq1+\frac{b}{a}+\frac{c}{b}+\frac{a}{c}}$, *applying which on a triangle with sides*

$\frac{2m_a}{3}, \frac{2m_b}{3}, \frac{2m_c}{3}$ whose *area* subsequently $= \frac{F}{3}$, we *arrive at*

$$: \frac{m_a}{m_b}+\frac{m_b}{m_c}+\frac{m_c}{m_a}+\frac{\frac{8m_am_bm_c}{27}\cdot\frac{2(m_a+m_b+m_c)}{3}}{\frac{16F^2}{9}} \geq 1+\frac{m_b}{m_a}+\frac{m_c}{m_b}+\frac{m_a}{m_c}$$

$$\Rightarrow 1+\frac{m_b}{m_a}+\frac{m_c}{m_b}+\frac{m_a}{m_c}-\left(\frac{m_a}{m_b}+\frac{m_b}{m_c}+\frac{m_c}{m_a}\right)$$

$$\leq \frac{m_am_bm_c(m_a+m_b+m_c)}{9F^2} \overset{\because m_am_bm_c \leq \frac{Rs^2}{2} \text{ and via Bager}}{\leq} \frac{\frac{Rs^2}{2}(4R+r)}{9r^2s^2} \overset{\text{Euler}}{\leq} \frac{\frac{R}{2}\left(4R+\frac{R}{2}\right)}{9r^2} = \frac{R^2}{4r^2}$$

$$\Rightarrow \frac{m_a}{m_b}+\frac{m_b}{m_c}+\frac{m_c}{m_a}+\frac{R^2}{4r^2} \geq 1+\frac{m_b}{m_a}+\frac{m_c}{m_b}+\frac{m_a}{m_c} \text{ (QED)}$$

Proof of $m_am_bm_c \leq \frac{Rs^2}{2}$

$$m_a^2m_b^2m_c^2 = \frac{1}{64}(2b^2+2c^2-a^2)(2c^2+2a^2-b^2)(2a^2+2b^2-c^2) \overset{(1)}{=} \frac{1}{64}\left\{-4\sum a^6 + 6\left(\sum a^4b^2+\sum a^2b^4\right)+3a^2b^2c^2\right\}$$

$$\text{Now, } \sum a^6 = \left(\sum a^2\right)^3 - 3(a^2+b^2)(b^2+c^2)(c^2+a^2)$$

$$= \left(\sum a^2\right)^3 - 3\left(2a^2b^2c^2+\sum a^2b^2\left(\sum a^2 - c^2\right)\right)$$

$$= \left(\sum a^2\right)^3 + 3a^2b^2c^2 - 3\left(\sum a^2b^2\right)\sum a^2$$

$$\therefore \sum a^6 \overset{(2)}{=} \left(\sum a^2\right)^3 + 3a^2b^2c^2 - 3\left(\sum a^2b^2\right)\sum a^2$$

$$\sum a^4b^2+\sum a^2b^4 = \sum a^2b^2\left(\sum a^2 - c^2\right) \overset{(3)}{=} \left(\sum a^2b^2\right)\sum a^2 - 3a^2b^2c^2$$

$$\therefore (1),(2),(3) \Rightarrow m_a^2m_b^2m_c^2$$

$$= \frac{1}{64}\left(-4\left(\sum a^2\right)^3 - 12a^2b^2c^2 + 12\left(\sum a^2b^2\right)\sum a^2 + 6\left(\sum a^2b^2\right)\sum a^2 - 18a^2b^2c^2 + 3a^2b^2c^2\right)$$

$$= \frac{1}{64}\left(-4\left(\sum a^2\right)^3 + 18\left(\sum a^2b^2\right)\sum a^2 - 27a^2b^2c^2\right)$$

$$= \frac{1}{64}\left(-4\left(\sum a^2\right)^3 + 18\left(\left(\sum ab\right)^2 - 16Rrs^2\right)\left(\sum a^2\right) - 27a^2b^2c^2\right)$$

$$= \frac{1}{64}\left\{-32(s^2-4Rr-r^2)^3 + 36(s^2-4Rr-r^2)(s^2+4Rr+r^2)^2 - 576Rrs^2(s^2-4Rr-r^2) - 432R^2r^2s^2\right\}$$

$$= \frac{1}{16}\left\{s^6 - s^4(12Rr-33r^2) - s^2(60R^2r^2+120Rr^3+33r^4) - r^3(4R+r)^3\right\} \leq \frac{R^2s^4}{4}$$

$$\Leftrightarrow s^6 - s^4(4R^2+12Rr-33r^2) - s^2(60R^2r^2+120Rr^3+33r^4) - r^3(4R+r)^3 \overset{(4)}{\leq} 0$$

$$\text{Now, LHS of (4)} \overset{\text{Gerretsen}}{\leq} -s^4(8Rr-36r^2)-s^2(60R^2r^2+120Rr^3+33r^4)$$

$$-r^3(4R+r)^3 \overset{?}{\leq} 0$$

$$\Leftrightarrow s^4(8R-16r)+s^2(60R^2r+120Rr^2+33r^3)+r^2(4R+r)^3 \underset{(5)}{\overset{?}{\geq}} 20rs^4$$

$$\text{Now, LHS of (5)} \underbrace{\overset{\text{Gerretsen}}{\geq}}_{(a)} s^2(16Rr-5r^2)(8R-16r)+s^2(60R^2r+120Rr^2+33r^3)$$

$$+r^2(4R+r)^3 \text{ and RHS of (5)} \underbrace{\overset{\text{Geretsen}}{\leq}}_{(b)} 20rs^2(4R^2+4Rr+3r^2)$$

$(a),(b) \Rightarrow$ in order to prove (5), it suffices to prove

$$: s^2(16Rr-5r^2)(8R-16r)+s^2(60R^2r+120Rr^2+33r^3)+r^2(4R+r)^3$$

$$\geq 20rs^2(4R^2+4Rr+3r^2)$$

$$\Leftrightarrow s^2(108R^2-256Rr+53r^2)+r(4R+r)^3 \geq 0$$

$$\Leftrightarrow s^2(108R^2-256Rr+80r^2)+r(4R+r)^3 \overset{(6)}{\geq} 27r^2s^2$$

$$\text{Now, LHS of (6)} \underbrace{\overset{\text{Gerretsen}}{\geq}}_{(c)} (108R^2-256Rr+80r^2)(16Rr-5r^2)$$

$$+r(4R+r)^3 \text{ and RHS of (6)} \underbrace{\overset{\text{Geretsen}}{\leq}}_{(d)} 27r^2(4R^2+4Rr+3r^2)$$

$(c),(d) \Rightarrow$ in order to prove (6), it suffices to prove

$$: (108R^2-256Rr+80r^2)(16Rr-5r^2)+r(4R+r)^3$$

$$\geq 27r^2(4R^2+4Rr+3r^2) \Leftrightarrow 224t^3-587t^2+308t-60 \geq 0$$

$$\left(\text{where } t=\frac{R}{r}\right) \Leftrightarrow (t-2)\{(t-2)(224t+309)+648\} \geq 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (6) \Rightarrow (5)$$

$$\Rightarrow (4) \text{ is true} \Rightarrow m_a^2m_b^2m_c^2 \leq \frac{R^2s^4}{4} \Rightarrow m_am_bm_c \leq \frac{Rs^2}{2} \text{ (Done)}$$

679. **In ΔABC the following relationship holds:**

$$\sum_{cyc} \frac{\cot^2 \frac{A}{2}}{\sin A} \geq 9 \sum_{cyc} \frac{\tan^2 \frac{A}{2}}{\sin A}$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:\ \cot\frac{A}{2}=\frac{s-a}{r}\ (and\ analogs)\ and\ \sin A=\frac{2R}{a}\ (and\ analogs).$$

$$Then: \sum_{cyc}\frac{\cot^2\frac{A}{2}}{\sin A}=\sum_{cyc}\left(\frac{s-a}{r}\right)^2\cdot\frac{2R}{a}=\frac{2R}{r^2}\sum_{cyc}\left(\frac{s^2}{a}-2s+a\right)$$

$$=\frac{2R}{r^2}\left(\frac{s(s^2+r^2+4Rr)}{4Rr}-4s\right)=$$

$$=\frac{s(s^2+r^2-12Rr)}{2r^3}\overset{Gerretsen}{\geq}\frac{s(4Rr-4r^2)}{2r^3}=\frac{2s(R-r)}{r^2}\quad(1)$$

$$Also, we\ have: \sum_{cyc}\frac{\tan^2\frac{A}{2}}{\sin A}=\sum_{cyc}\left(\frac{r}{s-a}\right)^2\cdot\frac{2R}{a}=\frac{2R}{s}\sum_{cyc}\frac{sr^2}{a(s-a)^2}$$

$$=\frac{2R}{s}\sum_{cyc}\frac{(s-b)(s-c)}{a(s-a)}\leq$$

$$\overset{AM-GM}{\leq}\frac{2R}{s}\sum_{cyc}\frac{a^2}{4a(s-a)}\leq\frac{R}{2s}\sum_{cyc}\left(\frac{s}{s-a}-1\right)=\frac{R}{2s}\left(\frac{4R+r}{r}-3\right)=\frac{R(2R-r)}{sr}\quad(2)$$

$$From\ (1)\ and\ (2)\ it\ suffices\ to\ prove: \frac{2s(R-r)}{r^2}\geq\frac{9R(2R-r)}{sr}\quad or$$

$$2s^2(R-r)\geq 9Rr(2R-r)$$

$$We\ have: 2s^2(R-r)\overset{Cosnita\ \&\ Turtoiu}{\geq}27Rr(R-r)$$

$$=9Rr(3R-3r)\overset{Euler}{\geq}9Rr(2R-r).$$

So the proof is completed. Equality holds iff ΔABC is equilateral.

680. " In any ΔABC holds:

$$\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}+\frac{R^3}{8r^3}\geq 1+\frac{b+c}{a+b}+\frac{c+a}{b+c}+\frac{a+b}{c+a}\text{ ".}$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have:

$$\sum_{cyc}\frac{b+c}{a+b}\overset{CBS}{\leq}\sqrt{\left(\sum(b+c)^2\right)\left(\sum\frac{1}{(a+b)^2}\right)}\overset{CBS\ \&\ AM-GM}{\leq}\sqrt{\left(\sum 2\left(b^2+c^2\right)\right)\left(\sum\frac{1}{4ab}\right)}=$$

$$=\sqrt{\left(\sum a^2\right)\cdot\frac{2s}{4Rsr}}\overset{Leibniz}{\leq}\sqrt{9R^2\cdot\frac{1}{2Rr}}=3\sqrt{\frac{R}{2r}}\overset{Euler}{\leq}3\cdot\frac{R}{2r}\overset{AM-GM}{\leq}\frac{R^3}{8r^3}+1+1=$$

$$=3+\frac{R^3}{8r^3}-1\overset{AM-GM}{\leq}\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}+\frac{R^3}{8r^3}-1.$$

$$Therefore,$$

$$\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}+\frac{R^3}{8r^3}\geq 1+\frac{b+c}{a+b}+\frac{c+a}{b+c}+\frac{a+b}{c+a}\text{ is true in any }\Delta ABC.$$

Solution 2 by Soumava Chakraborty-Kolkata-India

Let $s-a=x, s-b=y$ and $s-c=z \therefore s=x+y+z \Rightarrow a=y+z, b=z+x$ and $c=x+y$

$$\text{Now, }\frac{s^2}{r^2}=\frac{s^4}{\Delta^2}=\frac{s^4}{s(s-a)(s-b)(s-c)}\overset{(\bullet)}{=}\frac{\left(\sum_{cyc}x\right)^3}{xyz}\text{ and }1+\frac{4R}{r}$$

$$=1+\frac{4sabc}{4s(s-a)(s-b)(s-c)}=1+\frac{\prod_{cyc}(y+z)}{xyz}$$

$$\Rightarrow 1+\frac{4R}{r}\overset{(\bullet\bullet)}{=}\frac{xyz+\prod_{cyc}(y+z)}{xyz}$$

$$\text{Now, }\sum_{cyc}\frac{b}{a}=\sum\frac{z+x}{y+z}\Rightarrow\sum_{cyc}\frac{b}{a}\overset{(\bullet\bullet\bullet)}{=}\frac{\sum_{cyc}(x+y)^2(y+z)}{\prod_{cyc}(y+z)}\therefore(\bullet),(\bullet\bullet),(\bullet\bullet\bullet)\Rightarrow\frac{s^2}{r^2}$$

$$\geq\left(\sum_{cyc}\frac{b}{a}\right)\left(1+\frac{4R}{r}\right)\Leftrightarrow\frac{\left(\sum_{cyc}x\right)^3}{xyz}\geq\left(\frac{xyz+\prod_{cyc}(y+z)}{xyz}\right)\left(\frac{\sum_{cyc}(x+y)^2(y+z)}{\prod_{cyc}(y+z)}\right)$$

$$\Leftrightarrow\left(\prod_{cyc}(y+z)\right)\left(\sum_{cyc}x\right)^3\geq\left(xyz+\prod_{cyc}(y+z)\right)\left(\sum_{cyc}(x+y)^2(y+z)\right)$$

$$\Leftrightarrow\sum_{cyc}x^4y^2+\sum_{cyc}x^3y^3\overset{(i)}{\geq}xyz\left(\sum_{cyc}xy^2\right)+3x^2y^2z^2$$

$$\text{Now, }\forall u,v,w>0, u^3+u^3+v^3\overset{A-G}{\geq}3u^2v, v^3+v^3+w^3\overset{A-G}{\geq}3v^2w\text{ and }w^3+w^3$$
$$+u^3\overset{A-G}{\geq}3w^2u\text{ and summing up}: \sum_{cyc}u^3\geq\sum_{cyc}u^2v\text{ and choosing }u=xy, v$$
$$=yz\text{ and }w=zx,$$

$$\sum_{cyc} x^3y^3 \overset{(*)}{\underbrace{\ge}} xyz\left(\sum_{cyc} xy^2\right) \text{ and } \sum_{cyc} x^4y^2 \underset{(**)}{\overset{A-G}{\underbrace{\ge}}} 3x^2y^2z^2 \therefore (*)+(**) \Rightarrow \text{(i) is true} \Rightarrow \frac{s^2}{r^2}$$

$$\ge \left(\sum_{cyc}\frac{b}{a}\right)\left(1+\frac{4R}{r}\right) \Rightarrow \boxed{\sum_{cyc}\frac{b}{a} \overset{(\bullet\bullet\bullet\bullet)}{\le} \frac{s^2}{r(4R+r)}}$$

$$\text{Now, } \frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{R}{2r} \overset{?}{\ge} 1+\frac{b}{a}+\frac{c}{b}+\frac{a}{c} \Leftrightarrow \sum_{cyc}\frac{a}{b}+\sum_{cyc}\frac{b}{a}+\frac{R-2r}{2r} \overset{?}{\ge} 2\sum_{cyc}\frac{b}{a}$$

$$\Leftrightarrow \frac{\sum_{cyc}\left(ab\left(\sum_{cyc}a-c\right)\right)}{4Rrs}+\frac{R-2r}{2r} \overset{?}{\ge} 2\sum_{cyc}\frac{b}{a}$$

$$\Leftrightarrow \frac{2s(s^2+4Rr+r^2)-12Rrs}{4Rrs}+\frac{R-2r}{2r} \overset{?}{\ge} 2\sum_{cyc}\frac{b}{a}$$

$$\Leftrightarrow \boxed{\frac{s^2-2Rr+r^2+R(R-2r)}{2Rr} \underset{(***)}{\overset{?}{\underbrace{\ge}}} 2\sum_{cyc}\frac{b}{a}} \text{ and } \because 2\sum_{cyc}\frac{b}{a} \overset{\text{via }(\bullet\bullet\bullet\bullet)}{\le} \frac{2s^2}{r(4R+r)}$$

$$\therefore \text{in order to prove } (***), \text{it suffices to prove : } \frac{s^2-2Rr+r^2+R(R-2r)}{2Rr}$$

$$\ge \frac{2s^2}{r(4R+r)}$$

$$\Leftrightarrow \boxed{rs^2+R(R-2r)(4R+r) \overset{(****)}{\ge} r(2R-r)(4R+r)}$$

$$\text{Now, LHS of } (****) \overset{\text{Gerretsen}}{\ge} r(16Rr-5r^2)+R(R-2r)(4R+r) \overset{?}{\ge} r(2R-r)(4R+r)$$

$$\Leftrightarrow 4t^3-15t^2+16t-4 \overset{?}{\ge} 0 \left(t=\frac{R}{r}\right) \Leftrightarrow (t-2)(4t(t-2)+t+2) \overset{?}{\ge} 0$$

$$\to \text{true} \because t \overset{\text{Euler}}{\ge} 2 \Rightarrow (****) \Rightarrow (***) \text{ is true} \Rightarrow \frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{R}{2r} \ge 1+\frac{b}{a}+\frac{c}{b}+\frac{a}{c}$$

$$\Rightarrow \frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{abc(a+b+c)}{16F^2} \ge 1+\frac{b}{a}+\frac{c}{b}+\frac{a}{c}$$

$$\Rightarrow \boxed{1+\frac{b}{a}+\frac{c}{b}+\frac{a}{c}-\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) \overset{(l)}{\le} \frac{abc(a+b+c)}{2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)}}$$

$\because b+c, c+a, a+b$ form sides of a triangle,

$\therefore$ via (l) on triangle with sides $b+c, c+a, a+b$, we arrive at :

$$1+\frac{b+c}{a+b}+\frac{c+a}{b+c}+\frac{a+b}{c+a}-\left(\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}\right) \le \frac{\left(\prod_{cyc}(b+c)\right)\left(\sum_{cyc}(b+c)\right)}{2\sum_{cyc}(b+c)^2(c+a)^2-\sum_{cyc}(b+c)^4}$$

$$= \frac{2\left(\sum_{cyc}a\right)\left(\prod_{cyc}(b+c)\right)}{16abc\left(\sum_{cyc}a\right)} = \frac{\prod_{cyc}(b+c)}{8abc}$$

$$\therefore \text{ for original } \Delta\, ABC, 1+\frac{b+c}{a+b}+\frac{c+a}{b+c}+\frac{a+b}{c+a}-\left(\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}\right)$$

$$\le \frac{\prod_{cyc}(b+c)}{8abc} \overset{?}{\le} \frac{R}{2r} \Leftrightarrow \frac{R}{2r} \ge \frac{2s(s^2+2Rr+r^2)}{32Rrs} \Leftrightarrow s^2+2Rr+r^2 \overset{(*****)}{\le} 8R^2$$

$$\text{Now, LHS of } (*****) \overset{\text{Gerretsen}}{\le} 4R^2+6Rr+4r^2 \overset{?}{\le} 8R^2 \Leftrightarrow 2R^2-3Rr-2r^2 \overset{?}{\ge} 0$$

$$\Leftrightarrow (R-2r)(2R+r) \overset{?}{\ge} 0 \to \text{true} \because R \overset{\text{Euler}}{\ge} 2r \Rightarrow (*****) \text{ is true}$$

$$\Rightarrow 1+\frac{b+c}{a+b}+\frac{c+a}{b+c}+\frac{a+b}{c+a}-\left(\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}\right) \le \frac{R}{2r} \overset{\text{Euler}}{\le} \frac{R^3}{8r^3}$$

$$\therefore \text{ in any } \Delta\, ABC, \frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}+\frac{R^3}{8r^3}$$

$$\ge 1+\frac{b+c}{a+b}+\frac{c+a}{b+c}+\frac{a+b}{c+a} \text{ is true (QED)}$$

681. In any ΔABC the following relationship holds:

$$3\left(\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\right)+\frac{R^2-4r^2}{r^2} \ge 12\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution 1 by Alex Szoros-Romania

$$\sum_{cyc}\frac{b+c}{a} = \sum_{cyc}\left(\frac{c}{a}+\frac{c}{b}\right) = \sum_{cyc}\frac{c(a+b)}{ab} \ge \sum_{cyc}\frac{4c}{a+b}$$

$$\Rightarrow \sum_{cyc}\frac{b+c}{a} \ge 4\sum_{cyc}\frac{c}{a+b}$$

$$\Rightarrow 3\sum_{cyc}\frac{b+c}{a} \ge 12\sum_{cyc}\frac{c}{a+b}; (1)$$

$$R \ge 2r\ (Euler) \Rightarrow R^2 \ge 4r^2 \Rightarrow \frac{R^2-4r^2}{r^2} \ge 0; (2)$$

From (1) and (2), it follows that:

$$3\left(\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\right)+\frac{R^2-4r^2}{r^2} \ge 12\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)$$

Solution 2 by Tapas Das-India

$$\frac{a}{b+c} = \frac{a}{2}\cdot\frac{2}{b+c} \le \frac{a}{4}\left(\frac{1}{b}+\frac{1}{c}\right)$$

$$\frac{b}{c+a} \le \frac{b}{4}\left(\frac{1}{c}+\frac{1}{a}\right) \text{ and } \frac{c}{a+b} \le \frac{c}{4}\left(\frac{1}{a}+\frac{1}{b}\right)$$

By adding, we get:

$$\sum_{cyc}\frac{a}{b+c}\le\frac{1}{4}\sum_{cyc}\frac{b+c}{a}$$

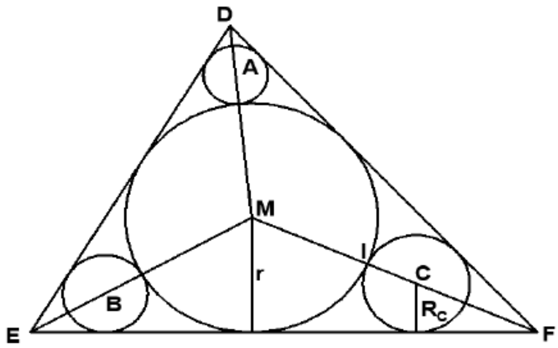
$$12\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)\le 12\cdot\frac{1}{4}\sum_{cyc}\frac{b+c}{a}=3\sum_{cyc}\frac{b+c}{a};(1)$$

$$R\ge 2r\ (Euler)\Rightarrow R^2\ge 4r^2\Rightarrow\frac{R^2-4r^2}{r^2}\ge 0;(2)$$

From (1) and (2), it follows that:

$$3\left(\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\right)+\frac{R^2-4r^2}{r^2}\ge 12\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)$$

682.



Let be an acute triangle DEF with r_1,r_2,r_3 –exradii, R_A,R_B,R_c –radii of circles with centers A,B,C. Prove that:

$$\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\ge\frac{1}{R_A+R_B+R_c}$$

Proposed by Marian Ursărescu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Let\ r\ be\ the\ inradius\ of\ \Delta DEF. We\ have:\ \frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=\frac{1}{r}.$$

$$So\ the\ problem\ becomes\ to\ prove:\ R_A+R_B+R_c\ge r.$$

$$We\ have:\ FM=FC+CI+IM\ with:\ FM=\frac{r}{\sin\frac{F}{2}},\ FC=\frac{R_c}{\sin\frac{F}{2}},\ CI=R_c,\ IM=r.$$

$$Then:\ \frac{r}{\sin\frac{F}{2}}=\frac{R_c}{\sin\frac{F}{2}}+R_c+r\Rightarrow R_c=\frac{1-\sin\frac{F}{2}}{1+\sin\frac{F}{2}}.r\ (and\ analogs)$$

$$Now,\ R_A+R_B+R_c=\left(\frac{1-\sin\frac{D}{2}}{1+\sin\frac{D}{2}}+\frac{1-\sin\frac{E}{2}}{1+\sin\frac{E}{2}}+\frac{1-\sin\frac{F}{2}}{1+\sin\frac{F}{2}}\right).r$$

$$=\left(\frac{2}{1+\sin\frac{D}{2}}+\frac{2}{1+\sin\frac{E}{2}}+\frac{2}{1+\sin\frac{F}{2}}-3\right).r\ge$$

$$\overset{CBS}{\ge}\left(2.\frac{9}{3+\sin\frac{D}{2}+\sin\frac{E}{2}+\sin\frac{F}{2}}-3\right).r\overset{Jensen}{\ge}\left(\frac{18}{3+3\sin\frac{\pi}{6}}-3\right).r=r.$$

$$\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\ge\frac{1}{R_A+R_B+R_c}.$$

Equality holds iff ΔDEF is equilateral.

683. Prove that in any triangle ABC with usual notations holds:

$$\frac{ab+bc+ca}{4\sqrt{3}F}\ge\sqrt{\frac{m_a}{s_a}}$$

Proposed by Marius Drăgan, Neculai Stanciu-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:\ \frac{m_a}{s_a}=\frac{b^2+c^2}{2bc}=\frac{1}{2}\left(\frac{b}{c}+\frac{c}{b}\right)\le\frac{R}{2r}\ (Bandila's\ inequality).$$

$$So\ it\ suffices\ to\ prove\ that:\ \frac{ab+bc+ca}{4\sqrt{3}F}\ge\sqrt{\frac{R}{2r}}.$$

$$We\ have:\ \frac{ab+bc+ca}{4\sqrt{3}F}=\frac{s^2+r(4R+r)}{4\sqrt{3}sr}=f(s),\ \ with$$

$$f'(s)=\frac{s^2-r(4R+r)}{4\sqrt{3}rs^2}\overset{Doucet}{>}0.$$

Then f is increasing and by Gerretsen's inequality we have:

$$\frac{ab+bc+ca}{4\sqrt{3}F}=f(s)\geq f\left(\sqrt{16Rr-5r^2}\right)=\frac{(16Rr-5r^2)+r(4R+r)}{4r\sqrt{3(16Rr-5r^2)}}$$

$$=\frac{5R-r}{\sqrt{3(16Rr-5r^2)}}\overset{?}{\underset{(1)}{\geq}}\sqrt{\frac{R}{2r}}$$

$$\Leftrightarrow 2(5R-r)^2\geq 3R(16R-5r)\ \Leftrightarrow 2R^2-5Rr+2r^2\geq 0\ \Leftrightarrow\ (R-2r)(2R-r)\geq 0$$

Which is true by Euler's inequality $(R\geq 2r)$.

Equality holds iff ΔABC *is equilateral.*

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sqrt{\frac{m_a}{s_a}}\leq\sqrt{\frac{m_a}{h_a}}\overset{Panaitopol}{\leq}\sqrt{\frac{R}{2r}}\overset{?}{\leq}\frac{ab+bc+ca}{4\sqrt{3}F}\Leftrightarrow\frac{(s^2+4Rr+r^2)^2}{48r^2s^2}\overset{?}{\geq}\frac{R}{2r}$$

$$\Leftrightarrow s^4+r^2(4R+r)^2+2s^2(4Rr+r^2)\overset{?}{\geq}24Rrs^2$$

$$\Leftrightarrow s^4+r^2(4R+r)^2\overset{?}{\underset{(i)}{\geq}}(16Rr-2r^2)s^2$$

$$\text{Now, LHS of (i)}\overset{Gerretsen+Trucht}{\geq}(16Rr-5r^2)s^2+3r^2s^2=(16Rr-2r^2)s^2\Rightarrow\text{(i) is true}$$

$$\Rightarrow\frac{ab+bc+ca}{4\sqrt{3}F}\geq\sqrt{\frac{m_a}{s_a}},\text{ equality iff }\Delta\text{ ABC is equilateral (QED)}$$

684. In any ΔABC the following relationship holds:

$$12\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)+\frac{R^2-4r^2}{r^2}\geq 3\left(\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\right).$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\text{We have}:\ \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\overset{CBS}{\geq}\frac{(a+b+c)^2}{2(ab+bc+ca)}=\frac{2s^2}{s^2+r^2+4Rr}$$

$$=2-\frac{2(r^2+4Rr)}{s^2+r^2+4Rr}\geq$$

$$\overset{Gerretsen}{\geq}\ 2-\frac{2(r^2+4Rr)}{(16Rr-5r^2)+r^2+4Rr}=2-\frac{4R+r}{2(5R-r)}=\frac{16R-5r}{2(5R-r)}.$$

$$\textit{Also we have}: \frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}=\frac{(a+b+c)(ab+bc+ca)}{abc}-3$$

$$=\frac{2s(s^2+r^2+4Rr)}{4Rsr}-3=$$

$$=\frac{s^2+r^2-2Rr}{2Rr}\overset{Gerretsen}{\leq}\frac{(4R^2+4Rr+3r^2)+r^2-2Rr}{2Rr}=\frac{2R^2+Rr+2r^2}{Rr}.$$

$$\textit{So it suffices to prove}: \ 12.\frac{16R-5r}{2(5R-r)}+\frac{R^2-4r^2}{r^2}\geq 3.\frac{2R^2+Rr+2r^2}{Rr}$$

$$\Leftrightarrow 6Rr^2(16R-5r)+R(5R-r)(R^2-4r^2)\geq 3r(5R-r)(2R^2+Rr+2r^2)$$

$$\Leftrightarrow 5R^4-31R^3r+67R^2r^2-53Rr^3+6r^4\geq 0$$

$$\Leftrightarrow (R-2r)\{(R-2r)[(R-2r)(5R-r)+r^2]+3r^2\}\geq 0$$

Which is true by Euler's inequality $(R\geq 2r)$.

So the proof is completed. Equality holds iff ΔABC *is equilateral.*

Solution 2 by Soumava Chakraborty-Kolkata-India

$$12\sum_{cyc}\frac{a}{b+c}=12\sum_{cyc}\frac{2s-(b+c)}{b+c}=\frac{24s}{\prod_{cyc}(b+c)}\sum_{cyc}(c+a)(a+b)-36$$

$$=\left(\frac{24s}{2s(s^2+2Rr+r^2)}\right)\left(\left(\sum_{cyc}a^2+2\sum_{cyc}ab\right)+\sum_{cyc}ab\right)-36$$

$$=\frac{24s(4s^2+s^2+4Rr+r^2)}{2s(s^2+2Rr+r^2)}-36\Rightarrow 12\sum_{cyc}\frac{a}{b+c}\overset{(i)}{=}\frac{24s^2-24Rr-24r^2}{s^2+2Rr+r^2}$$

$$\text{Again}, 3\sum_{cyc}\frac{b+c}{a}=3\sum_{cyc}\frac{2s-a}{a}=\frac{6s(s^2+4Rr+r^2)}{4Rrs}-9\Rightarrow 3\sum_{cyc}\frac{b+c}{a}\overset{(ii)}{=}\frac{3(s^2-2Rr+r^2)}{2Rr}$$

$\therefore$ via (i), (ii),

$$12\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)+\frac{R^2-4r^2}{r^2}\geq 3\left(\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\right)$$

$$\Leftrightarrow\frac{24s^2-24Rr-24r^2}{s^2+2Rr+r^2}+\frac{R^2-4r^2}{r^2}\geq\frac{3(s^2-2Rr+r^2)}{2Rr}$$

$$\Leftrightarrow\frac{r^2(24s^2-24Rr-24r^2)+(R^2-4r^2)(s^2+2Rr+r^2)}{r^2(s^2+2Rr+r^2)}\geq\frac{3(s^2-2Rr+r^2)}{2Rr}$$

$$\Leftrightarrow 3rs^4-(2R^3+40Rr^2-6r^3)s^2$$

$$-r(4R^4+2R^3r-52R^2r^2-56Rr^3-3r^4)\overset{(*)}{\leq} 0$$

$$\text{Now, LHS of } (*) \overset{\text{Gerretsen}}{\leq} 3r(4R^2+4Rr+3r^2)-(2R^3+40Rr^2-6r^3)s^2$$
$$-r(4R^4+2R^3r-52R^2r^2-56Rr^3-3r^4) \overset{?}{\leq} 0$$
$$\Leftrightarrow (2R^3-12R^2r+28Rr^2-15r^3)s^2+r(4R^4+2R^3r-52R^2r^2-56Rr^3-3r^4) \underbrace{\overset{?}{\geq}}_{(**)} 0$$
$$\because 2R^3-12R^2r+28Rr^2-15r^3=(R-2r)\left(2(R-2r)^2+4r^2\right)+9r^3 \overset{\text{Euler}}{\geq} 9r^3>0$$
$$\therefore \text{LHS of } (**)$$
$$\overset{\text{Gerretsen}}{\geq} (2R^3-12R^2r+28Rr^2-15r^3)(16Rr-5r^2)$$
$$+r(4R^4+2R^3r-52R^2r^2-56Rr^3-3r^4) \overset{?}{\geq} 0$$
$$\Leftrightarrow 9t^4-50t^3+114t^2-109t+18 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right)$$
$$\Leftrightarrow (t-2)\left((t-2)(2t^2+7t(t-2)+22)+35\right) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (**) \Rightarrow (*) \text{ is true}$$
$$\Rightarrow 12\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)+\frac{R^2-4r^2}{r^2} \geq 3\left(\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\right) \text{ (QED)}$$

685. In any Δ ABC holds:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{s^2}{27r^2} \geq 1+\frac{b}{a}+\frac{c}{b}+\frac{a}{c}$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Let $s-a=x, s-b=y$ and $s-c=z \therefore s=x+y+z \Rightarrow a=y+z$, $b=z+x$ and $c=x+y$

$$\text{Now, } \frac{s^2}{r^2}=\frac{s^4}{\Delta^2}=\frac{s^4}{s(s-a)(s-b)(s-c)} \overset{(\bullet)}{=} \frac{\left(\sum_{cyc} x\right)^3}{xyz} \text{ and } 1+\frac{4R}{r}$$
$$=1+\frac{4sabc}{4s(s-a)(s-b)(s-c)}=1+\frac{\prod_{cyc}(y+z)}{xyz}$$
$$\Rightarrow 1+\frac{4R}{r} \overset{(\bullet\bullet)}{=} \frac{xyz+\prod_{cyc}(y+z)}{xyz}$$
$$\text{Now, } \sum_{cyc}\frac{b}{a}=\sum\frac{z+x}{y+z} \Rightarrow \sum_{cyc}\frac{b}{a} \overset{(\bullet\bullet\bullet)}{=} \frac{\sum_{cyc}(x+y)^2(y+z)}{\prod_{cyc}(y+z)} \therefore (\bullet),(\bullet\bullet),(\bullet\bullet\bullet) \Rightarrow \frac{s^2}{r^2}$$
$$\geq \left(\sum_{cyc}\frac{b}{a}\right)\left(1+\frac{4R}{r}\right) \Leftrightarrow \frac{\left(\sum_{cyc} x\right)^3}{xyz} \geq \left(\frac{xyz+\prod_{cyc}(y+z)}{xyz}\right)\left(\frac{\sum_{cyc}(x+y)^2(y+z)}{\prod_{cyc}(y+z)}\right)$$

$$\Leftrightarrow\left(\prod_{cyc}(y+z)\right)\left(\sum_{cyc}x\right)^3\geq\left(xyz+\prod_{cyc}(y+z)\right)\left(\sum_{cyc}(x+y)^2(y+z)\right)$$

$$\Leftrightarrow\sum_{cyc}x^4y^2+\sum_{cyc}x^3y^3\overset{(i)}{\geq}xyz\left(\sum_{cyc}xy^2\right)+3x^2y^2z^2$$

Now, $\forall\, u,v,w>0, u^3+u^3+v^3\overset{A-G}{\geq}3u^2v, v^3+v^3+w^3\overset{A-G}{\geq}3v^2w$ and $w^3+w^3+u^3\overset{A-G}{\geq}3w^2u$ and summing up : $\sum_{cyc}u^3\geq\sum_{cyc}u^2v$ and choosing $u=xy, v=yz$ and $w=zx$,

$$\sum_{cyc}x^3y^3\underbrace{\geq}_{(*)}xyz\left(\sum_{cyc}xy^2\right)\text{ and }\sum_{cyc}x^4y^2\underbrace{\overset{A-G}{\geq}}_{(**)}3x^2y^2z^2\ \therefore (*)+(**)\Rightarrow(i)\text{ is true}\Rightarrow\frac{s^2}{r^2}\geq\left(\sum_{cyc}\frac{b}{a}\right)\left(1+\frac{4R}{r}\right)\Rightarrow\boxed{\sum_{cyc}\frac{b}{a}\overset{(\bullet\bullet\bullet\bullet)}{\leq}\frac{s^2}{r(4R+r)}}$$

Now, $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{R}{2r}\overset{?}{\geq}1+\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\Leftrightarrow\sum_{cyc}\frac{a}{b}+\sum_{cyc}\frac{b}{a}+\frac{R-2r}{2r}\overset{?}{\geq}2\sum_{cyc}\frac{b}{a}$

$$\Leftrightarrow\frac{\sum_{cyc}\left(ab\left(\sum_{cyc}a-c\right)\right)}{4Rrs}+\frac{R-2r}{2r}\overset{?}{\geq}2\sum_{cyc}\frac{b}{a}$$

$$\Leftrightarrow\frac{2s\left(s^2+4Rr+r^2\right)-12Rrs}{4Rrs}+\frac{R-2r}{2r}\overset{?}{\geq}2\sum_{cyc}\frac{b}{a}$$

$$\Leftrightarrow\boxed{\frac{s^2-2Rr+r^2+R(R-2r)}{2Rr}\underbrace{\overset{?}{\geq}}_{(***)}2\sum_{cyc}\frac{b}{a}}\text{ and }\because 2\sum_{cyc}\frac{b}{a}\overset{\text{via }(\bullet\bullet\bullet\bullet)}{\leq}\frac{2s^2}{r(4R+r)}$$

$\therefore$ in order to prove $(***)$, it suffices to prove : $\frac{s^2-2Rr+r^2+R(R-2r)}{2Rr}\geq\frac{2s^2}{r(4R+r)}$

$$\Leftrightarrow\boxed{rs^2+R(R-2r)(4R+r)\overset{(****)}{\geq}r(2R-r)(4R+r)}$$

Now, LHS of $(****)\overset{\text{Gerretsen}}{\geq}r\left(16Rr-5r^2\right)+R(R-2r)(4R+r)\overset{?}{\geq}r(2R-r)(4R+r)$

$$\Leftrightarrow 4t^3-15t^2+16t-4\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)\Leftrightarrow(t-2)(4t(t-2)+t+2)\overset{?}{\geq}0$$

$\rightarrow$ true $\because t\overset{\text{Euler}}{\geq}2\Rightarrow(****)\Rightarrow(***)$ is true $\Rightarrow\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{R}{2r}\geq 1+\frac{b}{a}+\frac{c}{b}+\frac{a}{c}$

$$\Rightarrow 1+\frac{b}{a}+\frac{c}{b}+\frac{a}{c}-\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\leq\frac{R}{2r}\overset{?}{\leq}\frac{s^2}{27r^2}\Leftrightarrow 2s^2\overset{?}{\geq}27Rr$$

$$\Leftrightarrow 2(s^2-16Rr+5r^2)+5r(R-2r)\overset{?}{\geq}0\to \text{true}$$

$$\because s^2-16Rr+5r^2\overset{\text{Gerretsen}}{\geq}0 \text{ and } 5r(R-2r)\overset{\text{Euler}}{\geq}0$$

$$\therefore 1+\frac{b}{a}+\frac{c}{b}+\frac{a}{c}-\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\leq\frac{s^2}{27r^2}$$

$$\Rightarrow \frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{s^2}{27r^2}\geq 1+\frac{b}{a}+\frac{c}{b}+\frac{a}{c} \text{ is true } \forall \text{ triangles (QED)}$$

686. In any Δ ABC holds:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{Rs^2}{54r^3}\geq 1+\frac{b}{a}+\frac{c}{b}+\frac{a}{c}$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Proof : Let $s-a=x, s-b=y$ and $s-c=z \therefore s=x+y+z\Rightarrow a=y+z, b=z+x$ and $c=x+y$

$$\text{Now, }\frac{s^2}{r^2}=\frac{s^4}{\Delta^2}=\frac{s^4}{s(s-a)(s-b)(s-c)}\overset{(\bullet)}{=}\frac{\left(\sum_{cyc}x\right)^3}{xyz} \text{ and } 1+\frac{4R}{r}$$

$$=1+\frac{4sabc}{4s(s-a)(s-b)(s-c)}=1+\frac{\prod_{cyc}(y+z)}{xyz}$$

$$\Rightarrow 1+\frac{4R}{r}\overset{(\bullet\bullet)}{=}\frac{xyz+\prod_{cyc}(y+z)}{xyz}$$

$$\text{Now, }\sum_{cyc}\frac{b}{a}=\sum\frac{z+x}{y+z}\Rightarrow\sum_{cyc}\frac{b}{a}\overset{(\bullet\bullet\bullet)}{=}\frac{\sum_{cyc}(x+y)^2(y+z)}{\prod_{cyc}(y+z)}\therefore(\bullet),(\bullet\bullet),(\bullet\bullet\bullet)\Rightarrow\frac{s^2}{r^2}$$

$$\geq\left(\sum_{cyc}\frac{b}{a}\right)\left(1+\frac{4R}{r}\right)\Leftrightarrow\frac{\left(\sum_{cyc}x\right)^3}{xyz}\geq\left(\frac{xyz+\prod_{cyc}(y+z)}{xyz}\right)\left(\frac{\sum_{cyc}(x+y)^2(y+z)}{\prod_{cyc}(y+z)}\right)$$

$$\Leftrightarrow\left(\prod_{cyc}(y+z)\right)\left(\sum_{cyc}x\right)^3\geq\left(xyz+\prod_{cyc}(y+z)\right)\left(\sum_{cyc}(x+y)^2(y+z)\right)$$

$$\Leftrightarrow\sum_{cyc}x^4y^2+\sum_{cyc}x^3y^3\overset{(i)}{\geq}xyz\left(\sum_{cyc}xy^2\right)+3x^2y^2z^2$$

Now, $\forall u,v,w>0, u^3+u^3+v^3\overset{A-G}{\geq}3u^2v, v^3+v^3+w^3\overset{A-G}{\geq}3v^2w$ and $w^3+w^3+u^3\overset{A-G}{\geq}3w^2u$ and summing up : $\sum_{cyc}u^3\geq\sum_{cyc}u^2v$ and choosing $u=xy, v=yz$ and $w=zx$,

$$\sum_{cyc} x^3y^3 \overset{(*)}{\gtrsim} xyz\left(\sum_{cyc} xy^2\right) \text{ and } \sum_{cyc} x^4y^2 \underbrace{\overset{A-G}{\geq}}_{(**)} 3x^2y^2z^2 \therefore (*) + (**) \Rightarrow \text{(i) is true} \Rightarrow \frac{s^2}{r^2}$$

$$\geq \left(\sum_{cyc} \frac{b}{a}\right)\left(1 + \frac{4R}{r}\right) \Rightarrow \boxed{\sum_{cyc} \frac{b}{a} \overset{(\bullet\bullet\bullet\bullet)}{\leq} \frac{s^2}{r(4R+r)}}$$

Now, $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} + \frac{R}{2r} \overset{?}{\geq} 1 + \frac{b}{a} + \frac{c}{b} + \frac{a}{c} \Leftrightarrow \sum_{cyc} \frac{a}{b} + \sum_{cyc} \frac{b}{a} + \frac{R-2r}{2r} \overset{?}{\geq} 2\sum_{cyc} \frac{b}{a}$

$$\Leftrightarrow \frac{\sum_{cyc}\left(ab\left(\sum_{cyc} a - c\right)\right)}{4Rrs} + \frac{R-2r}{2r} \overset{?}{\geq} 2\sum_{cyc} \frac{b}{a}$$

$$\Leftrightarrow \frac{2s\left(s^2 + 4Rr + r^2\right) - 12Rrs}{4Rrs} + \frac{R-2r}{2r} \overset{?}{\geq} 2\sum_{cyc} \frac{b}{a}$$

$$\Leftrightarrow \boxed{\frac{s^2 - 2Rr + r^2 + R(R-2r)}{2Rr} \underset{(***)}{\overset{?}{\geq}} 2\sum_{cyc} \frac{b}{a}} \text{ and } \because 2\sum_{cyc} \frac{b}{a} \overset{\text{via } (\bullet\bullet\bullet\bullet)}{\leq} \frac{2s^2}{r(4R+r)}$$

$\therefore$ in order to prove $(***)$, it suffices to prove : $\frac{s^2 - 2Rr + r^2 + R(R-2r)}{2Rr}$

$$\geq \frac{2s^2}{r(4R+r)}$$

$$\Leftrightarrow \boxed{rs^2 + R(R-2r)(4R+r) \overset{(****)}{\geq} r(2R-r)(4R+r)}$$

Now, LHS of $(****) \overset{\text{Gerretsen}}{\geq} r\left(16Rr - 5r^2\right) + R(R-2r)(4R+r) \overset{?}{\geq} r(2R-r)(4R+r)$

$$\Leftrightarrow 4t^3 - 15t^2 + 16t - 4 \overset{?}{\geq} 0 \; \left(t = \frac{R}{r}\right) \Leftrightarrow (t-2)(4t(t-2) + t + 2) \overset{?}{\geq} 0$$

$\rightarrow$ true $\because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (****) \Rightarrow (***)$ is true $\Rightarrow \frac{a}{b} + \frac{b}{c} + \frac{c}{a} + \frac{R}{2r} \geq 1 + \frac{b}{a} + \frac{c}{b} + \frac{a}{c}$

$$\Rightarrow 1 + \frac{b}{a} + \frac{c}{b} + \frac{a}{c} - \left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) \leq \frac{R}{2r} \overset{?}{\leq} \frac{Rs^2}{54r^3} \Leftrightarrow 2s^2 \overset{?}{\geq} 27Rr$$

$\Leftrightarrow 2\left(s^2 - 16Rr + 5r^2\right) + 5r(R-2r) \overset{?}{\geq} 0 \rightarrow$ true

$$\because s^2 - 16Rr + 5r^2 \overset{\text{Gerretsen}}{\geq} 0 \text{ and } 5r(R-2r) \overset{\text{Euler}}{\geq} 0$$

$$\therefore 1 + \frac{b}{a} + \frac{c}{b} + \frac{a}{c} - \left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) \leq \frac{Rs^2}{54r^3}$$

$\Rightarrow \frac{a}{b} + \frac{b}{c} + \frac{c}{a} + \frac{Rs^2}{54r^3} \geq 1 + \frac{b}{a} + \frac{c}{b} + \frac{a}{c}$ is true $\forall$ triangles (QED)

687. In ΔABC the following relationship holds:

$$\frac{R}{2r} + \sqrt{\frac{a}{b} + \frac{b}{c} + \frac{c}{a}} \geq 1 + \sqrt{\frac{b}{a} + \frac{c}{b} + \frac{a}{c}}$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution 1 by Alex Szoros-Romania

$$\text{Let: } M=\max\left\{\frac{a}{b}+\frac{b}{c}+\frac{c}{a},\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right\}\ m=\min\left\{\frac{a}{b}+\frac{b}{c}+\frac{c}{a},\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right\}$$

$$\Rightarrow M\geq m\geq 3$$

$$I)\text{ Suppose: } M=\frac{a}{b}+\frac{b}{c}+\frac{c}{a}.\text{ How }\frac{R}{2r}\geq 1\text{ and }\sqrt{M}\geq\sqrt{m}\Rightarrow$$

$$\frac{R}{2r}+\sqrt{M}\geq 1+\sqrt{m}$$

$$II)\text{ Suppose: } M=\frac{b}{a}+\frac{c}{b}+\frac{a}{c}$$

$$\frac{R}{2r}+\sqrt{m}\geq 1+\sqrt{M}(to\ prove)\Leftrightarrow\frac{R}{2r}-1\geq\sqrt{M}-\sqrt{m};(1)$$

We know that:

$$\frac{R}{2r}+\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq 1+\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\Rightarrow\frac{R}{2r}-1\geq M-m$$

It is enough to prove:

$$M-m\geq\sqrt{m}-\sqrt{m}\Leftrightarrow M-\sqrt{M}\geq m-\sqrt{m};(2)$$

$$\text{Let } f:[3,\infty)\to\mathbb{R}, f(x)=x-\sqrt{x}$$

$$f'(x)=1-\frac{1}{2\sqrt{x}}=\frac{2\sqrt{x}-1}{2\sqrt{x}}>0;\forall x\geq 3\Rightarrow$$

$$f\ -\text{increasing. How } M\geq m\Rightarrow f(M)\geq f(m)\Rightarrow(2)$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:$$

$$\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\overset{CBS}{\leq}\sqrt{(b^2+c^2+a^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)}\overset{Leibniz\ \&\ Steining}{\leq}\sqrt{9R^2.\frac{1}{4r^2}}=\frac{3R}{2r}.$$

$$Also\ by\ AM-GM\ inequality\ we\ have:\ \frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq 3$$

$$So\ it\ suffices\ to\ prove\ that:$$

$$\frac{R}{2r}+\sqrt{3}\geq 1+\sqrt{\frac{3R}{2r}}\ or\ \left(\sqrt{\frac{R}{2r}}-1\right)\left(\sqrt{\frac{R}{2r}}+1-\sqrt{3}\right)\geq 0$$

$$Which\ is\ true\ by\ Euler's\ inequality\ \ R \geq 2r.$$

$$Equality\ holds\ iff\ \Delta ABC\ is\ equilateral.$$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$\text{Let } s-a=x, s-b=y \text{ and } s-c=z \therefore s=x+y+z \Rightarrow a=y+z, b=z+x \text{ and } c=x+y$$

$$\text{Now, } \frac{s^2}{r^2}=\frac{s^4}{\Delta^2}=\frac{s^4}{s(s-a)(s-b)(s-c)} \overset{(\bullet)}{=} \frac{\left(\sum_{cyc}x\right)^3}{xyz} \text{ and } 1+\frac{4R}{r}$$
$$=1+\frac{4sabc}{4s(s-a)(s-b)(s-c)}=1+\frac{\prod_{cyc}(y+z)}{xyz}$$
$$\Rightarrow 1+\frac{4R}{r} \overset{(\bullet\bullet)}{=} \frac{xyz+\prod_{cyc}(y+z)}{xyz}$$

$$\text{Now, } \sum_{cyc}\frac{b}{a}=\sum\frac{z+x}{y+z} \Rightarrow \sum_{cyc}\frac{b}{a} \overset{(\bullet\bullet\bullet)}{=} \frac{\sum_{cyc}(x+y)^2(y+z)}{\prod_{cyc}(y+z)} \therefore (\bullet),(\bullet\bullet),(\bullet\bullet\bullet) \Rightarrow \frac{s^2}{r^2}$$
$$\geq \left(\sum_{cyc}\frac{b}{a}\right)\left(1+\frac{4R}{r}\right) \Leftrightarrow \frac{\left(\sum_{cyc}x\right)^3}{xyz} \geq \left(\frac{xyz+\prod_{cyc}(y+z)}{xyz}\right)\left(\frac{\sum_{cyc}(x+y)^2(y+z)}{\prod_{cyc}(y+z)}\right)$$
$$\Leftrightarrow \left(\prod_{cyc}(y+z)\right)\left(\sum_{cyc}x\right)^3 \geq \left(xyz+\prod_{cyc}(y+z)\right)\left(\sum_{cyc}(x+y)^2(y+z)\right)$$
$$\Leftrightarrow \sum_{cyc}x^4y^2+\sum_{cyc}x^3y^3 \overset{(i)}{\geq} xyz\left(\sum_{cyc}xy^2\right)+3x^2y^2z^2$$

$$\text{Now, } \forall\, u,v,w>0, u^3+u^3+v^3 \overset{A-G}{\geq} 3u^2v, v^3+v^3+w^3 \overset{A-G}{\geq} 3v^2w \text{ and } w^3+w^3$$
$$+u^3 \overset{A-G}{\geq} 3w^2u \text{ and summing up}: \sum_{cyc}u^3 \geq \sum_{cyc}u^2v \text{ and choosing } u=xy, v$$
$$=yz \text{ and } w=zx,$$
$$\sum_{cyc}x^3y^3 \underbrace{\geq}^{(*)} xyz\left(\sum_{cyc}xy^2\right) \text{ and } \sum_{cyc}x^4y^2 \underbrace{\overset{A-G}{\geq}}_{(**)} 3x^2y^2z^2 \therefore (*)+(**) \Rightarrow (i) \text{ is true} \Rightarrow \frac{s^2}{r^2}$$
$$\geq \left(\sum_{cyc}\frac{b}{a}\right)\left(1+\frac{4R}{r}\right) \Rightarrow \boxed{\sum_{cyc}\frac{b}{a} \overset{(\bullet\bullet\bullet\bullet)}{\leq} \frac{s^2}{r(4R+r)}}$$

$$\text{Now, } \frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{R}{2r} \overset{?}{\geq} 1+\frac{b}{a}+\frac{c}{b}+\frac{a}{c} \Leftrightarrow \sum_{cyc}\frac{a}{b}+\sum_{cyc}\frac{b}{a}+\frac{R-2r}{2r} \overset{?}{\geq} 2\sum_{cyc}\frac{b}{a}$$
$$\Leftrightarrow \frac{\sum_{cyc}\left(ab\left(\sum_{cyc}a-c\right)\right)}{4Rrs}+\frac{R-2r}{2r} \overset{?}{\geq} 2\sum_{cyc}\frac{b}{a}$$
$$\Leftrightarrow \frac{2s\left(s^2+4Rr+r^2\right)-12Rrs}{4Rrs}+\frac{R-2r}{2r} \overset{?}{\geq} 2\sum_{cyc}\frac{b}{a}$$

$$\Leftrightarrow \frac{s^2-2Rr+r^2+R(R-2r)}{2Rr} \underset{(***)}{\overset{?}{\geq}} 2\sum_{cyc}\frac{b}{a} \text{ and } \because 2\sum_{cyc}\frac{b}{a} \overset{\text{via }(\bullet\bullet\bullet\bullet)}{\leq} \frac{2s^2}{r(4R+r)}$$

$\therefore$ in order to prove $(***)$, it suffices to prove : $\frac{s^2-2Rr+r^2+R(R-2r)}{2Rr}$
$\geq \frac{2s^2}{r(4R+r)}$

$$\Leftrightarrow rs^2+R(R-2r)(4R+r) \overset{(****)}{\geq} r(2R-r)(4R+r)$$

Now, LHS of $(****) \overset{\text{Gerretsen}}{\geq} r(16Rr-5r^2)+R(R-2r)(4R+r) \overset{?}{\geq} r(2R-r)(4R+r)$

$$\Leftrightarrow 4t^3-15t^2+16t-4 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right) \Leftrightarrow (t-2)(4t(t-2)+t+2) \overset{?}{\geq} 0$$

$\rightarrow$ true $\because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (****) \Rightarrow (***)$ is true $\Rightarrow \frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{R}{2r} \geq 1+\frac{b}{a}+\frac{c}{b}+\frac{a}{c}$

$$\Rightarrow 1+\frac{b}{a}+\frac{c}{b}+\frac{a}{c}-\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) \leq \frac{R}{2r} \overset{\text{Euler}}{\leq} \frac{R^2}{4r^2}$$

$$\Rightarrow \boxed{1+\frac{b}{a}+\frac{c}{b}+\frac{a}{c}-\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) \overset{(l)}{\leq} \frac{R^2}{4r^2}}$$

Main inequality, $\frac{R}{2r}+\sqrt{\frac{a}{b}+\frac{b}{c}+\frac{c}{a}} \geq 1+\sqrt{\frac{b}{a}+\frac{c}{b}+\frac{a}{c}} \Leftrightarrow \frac{R^2}{4r^2}+\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{R}{r}\cdot\sqrt{\sum_{cyc}\frac{a}{b}}$
$\geq 1+\frac{b}{a}+\frac{c}{b}+\frac{a}{c}+2\cdot\sqrt{\sum_{cyc}\frac{b}{a}}$

$$\Leftrightarrow \boxed{1+\frac{b}{a}+\frac{c}{b}+\frac{a}{c}-\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+2\cdot\sqrt{\sum_{cyc}\frac{b}{a}} \overset{(1)}{\leq} \frac{R^2}{4r^2}+\frac{R}{r}\cdot\sqrt{\sum_{cyc}\frac{a}{b}}}$$

Now, via $(\bullet\bullet\bullet\bullet)$ and (l), LHS of $(1) \overset{(i)}{\leq} \frac{R^2}{4r^2}+2\cdot\sqrt{\frac{s^2}{r(4R+r)}}$ and via A − G, RHS of $(1) \overset{(ii)}{\geq} \frac{R^2}{4r^2}$
$+\frac{R}{r}\cdot\sqrt{3} \therefore$ (i), (ii) $\Rightarrow$ in order to prove (1), it suffices to prove :

$$\frac{R^2}{4r^2}+\frac{R}{r}\cdot\sqrt{3} \geq \frac{R^2}{4r^2}+2\cdot\sqrt{\frac{s^2}{r(4R+r)}} \Leftrightarrow \frac{3R^2}{r^2} \geq \frac{4s^2}{r(4R+r)} \Leftrightarrow 4rs^2 \overset{(2)}{\leq} 3(4R+r)R^2$$

Now, $4rs^2 \overset{\text{Gerretsen}}{\leq} 4r(4R^2+4Rr+3r^2) \overset{?}{\leq} 3(4R+r)R^2 \Leftrightarrow 12t^3-13t^2-16t-12 \overset{?}{\geq} 0$

$$\Leftrightarrow (t-2)(12t^2+11t+6) \overset{?}{\geq} 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (2) \Rightarrow (1) \text{ is true}$$

$\therefore$ in any Δ ABC, $\frac{R}{2r}+\sqrt{\frac{a}{b}+\frac{b}{c}+\frac{c}{a}}$
$\geq 1+\sqrt{\frac{b}{a}+\frac{c}{b}+\frac{a}{c}}$, equality iff Δ ABC is equilateral (QED)

688. In Δ ABC the following relationship holds:

$$\frac{R}{2r}+\sqrt{\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}}\ge 1+\sqrt{\frac{b+c}{a+b}+\frac{c+a}{b+c}+\frac{a+b}{c+a}}$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

Proof : Let $s-a=x, s-b=y$ and $s-c=z \therefore s=x+y+z \Rightarrow a=y+z, b=z+x$ and $c=x+y$

$$\text{Now, } \frac{s^2}{r^2}=\frac{s^4}{\Delta^2}=\frac{s^4}{s(s-a)(s-b)(s-c)}\overset{(\bullet)}{=}\frac{\left(\sum_{cyc}x\right)^3}{xyz} \text{ and } 1+\frac{4R}{r}$$

$$=1+\frac{4sabc}{4s(s-a)(s-b)(s-c)}=1+\frac{\prod_{cyc}(y+z)}{xyz}$$

$$\Rightarrow 1+\frac{4R}{r}\overset{(\bullet\bullet)}{=}\frac{xyz+\prod_{cyc}(y+z)}{xyz}$$

$$\text{Now, } \sum_{cyc}\frac{b}{a}=\sum\frac{z+x}{y+z}\Rightarrow\sum_{cyc}\frac{b}{a}\overset{(\bullet\bullet\bullet)}{=}\frac{\sum_{cyc}(x+y)^2(y+z)}{\prod_{cyc}(y+z)} \therefore (\bullet),(\bullet\bullet),(\bullet\bullet\bullet)\Rightarrow\frac{s^2}{r^2}$$

$$\ge\left(\sum_{cyc}\frac{b}{a}\right)\left(1+\frac{4R}{r}\right)\Leftrightarrow\frac{\left(\sum_{cyc}x\right)^3}{xyz}\ge\left(\frac{xyz+\prod_{cyc}(y+z)}{xyz}\right)\left(\frac{\sum_{cyc}(x+y)^2(y+z)}{\prod_{cyc}(y+z)}\right)$$

$$\Leftrightarrow\left(\prod_{cyc}(y+z)\right)\left(\sum_{cyc}x\right)^3\ge\left(xyz+\prod_{cyc}(y+z)\right)\left(\sum_{cyc}(x+y)^2(y+z)\right)$$

$$\Leftrightarrow\sum_{cyc}x^4y^2+\sum_{cyc}x^3y^3\overset{(i)}{\ge}xyz\left(\sum_{cyc}xy^2\right)+3x^2y^2z^2$$

Now, $\forall\, u,v,w>0, u^3+u^3+v^3\overset{A-G}{\ge}3u^2v, v^3+v^3+w^3\overset{A-G}{\ge}3v^2w$ and $w^3+w^3+u^3\overset{A-G}{\ge}3w^2u$ and summing up : $\sum_{cyc}u^3\ge\sum_{cyc}u^2v$ and choosing $u=xy, v=yz$ and $w=zx$,

$$\sum_{cyc}x^3y^3\overset{(*)}{\ge}xyz\left(\sum_{cyc}xy^2\right) \text{ and } \sum_{cyc}x^4y^2\underset{(**)}{\overset{A-G}{\ge}}3x^2y^2z^2 \therefore (*)+(**)\Rightarrow(i)\text{ is true}\Rightarrow\frac{s^2}{r^2}$$

$$\ge\left(\sum_{cyc}\frac{b}{a}\right)\left(1+\frac{4R}{r}\right)\Rightarrow\boxed{\sum_{cyc}\frac{b}{a}\overset{(\bullet\bullet\bullet\bullet)}{\le}\frac{s^2}{r(4R+r)}}$$

$$\text{Now,}\ \frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{R}{2r}\overset{?}{\geq}1+\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\Leftrightarrow\sum_{cyc}\frac{a}{b}+\sum_{cyc}\frac{b}{a}+\frac{R-2r}{2r}\overset{?}{\geq}2\sum_{cyc}\frac{b}{a}$$

$$\Leftrightarrow\frac{\sum_{cyc}\left(ab(\sum_{cyc}a-c)\right)}{4Rrs}+\frac{R-2r}{2r}\overset{?}{\geq}2\sum_{cyc}\frac{b}{a}$$

$$\Leftrightarrow\frac{2s(s^2+4Rr+r^2)-12Rrs}{4Rrs}+\frac{R-2r}{2r}\overset{?}{\geq}2\sum_{cyc}\frac{b}{a}$$

$$\Leftrightarrow\frac{s^2-2Rr+r^2+R(R-2r)}{2Rr}\underset{(***)}{\overset{?}{\geq}}2\sum_{cyc}\frac{b}{a}\ \text{and}\ \because 2\sum_{cyc}\frac{b}{a}\overset{via\ (\bullet\bullet\bullet\bullet)}{\leq}\frac{2s^2}{r(4R+r)}$$

$\therefore$ in order to prove $(***)$, it suffices to prove : $\frac{s^2-2Rr+r^2+R(R-2r)}{2Rr}$

$$\geq\frac{2s^2}{r(4R+r)}$$

$$\Leftrightarrow rs^2+R(R-2r)(4R+r)\overset{(****)}{\geq}r(2R-r)(4R+r)$$

$$\text{Now, LHS of }(****)\overset{Gerretsen}{\geq}r(16Rr-5r^2)+R(R-2r)(4R+r)\overset{?}{\geq}r(2R-r)(4R+r)$$

$$\Leftrightarrow 4t^3-15t^2+16t-4\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)\Leftrightarrow(t-2)(4t(t-2)+t+2)\overset{?}{\geq}0$$

$$\to\text{true}\because t\overset{Euler}{\geq}2\Rightarrow(****)\Rightarrow(***)\text{ is true}\Rightarrow\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{R}{2r}\geq1+\frac{b}{a}+\frac{c}{b}+\frac{a}{c}$$

$$\Rightarrow\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{abc(a+b+c)}{16F^2}\geq1+\frac{b}{a}+\frac{c}{b}+\frac{a}{c}$$

$$\Rightarrow\boxed{1+\frac{b}{a}+\frac{c}{b}+\frac{a}{c}-\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\overset{(l)}{\leq}\frac{abc(a+b+c)}{2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)}}$$

$\because b+c, c+a, a+b$ form sides of a triangle,

$\therefore$ via (l) on triangle with sides $b+c, c+a, a+b$, we arrive at :

$$1+\frac{b+c}{a+b}+\frac{c+a}{b+c}+\frac{a+b}{c+a}-\left(\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}\right)\leq\frac{\left(\prod_{cyc}(b+c)\right)\left(\sum_{cyc}(b+c)\right)}{2\sum_{cyc}(b+c)^2(c+a)^2-\sum_{cyc}(b+c)^4}$$

$$=\frac{2(\sum_{cyc}a)(\prod_{cyc}(b+c))}{16abc(\sum_{cyc}a)}=\frac{\prod_{cyc}(b+c)}{8abc}$$

$$\therefore\text{ for original }\Delta\ ABC,\ 1+\frac{b+c}{a+b}+\frac{c+a}{b+c}+\frac{a+b}{c+a}-\left(\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}\right)$$

$$\leq\frac{\prod_{cyc}(b+c)}{8abc}\overset{?}{\leq}\frac{R}{2r}\Leftrightarrow\frac{R}{2r}\geq\frac{2s(s^2+2Rr+r^2)}{32Rrs}\Leftrightarrow s^2+2Rr+r^2\overset{(*****)}{\leq}8R^2$$

$$\text{Now, LHS of }(*****)\overset{Gerretsen}{\leq}4R^2+6Rr+4r^2\overset{?}{\leq}8R^2\Leftrightarrow 2R^2-3Rr-2r^2\overset{?}{\geq}0$$

$$\Leftrightarrow(R-2r)(2R+r)\overset{?}{\geq}0\to\text{true}\because R\overset{Euler}{\geq}2r\Rightarrow(*****)\text{ is true}$$

$$\Rightarrow 1+\frac{b+c}{a+b}+\frac{c+a}{b+c}+\frac{a+b}{c+a}$$

$$-\left(\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}\right)\overset{(m)}{\leq}\frac{R}{2r}\overset{Euler}{\leq}\frac{R^2}{4r^2}\quad \boxed{\therefore 1+\frac{b+c}{a+b}+\frac{c+a}{b+c}+\frac{a+b}{c+a}-\left(\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}\right)\overset{(m)}{\leq}\frac{R^2}{4r^2}}$$

$$\text{Now, }\frac{R}{2r}+\sqrt{\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}}\geq 1+\sqrt{\frac{b+c}{a+b}+\frac{c+a}{b+c}+\frac{a+b}{c+a}}$$

$$\Leftrightarrow \frac{R^2}{4r^2}+\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}+\frac{R}{r}\cdot\sqrt{\sum_{cyc}\frac{a+b}{b+c}}$$

$$\geq 1+\frac{b+c}{a+b}+\frac{c+a}{b+c}+\frac{a+b}{c+a}+2\cdot\sqrt{\sum_{cyc}\frac{b+c}{a+b}}$$

$$\Leftrightarrow \boxed{1+\frac{b+c}{a+b}+\frac{c+a}{b+c}+\frac{a+b}{c+a}-\left(\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}\right)+2\cdot\sqrt{\sum_{cyc}\frac{b+c}{a+b}}\overset{(1)}{\leq}\frac{R^2}{4r^2}+\frac{R}{r}\cdot\sqrt{\sum_{cyc}\frac{a+b}{b+c}}}$$

$$\text{Now, via (m), LHS of (1)}\leq\frac{R^2}{4r^2}+2\cdot\sqrt{\sum_{cyc}\frac{b+c}{a+b}}\overset{?}{\leq}\frac{R^2}{4r^2}+\frac{R}{r}\cdot\sqrt{\sum_{cyc}\frac{a+b}{b+c}}$$

$$\Leftrightarrow \boxed{\frac{R^2}{r^2}\cdot\sum_{cyc}\frac{a+b}{b+c}\underset{(2)}{\overset{?}{\geq}}4\sum_{cyc}\frac{b+c}{a+b}}$$

$$\text{Now, }\sum_{cyc}\frac{a+b}{b+c}=\sum_{cyc}\frac{(a+b)^2}{(b+c)(a+b)}\overset{Bergstrom}{\geq}\frac{16s^2}{\left(\sum_{cyc}a^2+2\sum_{cyc}ab\right)+\sum_{cyc}ab}$$

$$=\frac{16s^2}{4s^2+s^2+4Rr+r^2}\Rightarrow \text{LHS of (2)}\overset{(i)}{\geq}\frac{R^2}{r^2}\cdot\frac{16s^2}{5s^2+4Rr+r^2}$$

$$\text{Again, }\sum_{cyc}\frac{b+c}{a+b}=\sum_{cyc}\frac{(b+c)(a+b)}{(a+b)^2}\overset{A-G}{\leq}\sum_{cyc}\frac{c(b+c)(a+b)}{4abc}=\frac{1}{4abc}\sum_{cyc}\left(c\left(b^2+\sum_{cyc}ab\right)\right)$$

$$=\frac{1}{4abc}\left(\sum_{cyc}b^2c+2s\sum_{cyc}ab\right)\leq\frac{1}{4abc}\left(\sum_{cyc}a^3+2s\sum_{cyc}ab\right)$$

$$=\frac{2s}{16Rrs}\cdot\left(s^2-6Rr-3r^2+s^2+4Rr+r^2\right)=\frac{s^2-Rr-r^2}{4Rr}\overset{Gerretsen}{\leq}\frac{4R^2+3Rr+2r^2}{4Rr}$$

$$\Rightarrow \text{RHS of (2)}\overset{(ii)}{\leq}\frac{4R^2+3Rr+2r^2}{Rr}\ \therefore (i),(ii)$$

$$\Rightarrow \text{in order to prove (2), it suffices}$$

$$\text{to prove}:\frac{R^2}{r^2}\cdot\frac{16s^2}{5s^2+4Rr+r^2}\geq\frac{4R^2+3Rr+2r^2}{Rr}\Leftrightarrow 16R^3s^2$$

$$\geq 5r\left(4R^2+3Rr+2r^2\right)s^2+r^2(4R+r)\left(4R^2+3Rr+2r^2\right)$$

$$\Leftrightarrow \boxed{(16R^3-20R^2r-15Rr^2-10r^3)s^2 \overset{(3)}{\geq} r^2(4R+r)(4R^2+3Rr+2r^2)}$$

$$\because 16R^3-20R^2r-15Rr^2-10r^3=(R-2r)(16R^2+12Rr+9r^2)+8r^3 \overset{\text{Euler}}{\geq} 8r^3>0$$

$$\therefore \text{LHS of } (3) \overset{\text{Gerretsen}}{\geq} (16R^3-20R^2r-15Rr^2-10r^3)(16Rr-5r^2)$$

$$\overset{?}{\geq} r^2(4R+r)(4R^2+3Rr+2r^2) \Leftrightarrow 64t^4-104t^3-39t^2-24t+12 \overset{?}{\geq} 0$$

$$\Leftrightarrow (t-2)\left(64t^3+24t^2+6t+3(t-2)\right) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (3) \Rightarrow (2)$$

$$\Rightarrow (1) \text{ is true}$$

$$\Rightarrow \frac{R}{2r}+\sqrt{\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}} \geq 1+\sqrt{\frac{b+c}{a+b}+\frac{c+a}{b+c}+\frac{a+b}{c+a}} \ (\text{QED})$$

Solution 2 by Mohamed Amine ben Ajiba-Tanger-Morocco

$$We\ have:$$

$$\sum_{cyc}\frac{b+c}{a+b} \overset{CBS}{\leq} \sqrt{\left(\sum(b+c)^2\right)\left(\sum\frac{1}{(a+b)^2}\right)} \overset{CBS\ \&\ AM-GM}{\leq} \sqrt{\left(\sum 2(b^2+c^2)\right)\left(\sum\frac{1}{4ab}\right)}$$

$$=\sqrt{\left(\sum a^2\right)\cdot\frac{2s}{4Rsr}} \overset{Leibniz}{\leq} \sqrt{9R^2\cdot\frac{1}{2Rr}}=3\sqrt{\frac{R}{2r}}.$$

$$Then\ we\ have:$$

$$1+\sqrt{\frac{b+c}{a+b}+\frac{c+a}{b+c}+\frac{a+b}{c+a}} \leq 1+\sqrt{3\sqrt{\frac{R}{2r}}} \overset{AM-GM}{\leq} 1+\frac{\sqrt{3}}{4}\left(\frac{R}{2r}+1+1+1\right)=$$

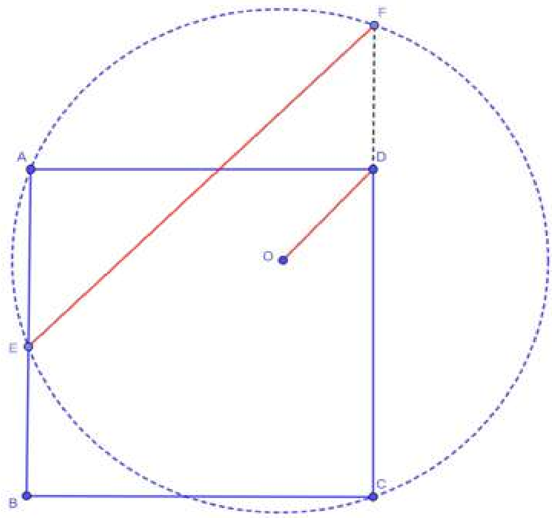
$$=\frac{R}{2r}+\sqrt{3}-\left(1-\frac{\sqrt{3}}{4}\right)\left(\frac{R}{2r}-1\right) \overset{Euler}{\leq} \frac{R}{2r}+\sqrt{3} \overset{AM-GM}{\leq} \frac{R}{2r}$$

$$+\sqrt{\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}},\ as\ desired.$$

$$Equality\ holds\ iff\ \Delta ABC\ is\ equilateral.$$

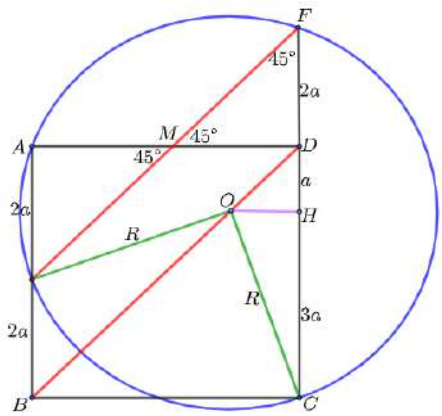
689. $ABCD$ −square, E is the midpoint of AB, $CD \cap (AEC) = F$, O is the center of (AEC). Prove that:

$$FE \parallel DO \text{ and } EF:OD = 4$$



Proposed by Eldeniz Hesenov-Georgia

Solution by Mansur Mansurov-Azerbaijan



$$FC \parallel AB \Rightarrow AF = EC, FD = EB \Rightarrow AM = MD = DF$$

$$\Rightarrow \mu(\widehat{EFC}) = 45^\circ \Rightarrow \mu(\widehat{EOC}) = 90^\circ \Rightarrow B, E, O, C \text{ –cyclic}$$

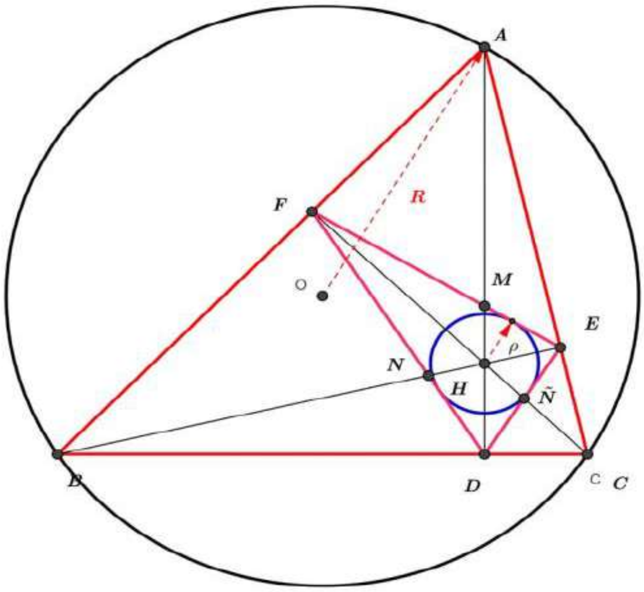
$$EO = OC = R \Rightarrow \mu(\widehat{EBO}) = \mu(\widehat{CBO}) = 45^\circ$$

$$\Rightarrow B, D, O \text{ –collinear} \Rightarrow \mu(\widehat{ODC}) = 45^\circ \Rightarrow OD \parallel EF; (1)$$

$$\frac{EF}{OD} = \frac{BD}{OD} = \frac{DC}{DH} = 4$$

690. H **–orthocenter of** $\Delta ABC, \Delta DEF$ **–is orthic triangle of** $\Delta ABC, R$ **–circumradius of** $\Delta ABC, \rho$ **–is inradius of** ΔDEF**. Prove that:**

$$\frac{DM}{MA} = \frac{(a^2 - b^2 + c^2)(a^2 + b^2 - c^2)}{a^2(-a^2 + b^2 + c^2)}, \qquad \frac{DM}{MA} \cdot \frac{EN}{NB} \cdot \frac{FN}{NC} = \frac{4\rho}{R}$$



Proposed by Juan Jose Isach Mayo-Valencia-Spain

Solution by Jose Ferreira Queiroz-Olinda-Brazil

$$\cos A=\frac{b^2+c^2-a^2}{2bc},\cos B=\frac{a^2+c^2-b^2}{2ac},\cos C=\frac{a^2+b^2-c^2}{2ab}$$

In ΔDME and ΔAME, we have:

$$\frac{DN}{\sin(\pi-2B)}=\frac{EN}{\sin\left(\frac{\pi}{2}-A\right)}\text{ and }\frac{EM}{\sin\left(\frac{\pi}{2}-C\right)}=\frac{AM}{\sin B}$$

$$\frac{DM\cdot\cos A}{\sin 2B}=\frac{AM\cdot\cos C}{\sin B}\Rightarrow\frac{DM}{AM}=\frac{2\cos C\cdot\cos B}{\cos A}$$

$$\frac{DM}{AM}=\frac{2\cdot\frac{a^2+b^2-c^2}{2ab}\cdot\frac{a^2+c^2-b^2}{2ac}}{\frac{b^2+c^2-a^2}{2bc}}=\frac{(a^2-b^2+c^2)(a^2+b^2-c^2)}{a^2(-a^2+b^2+c^2)}$$

Similarly, we get:

$$\frac{EN}{NB}=\frac{2\cos A\cdot\cos C}{\cos B}\text{ and }\frac{FN}{NC}=\frac{2\cos A\cdot\cos B}{\cos C}$$

In ΔDCE we have:

$$\frac{DE}{\sin C}=\frac{a\cdot\cos C}{\sin A}\Rightarrow DE=\frac{a\cdot\sin C\cos C}{\sin A}$$

Now, let's calculate the inradius of ΔDEF:

$$\rho=\frac{DE\cdot\sin\left(\frac{FDE}{2}\right)\cdot\sin\left(\frac{FED}{2}\right)}{\cos\left(\frac{DFE}{2}\right)}=\frac{\frac{a\cdot\sin C\cos C}{\sin A}\cdot\sin\left(\frac{\pi}{2}-A\right)\cdot\sin\left(\frac{\pi}{2}-B\right)}{\cos\left(\frac{\pi}{2}-C\right)}=$$

$$=\frac{a\cdot\cos A\cos B\cos C}{\sin A}$$

$$\frac{DM}{MA}\cdot\frac{EN}{NB}\cdot\frac{FN}{NC}=8\cos A\cos B\cos C=\frac{8\rho\cdot\sin A}{a}=\frac{8\rho\cdot\sin A}{2R\cdot\sin A}$$

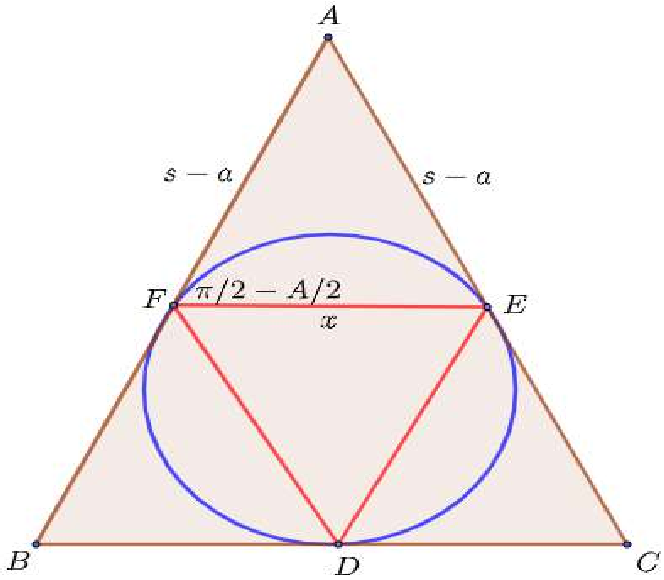
$$\frac{DM}{MA}\cdot\frac{EN}{NB}\cdot\frac{FN}{NC}=\frac{4\rho}{R}$$

691. Let $I,\Delta DEF$ be the incenter and the intouch triangle in ΔABC. Let R_1,R_2,R_3 −be circumradii of $\Delta AEF,\Delta ADF,\Delta CDE$. Prove that:

$$8R_1R_2R_3=AI\cdot BI\cdot CI$$

Proposed by Mehmet Şahin-Ankara-Turkiye

Solution by Ertan Yildirim-Izmir-Turkiye



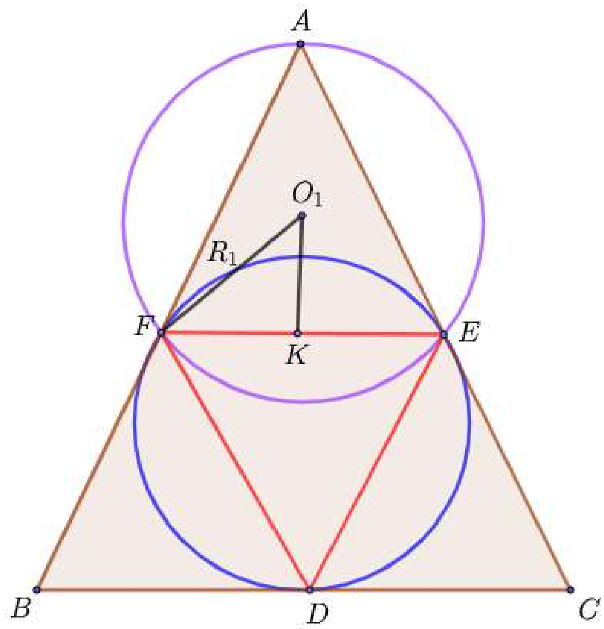
Lemma 1: $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\frac{r}{4R}$

Lemma 2: $\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=\frac{s}{4R}$

Lemma 3: $AI=\frac{r}{\sin\frac{A}{2}}$

In ΔAFE, from Law of Sines:

$$\frac{x}{\sin A}=\frac{s-a}{\sin\left(\frac{\pi}{2}-\frac{A}{2}\right)}\Rightarrow\frac{x}{2\sin\frac{A}{2}\cos\frac{A}{2}}=\frac{s-a}{\cos\frac{A}{2}}\Rightarrow x=2(s-a)\sin\frac{A}{2}$$



$$\sin A=\frac{FK}{R_1}\Rightarrow R_1=\frac{(s-a)\sin\frac{A}{2}}{\sin A}=\frac{(s-a)\sin\frac{A}{2}}{2\sin\frac{A}{2}\cos\frac{A}{2}}=\frac{s-a}{2\cos\frac{A}{2}}$$

Similarly,

$$R_2=\frac{s-b}{2\cos\frac{B}{2}}\text{ and }R_3=\frac{s-c}{2\cos\frac{C}{2}}$$

$$RHS=AI\cdot BI\cdot CI=\frac{r}{\sin\frac{A}{2}}\cdot\frac{r}{\sin\frac{B}{2}}\cdot\frac{r}{\sin\frac{C}{2}}=\frac{r^3}{\frac{r}{8R}}=4Rr^2$$

$$LHS=8R_1R_2R_3=8\frac{s-a}{2\cos\frac{A}{2}}\cdot\frac{s-b}{2\cos\frac{B}{2}}\cdot\frac{s-c}{2\cos\frac{C}{2}}=\frac{sr^2}{\frac{s}{4R}}=4Rr^2$$

692. In ΔABC the following relationship holds:

$$min\left\{\frac{4R+r}{5R-r}+\frac{a}{b}+\frac{b}{c}+\frac{c}{a},\frac{6R+4r}{9R-2r}+2\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)\right\}\geq 4.$$

Proposed by Alex Szoros-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have: \frac{a}{b}+\frac{b}{c}+\frac{c}{a} \overset{CBS}{\geq} \frac{(a+b+c)^2}{ab+bc+ca} = \frac{4s^2}{s^2+r^2+4Rr} = 4-\frac{4r(4R+r)}{s^2+r^2+4Rr} \geq$$

$$\overset{Gerretsen}{\geq} 4-\frac{4r(4R+r)}{(16Rr-5r^2)+r^2+4Rr} = 4-\frac{4R+r}{5R-r}.$$

$$Then: \frac{4R+r}{5R-r}+\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq 4 \quad (1)$$

$$Now,\ we\ have: \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} = \frac{(a+b+c)(a^2+b^2+c^2)+3abc}{(a+b+c)(ab+bc+ca)-abc} =$$

$$= \frac{2s.2(s^2-r^2-4Rr)+3.4Rsr}{2s.(s^2+r^2+4Rr)-4Rsr} = \frac{2(s^2-r^2-Rr)}{s^2+r^2+2Rr} = 2-\frac{r(6R+4r)}{s^2+r^2+2Rr} \geq$$

$$\overset{Gerretsen}{\geq} 2-\frac{r(6R+4r)}{(16Rr-5r^2)+r^2+2Rr} = 2-\frac{6R+4r}{2(9R-2r)}.$$

$$Then: \frac{6R+4r}{9R-2r}+2\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right) \geq 4 \quad (2)$$

$$From\ (1)\ and\ (2)\ we\ get:$$

$$min\left\{\frac{4R+r}{5R-r}+\frac{a}{b}+\frac{b}{c}+\frac{c}{a}, \frac{6R+4r}{9R-2r}+2\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)\right\} \geq 4.$$

Equality holds iff ΔABC is equilateral.

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq \frac{9(a^2+b^2+c^2)}{(a+b+c)^2} \overset{?}{\geq} 4-\frac{4R+r}{5R-r} = \frac{16R-5r}{5R-r} \Leftrightarrow \frac{9(s^2-4Rr-r^2)}{2s^2} \overset{?}{\geq} \frac{16R-5r}{5R-r}$$

$$\Leftrightarrow 9(5R-r)(s^2-4Rr-r^2) \overset{?}{\geq} 2(16R-5r)s^2$$

$$\Leftrightarrow (13R+r)s^2 \underset{(i)}{\overset{?}{\geq}} r(180R^2+9Rr-9r^2)$$

$$\text{Now, LHS of (i)} \overset{Gerretsen}{\geq} (13R+r)(16Rr-5r^2) \overset{?}{\geq} r(180R^2+9Rr-9r^2)$$

$$\Leftrightarrow 14R^2-29Rr+2r^2 \overset{?}{\geq} 0 \Leftrightarrow (14R-r)(R-2r) \overset{?}{\geq} 0 \to \text{true} \because R \overset{Euler}{\geq} 2r$$

$$\Rightarrow \text{(i) is true}$$

$$\therefore \frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq 4-\frac{4R+r}{5R-r} \Rightarrow \boxed{\frac{4R+r}{5R-r}+\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \overset{(*)}{\geq} 4}$$

$$\text{Again, } 2\sum_{cyc}\frac{a}{b+c}=2\sum_{cyc}\frac{2s-(b+c)}{b+c}=\frac{4s}{\prod_{cyc}(b+c)}\cdot\left(\left(\sum_{cyc}a^2+2\sum_{cyc}ab\right)+\sum_{cyc}ab\right)-6$$

$$=\frac{4s}{2s(s^2+2Rr+r^2)}\cdot(4s^2+s^2+4Rr+r^2)-6\overset{?}{\geq}4-\frac{6R+4r}{9R-2r}$$

$$\Leftrightarrow\frac{2(5s^2+4Rr+r^2)}{s^2+2Rr+r^2}\overset{?}{\geq}\frac{84R-24r}{9R-2r}$$

$$\Leftrightarrow(9R-2r)(5s^2+4Rr+r^2)\overset{?}{\geq}(42R-12r)(s^2+2Rr+r^2)$$

$$\Leftrightarrow(3R+2r)s^2\overset{?}{\geq}r(48R^2+17Rr-10r^2)\to\text{true}$$

$$\because(3R+2r)s^2\overset{\text{Gerretsen}}{\geq}(3R+2r)(16Rr-5r^2)=r(48R^2+17Rr-10r^2)\therefore 2\sum_{cyc}\frac{a}{b+c}$$

$$\geq 4-\frac{6R+4r}{9R-2r}\Rightarrow\boxed{\frac{6R+4r}{9R-2r}+2\sum_{cyc}\frac{a}{b+c}\overset{(**)}{\geq}4}$$

$$\therefore(*),(**)\Rightarrow\min\left\{\frac{4R+r}{5R-r}+\frac{a}{b}+\frac{b}{c}+\frac{c}{a},\frac{6R+4r}{9R-2r}+2\sum_{cyc}\frac{a}{b+c}\right\}\geq 4\ (\text{QED})$$

693. In acute ΔABC the following relationship holds:

$$a\cos B\cos C+b\cos C\cos A+c\cos A\cos B<\sqrt{\frac{2}{3}(a^2+b^2+c^2)}.$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By CBS inequality we have :

$$LHS\leq\sqrt{(a^2+b^2+c^2)(\cos^2 B.\cos^2 C+\cos^2 C.\cos^2 A+\cos^2 A.\cos^2 B)}.$$

$$\sum_{cyc}\cos^2 B.\cos^2 C=\sum_{cyc}\frac{1}{(1+\tan^2 B)(1+\tan^2 C)}\overset{AM-GM}{\leq}\sum_{cyc}\frac{1}{2\tan B.2\tan C}=\frac{1}{4}<\frac{2}{3}.$$

$$\textit{Therefore, }\ a\cos B\cos C+b\cos C\cos A+c\cos A\cos B<\sqrt{\frac{2}{3}(a^2+b^2+c^2)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\textit{We will prove the following inequality :}$$

$$a.\cos B.\cos C+b.\cos C.\cos A+c.\cos A.\cos B\le\frac{\sqrt{3}}{4}.\sqrt{a^2+b^2+c^2}<\sqrt{\frac{2}{3}(a^2+b^2+c^2)}$$

$$\textit{Since } a=2R\sin A\ (\textit{and analogs})\textit{ then we have :}$$

$$a.\cos B.\cos C+b.\cos C.\cos A+c.\cos A.\cos B=2R.\cos A.\cos B.\cos C.(\tan A+\tan B+\tan C)=$$

$$=2R.\cos A.\cos B.\cos C.\tan A.\tan B.\tan C=2R.\sin A.\sin B.\sin C=2R.\frac{sr}{2R^2}$$

$$=\frac{sr}{R}\overset{Euler}{\le}\frac{s}{2}=\frac{a+b+c}{4}\overset{CBS}{\le}\frac{\sqrt{3}}{4}.\sqrt{a^2+b^2+c^2}.$$

$$\textit{Therefore, } a.\cos B.\cos C+b.\cos C.\cos A+c.\cos A.\cos B\le\frac{\sqrt{3}}{4}.\sqrt{a^2+b^2+c^2}<\sqrt{\frac{2}{3}(a^2+b^2+c^2)}$$

694. Prove or disprove : In any ΔABC :

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{R^2}{4r^2}\ge 1+\frac{b^2}{a^2}+\frac{c^2}{b^2}+\frac{a^2}{c^2}.$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\textit{For a } \Delta ABC \textit{ with sides : } a=5,\ \ b=c=4\ \textit{ we have :}$$

$$\frac{R}{2r}=\frac{abc}{(a+b-c)(a-b+c)(-a+b+c)}=\frac{5\times 4^2}{5^2\times 3}=\frac{16}{15}$$

$$\textit{Then : }\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{R^2}{4r^2}=\frac{5}{4}+1+\frac{4}{5}+\left(\frac{16}{15}\right)^2=\frac{3769}{900}.$$

Also we have : $1+\frac{b^2}{a^2}+\frac{c^2}{b^2}+\frac{a^2}{c^2}=1+\frac{16}{25}+1+\frac{25}{16}=\frac{1681}{400}$.

Since : $9\times 1681=15129>15076=4\times 3769$ *then we get* :

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{R^2}{4r^2}<1+\frac{b^2}{a^2}+\frac{c^2}{b^2}+\frac{a^2}{c^2}.$$

Therefore, $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{R^2}{4r^2}\ge 1+\frac{b^2}{a^2}+\frac{c^2}{b^2}+\frac{a^2}{c^2}$ *is not true for all* ΔABC.

695. In ΔABC the following relationship holds:

$$r\left(5+\frac{2R}{r}\right)^2\ge\left(\sum_{cyc}a\right)\left(\sum_{cyc}\frac{1}{a}\right)\left(\sum_{cyc}h_a\right)\ge 81r$$

Proposed by Alex Szoros-Romania

Solution by Tapas Das-India

We know that:

$$h_a+h_b+h_c=2F\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\text{ and }9r\le h_a+h_b+h_c\le\frac{9R}{2}\Leftrightarrow$$

$$\frac{9r}{2F}\le\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\le\frac{9R}{4F}$$

$$\left(\sum_{cyc}a\right)\left(\sum_{cyc}\frac{1}{a}\right)\left(\sum_{cyc}h_a\right)=2F\cdot\sum_{cyc}a\cdot\left(\sum_{cyc}\frac{1}{a}\right)^2=$$

$$=2F\cdot 2s\left(\sum_{cyc}\frac{1}{a}\right)^2\ge 2F\cdot 2s\cdot\frac{9r}{2F}\cdot\frac{9r}{2F}=\frac{2s\cdot 9r\cdot 9r}{2rs}=81r$$

$$\left(\sum_{cyc}a\right)\left(\sum_{cyc}\frac{1}{a}\right)\left(\sum_{cyc}h_a\right)=2F\cdot 2s\cdot\left(\sum_{cyc}\frac{1}{a}\right)^2=$$

$$=2rs\cdot 2s\left(\sum_{cyc}\frac{1}{a}\right)^2=r\cdot 4s^2\left(\sum_{cyc}\frac{1}{a}\right)^2$$

We need show:

$$r\left(5+\frac{2R}{r}\right)^2 \ge r\cdot 4s^2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2$$

$$5+\frac{2R}{r} \ge (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

$$5+\frac{2R}{r} \ge 2s\cdot\frac{ab+bc+ca}{abc}$$

$$2s\cdot\frac{ab+bc+ca}{abc}-2s\cdot\frac{s^2+r^2+4Rr}{4Rrs}=\frac{s^2+r^2+4Rr}{2Rr}$$

$$s^2 \le 4R62+4Rr+3r^2(Gerretsen)$$

$$2s\cdot\frac{ab+bc+ca}{abc} \le \frac{4R^2+4Rr+3r^2+r^2+4Rr}{2Rr}$$

$$5+\frac{2R}{r}-\frac{4R^2+8Rr+4r^2}{2Rr}=\frac{10Rr+4R^2-4R^2-8Rr-4r^2}{2Rr}=$$

$$=\frac{2Rr-4r^2}{2Rr} \ge 0;(R \ge 2r)$$

$$5+\frac{2R}{r} \ge \frac{4R^2+8Rr+4r^2}{2Rr}$$

$$5+\frac{2R}{r} \ge 2s\cdot\frac{ab+bc+ca}{abc}$$

$$r\left(5+\frac{2R}{r}\right)^2 \ge \left(\sum_{cyc} a\right)\left(\sum_{cyc}\frac{1}{a}\right)\left(\sum_{cyc} h_a\right)$$

696. In ΔABC the following relationship holds:

$$\frac{R+r}{3r} \ge \sum_{cyc}\frac{2a^2+b^2}{(2a+b)^2} \ge 1.$$

Proposed by Alex Szoros-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Lemma: \ If\ x,y>0\ then:$$

$$\frac{1}{9}\left(\frac{x}{y}+\frac{y}{x}+1\right) \ge \frac{2x^2+y^2}{(2x+y)^2} \ge \frac{1}{3}.$$

$$\textit{Proof}: \ \textit{By CBS inequality we have}:$$

$$(2+1)(2x^2+y^2) \geq (2x+y)^2 \ \textit{then}: \ \frac{2x^2+y^2}{(2x+y)^2} \geq \frac{1}{3}.$$

$$\textit{Now, we have}:$$

$$\frac{1}{9}\left(\frac{x}{y}+\frac{y}{x}+1\right) \geq \frac{2x^2+y^2}{(2x+y)^2} \Leftrightarrow (x^2+y^2+xy)(2x+y)^2 \geq 9xy(2x^2+y^2)$$

$$\Leftrightarrow 4x^4-10x^3y+9x^2y^2-4xy^3+y^4 \geq 0 \Leftrightarrow$$

$$(x-y)^2.[3x^2+(x-y)^2] \geq 0 \ \textit{which is true}.$$

$$\textit{Then}: \ \frac{1}{9}\left(\frac{x}{y}+\frac{y}{x}+1\right) \geq \frac{2x^2+y^2}{(2x+y)^2} \geq \frac{1}{3}, \ \ \forall x,y>0.$$

$$\textit{Using the lemma and Bandila's inequality} \left(\therefore \frac{a}{b}+\frac{b}{a} \leq \frac{R}{r}\right) \textit{ we get}:$$

$$\frac{1}{9}\left(\frac{R}{r}+1\right) \geq \frac{2a^2+b^2}{(2a+b)^2} \geq \frac{1}{3} \ \ (\textit{and analogs})$$

Summing up this inequality with similar ones yields the desired inequality.

Equality holds iff ΔABC is equilateral.

Solution 2 by Namig Mammadov-Azerbaijan

We will use some inequalities:

$$1)\ 2\sum a^3 \geq \sum ab(a+b)$$

$$2)\ 3abc+\sum a^3 \geq \sum ab(a+b)\ \ (\textit{Schur})$$

$$3)\ \frac{R}{r} \geq \frac{a^3+b^3+c^3+abc}{2abc}$$

$$\text{Let } h=\sum \frac{2a^2+b^2}{(2a+b)^2} \Rightarrow h-1=\sum\left(\frac{2a^2+b^2}{(2a+b)^2}-\frac{1}{3}\right)=\frac{2}{3}\sum\frac{(a-b)^2}{(2a+b)^2} \geq 0$$

$$\Rightarrow h \geq 1$$

$$h-1=\frac{2}{3}\sum\frac{(a-b)^2}{ab}\cdot\frac{ab}{(2a+b)^2} \leq \frac{2}{3}\sum\frac{(a-b)^2}{ab}\cdot\frac{1}{6}=\frac{1}{9}\sum\frac{(a-b)^2}{ab}=$$

$$=\frac{1}{9abc}\sum(a-b)^2c$$

$$\frac{R+r}{3r}-1=\frac{R-2r}{3r}=\frac{R}{3r}-\frac{2}{3} \geq \frac{a^3+b^3+c^3+abc}{6abc}-\frac{2}{3}=\frac{a^3+b^3+c^3-3abc}{6abc}$$

So, we need to prove:

$$\frac{1}{9abc}\sum(a-b)^2c \le \frac{a^3+b^3+c^3-3abc}{6abc} \Leftrightarrow$$

$$3\sum a^3 - 9abc \ge 2\sum(a-b)^2c\,;(4)$$

The difference $LHS - RHS$ of (4)=

$$=\left[2\sum a^3 - \sum ab(a+b)\right] + \left[3abc + \sum a^3 - \sum ab(a+b)\right] \ge 0$$

which is true due the inequalities (1) and (2).

697. In ΔABC the following relationship holds:

$$F\sum_{cyc}\frac{1}{h_a}\cot\frac{A}{2} \ge 4R+r$$

Proposed by Marian Ursărescu-Romania

Solution 1 by Alex Szoros-Romania

$$ah_a = 2F \Rightarrow \frac{1}{h_a} = \frac{a}{2F}$$

$$F\sum_{cyc}\frac{1}{h_a}\cot\frac{A}{2} = F\sum_{cyc}\frac{a}{2F}\cot\frac{A}{2} = \frac{1}{2}\sum_{cyc}a\sqrt{\frac{s(s-a)}{(s-b)(s-c)}} =$$

$$=\frac{1}{2}\sum_{cyc}\frac{as(s-a)}{S} = \frac{s}{F}\sum_{cyc}(as-a^2) = \frac{1}{2r}\left(s\sum_{cyc}a - \sum_{cyc}a^2\right) =$$

$$=\frac{1}{2r}\left(s\cdot 2s - (2s^2-2r^2-8Rr)\right) = \frac{2r^2+8Rr}{2r} = 4R+r \ge 4R+r$$

Solution 2 by Tapas Das-India

$$\sum_{cyc}a^2 = 2(s^2-r^2-4Rr)$$

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{r_a}{s} \Rightarrow \cot\frac{A}{2} = \frac{s}{r_a}$$

$$r_a = \frac{F}{s-a} \Rightarrow F\sum_{cyc}\frac{1}{h_a}\cot\frac{A}{2} = \frac{1}{2}\sum_{cyc}a\cot\frac{A}{2} = \frac{1}{2}\sum_{cyc}a\cdot\frac{F}{r_a} =$$

$$=\frac{s}{2}\sum_{cyc}a\cdot\frac{s-a}{F}=\frac{s}{2F}\left(s\sum_{cyc}a-\sum_{cyc}a^2\right)=$$

$$=\frac{s}{2F}(2r^2+8Rr)=\frac{s(r^2+4Rr)}{rs}=4R+r$$

Solution 3 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$F\sum_{cyc}\frac{1}{h_a}\cot\frac{A}{2}=\sum_{cyc}\frac{ah_a}{2h_a}\cdot\cot\frac{A}{2}=\sum_{cyc}\frac{2R\sin A}{2}\cdot\cot\frac{A}{2}=$$

$$=R\sum_{cyc}2\sin\frac{A}{2}\cos\frac{A}{2}\cdot\frac{\cos\frac{A}{2}}{\sin\frac{A}{2}}=R\sum_{cyc}2\cos^2\frac{A}{2}=R\sum_{cyc}(1+\cos A)=$$

$$=3R+R\sum_{cyc}\cos A=3R+R\left(1+\frac{r}{R}\right)$$

698. In ΔABC the following relationship holds:

$$\sqrt{(r_a+r_b)(r_a+r_c)}+\sqrt{(r_b+r_a)(r_b+r_c)}+\sqrt{(r_c+r_b)(r_c+r_a)}\ge 18r$$

Proposed by Ertan Yildirim-Izmir-Turkiye

Solution 1 by Tapas Das-India

$$(r_a+r_b)(r_a+r_c)\ge 2\sqrt{r_ar_b}\cdot 2\sqrt{r_ar_c}\ge 4w_bw_c\ge 4h_bh_c=$$

$$=4\cdot\frac{2F}{b}\cdot\frac{2F}{c}=\frac{16F^2}{bc}$$

$$\sqrt{(r_a+r_b)(r_a+r_c)}\ge\frac{4F}{\sqrt{bc}}$$

Analogous,

$$\sqrt{(r_b+r_a)(r_b+r_c)}\ge\frac{4F}{\sqrt{ac}}$$

$$\sqrt{(r_c+r_b)(r_c+r_a)}\ge\frac{4F}{\sqrt{ab}}$$

By adding, we have:

$$\sqrt{(r_a+r_b)(r_a+r_c)}+\sqrt{(r_b+r_a)(r_b+r_c)}+\sqrt{(r_c+r_b)(r_c+r_a)}\ge$$

$$\geq 4F\left(\frac{1}{\sqrt{bc}}+\frac{1}{\sqrt{ca}}+\frac{1}{\sqrt{ab}}\right)\geq 4F\left(\frac{1}{\frac{b+c}{2}}+\frac{1}{\frac{c+a}{2}}+\frac{1}{\frac{a+b}{2}}\right)=$$

$$=8F^2\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right)\geq 8F^2\cdot\frac{(1+1+1)^2}{2(a+b+c)}=\frac{8F\cdot 9}{4s}=\frac{9\cdot 8rs}{4s}=18r$$

Solution 2 by Alex Szoros-Romania

$$\sum_{cyc} r_a r_b=\sum_{cyc}\frac{F^2}{(s-a)(s-b)}=\sum_{cyc} s(s-c)=s\sum_{cyc}(s-c)=s^2$$

$$(r_a+r_b)(r_a+r_c)=r_a^2+\sum_{cyc} r_a r_b=r_a^2+s^2$$

$$\sqrt{(r_a+r_b)(r_a+r_c)}\geq\sqrt{r_a^2+s^2}$$

$$\sum_{cyc}\sqrt{(r_a+r_b)(r_a+r_c)}\geq\sum_{cyc}\sqrt{r_a^2+s^2}\geq\sqrt{\left(\sum_{cyc} r_a\right)^2+(3s)^2}=$$

$$=\sqrt{(4R+r)^2+9s^2}\geq\sqrt{(9r)^2+9\cdot 27r^2}=\sqrt{324r^2}=18r$$

699. In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{(2a)^2}{(s-b)(s-c)\big(3(b+c)\big)^2}\geq\frac{1}{r(4R+r)}$$

Proposed by Neculai Stanciu-Romania

Solution by Tapas Das-India

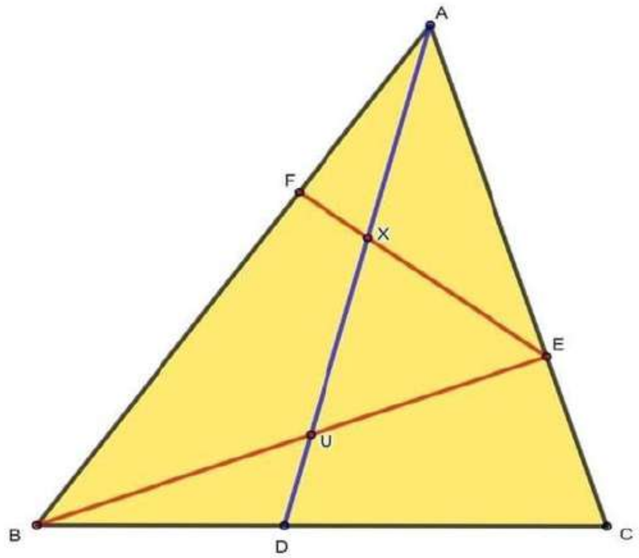
$$\sum_{cyc}\frac{(2a)^2}{(s-b)(s-c)\big(3(b+c)\big)^2}=\frac{4}{9}\sum_{cyc}\frac{\left(\frac{a}{b+c}\right)^2}{(s-b)(s-c)}\geq$$

$$\geq\frac{4}{9}\cdot\frac{\left(\sum\frac{a}{b+c}\right)^2}{\sum(s-b)(s-c)}\overset{Nesbitt}{\geq}\frac{4}{9}\cdot\frac{\left(\frac{3}{2}\right)^2}{\sum(s-b)(s-c)}=$$

$$=\frac{4}{9}\cdot\frac{9}{4}\cdot\frac{1}{\sum(s-b)(s-c)}=\frac{1}{3s^2-2s\sum a+\sum ab}=$$

$$=\frac{1}{3s^2-4s^2+s^2+r^2+4Rr}=\frac{1}{r(4R+r)}$$

700.



$$\frac{BD}{DC}=x,\frac{CE}{EA}=y,\frac{BF}{FA}=z.\ \text{Prove: }\ \frac{FX}{XE}\cdot\frac{BU}{UE}=\frac{x^2(y+1)^2}{z+1}$$

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution 1 by Rajarshi Chakraborty-India

$$\{G\}=FE\cap BC$$

$$\frac{BD}{DC}=x,\frac{CE}{EA}=y,\frac{BF}{FA}=z\Rightarrow\frac{CA}{EA}=y+1,\frac{BA}{FA}=z+1$$

From Menelaus' theorem:

$$\frac{BU}{UE}\cdot\frac{EA}{CA}\cdot\frac{CD}{BD}=1\Rightarrow\frac{BU}{UE}=\frac{CA}{EA}\cdot\frac{BD}{CD}=(y+1)\cdot\frac{BD}{CD}=(y+1)x$$

$$\text{Also,}\frac{BU}{UE}\cdot\frac{EX}{FX}\cdot\frac{FA}{BA}=1\Rightarrow\frac{BU}{UE}\cdot\frac{FA}{BA}=\frac{FX}{EX}\Rightarrow\frac{x(y+1)}{z+1}=\frac{FX}{EX}$$

$$\text{So,}\frac{BU}{EU}\cdot\frac{FX}{EX}=x(y+1)\cdot\frac{x(y+1)}{z+1}=\frac{x^2(y+1)^2}{z+1}$$

Solution 2 by Hikmat Mammadov-Azerbaijan

$$\frac{UE}{BU}=\frac{pq}{x(q+qy)}\Rightarrow\frac{BU}{UE}=x(y+1),\qquad\frac{FX}{XE}=\frac{BU\cdot r}{UE(r+rz)}\Rightarrow\frac{FX}{XE}=\frac{BU}{UE(1+z)}$$

$$\frac{FX}{XE}\cdot\frac{BU}{UE}=\left(\frac{BU}{UE}\right)^2\cdot\frac{1}{z+1}=\frac{x^2(y+1)^2}{z+1}$$

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru