ROMANIAN MATHEMATICAL SOCIETY

Mehedinţi Branch

R. M. M. - 35

ROMANIAN MATHEMATICAL MAGAZINE

WINTER EDITION 2022

ISSN 2501-0099

ROMANIAN MATHEMATICAL SOCIETY

Mehedinți Branch

ROMANIAN MATHEMATICAL MAGAZINE

R.M.M.

Nr.35-WINTER EDITION 2022

ROMANIAN MATHEMATICAL SOCIETY

Mehedinți Branch

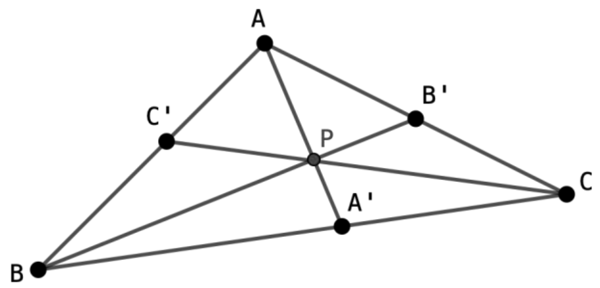
CONTENT

6 AREAS OF 6 FAMOUS PEDAL TRIANGLES

By Daniel Sitaru, Claudia Nănuți – Romania

Abstract: In this paper is illustrated the way to find areas of pedal triangles of centroid, incenter, orthocenter (for an acute triangle), symmedian point, Gergonne's point and Nagel's point in a triangle in terms of a, b, c – sides of a given triangle.

Let AA', BB', CC' be concurrent cevians in ΔABC and $\{P\} = AA' \cap BB' \cap CC'$.



By Ceva's theorem:

$$\frac{C'A}{C'B} \cdot \frac{A'B}{A'C} \cdot \frac{B'C}{B'A} = 1$$

Let be $u, v, w > 0$ such that:

$$\frac{C'A}{C'B} = \frac{v}{u}; \frac{A'B}{A'C} = \frac{w}{v}; \frac{B'C}{B'A} = \frac{u}{w}$$

We will find a relationship which can be used to find the distances $AP, BP, CP, A'P, B'P, C'P$ in terms of AA', BB', CC'.

$$\frac{A'B}{A'C} = \frac{w}{v} \Rightarrow \frac{A'B}{A'C + A'B} = \frac{w}{v+w} \Rightarrow \frac{A'B}{a} = \frac{w}{v+w}$$

$$A'B = \frac{aw}{v+w} \qquad (1)$$

$$A'C = BC - A'B = a - \frac{aw}{v+w} = \frac{av}{v+w}$$

$$A'C = \frac{av}{v+w} \qquad (2)$$

By Van Aubel's theorem:

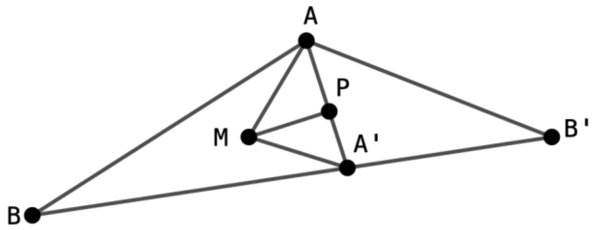
$$\frac{PA}{PA'} = \frac{C'A}{C'B} + \frac{B'A}{B'C} = \frac{v}{u} + \frac{w}{u} = \frac{v+w}{u}$$

$$\frac{PA}{PA + PA'} = \frac{v+w}{v+w+u} \Rightarrow \frac{PA}{AA'} = \frac{v+w}{u+v+w}$$

$$PA = \frac{(v+w)AA'}{u+v+w}$$

$$PA' = AA' - PA = AA' - \frac{(v+w)AA'}{u+v+w} = \frac{u \cdot AA'}{u+v+w}$$

Let be $M \in Int\ (\Delta ABC)$.



By Stewart's theorem in $\Delta AMA'$:

$$MP^2 \cdot AA' = MA^2 \cdot PA' + MA'^2 \cdot PA - PA \cdot PA \cdot AA'$$

$$MP^2 \cdot AA' = MA^2 \cdot \frac{u \cdot AA'}{u+v+w} + MA'^2 \cdot \frac{(v+w)AA'}{u+v+w} - \frac{u(v+w)}{(u+v+w)^2} \cdot AA'^3$$

$$MP^2 = \frac{u \cdot MA^2}{u+v+w} + \frac{(v+w) \cdot MA'^2}{u+v+w} - \frac{u(v+w)}{(u+v+w)^2} \cdot AA'^2 \qquad (3)$$

By Stewart's theorem in ΔBMC:

$$MA'^2 \cdot BC = MB^2 \cdot A'C + MC^2 \cdot A'B - A'C \cdot A'B \cdot BC$$

By (1); (2):

$$MA'^2 \cdot a = MB^2 \cdot \frac{av}{v+w} + MC^2 \cdot \frac{aw}{v+w} - \frac{av}{v+w} \cdot \frac{aw}{v+w} \cdot a$$

$$MA'^2 = MB^2 \cdot \frac{v}{v+w} + MC^2 \cdot \frac{w}{v+w} - \frac{a^2vw}{(v+w)^2} \qquad (4)$$

By Stewart's theorem in ΔABC:

$$AA'^2 \cdot BC = AB^2 \cdot A'C + AC^2 \cdot A'B - A'B \cdot A'C \cdot BC$$

$$AA'^2 \cdot a = c^2 \cdot \frac{av}{v+w} + b^2 \cdot \frac{aw}{v+w} - \frac{av}{v+w} \cdot \frac{aw}{v+w} \cdot a$$

$$AA'^2 = \frac{c^2v + b^2w}{v+w} - \frac{a^2vw}{(v+w)^2} \qquad (5)$$

Replace (4); (5) in (3):

$$MP^2 = \frac{u}{u+v+w} MA^2 + \frac{v+w}{u+v+w}\left(MB^2 \cdot \frac{v}{v+w} + MC^2 \cdot \frac{w}{v+w} - \frac{a^2vw}{(v+w)^2}\right) -$$

$$-\frac{u(v+w)}{(u+v+w)^2}\left(\frac{c^2v + b^2w}{v+w} - \frac{a^2vw}{(v+w)^2}\right)$$

$$MP^2 = MA^2 \cdot \frac{u}{u+v+w} + MB \cdot \frac{v}{u+v+w} + MC^2 \cdot \frac{w}{u+v+w} - \frac{c^2vu + b^2uw}{(u+v+w)^2} +$$

$$+\frac{a^2vw}{(u+v+w)(v+w)}\left(\frac{u}{u+v+w} - 1\right)$$

$$MP^2 = MA^2 \cdot \frac{u}{u+v+w} + MB^2 \cdot \frac{v}{u+v+w} + MC^2 \cdot \frac{w}{u+v+w} -$$

$$-\frac{c^2vu + b^2uw}{(u+v+w)^2} + \frac{a^2v \cdot w(u-u-v-w)}{(u+v+w)^2(v+w)}$$

$$MP^2 = MA^2 \cdot \frac{u}{u+v+w} + MB^2 \cdot \frac{v}{u+v+w} + MC^2 \cdot \frac{w}{u+v+w} - \frac{a^2vw+b^2wu+c^2uv}{(u+v+w)^2} \quad (6)$$

Let be $M = O$ – circumcenter. By (6):

$$OP^2 = OA^2 \cdot \frac{u}{u+v+w} + OB^2 \cdot \frac{v}{u+v+w} + OC^2 \cdot \frac{w}{u+v+w} - \frac{a^2vw + b^2wu + c^2uv}{(u+v+w)^2}$$

$$OA = OB = OC = R \text{ – circumradii}$$

$$OP^2 = R^2 \cdot \frac{u+v+w}{u+v+w} - \frac{a^2vw + b^2wu + c^2uv}{(u+v+w)^2}$$

$$OP^2 = R^2 - \frac{a^2vw+b^2wu+c^2uv}{(u+v+w)^2} \quad (7)$$

1a. For $P = G$ – centroid; $u = v = w = 1; u + v + w = 3$, replaced in (7):

$$OG^2 = R^2 - \frac{a^2+b^2+c^2}{9} \quad (8)$$

2a. For $P = I$ – incenter; $u = a; v = b; w = c; u + v = w = 2s$, replaced in (7):

$$OI^2 = R^2 - \frac{a^2bc + b^2ca + c^2ab}{4s^2} = R^2 - \frac{abc(a+b+c)}{4s^2} = R^2 - \frac{abc \cdot 2s}{4s^2}$$

$$OI^2 = R^2 - \frac{abc}{2s} = R^2 - \frac{4Rrs}{2s} = R^2 - 2Rr = R(R-2r) \quad (9)$$

3a. For $P = H$ – orthocenter (ΔABC – acute), $u = \tan A, v = \tan B, w = \tan C$

$$u + v + w = \tan A + \tan B + \tan C = \tan A \tan B \tan C$$

Replaced in (7):

$$OH^2 = R^2 - \frac{a^2 \tan B \tan C + b^2 \tan C \tan A + c^2 \tan A \tan B}{(\tan A + \tan B + \tan C)^2}$$

By [1]:

$$\sum_{cyc} a^2 \tan B \tan C = \frac{2F^2}{R^2 \cos A \cos B \cos C}$$

$$OH^2 = R^2 - \frac{1}{\left(\prod_{cyc} \tan A\right)^2} \cdot \sum_{cyc} a^2 \tan B \tan C$$

$$OH^2 = R^2 - \frac{1}{\left(\prod_{cyc} \frac{\sin A}{\cos A}\right)^2} \cdot \frac{2F^2}{R^2 \cdot \prod_{cyc} \cos A}$$

We replace F – area by:

$$F = 2R^2 \prod_{cyc} \sin A$$

$$OH^2 = R^2 - \frac{\left(\prod_{cyc} \cos A\right)^2}{\left(\prod_{cyc} \sin A\right)^2} \cdot \frac{2 \cdot 4R^4 \cdot \left(\prod_{cyc} \sin A\right)^2}{R^2 \cdot \prod_{cyc} \cos A}$$

$$OH^2 = R^2 - 8R^2 \prod_{cyc} \cos A, \quad OH^2 = R^2\left(1 - 8\prod_{cyc} \cos A\right)$$

By [1]:

$$\prod_{cyc} \cos A = \frac{a^2 + b^2 + c^2 - 8R^2}{8R^2}$$

$$OH^2 = R^2\left(1 - 8 \cdot \frac{a^2 + b^2 + c^2 - 8R^2}{8R^2}\right)$$

$$OH^2 = R^2 - (a^2 + b^2 + c^2 - 8R^2),\ OH^2 = 9R^2 - (a^2 + b^2 + c^2) \quad (10)$$

4a. For $P = K$ – symmedian point; $u = a^2; v = b^2; w = c^2$.

$$u + v + w = a^2 + b^2 + c^2 = 2(s^2 - r^2 - 4Rr)$$

Replaced in (7):

$$OK^2 = R^2 - \frac{a^2b^2c^2 + a^2b^2c^2 + a^2b^2c^2}{(a^2 + b^2 + c^2)^2}$$

$$OK^2 = R^2 - \frac{3a^2b^2c^2}{(a^2+b^2+c^2)^2} \quad (11)$$

5a. For $P = \Gamma$ – Gergonne's point:

$$u = \frac{1}{s-a}; v = \frac{1}{s-b}; w = \frac{1}{s-c}$$

By [1]:

$$\sum_{cyc} \frac{1}{s-a} = \frac{4R + r}{rs}$$

$$u + v + w = \frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} = \frac{4R + r}{rs}$$

Replaced in (7):

$$O\Gamma^2 = R^2 - \frac{\frac{a^2}{(s-b)(s-c)} + \frac{b^2}{(s-c)(s-a)} + \frac{c^2}{(s-a)(s-b)}}{\left(\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c}\right)^2}$$

$$O\Gamma^2 = R^2 - \left(\frac{rs}{4R + r}\right)^2 \cdot \frac{1}{(s-a)(s-b)(s-c)} \cdot \sum_{cyc} a^2(s-a)$$

$$\sum_{cyc} a^2(s-a) = s\sum_{cyc} a^2 - \sum_{cyc} a^3 =$$

$$= 2s(s^2-r^2-4Rr) - 2s(s^2-3r^2-6Rr) =$$

$$= 2s(s^2-r^2-4Rr-s^2+3r^2+6Rr) = 2s(2r^2+2Rr) = 4rs(R+r) = 4F(R+r)$$

$$O\Gamma^2 = R^2 - \frac{r^2s^2}{(4R+r)^2}\cdot\frac{s}{s(s-a)(s-b)(s-c)}\cdot 4F(R+r)$$

$$O\Gamma^2 = R^2 - \frac{F^2}{(4R+r)^2}\cdot\frac{s}{F^2}\cdot 4F(R+r)$$

$$O\Gamma^2 = R^2 - \frac{4Fs(R+r)}{(4R+r)^2},\ O\Gamma^2 = R^2 - \frac{4rs^2(R+r)}{(4R+r)^2} \quad (12)$$

6a. For $P = N$ – Nagel's point: $u = s-a; v = s-b; w = s-c$

$$u+v+w = 3s-(a+b+c) = 3s-2s = s$$

Replaced in (7):

$$ON^2 = R^2 - \frac{a^2(s-b)(s-c)+b^2(s-c)(s-a)+c^2(s-a)(s-b)}{s^2}$$

$$ON^2 = R^2 - \frac{1}{s^2}\sum_{cyc} a^2(s-b)(s-c)$$

By [1]:

$$\sum_{cyc} a^2\,(s-b)(s-c) = 4rs^2(R-r)$$

$$ON^2 = R^2 - \frac{1}{s^2}\cdot 4rs^2(R-r),\ ON^2 = R^2-4rR+4r^2 = (R-2r)^2 \quad (13)$$

If ΔDEF is pedal triangle of P then ([2]):

$$[DEF] = \frac{1}{4}(R^2-OP^2)\cdot\frac{F}{R^2}; F = [ABC]$$

1b. Pedal triangle of centroid:

$$[DEF] = \frac{1}{4R^2}(R^2-OG^2)\cdot F \overset{(8)}{=} \frac{1}{4R^2}\left(R^2-R^2+\frac{a^2+b^2+c^2}{9}\right)\cdot F$$

$$[DEF] = \frac{a^2+b^2+c^2}{36R^2}\cdot F$$

2b. Pedal triangle of incenter:

$$[DEF] = \frac{1}{4R^2}(R^2-OI^2)\cdot F \overset{(9)}{=} \frac{1}{4R^2}(R^2-R^2+2Rr)F$$

$$[DEF] = \frac{r}{2R}\cdot F$$

3b. Pedal triangle of orthocenter (in acute triangle):

$$[DEF]=\frac{1}{4R^2}(R^2-OH^2)F\overset{(10)}{=}\frac{1}{4R^2}(R^2-9R^2+a^2+b^2+c^2)F=$$

$$=\frac{1}{4R^2}(a^2+b^2+c^2-8R^2)F$$

4b. Pedal triangle of symmedian point:

$$[DEF]=\frac{1}{4R^2}(R^2-OK^2)F\overset{(11)}{=}\frac{1}{4R^2}\left(R^2-R^2+\frac{3a^2b^2c^2}{(a^2+b^2+c^2)^2}\right)F$$

$$[DEF]=\frac{3a^2b^2c^2}{4R^2(a^2+b^2+c^2)^2}\cdot F$$

5b. Pedal triangle of Gergonne's point:

$$[DEF]=\frac{1}{4R^2}(R^2-O\Gamma^2)F\overset{(12)}{=}\frac{1}{4R^2}\left(R^2-R^2+\frac{4rs^2(R+r)}{(4R+r)^2}\right)F$$

$$[DEF]=\frac{rs^2(R+r)F}{R^2(4R+r)^2}$$

6b. Pedal triangle of Nagel's point:

$$[DEF]=\frac{1}{4R^2}(R^2-ON^2)F\overset{(13)}{=}\frac{1}{4R^2}(R^2-(R-2r)^2)F=$$

$$[DEF]=\frac{1}{4R^2}(R^2-R^2+4Rr+4r^2)F,[DEF]=\frac{r(R+r)F}{R^2}$$

Reference:

[1] Marin Chirciu, ***Inequalities in triangle***. Paralela 45 Publishing House, Pitesti, Romania, 2018, ISBN 978-973-47-2692-9

[2] www.cut-the-knot.org/Pedaltriangle/shtml

[3] **Romanian Mathematical Magazine** – www.ssmrmh.ro

REFINEMENT FOR SOME GEOMETRICAL INEQUALITIES

By Mihály Bencze, Daniel Sitaru – Romania

Abstract. In this paper we present some refinement for the classical geometrical inequality.

Theorem 1. If $x,y,z>0$ then:

$$\sum xy\le\sum F(x^2,y^2)\le\sum x^2$$

where $F(x,y)=\frac{1}{e}\left(\frac{y^y}{x^x}\right)^{\frac{1}{y-x}}$.

Proof. Using Hadamard – Hermite inequality:

$$f\left(\frac{x+y}{2}\right) \le \frac{1}{y-x}\int_x^y f(t)\,dt \le \frac{f(x)+f(y)}{2}$$

$(x, y > 0)$ for the convex function $f(t) = -\ln t$. We obtain:

$$-\ln\left(\frac{x+y}{2}\right) \le -\frac{(y\ln y - y)-(x\ln x - x)}{y-x} \le \frac{-\ln x - \ln y}{2} \text{ or}$$

$$\sqrt{xy} \le \frac{1}{e}\left(\frac{y^y}{x^x}\right)^{\frac{1}{y-x}} = F(x,y) \le \frac{x+y}{2}$$

or $xy \le F(x^2, y^2) \le \frac{x^2+y^2}{2}$ and

$$\sum xy \le \sum F(x^2, y^2) \le \sum x^2$$

Corollary 1.1. In all ΔABC holds:

1) $\boldsymbol{s^2 + r^2 + 4Rr \le \sum F(a^2, b^2) \le 2(s^2 - r^2 - 4Rr)}$

a refinement of the inequality $s^2 \ge 3r(4Rr + r)$.

2) $\boldsymbol{r(4R + r) \le \sum f\left((s-a)^2, (s-b)^2\right) \le s^2 - 2r^2 - 8Rr}$

a refinement of the inequality $s^2 \ge 3r(4R + r)$.

3) $\boldsymbol{s^2 \le \sum F(r_a^2, r_b^2) \le (4R + r)^2 - 2s^2}$**,** a refinement of the inequality $4R + r \ge s\sqrt{3}$.

4) $\boldsymbol{\frac{s^2+r^2-8Rr}{16R^2} \le \sum F\left(\sin^4\frac{A}{2}, \sin^4\frac{B}{2}\right) \le \frac{8R^2+r^2-s^2}{8R^2}}$

a refinement of the inequality $4R + r \ge s\sqrt{3}$.

5) $\boldsymbol{\frac{s^2+(4R+r)^2}{16R^2} \le \sum F\left(\cos^4\frac{A}{2}, \cos^4\frac{B}{2}\right) \le \frac{(4R+r)^2-s^2}{8R^2}}$, a refinement of the inequality $4R + r \ge s\sqrt{3}$.

Theorem 2. If $x, y, z > 0$ then: $8xyz \le G(x,y,z) \le (x+y)(y+z)(z+x)$

where $G(x,y,z) = 8F(x,y)F(y,z)F(z,x)$.

Proof. We multiply the inequality

$$\sqrt{xy} \le F(x,y) \le \frac{x+y}{2}$$

Corollary 2.1 In all ΔABC the inequalities hold:

1) $\boldsymbol{32sRr \le G(a,b,c) \le 2s(s^2 + r^2 + 2Rr)}$, **a refinement of the inequality** $\boldsymbol{s^2 + r^2 \ge 14Rr}$.

2) $\boldsymbol{8sr^2 \le G(s-a, s-b, s-c) \le 4sRr}$ a refinement of Euler's $R \ge 2r$ inequality:

3) $\boldsymbol{8s^2r \le G(r_a, r_b, r_c) \le 4s^2R}$ a refinement of Euler's $R \ge 2r$ inequality.

4) $\frac{r^2}{2R^2} \leq G\left(\sin^2\frac{A}{2}, \sin^2\frac{B}{2}, \sin^2\frac{C}{2}\right) \leq \frac{(2R-r)(s^2+r^2-8Rr)-2Rr^2}{32R^3}$

a refinement of the inequality $18Rr^2 \leq (2R-r)(s^2+r^2-8Rr)$.

5) $\frac{s^2}{2R^2} \leq G\left(\cos^2\frac{A}{2}, \cos^2\frac{B}{2}, \cos^2\frac{C}{2}\right) \leq \frac{(4R+r)^3+s^2(2R+r)}{32R^3}$

a refinement of the inequality $s^2(14R-r) \leq (4Rr+r)^3$.

Reference:

[1] **Octogon Mathematical Magazine** (1992-2021)

[2] **Romanian Mathematical Magazine** , www.ssmrmh.ro

ABOUT A RMM INEQUALITY-II

By Marin Chirciu – Romania

1) If $a, b, c, m, n > 0$ then:

$$\frac{a^2}{ma+n\sqrt{bc}} + \frac{b^2}{mb+n\sqrt{ca}} + \frac{c^2}{mc+n\sqrt{ab}} \geq \frac{a+b+c}{m+n}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution: *Using means inequality and Berström we obtain:*

$$\sum \frac{a^2}{ma+n\sqrt{bc}} \geq \sum \frac{a^2}{ma+n\cdot\frac{b+c}{2}} \geq \frac{(\sum a)^2}{\sum\left(ma+n\cdot\frac{b+c}{2}\right)} = \frac{(\sum a)^2}{(m+n)\sum a} = \frac{a+b+c}{m+n}$$

Equality holds if and only if $a=b=c$.

Remark.*The inequality can be developed.*

2). If $a, b, c, m, n > 0$ then:

$$\frac{a^3}{ma+n\sqrt{bc}} + \frac{b^3}{mb+n\sqrt{ca}} + \frac{c^2}{mc+n\sqrt{ab}} \geq \frac{(a+b+c)^2}{3(m+n)}$$

Proposed by Marin Chirciu – Romania

Solution *Using means inequality and Hölder's inequality we obtain:*

$$\sum \frac{a^3}{ma+n\sqrt{bc}} \geq \sum \frac{a^3}{ma+n\cdot\frac{b+c}{2}} \geq \frac{(\sum a)^3}{3\sum\left(ma+n\cdot\frac{b+c}{2}\right)} = \frac{(a+b+c)^2}{3(m+n)}$$

Equality holds if and only if $a=b=c$.

3) If $a, b, c, m, n > 0$ then:

$$\frac{a^4}{ma+n\sqrt{bc}} + \frac{b^4}{mb+n\sqrt{ca}} + \frac{c^4}{mc+n\sqrt{ab}} \geq \frac{(a+b+c)^3}{9(m+n)}$$

Proposed by Marin Chirciu – Romania

Solution *Using means inequality and Hölder's inequality we obtain:*

$\sum\frac{a^4}{ma+n\sqrt{bc}}\ge\sum\frac{a^4}{ma+n\cdot\frac{b+c}{2}}\ge\frac{(\sum a)^4}{9\sum\left(ma+n\cdot\frac{b+c}{2}\right)}=\frac{(\sum a)}{9(m+n)\sum a}=\frac{(a+b+c)^3}{9(m+n)}$. *Equality holds if and only if*

$$a=b=c.$$

4) If $a,b,c,m,n>0$ and $k\ge 2,k\in\mathbb{N}$ then:

$$\frac{a^k}{ma+n\sqrt{bc}}+\frac{b^k}{mb+n\sqrt{ca}}+\frac{c^k}{mc+n\sqrt{ab}}\ge\frac{(a+b+c)^{k-1}}{3^{k-2}(m+n)}$$

Proposed by Marin Chirciu – Romania

Solution: *Using means inequality and Hölder's inequality we obtain:*

$\sum\frac{a^k}{ma+n\sqrt{bc}}\ge\sum\frac{a^k}{ma+n\cdot\frac{b+c}{2}}\ge\frac{(\sum a)^k}{3^{k-2}\sum\left(ma+n\cdot\frac{b+c}{2}\right)}=\frac{(\sum a)^k}{3^{k-2}(m+n)\sum a}=\frac{(a+b+c)^{k-1}}{3^{k-2}(m+n)}$. *Equality holds if and only if* $a=b=c$.

Note.For $k=2$ we obtain VIII.22, from RMM-24, Spring Edition 2020, D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Reference:

ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

ABOUT A RMM INEQUALITY-V

By Marin Chirciu-Romania

1) If $a,b>0$ then:

$$\sqrt[3]{\frac{a^3+b^3}{2}}\sqrt[4]{\frac{a^4+b^4}{2}}\sqrt[5]{\frac{a^5+b^5}{2}}\le\frac{a^5+b^5}{a^2+b^2}$$

Proposed by Daniel Sitaru-Romania

Solution: Using power means inequality:

If $x_1,x_2,\dots,x_n>0,r\ge s>0$ then:

$$\left(\frac{x_1^r+x_2^r+\cdots+x_n^r}{n}\right)^{\frac{1}{r}}\ge\left(\frac{x_1^s+x_2^s+\cdots+x_n^s}{n}\right)^{\frac{1}{s}}\ge\sqrt[n]{x_1x_2\dots x_n}$$

$$\sqrt[r]{\frac{x_1^r+x_2^r+\cdots+x_n^r}{n}}\ge\sqrt[s]{\frac{x_1^s+x_2^s+\cdots+x_n^s}{n}}\ge\sqrt[n]{x_1x_2\dots x_n},;r,s\in\mathbb{N},r\ge s\ge 2$$

We get: $\sqrt[3]{\frac{a^3+b^3}{2}}\le\sqrt[4]{\frac{a^4+b^4}{2}}\le\sqrt[5]{\frac{a^5+b^5}{2}}$, thus

$$LHS=\sqrt[3]{\frac{a^3+b^3}{2}}\sqrt[4]{\frac{a^4+b^4}{2}}\sqrt[5]{\frac{a^5+b^5}{2}}\le\sqrt[5]{\frac{a^5+b^5}{2}}\sqrt[5]{\frac{a^5+b^5}{2}}\sqrt[5]{\frac{a^5+b^5}{2}}=$$

$$\le\left(\sqrt[5]{\frac{a^5+b^5}{2}}\right)^3\overset{(1)}{\le}\frac{a^5+b^5}{a^2+b^2}=RHD,$$

Where $\left(\sqrt[5]{\frac{a^5+b^5}{2}}\right)^3\le\frac{a^5+b^5}{a^2+b^2}\Leftrightarrow\left(\frac{a^5+b^5}{2}\right)^3\le\left(\frac{a^5+b^5}{a^2+b^2}\right)^5\Leftrightarrow$

$(a^2+b^2)^5\le 8(a^5+b^5)^2$which is true from Holder inequality:

$$8(a^5+b^5)^2=(1+1)(1+1)(1+1)(a^5+b^5)(a^5+b^5)\ge(a^2+b^2)^5$$

Equality if and only if $a=b$. **Remark.** Inequality can be developed.

2) If $\boldsymbol{a},\boldsymbol{b}>0,n\in\mathbb{N},\boldsymbol{n}\ge\mathbf{2}$ then:

$$\sqrt[n]{\frac{\boldsymbol{a}^{\boldsymbol{n}}+\boldsymbol{b}^{\boldsymbol{n}}}{\mathbf{2}}}\sqrt[n+1]{\frac{\boldsymbol{a}^{\boldsymbol{n}+\mathbf{1}}+\boldsymbol{b}^{\boldsymbol{n}+\mathbf{1}}}{\mathbf{2}}}\sqrt[n+2]{\frac{\boldsymbol{a}^{\boldsymbol{n}+\mathbf{2}}+\boldsymbol{b}^{\boldsymbol{n}+\mathbf{2}}}{\mathbf{2}}}\le\frac{\boldsymbol{a}^{\boldsymbol{n}+\mathbf{2}}+\boldsymbol{b}^{\boldsymbol{n}+\mathbf{2}}}{\boldsymbol{a}^{\boldsymbol{n}-\mathbf{1}}+\boldsymbol{b}^{\boldsymbol{n}-\mathbf{1}}}$$

Proposed by Marin Chirciu-Romania

Solution: Using power means inequality:

If $x_1,x_2,\dots,x_n>0,r\ge s>0$ then:

$$\left(\frac{x_1^r+x_2^r+\cdots+x_n^r}{n}\right)^{\frac{1}{r}}\ge\left(\frac{x_1^s+x_2^s+\cdots+x_n^s}{n}\right)^{\frac{1}{s}}\ge\sqrt[n]{x_1x_2\dots x_n}$$

$$\sqrt[r]{\frac{x_1^r+x_2^r+\cdots+x_n^r}{n}}\ge\sqrt[s]{\frac{x_1^s+x_2^s+\cdots+x_n^s}{n}}\ge\sqrt[n]{x_1x_2\dots x_n},;r,s\in\mathbb{N},r\ge s\ge 2$$

We get: $\sqrt[n]{\frac{a^n+b^n}{2}}\le\sqrt[n+1]{\frac{a^{n+1}+b^{n+1}}{2}}\le\sqrt[n+2]{\frac{a^{n+2}+b^{n+2}}{2}}$, thus

$$LHS=\sqrt[n]{\frac{a^n+b^n}{2}}\sqrt[n+1]{\frac{a^{n+1}+b^{n+1}}{2}}\sqrt[n+2]{\frac{a^{n+2}+b^{n+2}}{2}}\le$$

$$\le\sqrt[n+2]{\frac{a^{n+2}+b^{n+2}}{2}}\sqrt[n+2]{\frac{a^{n+2}+b^{n+2}}{2}}\sqrt[n+2]{\frac{a^{n+2}+b^{n+2}}{2}}=$$

$$\le\left(\sqrt[n+2]{\frac{a^{n+2}+b^{n+2}}{2}}\right)^3\overset{(1)}{\le}\frac{a^{n+2}+b^{n+2}}{a^{n-1}+b^{n-1}}=RHD,$$

Where $\left(\sqrt[n+2]{\frac{a^{n+2}+b^{n+2}}{2}}\right)^3 \le \frac{a^{n+2}+b^{n+2}}{a^{n-1}+b^{n-1}} \Leftrightarrow \left(\frac{a^{n+2}+b^{n+2}}{2}\right)^3 \le \left(\frac{a^{n+2}+b^{n+2}}{a^{n-1}+b^{n-1}}\right)^5 \Leftrightarrow$

$(a^{n-1}+b^{n-1})^{n+2} \le 8(a^{n+2}+b^{n+2})^{n-1}$which is true from Holder inequality:

$$8(a^{n+2}+b^{n+2})^{n-1} = (1+1)(1+1)(1+1)(a^{n+2}+b^{n+2})(a^{n+2}+b^{n+2}) \ge$$

$$\ge \left(\sqrt[n+2]{a^{(n+2)(n-1)}} + \sqrt[n+2]{b^{(n+2)(n-1)}}\right)^{n+2} \ge (a^{n-1}+b^{n-1})^{n+1}$$

Equality if and only if $a=b$.

Note. For $n=2$ it follows Cyclic Inequality-730 proposed by Daniel Sitaru-R.M.M 4/2020.

Remark. Inequality can be developed for three variables.

3) If $a,b,c>0, n\in\mathbb{N}, n\ge 2$ then:

$$\sqrt[n]{\frac{a^n+b^n+c^n}{3}}\sqrt[n+1]{\frac{a^{n+1}+b^{n+1}+c^{n+1}}{3}}\sqrt[n+2]{\frac{a^{n+2}+b^{n+2}+c^{n+2}}{3}}$$

$$\le \frac{a^{n+2}+b^{n+2}+c^{n+2}}{a^{n-1}+b^{n-1}+c^{n-1}}$$

Proposed by Marin Chirciu-Romania

Solution: Using power means inequality:

If $x_1, x_2, \dots, x_n > 0, r \ge s > 0$ then:

$$\left(\frac{x_1^r+x_2^r+\dots+x_n^r}{n}\right)^{\frac{1}{r}} \ge \left(\frac{x_1^s+x_2^s+\dots+x_n^s}{n}\right)^{\frac{1}{s}} \ge \sqrt[n]{x_1x_2\dots x_n}$$

$$\sqrt[r]{\frac{x_1^r+x_2^r+\dots+x_n^r}{n}} \ge \sqrt[s]{\frac{x_1^s+x_2^s+\dots+x_n^s}{n}} \ge \sqrt[n]{x_1x_2\dots x_n},; r,s\in\mathbb{N}, r\ge s\ge 2$$

We get: $\sqrt[n]{\frac{a^n+b^n+c^n}{3}} \le \sqrt[n+1]{\frac{a^{n+1}+b^{n+1}+c^{n+1}}{3}} \le \sqrt[n+2]{\frac{a^{n+2}+b^{n+2}+c^{n+2}}{3}}$

$$LHS = \sqrt[n]{\frac{a^n+b^n+c^n}{3}}\sqrt[n+1]{\frac{a^{n+1}+b^{n+1}+c^{n+1}}{3}}\sqrt[n+2]{\frac{a^{n+2}+b^{n+2}+c^{n+2}}{3}} \le$$

$$\le \sqrt[n+2]{\frac{a^{n+2}+b^{n+2}+c^{n+2}}{3}}\sqrt[n+2]{\frac{a^{n+2}+b^{n+2}+c^{n+2}}{3}}\sqrt[n+2]{\frac{a^{n+2}+b^{n+2}+c^{n+2}}{3}} =$$

$$= \left(\sqrt[n+2]{\frac{a^{n+2}+b^{n+2}+c^{n+2}}{3}}\right)^3 \overset{(1)}{\le} \frac{a^{n+2}+b^{n+2}+c^{n+2}}{a^{n-1}+b^{n-1}+c^{n-1}} = RHD$$

$$(1) \Leftrightarrow \left(\sqrt[n+2]{\frac{a^{n+2}+b^{n+2}+c^{n+2}}{3}}\right)^3 \le \frac{a^{n+2}+b^{n+2}+c^{n+2}}{a^{n-1}+b^{n-1}+c^{n-1}}$$

$$\Leftrightarrow \left(\frac{a^{n+2}+b^{n+2}+c^{n+2}}{3}\right)^3 \le \left(\frac{a^{n+2}+b^{n+2}+c^{n+2}}{a^{n-1}+b^{n-1}+c^{n-1}}\right)^{n+2}$$

$$\Leftrightarrow (a^{n-1}+b^{n-1}+c^{n-1})^{n+2} \le 27(a^{n+2}+b^{n+2}+c^{n+2})^{n-1}$$

Which follows from Holder inequality:

$$27(a^{n+2}+b^{n+2}+c^{n+2})^{n-1} =$$

$$= (1+1+1)(1+1+1)(1+1+1)(a^{n+2}+b^{n+2}+c^{n+2})\dots(a^{n+2}+b^{n+2}+c^{n+2})$$

$$\ge \left(\sqrt[n+2]{a^{(n+2)(n-1)}}+\sqrt[n+2]{b^{(n+2)(n-1)}}+\sqrt[n+2]{c^{(n+2)(n-1)}}\right)^{n+2}$$

$$= (a^{n-1}+b^{n-1}+c^{n-1})^{n+2}$$

Equality holds if and only if $a=b=c$.

REFERENCE:

ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

ABOUT A RMM INEQUALITY-VI

By Marin Chirciu-Romania

1) In $\Delta ABC, o_a$ −circumcevian, the following relationship holds:

$$\frac{8r}{R}-1 \le \sum_{cyc}\frac{h_a}{o_a} \le 1+\frac{3r}{R}+\frac{2r^2}{R^2}$$

Proposed by Daniel Sitaru-Romania

Solution. **Lemma. 2) In $\Delta ABC, o_a$ −circumcevian, the following relationship holds:**

$$\sum_{cyc}\frac{h_a}{o_a} = \frac{s^2+r^2+2Rr-2R^2}{2R^2}$$

Proof. Using $o_a = \frac{h_a}{cos(B-C)}, \sum cos(B-C) = \frac{s^2+r^2+2Rr-2R^2}{2R^2}$, it follows that:

$$\sum_{cyc}\frac{h_a}{o_a} = \sum_{cyc}\frac{h_a}{\frac{h_a}{cos(B-C)}} = \frac{s^2+r^2+2Rr-2R^2}{2R^2}$$

Let's get back to the main problem. For RHS using Lemma, we get:

$$\frac{s^2+r^2+2Rr-2R^2}{2R^2} \le 1+\frac{3r}{R}+\frac{2r^2}{R^2} \Leftrightarrow s^2 \le 4R^2+4Rr+3r^2(Gerretsen).$$

Equality holds if and only if triangle is equilateral. For LHS using Lemma, we get:

$\frac{s^2+r^2+2Rr-2R^2}{2R^2} \geq \frac{8r}{R} - 1 \Leftrightarrow s^2 \geq 14Rr - r^2$, which it follows from

$s^2 \geq 16Rr - 5r^2 (Gerretsen)$. Remains to prove $16Rr - 5r^2 \geq 14Rr - r^2$

$\Leftrightarrow R \geq 2r(Euler)$. Equality holds if and only if triangle is equilateral.

3) In $\Delta ABC, o_a -$ circumcevian, the following relationship holds:

$$\mathbf{1 + \frac{11r}{R} - \frac{2r^2}{R^2} \leq \sum_{cyc} \frac{h_b + h_c}{o_a} \leq 3 + \frac{5r}{R} + \frac{2r^2}{R^2}}$$

Proposed by Marin Chirciu-Romania

Solution. **Lemma. 4) In $\Delta ABC, o_a -$circumcevian, the following relationship holds:**

$$\sum_{cyc} \frac{h_b + h_c}{o_a} = \frac{1}{2R^2}(s^2 + r^2 + 6Rr + 2R^2)$$

Proof. Using $o_a = \frac{h_a}{cos(B-C)}, \sum cos(B - C) = \frac{s^2+r^2+2Rr-2R^2}{2R^2}$, it follows that:

$$\sum_{cyc} \frac{h_b + h_c}{o_a} = \sum_{cyc} \frac{h_b + h_c}{\frac{h_a}{cos(B-C)}} = \sum_{cyc} \frac{\frac{2F}{b} + \frac{2F}{c}}{\frac{\frac{2F}{a}}{cos(B-C)}} = \sum_{cyc} \frac{a(b+c)}{bc} cos(B - C) =$$

$$= \frac{1}{abc} \sum_{cyc} a^2(b + c)cos(B - C) = \frac{1}{4Rrs} \sum_{cyc} a^2(b + c)cos(B - C) =$$

$$= \frac{1}{4Rrs} \cdot \frac{2sr}{R}(s^2 + r^2 + 6Rr + 2R^2) = \frac{1}{2R^2}(s^2 + r^2 + 6Rr + 2R^2)$$

Which it follows from: $\sum_{cyc} a^2(b + c)cos(B - C) =$

$$= \sum_{cyc} a^2(a + b + c - a)cos(B - C) = \sum_{cyc} a^2(2s - a)cos(B - C) =$$

$$= 2s \sum_{cyc} a^2 cos(B - C) - \sum_{cyc} a^3 cos(B - C) = 2s \cdot \frac{r}{R}(s^2 + r^2 + 6Rr + 8R^2) - 3abc =$$

$$= 2s \cdot \frac{r}{R}(s^2 + r^2 + 6Rr + 8R^2) - 3 \cdot 4Rrs = \frac{2sr}{R}(s^2 + r^2 + 6Rr + 2R^2)$$

$$\sum_{cyc} a^2 \, (b + c)cos(B - C) = \frac{2sr}{R}(s^2 + r^2 + 6Rr + 2R^2)$$

$$\sum_{cyc} a^3 cos(B - C) = 3abc; \sum_{cyc} a^2 cos(B - C) = \frac{r}{R}(s^2 + r^2 + 6Rr + 8R^2)$$

Let's get back to the main problem. For RHS using Lemma, we get:

$$\frac{1}{2R^2}(s^2 + r^2 + 6Rr + 2R^2) \overset{Gerretsen}{\leq} \frac{1}{2R^2}(4R^2 + 4Rr + 3r^2 + r^2 + 6Rr + 2R^2)$$

$$= \frac{6R^2 + 10Rr + 4r^2}{2R^2} = \frac{3R^2 + 5Rr + 2r^2}{R^2} = 3 + \frac{5r}{R} + \frac{2r^2}{R^2}$$

Equality holds if and only if triangle is equilateral. For LHS using Lemma, we get:

$$\frac{1}{2R^2}(s^2 + r^2 + 6Rr + 2R^2) \overset{Gerretsen}{\geq} \frac{1}{2R^2}(16Rr - 5r^2 + r^2 + 6Rr + 2R^2) =$$

$$= \frac{2R^2 + 22Rr - 4r^2}{2R^2} = \frac{R^2 + 11Rr - 2r^2}{R^2} = 1 + \frac{11r}{R} - \frac{2r^2}{R^2}.$$

Equality holds if and only if triangle is equilateral.

5) In $\Delta ABC, o_a -$ circumcevian, the following relationship holds:

$$\mathbf{1 + \frac{11r}{R} - \frac{2r^2}{R^2} \leq \sum_{cyc} \frac{r_a}{o_a} \leq \frac{2R}{r} + \frac{4r}{R} - 3}$$

Proposed by Marin Chirciu-Romania

Solution. **Lemma. 6) In $\Delta ABC, o_a -$ circumcevian, the following relationship holds:**

$$\sum_{cyc} \frac{r_a}{o_a} = \frac{4R^2 + 10Rr + 3r^2 - s^2}{2Rr}$$

Proof. Using $o_a = \frac{h_a}{cos(B-C)}, \sum cos(B-C) = \frac{s^2+r^2+2Rr-2R^2}{2R^2}$, it follows that:

$$\sum_{cyc} \frac{r_a}{o_a} = \sum_{cyc} \frac{r_a}{\frac{h_a}{cos(B-C)}} = \sum_{cyc} \frac{r_a}{h_a} cos(B-C) = \sum_{cyc} \frac{\frac{F}{s-a}}{\frac{2F}{a}} cos(B-C) =$$

$$= \frac{1}{2}\sum_{cyc} \frac{a}{s-a} cos(B-C) = \frac{1}{2}\sum_{cyc} \frac{a}{s-a} cos(B-C) =$$

$$= \frac{1}{2} \cdot \frac{\sum a(s-b)(s-c)cos(B-C)}{\prod(s-a)} = \frac{1}{2} \cdot \frac{\frac{sr}{R}(4R^2 + 10R + 3r^2 - s^2)}{r^2 s} =$$

$$= \frac{1}{2Rr}(4R^2 + 10Rr + 3r^2 - s^2) = \frac{4R^2 + 10Rr + 3r^2 - s^2}{2Rr}$$

$$\sum_{cyc} a(s-b)(s-c)cos(B-C) = \sum_{cyc} a(s^2 - sc - sb + bc)cos(B-C) =$$

$$= s^2 \cdot \frac{4rs}{R} - s \cdot \frac{r}{R}(7s^2 - r^2 - 6Rr - 8R^2) + 4Rrs \cdot \frac{s^2 + r^2 + 2Rr - 2R^2}{2R^2} =$$

$$= s^2 \cdot \frac{4sr}{R} - s \cdot \frac{r}{R}(7s^2 - r^2 - 6Rr - 8R^2) + 4Rrs \cdot \frac{s^2 + r^2 + 2Rr - 2R^2}{2R^2} =$$

$= \frac{sr}{R}(4R^2 + 10R + 3r^2 - s^2)$, which it follows from:

$$\sum_{cyc} acos(B-C)=\frac{4sr}{R};\ \sum_{cyc} a^2cos(B-C)=\frac{r}{R}(s^2+r^2+6r+8R^2)$$

$$\sum_{cyc} a(b+c)cos(B-C)=\frac{r}{R}(7s^2-r^2-6Rr-8R^2),$$

$$\sum_{cyc} cos(B-C)=\frac{s^2+r^2+2Rr-2R^2}{2R^2}$$

Let's get back to the main problem. For RHS using Lemma, it follows that:

$$\frac{4R^2+10Rr+3r^2-s^2}{2Rr}\overset{Gerretsen}{\leq}\frac{4R^2+10Rr+3r^2-16Rr+5r^2}{2r}=$$

$$=\frac{4R^2-6Rr+8r^2}{2Rr}=\frac{R^2-3Rr+4r^2}{Rr}=\frac{2R}{r}+\frac{4r}{R}-3\Leftrightarrow$$

$s^2\leq 4R^2+4Rr+3r^2(Gerretsen)$. Equality holds if and only if triangle is equilateral. For LHS using Lemma, we get:

$\frac{4R^2+10Rr+3r^2-s^2}{2Rr}\overset{Gerretsen}{\geq}\frac{4R^2+10Rr+3r^2-4R^2-4Rr-3r^2}{2Rr}=\frac{6Rr}{2Rr}=3$. Equality if and only if triangle is equilateral.

7) In $\Delta ABC, o_a-$ circumcevian, the following relationship holds:

$$\mathbf{11-\frac{2r}{R}\leq\sum_{cyc}\frac{r_b+r_c}{o_a}\leq 5+\frac{2r}{R}}$$

Proposed by Marin Chirciu-Romania

Solution. **8) Lemma. In $\Delta ABC, o_a-$ circumcevian, the following relationship holds:**

$$\sum_{cyc}\frac{r_b+r_c}{o_a}=\frac{s^2+r^2+6Rr-4R^2}{2Rr}$$

Proof. Using $o_a=\frac{h_a}{cos(B-C)},\sum cos(B-C)=\frac{s^2+r^2+2Rr-2R^2}{2R^2}$, it follows that:

$$\sum_{cyc}\frac{r_b+r_c}{o_a}=\sum_{cyc}\frac{r_b+r_c}{\frac{h_a}{cos(B-C)}}=\sum_{cyc}\frac{r_b+r_c}{h_a}cos(B-C)=$$

$$=\sum_{cyc}\frac{\frac{F}{s-b}+\frac{F}{s-c}}{\frac{2F}{a}}cos(B-C)=\frac{1}{2}\sum_{cyc}\frac{a^2}{(s-B)(s-c)}cos(B-C)=$$

$$=\frac{1}{2}\cdot\frac{\prod a^2(s-a)cos(B-C)}{\prod(s-a)}=\frac{1}{2}\cdot\frac{\frac{sr}{R}(s^2+r^2+6Rr-4R^2)}{r^2s}=$$

$$=\frac{1}{2Rr}(s^2+r^2+6Rr-4R^2)=\frac{s^2+r^2+6Rr-4R^2}{2Rr}$$

Which follows from:

$$\sum_{cyc} a^2(s-a)cos(B-C) = s\sum_{cyc} a^2cos(B-C) - \sum_{cyc} a^3cos(B-C) =$$

$$= s\cdot\frac{r}{R}(s^2+r^2+6Rr+8R^2) - 3abc = s\cdot\frac{r}{R}(s^2+r^2+6Rr+8R^2) - 3\cdot 4Rrs =$$

$= \frac{sr}{R}(s^2+r^2+6Rr-4R^2)$, which is true from:

$$\sum_{cyc} a^2cos(B-C) = \frac{r}{R}(s^2+r^2+6r+8R^2);\ \sum_{cyc} a^3cos(B-C) = 3abc.$$

Let's get back to the main problem. For RHS using Lemma, we get:

$$\frac{s^2+r^2+6Rr-4R^2}{2Rr} \overset{Gerretsen}{\leq} \frac{4R^2+4Rr+3r^2+r^2+6Rr-4R^2}{2Rr} =$$

$= \frac{5R+2r}{R} = 5 + \frac{2r}{R}$. Equality holds if and only if triangle is equilateral.

For LHS using Lemma, it follows that: $\frac{s^2+r^2+6Rr-4R^2}{2Rr} \overset{Gerretsen}{\geq} \frac{16Rr-5r^2+r^2+6Rr-4R^2}{2Rr} =$

$$= \frac{22Rr-4r^2-4R^2}{2Rr} = \frac{11Rr-2r^2-2R^2}{Rr} = 11 - \frac{2r}{R} - \frac{2R}{r}.$$

Equality holds if and only if triangle is equilateral.

9) In $\Delta ABC, o_a$ –circumcevian, the following relationship holds:

$$\frac{2r^2}{R^2} + \frac{3r}{R} + 4 \leq \sum_{cyc}\frac{h_a+r_a}{o_a} \leq \frac{2R}{r} + \frac{13r}{R} - \frac{2r^2}{R^2} - 4$$

Proposed by Marin Chirciu-Romania

Solution. **Lemma 10) In $\Delta ABC, o_a$ –circumcevian, the following relationship holds:**

$$\sum_{cyc}\frac{h_a+r_a}{o_a} = \frac{4R^3+8R^2r+5Rr^2+r^3+(r-R)s^2}{2R^2r}$$

Proof. Using $o_a = \frac{h_a}{cos(B-C)}, \sum cos(B-C) = \frac{s^2+r^2+2Rr-2R^2}{2R^2}$, it follows that:

$$\sum_{cyc}\frac{h_a+r_a}{o_a} = \sum_{cyc}\frac{h_a+r_a}{\frac{h_a}{cos(B-C)}} = \sum_{cyc}\frac{h_a+r_a}{h_a}cos(B-C) =$$

$$= \sum_{cyc}\frac{\frac{2F}{a}+\frac{F}{s-a}}{\frac{2F}{a}}cos(B-C) = \frac{1}{2}\sum_{cyc}\frac{b+c}{s-a}cos(B-C) =$$

$$= \frac{1}{2}\sum_{cyc}\frac{b+c}{s-a}cos(B-C) = \frac{1}{2}\cdot\frac{\sum(b+c)(s-b)(s-c)cos(B-C)}{\prod(s-a)} =$$

$$= s^2\sum_{cyc} a(b+c)cos(B-C) - s\sum_{cyc} a(b+c)^2cos(B-C)$$

$$+ \sum_{cyc} bc(b+c)cos(B-C)$$

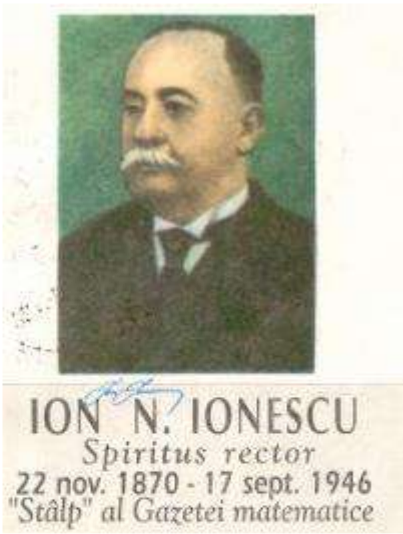
Let's get back to the main problem. For RHS using Lemma, it follows that:

$$\frac{4R^3+8R^2r+5Rr^2+r^3+(r-R)s^2}{2R^2r} \overset{Gerretsen}{\leq}$$

$$\frac{4R^3+8R^2r+5Rr^2+r^3+(r-R)(16Rr-5r^2)}{2R^2r}$$
$$=\frac{4R^3+8R^2r+5Rr^2+r^3+16Rr^2-16R^2r-5r^3+5Rr^2}{2R^2r}=$$
$$=\frac{4R^3-8R^2r+26Rr^2-4r^3}{2R^2r}=\frac{2R^3-4R^2r+13Rr^2-3r^3}{R^2r}=$$
$=\frac{2R}{r}+\frac{13r}{R}-\frac{2r^2}{R^2}-4$. For LHS using Lemma, it follows that:
$$\frac{4R^3+8R^2r+5Rr^2+r^3+(r-R)s^2}{2R^2r}\overset{Gerretsen}{\geq}$$
$$\frac{4R^3+8R^2r+5Rr^2+r^3+(r-R)(4R^2+4Rr+3r^2)}{2R^2r}$$
$$=\frac{4R^3+8R^2r+5Rr^2+r^3-4R^3-4R^2r-3Rr^2+4R^2r+4Rr^2+3r^3}{2R^2r}=$$
$$=\frac{8R^2r+6Rr^2+4r^3}{2R^2r}=\frac{4R^2+3Rr+2r^2}{R^2}=\frac{2r^2}{R^2}+\frac{3r}{R}+4.$$
Equality holds if and only if triangle is equilateral.

Reference:

ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro



TRIBUTE TO ION IONESCU – WHO DISCOVERED WITH 22 YEARS BEFORE WEITZENBÖCK THE INEQUALITY:

$$\boldsymbol{a^2+b^2+c^2\geq 4\sqrt{3}S}$$

By D.M.Bătinețu-Giurgiu,Neculai Stanciu-Romania

The authors of this proposed problem demonstrated in Romanian Mathematical Gazette , No. 1/2013, pp. 1-10, that the *Weitzenböck*'s inequality must be named the *Ionescu-*

Weitzenböck's inequality. Our proof is based on: Romanian Mathematical Gazette, Vol. III (15 September 1897 – 15 August 1898), No. 2 , 15 October 1897, on page 52, *Ion Ionescu*, the founder of Romanian Mathematical Gazette, published the problem:*273. Prove that there is no triangle for which the inequality: $4S\sqrt{3} > a^2 + b^2 + c^2$ be satisfied. The solution of the problem 273, apperead in Romanian Mathematical Gazette, Vol. III (15 September 1897 – 15 August 1898), No. 12 , 15 August 1898, on pages 281, 282 and 283. In the year 1919, *Roland Weitzenböck* published in Mathematische Zeitschrift, Vol. 5, No. 1-2, pp. 137-146 the article *Uber eine Ungleichung in der Dreiecksgeometrie*, where he proof that: In any triangle ABC, with usual notations holds the inequality: $a^2 + b^2 + c^2 \geq 4\sqrt{3}S$.

Proposed problem for RMM

If $m \in \mathbb{R}_+$, then in all triangle ABC, with usual notations (i.e. R =circumradius, r =inradius, the lengths of the sides are a, b, c and S = the area of triangle ABC) the following inequality holds:

$$\frac{a^{m+2}}{(b\cdot R + c\cdot r)^m} + \frac{b^{m+2}}{(c\cdot R + a\cdot r)^m} + \frac{c^{m+2}}{(a\cdot R + b\cdot r)^m} \geq \frac{4\sqrt{3}}{(R+r)^m}S$$

D.M. Bătineţu-Giurgiu, ''Matei Basarab'' National College, Bucharest, Romania and **Neculai Stanciu,** ''George Emil Palade'' School, Buzău, Romania

Solution. *We have:*

$$W = \sum_{cyc} \frac{a^{m+2}}{(b\cdot R + c\cdot r)^m} = \sum_{cyc} \frac{a^{2(m+1)}}{(abR + acr)^m} \geq 2^m \cdot \sum_{cyc} \frac{(a^2)^{m+1}}{\left((a^2+b^2)R + (a^2+c^2)r\right)^m},$$

and applying the inequality of J. Radon we obtain:

$$W \geq 2^m \cdot \frac{\left(\sum_{cyc} a^2\right)^{m+1}}{\left(\sum_{cyc} R(a^2+b^2) + \sum_{cyc} r(a^2+c^2)\right)^m} = 2^m \cdot \frac{\left(\sum_{cyc} a^2\right)^{m+1}}{2^m (R+r)^m \cdot \left(\sum_{cyc} a^2\right)^m} = \frac{\sum_{cyc} a^2}{(R+r)^m}.$$

By Ionescu- Weitzenböck's inequality, i.e.$\sum_{cyc} a^2 \geq 4S\sqrt{3}$, *hence* $W \geq \frac{4S\sqrt{3}}{(R+r)^m}$, *and the proof is complete.*

SECLĂMAN'S SEQUENCE REVISITED

Let $(s_n)_{n\geq 0}, s_0 > 1, s_{n+1} = \frac{s_n^2}{s_{n-1}}$ be Seclăman's sequence and

$f: [0,1] \to \mathbb{R}$ continuous and convex function. Find:

$$\Omega = \lim_{n\to\infty} \left(\frac{1}{n} \sum_{k=1}^{n} s_k \left(f\left(\frac{k}{n}\right) - n \int_{\frac{k-1}{n}}^{\frac{k}{n}} f(x)dx \right) \right)$$

By Florică Anastase-Romania

Solution: $s_{n+1} - 1 = \frac{s_n^2 - s_n + 1}{s_n - 1}, \forall n \in \mathbb{N} \to (s_{n+1} - 1)(s_n - 1) = s_n^2 - s_n + 1 > 0$

$$s_0 > 1, \text{from mathematical induction, we get } s_n > 1, \forall n \in \mathbb{N}.$$

$$s_{n+1} - s_n = \frac{s_n}{s_n - 1} > 0 \to (s_n)_{n\geq 0} \nearrow, \text{then } \exists s = \lim_{x\to\infty} s_n \in \mathbb{R} \text{ and from } s_n > 1$$

$$\to s > 1 \text{ and } s^2 - s = s^2 \to s = 0 \text{ (contradiction!)}$$

$$\lim_{n\to\infty} \frac{n}{s_n} \overset{L.C-S}{=} \lim_{n\to\infty} \frac{n+1-n}{s_{n+1}-s_n} = \lim_{n\to\infty} \frac{1}{s_{n+1}-s_n} = \lim_{n\to\infty} \frac{s_n - 1}{s_n} = 1 - \lim_{n\to\infty} \frac{1}{s_n} = 1; (1)$$

How f' have Darboux property and f' increase function, we obtain that f' is continuous function. We can write:

$$\sum_{k=1}^{n} s_k \left(f\left(\frac{k}{n}\right) - n \int_{\frac{k-1}{n}}^{\frac{k}{n}} f(x)\,dx \right) = n\left(\frac{1}{n}\sum_{k=1}^{n} s_k f\left(\frac{k}{n}\right) - \sum_{k=1}^{n} s_k \int_{\frac{k-1}{n}}^{\frac{k}{n}} f(x)dx \right) =$$

$$= n\left(\sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} s_k f\left(\frac{k}{n}\right) - \sum_{k=1}^{n} s_k \int_{\frac{k-1}{n}}^{\frac{k}{n}} f(x)dx \right) = n\left(\sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} s_k \left[f\left(\frac{k}{n}\right) - f(x) \right] dx \right) \overset{M.V.T}{=}$$

$$\overset{M.V.T}{=} n\left(\sum_{k=1}^{n} s_k \int_{\frac{k-1}{n}}^{\frac{k}{n}} \left(\frac{k}{n} - x\right) f'(c_{k,n,x})dx \right) \overset{f'-increase}{\geq}$$

$$= n \cdot \sum_{k=1}^{n} s_k f'\left(\frac{k-1}{n}\right) \int_{\frac{k-1}{n}}^{\frac{k}{n}} \left(\frac{k}{n} - x\right) dx = n \cdot l \cdot \sum_{k=1}^{n} s_k f'\left(\frac{k-1}{n}\right) \cdot \frac{1}{n^2} = \frac{1}{n}\sum_{k=1}^{n} s_k f'\left(\frac{k-1}{n}\right)$$

On the other hand, we have:

$$n\left(\frac{1}{n}\sum_{k=1}^{n} s_k f\left(\frac{k}{n}\right) - \sum_{k=1}^{n} s_k \int_{\frac{k-1}{n}}^{\frac{k}{n}} f(x)dx \right) =$$

$$= n\left(\sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} s_k f\left(\frac{k}{n}\right) dx - \sum_{k=1}^{n} s_k \int_{\frac{k-1}{n}}^{\frac{k}{n}} f(x)dx \right) =$$

$$= n\left(\sum_{k=1}^{n} s_k \left(f\left(\frac{k}{n}\right) - f(x) \right) dx \right) \overset{M.V.T}{=} n\left(\sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} \left(\frac{k}{n} - x\right) f'(c_{k,n,x})dx \right) \overset{f'-increase}{\leq}$$

$$\overset{f'-increase}{\leq} n\sum_{k=1}^{n} s_k f'\left(\frac{k}{n}\right) \int_{\frac{k-1}{n}}^{\frac{k}{n}} \left(\frac{k}{n} - x\right) dx = n\sum_{k=1}^{n} s_k f'\left(\frac{k}{n}\right) \cdot \frac{1}{n^2} = \frac{1}{n}\sum_{k=1}^{n} s_k f'\left(\frac{k}{n}\right)$$

So, it follows that:

$$\frac{1}{n}\sum_{k=1}^{n} s_k f'\left(\frac{k-1}{n}\right) \le \sum_{k=1}^{n} s_k\left(f\left(\frac{k}{n}\right) - n\int_{\frac{k-1}{n}}^{\frac{k}{n}} f(x)\,dx\right) \le \frac{1}{n}\sum_{k=1}^{n} s_k f'\left(\frac{k}{n}\right)$$

$$\frac{1}{n}\cdot\frac{1}{n}\sum_{k=1}^{n} s_k f'\left(\frac{k-1}{n}\right) \le \frac{1}{n}\sum_{k=1}^{n} s_k\left(f\left(\frac{k}{n}\right) - n\int_{\frac{k-1}{n}}^{\frac{k}{n}} f(x)\,dx\right) \le \frac{1}{n}\cdot\frac{1}{n}\sum_{k=1}^{n} s_k f'\left(\frac{k}{n}\right)$$

Therefore,

$$\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n} s_k\left(f\left(\frac{k}{n}\right) - n\int_{\frac{k-1}{n}}^{\frac{k}{n}} f(x)\,dx\right)\right) = \frac{\big(f(1)-f(0)\big)}{2}\cdot\lim_{n\to\infty}\frac{s_n}{n} \overset{(1)}{=} \frac{\big(f(1)-f(0)\big)}{2}$$

Reference:

ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

ABOUT NAGEL'S AND GERGONNE'S CEVIANS-(VI)

By Bogdan Fuștei-Romania

1. In ΔABC, n_a –Nagel's cevian, the following relationship holds:

$$\boldsymbol{n_a n_b + n_b n_c + n_c n_a \ge s^2}$$

$$\boldsymbol{n_a^2 + n_b^2 + n_c^2 = \frac{s^2(5R-r) - r(4R+r)^2}{R}}$$

So, we have the following relationships:

$$(n_a+n_b+n_c)^2 \ge \frac{s^2(5R-r)-r(4R+r)^2}{R} \Leftrightarrow$$

$$\boldsymbol{n_a+n_b+n_c \ge \sqrt{\frac{s^2(5R-r)-r(4R+r)^2}{R}}};\ (\mathbf{1})$$

We want to prove next relationships:

$$\frac{s^2(5R-r)-r(4R+r)^2}{R} \ge s^2\left(4-\frac{2r}{R}\right) \Leftrightarrow \frac{s^2(5R-r)-r(4R+r)^2}{R} \ge \frac{s^2(4R-2r)}{R} \Leftrightarrow$$

$$s^2(5R-r)-r(4R+r)^2 \ge s^2(4R-2r) \Leftrightarrow$$

$$s^2(5R-r-4R+2r) \ge r(4R+r)^2 \Leftrightarrow s^2(R+r) \ge r(4R+r)^2$$

But $s^2 \ge 16Rr - 5r^2$ $(Gerretsen)$, then

$$r(16R-5r)(R+r) \ge r(4R+r)^2 = r(16R^2+8Rr+r^2) \Leftrightarrow$$

$$16R^2 - 5Rr + 16Rr - 5r^2 \ge 16R^2 + 8Rr + r^2 \Leftrightarrow$$

$$-5Rr - 8Rr + 16Rr \ge 6r^2 \Leftrightarrow 3Rr \ge 6r^2 \Leftrightarrow R \ge 2r\ (Euler)$$

So, we get:

$$\boldsymbol{n_a+n_b+n_c\geq s\cdot\sqrt{4-\frac{2r}{R}};(2)}$$

Now, from Blundon-Gerretsen we have: $4R+r\geq s\sqrt{4-\frac{2r}{R}}$ and using $r_a+r_b+r_c=4R+r$, we get:

$$\boldsymbol{\sqrt{(n_a+n_b+n_c)(r_a+r_b+r_c)}\geq s\cdot\sqrt{4-\frac{2r}{R}};(3)}$$

From $R\geq 2r\ (Euler)$ and $n_a+n_b+n_c\geq s\cdot\sqrt{4-\frac{2r}{R}}$, we have:

$$\boldsymbol{n_a+n_b+n_c\geq s\sqrt{3};(4)}$$

2. In $\Delta ABC, n_a$ –Nagel's cevian, g_a –Gergonne's cevian, the following relationship holds:

$$\boldsymbol{|b-c|\geq n_a-g_a}$$

So, we have: $|a-b|+|b-c|+|c-a|\geq n_a+n_b+n_c-g_a-g_b-g_c$ and from $n_a+n_b+n_c\geq s\cdot\sqrt{4-\frac{2r}{R}}$ it follows that:

$$\boldsymbol{|a-b|+|b-c|+|c-a|\geq s\cdot\sqrt{4-\frac{2r}{R}}-g_a-g_b-g_c;(5)}$$

Now, we want to prove that: $s\geq\sqrt{r(4R+r)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)}$

$$s\cdot\sqrt{4-\frac{2r}{R}}\geq\sqrt{r(4R+r)\left(4-\frac{2r}{R}\right)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)}\Leftrightarrow$$

$$\boldsymbol{n_a+n_b+n_c\geq\sqrt{2r(4R+r)\left(2-\frac{r}{R}\right)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)};(6)}$$

Similarly, we obtain: $s\geq\sqrt{r(4R+r)\left(2-\frac{r}{R}\right)\left(\frac{a}{c}+\frac{c}{b}+\frac{b}{a}\right)}$, namely

$$\boldsymbol{n_a+n_b+n_c\geq\sqrt{2r(4R+r)\left(2-\frac{r}{R}\right)\left(\frac{a}{c}+\frac{c}{b}+\frac{b}{a}\right)};(7)}$$

$$\tan\frac{A}{2}=\frac{r_a}{s}\ (and\ analogs)\Rightarrow\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}=\frac{r_a+r_b+r_c}{s}$$

$$\boldsymbol{\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\geq\frac{r_a+r_b+r_c}{n_a+n_b+n_c}\sqrt{4-\frac{2r}{R}};(8)}$$

We know that: $n_a+g_a\geq 2m_a\ (and\ analogs)$ so, we get $n_a+n_b+n_c\geq 2(m_a+m_b+m_c)-g_a-g_b-g_c$ and from $n_a+n_b+n_c\geq s\cdot\sqrt{4-\frac{2r}{R}}$ we get

$$2(n_a+n_b+n_c)\geq 2(m_a+m_b+m_c)+s\cdot\sqrt{4-\frac{2r}{R}}-g_a-g_b-g_c$$

Finally, we get:

$$\boldsymbol{n_a+n_b+n_c\geq m_a+m_b+m_c+\frac{1}{2}\left(s\cdot\sqrt{4-\frac{2r}{R}}-g_a-g_b-g_c\right);(9)}$$

3. In $\Delta ABC, n_a$ –Nagel's cevian, the following relationship holds:

$$\boldsymbol{r_a+r_b+r_c\geq s\cdot\sqrt{4-\frac{2r}{R}}}$$

$$\boldsymbol{n_a+n_b+n_c\geq s\cdot\sqrt{4-\frac{2r}{R}}}$$

Adding these two up relations, we get

$$\boldsymbol{\frac{1}{2}(n_a+n_b+n_c+r_a+r_b+r_c)\geq s\cdot\sqrt{4-\frac{2r}{R}};(10)}$$

Now, from $\frac{R}{2r}\geq\frac{m_a}{h_a}$ $(Panaitopol's)\Rightarrow\frac{h_a}{m_a}\geq\frac{2r}{R}$ $(and\ analogs)\Rightarrow$

$$\boldsymbol{n_a+n_b+n_c\geq s\cdot\sqrt{4-\frac{h_a}{m_a}};(11)}$$

From $\cos\frac{B-C}{2}\geq\sqrt{\frac{2r}{R}}$ and $\cos\frac{B-C}{2}=\frac{h_a}{w_a}$ it follows that $\frac{h_a}{w_a}\geq\sqrt{\frac{2r}{R}}\Leftrightarrow\sqrt{\frac{R}{2r}}\geq\frac{w_a}{h_a}\Leftrightarrow$ $\frac{R}{2r}\geq\frac{w_a^2}{h_a^2}\Leftrightarrow\frac{h_a^2}{w_a^2}\geq\frac{2r}{R}$. So, we get:

$$\boldsymbol{n_a+n_b+n_c\geq s\cdot\sqrt{4-\frac{h_a^2}{w_a^2}};(12)}$$

We know that: $\frac{R}{r}\geq\frac{n_a+h_a}{h_a}$ $(and\ analogs)\Rightarrow\frac{R}{2r}\geq\frac{n_a+h_a}{2h_a}\Rightarrow\frac{2h_a}{n_a+h_a}\geq\frac{2r}{R}$, hence

$$\boldsymbol{n_a+n_b+n_c\geq s\cdot\sqrt{4-\frac{2h_a}{n_a+h_a}};(13)}$$

We know that: $m_a-h_a\geq\frac{(b-c)^2}{2a}$ $(and\ analogs)$ and is easy to show that

$$m_a^2-h_a^2=\left(\frac{b^2-c^2}{2a}\right)^2$$

Namely, $m_a-h_a=\sqrt{h_a^2+\left(\frac{b^2-c^2}{2}\right)^2}-h_a\Rightarrow\sqrt{h_a^2+\left(\frac{b^2-c^2}{2}\right)^2}\geq h_a+\frac{(b-c)^2}{2a}$

After some easy calculation, we get:

$$h_a\leq\frac{(b+c)^2}{4a}-\frac{(b-c)^2}{4a}=\frac{b+c}{a}\Rightarrow a\cdot h_a\leq bc$$

$$a\cdot h_a=2F=bc\cdot\sin A\leq bc\Rightarrow\sin A\leq 1\ (true).$$

4. In $\Delta ABC, n_a$ –Nagel's cevian, the following relationship holds:

$$\boldsymbol{n_a^2=s(s-a)+\frac{(b-c)^2}{a}\cdot s\ (and\ analogs)}$$

$$\frac{n_a^2-s(s-a)}{s}=\frac{(b-c)^2}{a}\Rightarrow\frac{n_a^2-s(s-a)}{2s}=\frac{(b-c)^2}{2a}$$

$$2(m_a-h_a)\ge\frac{n_a^2}{s}-(s-a)\Rightarrow$$

$$\boldsymbol{2(m_a-h_a)+s-a\ge\frac{n_a^2}{s};(14)}$$

$\frac{2(m_a-h_a)+s-a}{n_a}\ge\frac{n_a}{s}$ $(and\ analogs)$ Summing, we get:

$$\boldsymbol{\sum_{cyc}\frac{2(m_a-h_a)+s-a}{n_a}\ge\frac{n_a+n_b+n_c}{s};(15)}$$

But $\frac{n_a+n_b+n_c}{s}\ge\sqrt{4-\frac{2r}{R}}$ then we have

$$\boldsymbol{\sum_{cyc}\frac{2(m_a-h_a)+s-a}{n_a}\ge\sqrt{4-\frac{2r}{R}};(16)}$$

From $n_a+g_a\ge 2m_a\Rightarrow n_a\ge 2m_a-g_a$ and we have:

$$\boldsymbol{\sum_{cyc}\frac{2(m_a-h_a)+s-a}{2m_a-g_a}\ge\sqrt{4-\frac{2r}{R}};(17)}$$

$$\boldsymbol{\sum_{cyc}\frac{2(m_a-h_a)+s-a}{2m_a-g_a}\ge\frac{n_a+n_b+n_c}{s};(18)}$$

$$\boldsymbol{\sum_{cyc}\frac{2(m_a-h_a)+s-a}{n_a}\ge\frac{2(m_a+m_b+m_c)-g_a-g_b-g_c}{s};(19)}$$

But $n_a+n_b+n_c\ge m_a+m_b+m_c+\frac{1}{2}\left(s\cdot\sqrt{4-\frac{2r}{R}}-g_a-g_b-g_c\right)$

$$\sum_{cyc}\frac{2(m_a-h_a)+s-a}{n_a}\ge\frac{n_a+n_b+n_c}{s}$$

So, it follows that:

$$\sum_{cyc}\frac{2(m_a-h_a)+s-a}{n_a}\ge\frac{2(m_a+m_b+m_c)+s\cdot\sqrt{4-\frac{2r}{R}}-g_a-g_b-g_c}{2s}\overset{(14)}{\Longleftrightarrow}$$

$$\boldsymbol{2(m_a+m_b+m_c-h_a-h_b-h_c)+s\ge\frac{n_a^2+n_b^2+n_c^2}{s};(20)}$$

5. In ΔABC, T –Toricelli's point , $\angle ATB=\angle BTC=\angle CTA=120^\circ$

Applying Law of cosines in $\Delta BTC,\Delta CTA,\Delta ATB$ we have:

$a=BC=\sqrt{y^2+x^2+yz}$, where $TA=x$

$b=CA=\sqrt{z^2+x^2+zx}$, where $TB=y$

$c=AB=\sqrt{x^2+y^2+xy}$, where $TC=z$.

Is easy to prove that $y^2+yz+z^2=\frac{3}{4}(y+z)^2+\frac{1}{4}(y-z)^2$ $(and\ analogs)$

$$\Rightarrow a\ge\frac{\sqrt{3}}{2}(y+z)=\frac{\sqrt{3}}{2}(TB+TC)$$

$$\Rightarrow a+b+c\ge\frac{\sqrt{3}}{2}\cdot 2(AT+BT+CT)$$

$$\Rightarrow s \geq \frac{\sqrt{3}}{2}(AT+BT+CT) \Rightarrow s\cdot\sqrt{4-\frac{2r}{R}} \geq \sqrt{\frac{3}{4}\left(4-\frac{2r}{R}\right)}\cdot\sum_{cyc} AT$$

$$\Rightarrow s\cdot\sqrt{4-\frac{2r}{R}} \geq \sqrt{3\left(1-\frac{r}{2R}\right)}\cdot(AT+BT+CT) \text{ and } n_a+n_b+n_c \geq \sqrt{4-\frac{2r}{R}} \text{ then}$$

$$\boldsymbol{n_a+n_b+n_c \geq \sqrt{3\left(1-\frac{r}{2R}\right)}\cdot(AT+BT+CT); (21)}$$

Now, $s^2 = n_a^2 + 2r_a h_a \Rightarrow s^2 - n_a^2 = 2r_a h_a \Rightarrow (s-n_a)(s+n_a) = 2r_a h_a$

$$s-n_a = \frac{2r_a h_a}{s+n_a} \Rightarrow s = n_a + \frac{2r_a h_a}{s+n_a}$$

$$3s = n_a+n_b+n_c+\sum_{cyc}\frac{2r_a h_a}{s+n_a}$$

$$3(n_a+n_b+n_c) \geq 3s\cdot\sqrt{4-\frac{2r}{R}} = \left(n_a+n_b+n_c+\sum_{cyc}\frac{r_a h_a}{s+n_a}\right)\sqrt{4-\frac{2r}{R}}$$

Finally, we get:

$$\boldsymbol{\left(3-\sqrt{4-\frac{2r}{R}}\right)(n_a+n_b+n_c) \geq 2\sqrt{4-\frac{2r}{R}}\cdot\sum_{cyc}\frac{r_a h_a}{s+n_a}; (22)}$$

6. In $\Delta ABC, n_a$ –Nagel's cevian, g_a –Gergonne's cevian, the following relationship holds:
$n_a + g_a \geq 2m_a$ and $n_a + g_a \geq 2\sqrt{n_a g_a}$, then

$$2(n_a+g_a) \geq 2\left(m_a+\sqrt{n_a g_a}\right) \Leftrightarrow n_a+g_a \geq m_a+\sqrt{n_a g_a}$$

$$n_a \geq m_a + \sqrt{g_a}\left(\sqrt{n_a}-\sqrt{g_a}\right)$$

Adding, we have:

$$\sum_{cyc} n_a \geq \sum_{cyc} m_a + \sum_{cyc}\sqrt{g_a}\left(\sqrt{n_a}-\sqrt{g_a}\right) \text{ but } n_a+n_b+n_c \geq s\cdot\sqrt{4-\frac{2r}{R}}$$

$$\sum_{cyc} n_a \geq s\cdot\sqrt{4-\frac{2r}{R}} + \sum_{cyc} m_a + \sum_{cyc}\sqrt{g_a}\left(\sqrt{n_a}-\sqrt{g_a}\right)$$

$$\boldsymbol{n_a+n_b+n_c \geq s\cdot\sqrt{1-\frac{r}{2R}}+\frac{1}{2}\left(\sum_{cyc} m_a + \sum_{cyc}\sqrt{g_a}\left(\sqrt{n_a}-\sqrt{g_a}\right)\right); (23)}$$

If $A \geq B \geq \frac{\pi}{3} \geq C \Rightarrow s \geq (R+r)\sqrt{3}$

We prove that question, from $A \geq B \geq \frac{\pi}{3} \geq C$ we have:

$$\tan\frac{A}{2}-\frac{\sqrt{3}}{3} \geq 0, \tan\frac{B}{2}-\frac{\sqrt{3}}{3} \geq 0 \text{ and } \tan\frac{C}{2}-\frac{\sqrt{3}}{3} \leq 0 \text{ then}$$

$$\prod_{cyc}\left(\tan\frac{A}{2}-\frac{\sqrt{3}}{3}\right) \leq 0$$

$$\sum_{cyc}\tan\frac{B}{2}\tan\frac{C}{2} = 1, \sum_{cyc}\tan\frac{A}{2} = \frac{4R+r}{s}$$

$$\prod_{cyc}\left(\tan\frac{A}{2}-\frac{\sqrt{3}}{3}\right)\le 0\Rightarrow \prod_{cyc}\tan\frac{A}{2}-\frac{\sqrt{3}}{3}\sum_{cyc}\tan\frac{B}{2}\tan\frac{C}{2}+\frac{1}{3}\sum_{cyc}\tan\frac{A}{2}-\frac{\sqrt{3}}{9}\le 0$$

$$\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}=\frac{r}{s}\Rightarrow \frac{r}{s}-\frac{\sqrt{3}}{3}+\frac{1}{3}\cdot\frac{4R+r}{s}-\frac{\sqrt{3}}{9}\le 0$$

$$\Rightarrow 9r+3(4R+r)-4s\sqrt{3}\le 0\Leftrightarrow s\ge\sqrt{3}(R+r)$$

But $n_a+n_b+n_c\ge s\cdot\sqrt{4-\frac{2r}{R}}$. Finally, we get:

If $\boldsymbol{A\ge B\ge\frac{\pi}{3}\ge C\Rightarrow n_a+n_b+n_c\ge (R+r)\sqrt{3\left(4-\frac{2r}{R}\right)};(25)}$

Reference:

ROMANIAN MATHEMATICAL MAGAZINE- www.ssmrmh.ro

NECESSARY AND SUFFICIENT CONDITION SUCH THAT LINE l PASSES THROUGH A CENTER OF A TRIANGLE (LEMMAS)

By Thanasis Gakopoulos-Farsala-Greece

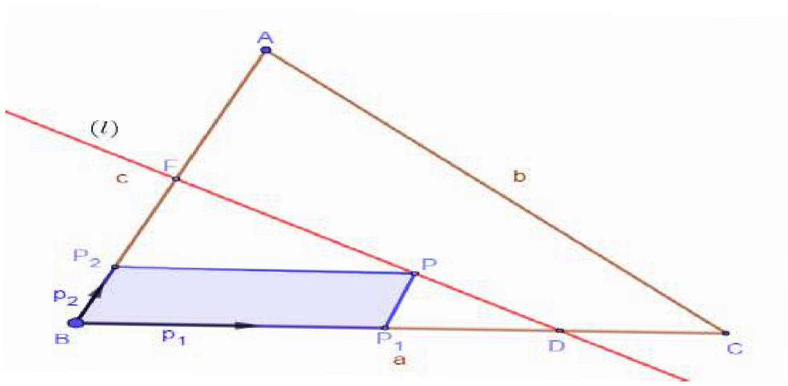
Abstract.

This article is a systematic documentation of the mathematical relationship holds and their proofs, when a line passes through the main centers of a triangle, as a necessary and sufficient condition.Based on a pre-existing relationship (lemma) of a line centroid, I have proved the mathematical relationship, when a line passes through other main centers of a triangle, as it intersects its two sides. More specifically, relationships created for the triangle incenter, excenter, circumcenter, orthocentre,nine-point center, Nagel's point, symmedian point, Spieker's point, Mittenpunkt and Gergonne's point.

At the end of this article, the reader can find two exercises that demonstrate the utility of the proven relationships. I strongly believe that the proposals generated through this study are of a great use, as they could lead us to solutions of similar exercises in a prompt and comprehensive way.

$\boldsymbol{A_0}$**. Line** $(\boldsymbol{l})$ **passes through a point** $\boldsymbol{P}$ **on the plane of the triangle** $\boldsymbol{ABC}$.

Figure-1



Plagiogonal system: $BC\equiv Bx, BA\equiv By, BD=d, BF=f, OP_1=p_1, OP_1=p_2$

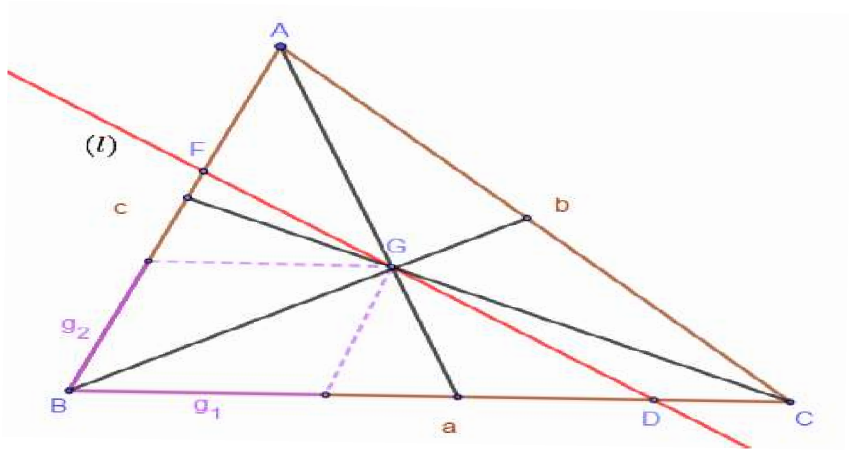
$$B(0,0), D(d,0), C(a,0), F(0,f), A(0,c), P(p_1,p_2)$$

$$D, F, P - \text{are collinear} \leftrightarrow \begin{vmatrix} 1 & 1 & 1 \\ d & 0 & p_1 \\ 0 & f & p_2 \end{vmatrix} = 0 \leftrightarrow fp_1 + dp_2 = df \leftrightarrow \frac{p_1}{d} + \frac{p_2}{f} = 1; (*)(Lemma)$$

Note: The proposal $(*)$ is basic and is used as Lemma to prove the relationship which follow below.

A_1. Line (l) passes through the centroid G of the triangle ABC.

Figure-2



$$g_1 = \frac{a}{3}, g_2 = \frac{c}{3}, BD = d, BF = f$$

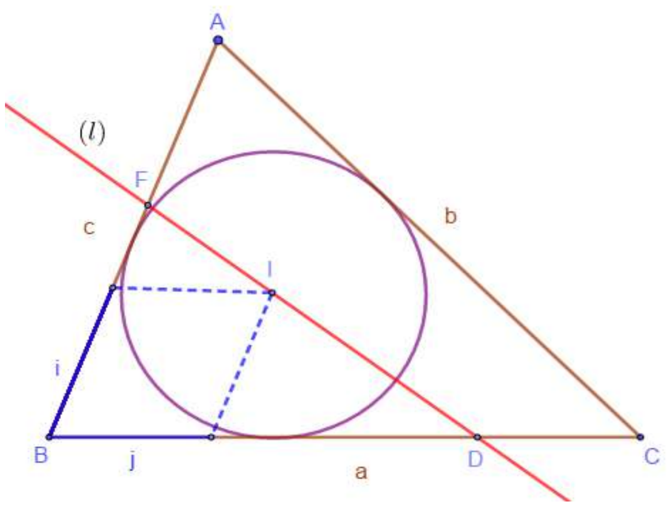
$$\text{From Lemma}(*): \frac{\frac{a}{3}}{d} + \frac{\frac{c}{3}}{f} = 1 \leftrightarrow \frac{a}{d} + \frac{c}{f} = 1 \leftrightarrow \frac{d+a-d}{d} + \frac{f+c-f}{f} = 3 \leftrightarrow \frac{a-d}{d} + \frac{c-f}{f} = 1$$

$$D, F, G - \text{are collinear} \leftrightarrow \frac{BC}{BD} + \frac{BA}{BF} = 3 \; or \; \frac{CD}{BC} + \frac{AF}{BF} = 1.$$

The proposal exists in the mathematical bibliography.

A_2. Line (l) passes through the incenter I of the triangle ABC.

Figure-3



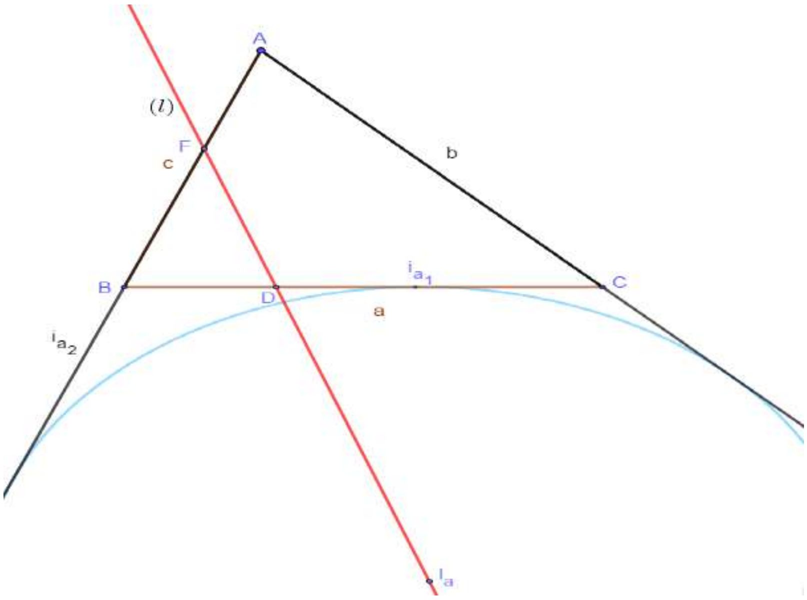
$$i=\frac{ac}{a+b+c},BD=d,BF=f$$

From Lemma $(*)$: $\frac{i}{d}+\frac{i}{f}=1\leftrightarrow\left(\frac{1}{d}-\frac{1}{a}\right)+\left(\frac{1}{f}-\frac{1}{c}\right)=\frac{b}{ac}$

D,F,I —are collinear $\leftrightarrow\left(\frac{1}{BD}-\frac{1}{BC}\right)+\left(\frac{1}{BF}-\frac{1}{BA}\right)=\frac{AC}{BC\cdot BA}$

A_3. Line (l) passes through the excenter I_a of the triangle ABC.

Figure-4



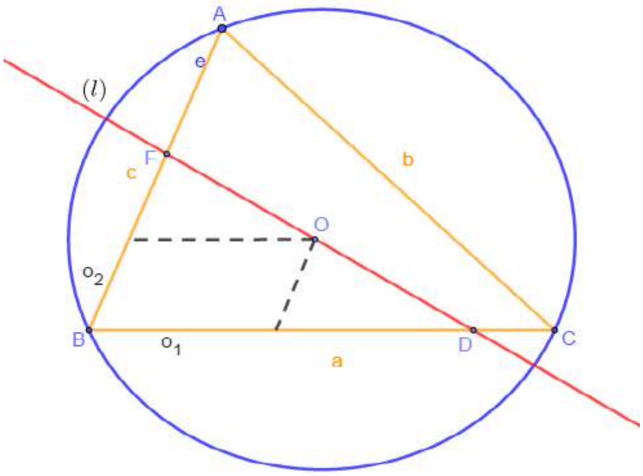
$$i_{a_1}=\frac{ac}{-a+b+c},i_{a_2}=-\frac{ac}{a+b+c},BD=d,BF=f$$

From Lemma $(*)$: $\frac{i_{a_1}}{d}+\frac{i_{a_2}}{f}=1\leftrightarrow\left(\frac{1}{d}-\frac{1}{a}\right)+\left(\frac{1}{f}-\frac{1}{c}\right)=\frac{b}{ac}$

D,F,I —are collinear $\leftrightarrow\left(\frac{1}{BD}-\frac{1}{BC}\right)+\left(-\frac{1}{BF}-\frac{1}{BA}\right)=\frac{AC}{BC\cdot BA}$

A_4. Line (l) passes through the circumcenter O of the triangle ABC.

Figure-5



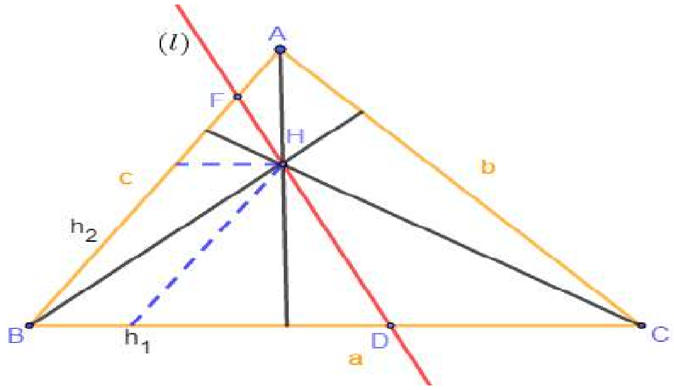
$$o_1=\frac{a-c\cdot\cos B}{2\sin^2 B}, o_2=\frac{c-a\cdot\cos B}{2\sin^2 B}, BD=d, BF=f$$

From Lemma $(*)$: $\frac{o_1}{d}+\frac{o_2}{f}=1\leftrightarrow\left(\frac{a}{d}+\frac{c}{f}\right)-\left(\frac{a}{f}+\frac{c}{d}\right)\cdot\cos B=2\sin^2 B$

$D,F,O-$ are collinear$\leftrightarrow\left(\frac{BC}{BD}+\frac{BA}{BF}\right)-\left(\frac{BC}{BF}+\frac{BA}{BD}\right)\cdot\cos B=2\sin^2 B.$

A_5.Line (l) passes through the orthocenter H of the triangle ABC.

Figure-6



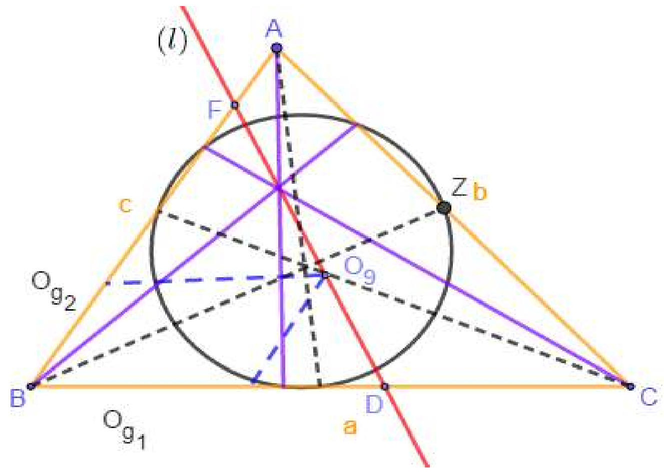
$$h_1=\frac{(c-a\cdot\cos B)\cos B}{\sin^2 B}, h_2=\frac{(a-c\cdot\cos B)\cos B}{\sin^2 B}, BD=d, BF=f$$

From Lemma $(*)$: $\frac{h_a}{a}+\frac{h_2}{f}=1\leftrightarrow\left(\frac{a}{f}+\frac{c}{d}\right)-\left(\frac{a}{d}+\frac{c}{f}\right)\cos B=\sin B\cdot\tan B$

$D,F,H-$collinear $\leftrightarrow\left(\frac{BC}{BF}+\frac{BA}{BD}\right)-\left(\frac{BC}{BD}+\frac{BA}{BF}\right)\cos B=\sin B\cdot\tan B.$

A_6. Line (l) passes through the nine point center O_9 of the triangle ABC.

Figure-7



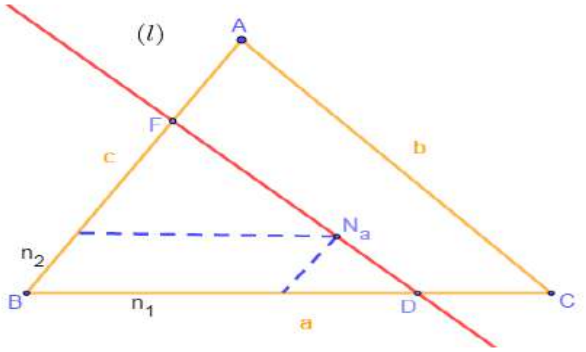
$$O_{9_1}=\frac{c\cdot\cos B-a\cdot\cos 2B}{4\sin^2 B}, O_{9_2}=\frac{a\cdot\cos B-c\cdot\cos 2B}{4\sin^2 B}, BD=d, BF=f$$

From Lemma $(*)$: $\frac{O_{9_1}}{d}+\frac{O_{9_2}}{f}=1\leftrightarrow\left(\frac{a}{f}+\frac{c}{f}\right)\cos B-\left(\frac{a}{d}+\frac{c}{f}-2\right)\cos 2B=2$

D,F,O_9-collinear $\leftrightarrow\left(\frac{BC}{BF}+\frac{BA}{BD}\right)\cos B-\left(\frac{BC}{BD}+\frac{BA}{BF}-2\right)\cos 2B=2.$

A_7. Line (l) passes through the Nagel's N_a of the triangle ABC.

Figure-8

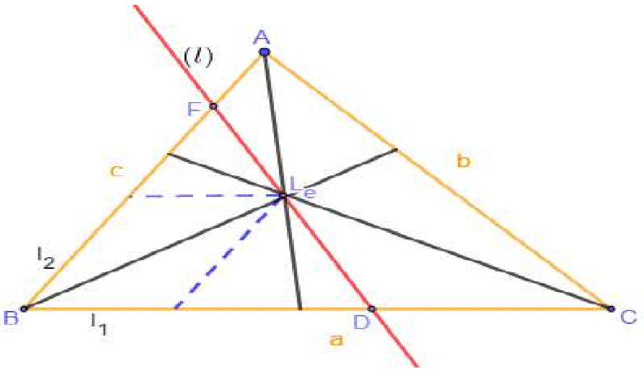


$$n_1=\frac{a+b-c}{a+b+c}\cdot a, n_2=\frac{-a+b+c}{a+b+c}\cdot c, BD=d, BF=f$$

From Lemma $(*)$: $\frac{n_1}{d}+\frac{n_2}{f}=1\leftrightarrow\frac{c}{f}\cdot\frac{s-c}{s}+\frac{BA}{BF}\cdot\frac{s-a}{a}=1.$

A_8. Line (l) passes through the Lemoine's point L_e of the triangle ABC.

Figure-9



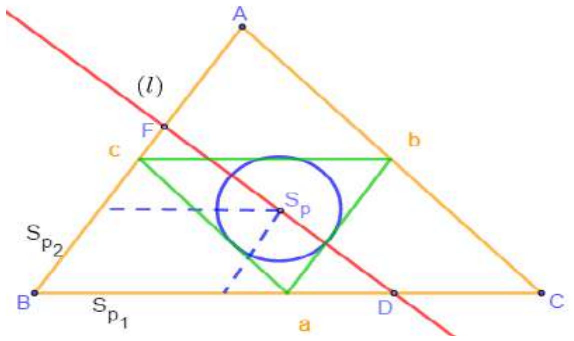
$$l_1=\frac{ac^2}{a^2+b^2+b^2}, l_2=\frac{a^2c}{a^2+b^2+c^2}, BD=d, BF=f$$

From Lemma $(*)$: $\frac{l_1}{d}+\frac{l_2}{f}=1\leftrightarrow\frac{a}{f}+\frac{c}{d}=\frac{a^2+b^2+c^2}{ac}$

B,D,L_e –collinear$\leftrightarrow\frac{BC}{BF}+\frac{BA}{BD}=\frac{AB^2+BC^2+CA^2}{BA\cdot BC}$

A_9. Line (l) passes through the Spieker's point S_p of the triangle ABC.

Figure-10



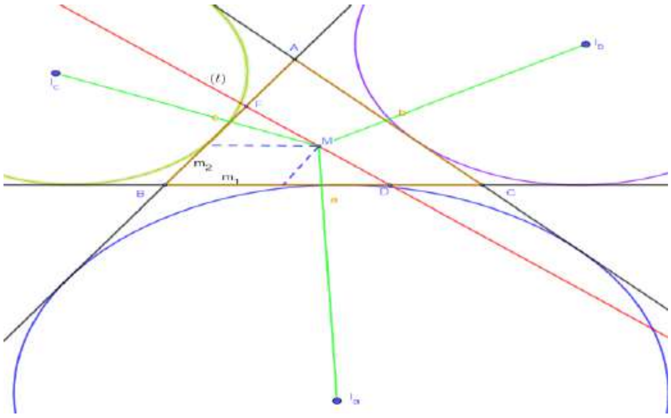
$$S_{p_1} = \frac{a(a+b)}{2(a+b+c)}, S_{p_2} = \frac{c(b+c)}{2(a+b+c)}, BD = d, BF = f$$

From Lemma $(*)$: $\frac{S_{p_1}}{d} + \frac{S_{p_2}}{f} = 1 \leftrightarrow \frac{a}{d}(a+b) + \frac{c}{f}(b+c) = 2(a+b+c)$

D, F, S_p $-$collinear $\leftrightarrow \frac{BC}{BD}(AC+BC) + \frac{BA}{BF}(AB+AC) = 2(AB+BC+CA)$

A_{10}. Line (l) passes through the Mittenpunkt's point of a triangle ABC.

Figure-11



$$m_1 = \frac{ac(a+b-c)}{(a+b+c)^2 - 2(a^2+b^2+c^2)}, m_2 = \frac{ac(-a+b+c)}{(a+b+c)^2 - 2(a^2+b^2+c^2)}$$

$$BD = d, BF = f$$

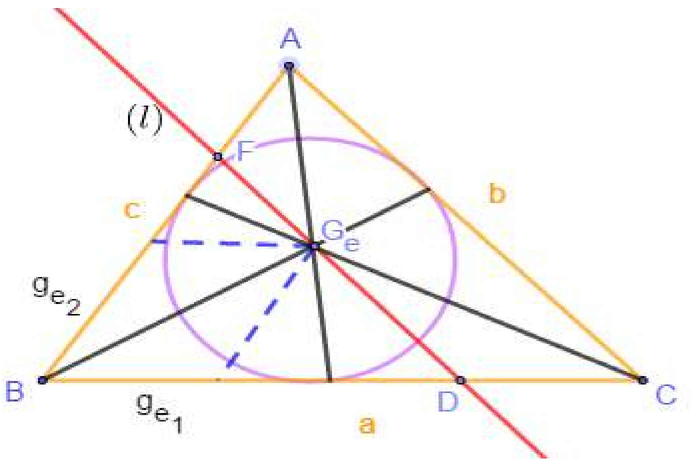
From Lemma $(*)$: $\frac{m_1}{d} + \frac{m_2}{f} = 1 \leftrightarrow \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - \frac{1}{2}\left(\frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab}\right) = \frac{1}{b}\left(\frac{s-a}{f} + \frac{s-c}{d}\right)$ or

$$\frac{a(s-a) + b(s-b) + c(s-c)}{abc} = \frac{1}{b}\left(\frac{s-a}{f} + \frac{s-c}{d}\right)$$

D, F, M $-$collinear $\leftrightarrow \frac{a(s-a)+b(s-b)+c(s-c)}{abc} = \frac{1}{b}\left(\frac{s-a}{BF} + \frac{s-c}{BD}\right)$

A_{11}. Line (l) passes therough the Gergonne's point $G_e$of the triangle ABC.

Figure-12



$$g_{e_1}=\frac{a(-a+b+c)(a-b+c)}{2(ab+bc+ca)-a^2-b^2-c^2},g_{e_2}=\frac{c(a+b-c)(a-b+c)}{2(ab+bc+ca)-a^2-b^2-c^2}$$

$$BD=d,BF=f$$

From Lemma $(*)$: $\frac{g_{e_1}}{d}+\frac{g_{e_2}}{f}=1\leftrightarrow\frac{1}{b}\left[\frac{a}{d}(s-a)+\frac{c}{f}(s-c)-\frac{(s-a)(s-c)}{s-b}\right]=1$

D,F,G_e −collinear $\leftrightarrow\frac{1}{b}\left[\frac{a}{BD}(s-a)+\frac{c}{BF}(s-c)-\frac{(s-a)(s-c)}{s-b}\right]=1.$

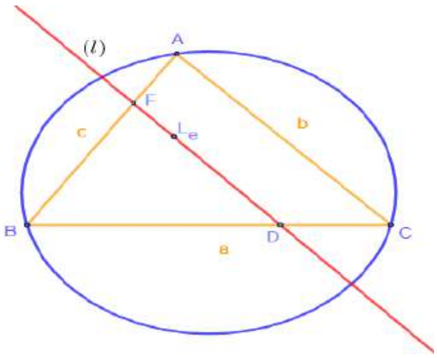
Applications.

Exercise 1. Let L_e −the Lemonine's point of a triangle ABC, line (l) passes through the L_e and intersects the sides BC,BA at the points D,F respectively.

Denote $BD=d,BF=f$ and R −the circumradius of ΔABC. Prove that: $\frac{a}{f}+\frac{c}{d}\geq\frac{b\sqrt{3}}{R}$

Solution.

Figure-13



From (A_8) we have: $\frac{a}{f}+\frac{c}{d}=\frac{a^2+b^2+c^2}{ac}$

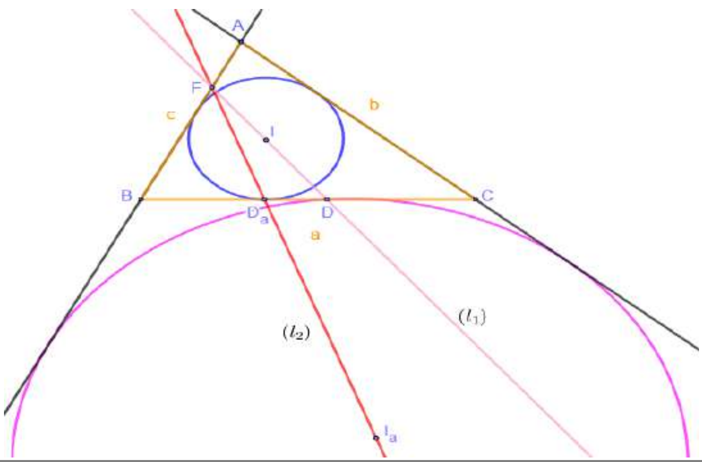
Is $a^2+b^2+c^2\geq4\sqrt{3}F=4\sqrt{3}\cdot\frac{1}{2}ac\cdot\sin B=\frac{4\sqrt{3}ac}{2}\cdot\frac{b}{2R}$

Hence, $\frac{a}{f}+\frac{c}{d}\geq\frac{b\sqrt{3}}{R}$.

Exercise 2. Line (l_1) passes from incenter I of a triangle ABC and intersects the sides BC,BA at the points D,F respectively. Line (l_2) passes from excenter I_a of the same triangle and intersects the sides BC,BA at the points D_a,F respectively. Denote $BD=d,BD_a=d_a,BF=f$. Prove that: $\frac{1}{d_a}-\frac{1}{d}=2\left(\frac{1}{f}-\frac{1}{c}\right)$.

Solution.

Figure-14



From (A_2) we have: $\left(\frac{1}{d}-\frac{1}{a}\right)+\left(\frac{1}{f}-\frac{1}{c}\right)=\frac{b}{ac}$ and from (A_3) we have:

$$\frac{1}{d_a}-\frac{1}{a}+\left(-\frac{1}{f}+\frac{1}{c}\right)=\frac{b}{ac}$$

Therefore, $\frac{1}{d_a}-\frac{1}{d}=2\left(\frac{1}{f}-\frac{1}{c}\right)$.

Reference:

ROMANIAN MATHEMATICAL MAGAZINE –www.ssmrmh.ro

A SIMPLE PROOF FOR MAHLER'S INEQUALITY

By Daniel Sitaru – Romania

If $\boldsymbol{x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n} > 0; n \in \mathbb{N}; \boldsymbol{n \geq 2}$ **then:**

$$\sqrt{(\boldsymbol{x_1+y_1})(\boldsymbol{x_2+y_2})} \geq \sqrt{\boldsymbol{x_1x_2}}+\sqrt{\boldsymbol{y_1y_2}}$$

$$\sqrt[3]{(\boldsymbol{x_1+y_1})(\boldsymbol{x_2+y_2})(\boldsymbol{x_3+y_3})} \geq \sqrt[3]{\boldsymbol{x_1x_2x_3}}+\sqrt[3]{\boldsymbol{y_1y_2y_3}}$$

$$\sqrt[4]{(\boldsymbol{x_1+y_1})(\boldsymbol{x_2+y_2})(\boldsymbol{x_3+y_3})(\boldsymbol{x_4+y_4})} \geq \sqrt[4]{\boldsymbol{x_1x_2x_3x_4}}+\sqrt[4]{\boldsymbol{y_1y_2y_3y_4}}$$

$$\sqrt[n]{(\boldsymbol{x_1+y_1})(\boldsymbol{x_2+y_2})\cdot\ldots\cdot(\boldsymbol{x_n+y_n})} \geq \sqrt[n]{\boldsymbol{x_1x_2\cdot\ldots\cdot x_n}}+\sqrt[n]{\boldsymbol{y_1y_2\cdot\ldots y_n}}$$

Proof:

$$\frac{x_1}{x_1+y_1}+\frac{x_2}{x_2+y_2}+\cdots+\frac{x_n}{x_n+y_n} \overset{AM-GM}{\geq} n\cdot\sqrt[n]{\frac{x_1x_2\cdot\ldots\cdot x_n}{(x_1+y_1)(x_2+y_2)\cdot\ldots\cdot(x_n+y_n)}}$$

$$\frac{x_1}{x_1+y_1}+\frac{x_2}{x_2+y_2}+\cdots+\frac{x_n}{x_n+y_n} \geq \frac{n\sqrt[n]{x_1x_2\cdot\ldots\cdot x_n}}{\sqrt[n]{(x_1+y_1)(x_2+y_2)\cdot\ldots\cdot(x_n+y_n)}} \quad (1)$$

Analogous:

$$\frac{y_1}{x_1+y_1}+\frac{y_2}{x_2+y_2}+\cdots+\frac{y_n}{x_n+y_n} \geq \frac{n\sqrt[n]{y_1y_2\cdot\ldots\cdot y_n}}{\sqrt[n]{(x_1+y_1)(x_2+y_2)\cdot\ldots\cdot(x_n+y_n)}} \quad (2)$$

By adding (1); (2):

$$\frac{x_1+y_1}{x_1+y_1}+\frac{x_2+y_2}{x_2+y_2}+\cdots+\frac{x_n+y_n}{x_n+y_n} \geq \frac{n\left(\sqrt[n]{x_1x_2\cdot\ldots\cdot x_n}+\sqrt[n]{y_1y_2\cdot\ldots\cdot y_n}\right)}{\sqrt[n]{(x_1+y_1)(x_2+y_2)\cdot\ldots\cdot(x_n+y_n)}}$$

$$\underbrace{1+1+\cdots+1}_{for\ "n"\ times} \geq \frac{n\left(\sqrt[n]{x_1x_2\cdot\ldots\cdot x_n}+\sqrt[n]{y_1y_2\cdot\ldots y_n}\right)}{\sqrt[n]{(x_1+y_1)(x_2+y_2)\cdot\ldots\cdot(x_n+y_n)}}$$

$$n \geq \frac{n\left(\sqrt[n]{x_1x_2\cdot\ldots\cdot x_n}+\sqrt[n]{y_1y_2\cdot\ldots\cdot y_n}\right)}{\sqrt[n]{(x_1+y_1)(x_2+y_2)\cdot\ldots\cdot(x_n+y_n)}}$$

$$\sqrt[n]{(x_1+y_1)(x_2+y_2)\cdot\ldots\cdot(x_n+y_n)} \geq \sqrt[n]{x_1x_2\cdot\ldots\cdot x_n}+\sqrt[n]{y_1y_2\cdot\ldots\cdot y_n}$$

Observations:

1. If $y_1 = y_2 = \cdots = y_n = 1$ then:

$$\sqrt[n]{(\mathbf{1}+\boldsymbol{x_1})(\mathbf{1}+\boldsymbol{x_2})\cdot\ldots\cdot(\mathbf{1}+\boldsymbol{x_n})} \geq \mathbf{1} + \sqrt[n]{\boldsymbol{x_1 x_2}\cdot\ldots\cdot\boldsymbol{x_n}}$$

$$(\mathbf{1}+\boldsymbol{x_1})(\mathbf{1}+\boldsymbol{x_2})\cdot\ldots\cdot(\mathbf{1}+\boldsymbol{x_n}) \geq \left(\mathbf{1} + \sqrt[n]{\boldsymbol{x_1 x_2}\cdot\ldots\cdot\boldsymbol{x_n}}\right)^{\boldsymbol{n}}$$

2. If $y_1 = y_2 = \cdots = y_n = a$ then:

$$\sqrt[n]{(\boldsymbol{a}+\boldsymbol{x_1})(\boldsymbol{a}+\boldsymbol{x_2})\cdot\ldots\cdot(\boldsymbol{a}+\boldsymbol{x_n})} \geq \boldsymbol{a} + \sqrt[n]{\boldsymbol{x_1 x_2}\cdot\ldots\cdot\boldsymbol{x_n}}$$

$$(\boldsymbol{a}+\boldsymbol{x_1})(\boldsymbol{a}+\boldsymbol{x_2})\cdot\ldots\cdot(\boldsymbol{a}+\boldsymbol{x_n}) \geq \left(\boldsymbol{a} + \sqrt[n]{\boldsymbol{x_1 x_2}\cdot\ldots\cdot\boldsymbol{x_n}}\right)^{\boldsymbol{n}}$$

3. If $y_1 = \frac{1}{x_1}; y_2 = \frac{1}{x_2}; \ldots; y_n = \frac{1}{x_n}$ then:

$$\sqrt[n]{\left(\boldsymbol{x_1}+\frac{\mathbf{1}}{\boldsymbol{x_1}}\right)\left(\boldsymbol{x_2}+\frac{\mathbf{1}}{\boldsymbol{x_2}}\right)\cdot\ldots\cdot\left(\boldsymbol{x_n}+\frac{\mathbf{1}}{\boldsymbol{x_n}}\right)} \geq \sqrt[n]{\boldsymbol{x_1 x_2}\cdot\ldots\cdot\boldsymbol{x_n}} + \frac{\mathbf{1}}{\sqrt[n]{\boldsymbol{x_1 x_2}\cdot\ldots\cdot\boldsymbol{x_n}}}$$

$$\left(\boldsymbol{x_1}+\frac{\mathbf{1}}{\boldsymbol{x_1}}\right)\left(\boldsymbol{x_2}+\frac{\mathbf{1}}{\boldsymbol{x_2}}\right)\cdot\ldots\cdot\left(\boldsymbol{x_n}+\frac{\mathbf{1}}{\boldsymbol{x_n}}\right) \geq \left(\sqrt[n]{\boldsymbol{x_1 x_2}\cdot\ldots\cdot\boldsymbol{x_n}} + \frac{\mathbf{1}}{\sqrt[n]{\boldsymbol{x_1 x_2}\cdot\ldots\cdot\boldsymbol{x_n}}}\right)^{\boldsymbol{n}}$$

4. Equality holds for:

$$x_1 = x_2 = \cdots = x_n; y_1 = y_2 = \cdots = y_n$$

Reference:

ROMANIAN MATHEMATICAL MAGAZINE – www.ssmrmh.ro

6 SOLUTIONS FOR A SSMA MATH PROBLEM

5620. Prove: If $\boldsymbol{a},\boldsymbol{b}\in[\mathbf{0},\mathbf{1}];\boldsymbol{a}\leq\boldsymbol{b}$, then:

$$\mathbf{4}\sqrt{\boldsymbol{ab}} \leq \boldsymbol{a}\left(\left(\frac{\boldsymbol{b}}{\boldsymbol{a}}\right)^{\sqrt{\boldsymbol{ab}}} + \sqrt{\left(\frac{\boldsymbol{b}}{\boldsymbol{a}}\right)^{\boldsymbol{a}+\boldsymbol{b}}}\right) + \boldsymbol{b}\left(\left(\frac{\boldsymbol{a}}{\boldsymbol{b}}\right)^{\sqrt{\boldsymbol{ab}}} + \sqrt{\left(\frac{\boldsymbol{a}}{\boldsymbol{b}}\right)^{\boldsymbol{a}+\boldsymbol{b}}}\right) \leq \mathbf{2}(\boldsymbol{a}+\boldsymbol{b})$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Moti Levy -Rehovot – Israel

Let $\alpha := \sqrt{ab} \leq 1, \beta := \frac{a+b}{2} \leq 1, r := \frac{b}{a}$.Then the original inequality can be reformulated as

$$\sqrt{r} \leq \frac{r^{\alpha} + r^{\beta} + r^{1-\alpha} + r^{1-\beta}}{4} \leq \frac{1}{2} + r$$

Since $f(x) := r^x$ is convex function, then $\frac{r^\alpha+r^\beta+r^{1-\alpha}+r^{1-\beta}}{4} \geq r^{\frac{\alpha+\beta+(1-\alpha)+(1-\beta)}{4}} = \sqrt{r}$

The Bernoulli's inequality is $(1+x)^\alpha \leq 1+\alpha x, 0 \leq \alpha \leq 1, x \geq -1$

Using the Bernoulli's inequality we get $r^\alpha \leq 1+\alpha(r-1), \quad r^{1-\alpha} \leq 1+(1-\alpha)(r-1)$

hence $r^\alpha + r^{1-\alpha} \leq 1+r$. It follows that

$$\frac{(r^\alpha + r^{1-\alpha}) + \left(r^\beta + r^{1-\beta}\right)}{4} \leq \frac{1}{2} + r$$

Solution 2 by Michel Bataille – Rouen – France

We suppose $a, b \in (0,1]$ and do not use the hypothesis $a \leq b$. Let

$$M = a\left(\left(\frac{b}{a}\right)^{\sqrt{ab}} + \sqrt{\left(\frac{b}{a}\right)^{a+b}}\right) + b\left(\left(\frac{a}{b}\right)^{\sqrt{ab}} + \sqrt{\left(\frac{a}{b}\right)^{a+b}}\right)$$

Since $\sqrt{ab}$ and $\frac{a+b}{2}$ are in $(0,1]$, the functions $x \to x^{\sqrt{ab}}$ and $x \to x^{\frac{a+b}{2}}$ are concave on $(0,\infty)$. It follows that

$$a\left(\frac{b}{a}\right)^{\sqrt{ab}} + b\left(\frac{a}{b}\right)^{\sqrt{ab}} \leq (a+b)\left(\frac{a}{a+b}\cdot\frac{b}{a} + \frac{b}{a+b}\cdot\frac{a}{b}\right)^{\sqrt{ab}} = a+b$$

and

$$a\left(\frac{b}{a}\right)^{\frac{a+b}{2}} + b\left(\frac{a}{b}\right)^{\frac{a+b}{2}} \leq (a+b)\left(\frac{a}{a+b}\cdot\frac{b}{a} + \frac{b}{a+b}\cdot\frac{a}{b}\right)^{\frac{a+b}{2}} = a+b$$

By addition, $M \leq 2(a+b)$. If m is a positive real number, the function $x \to m^x$ is convex on R. Taking successively $m = \frac{b}{a}$ and $m = \frac{a}{b}$ and setting $k = \frac{1}{2}\left(\sqrt{ab} + \frac{a+b}{2}\right)$, it follows that

$$\left(\frac{b}{a}\right)^{\sqrt{ab}} + \left(\frac{b}{a}\right)^{\frac{a+b}{2}} \geq 2\left(\frac{b}{a}\right)^k$$

and

$$\left(\frac{a}{b}\right)^{\sqrt{ab}} + \left(\frac{a}{b}\right)^{\frac{a+b}{2}} \geq 2\left(\frac{a}{b}\right)^k$$

Using $x+y \geq 2\sqrt{xy}$ for positive x, y, we deduce that

$$M \geq 2\left(a\left(\frac{b}{a}\right)^k + b\left(\frac{a}{b}\right)^k\right) \geq 2\cdot 2\left(a\left(\frac{b}{a}\right)^k \cdot b\left(\frac{a}{b}\right)^k\right)^{\frac{1}{2}}$$

and $M \geq 4\sqrt{ab}$ follows.

Solution 3 by Arkady Alt – San Jose – California

Applying inequality $x+y\geq 2\sqrt{xy}, x,y>0$ to $(x,y)=\left(a\left(\frac{b}{a}\right)^{\sqrt{ab}}, b\left(\frac{a}{b}\right)^{\sqrt{ab}}\right)$ and to

$(x,y)=\left(a\left(\frac{b}{a}\right)^{\frac{a+b}{2}}, b\left(\frac{a}{b}\right)^{\frac{a+b}{2}}\right)$ we obtain $a\left(\frac{b}{a}\right)^{\sqrt{ab}}+b\left(\frac{a}{b}\right)^{\sqrt{ab}}\geq 2\sqrt{a\left(\frac{b}{a}\right)^{\sqrt{ab}}\cdot b\left(\frac{a}{b}\right)^{\sqrt{ab}}}=2\sqrt{ab}$ and

$$a\left(\frac{b}{a}\right)^{\frac{a+b}{2}}+b\left(\frac{a}{b}\right)^{\frac{a+b}{2}}\geq 2\sqrt{a\left(\frac{b}{a}\right)^{\frac{a+b}{2}}\cdot b\left(\frac{a}{b}\right)^{\frac{a+b}{2}}}=2\sqrt{ab}$$

Thus, $a\left(\left(\frac{b}{a}\right)^{\sqrt{ab}}+\sqrt{\left(\frac{b}{a}\right)^{a+b}}\right)+b\left(\left(\frac{a}{b}\right)^{\sqrt{ab}}+\sqrt{\left(\frac{a}{b}\right)^{a+b}}\right)\geq 4\sqrt{ab}$

For function $f(t)=t^p$, which for $p\in[0,1]$ is concave down on $(0,\infty)$, holds inequality

$\frac{ax^p+by^p}{a+b}\leq\left(\frac{ax+by}{a+b}\right)^p$ for any $x,y>0$ .Since $\sqrt{ab},\frac{a+b}{2}\in[0,1]$ then applying this inequality to $(x,y,p)=\left(\frac{b}{a},\frac{a}{b},\sqrt{ab}\right)$ and $(x,y,p)=\left(\frac{b}{a},\frac{a}{b},\frac{a+b}{2}\right)$

we obtain $\frac{a\left(\frac{b}{a}\right)^{\sqrt{ab}}+b\left(\frac{a}{b}\right)^{\sqrt{ab}}}{a+b}\leq\left(\frac{a\cdot\frac{b}{a}+b\cdot\frac{a}{b}}{a+b}\right)^{\sqrt{ab}}=1\Leftrightarrow a\left(\frac{b}{a}\right)^{\sqrt{ab}}+b\left(\frac{a}{b}\right)^{\sqrt{ab}}\leq a+b$ and

$$\frac{a\left(\frac{b}{a}\right)^{\frac{a+b}{2}}+b\left(\frac{a}{b}\right)^{\frac{a+b}{2}}}{a+b}\leq\left(\frac{a\cdot\frac{b}{a}+b\cdot\frac{a}{b}}{a+b}\right)^{\frac{a+b}{2}}=1\Leftrightarrow a\left(\frac{b}{a}\right)^{\frac{a+b}{2}}+b\left(\frac{a}{b}\right)^{\frac{a+b}{2}}\leq a+b$$

Therefore, $a\left(\left(\frac{b}{a}\right)^{\sqrt{ab}}+\sqrt{\left(\frac{b}{a}\right)^{a+b}}\right)+b\left(\left(\frac{a}{b}\right)^{\sqrt{ab}}+\sqrt{\left(\frac{a}{b}\right)^{a+b}}\right)\leq 2(a+b)$

Solution 4 by Henry Ricardo – Westchester Area Math Circle – Purchase – NY

In the following proof, we use Heinz's inequality:

$\sqrt{ab}\leq\frac{a^{1-\alpha}b^{\alpha}+a^{\alpha}b^{1-\alpha}}{2}\leq\frac{a+b}{2}$ for $a,b>0,\alpha\in[0,1]$, first with $\alpha=\sqrt{ab}$ and then with $\alpha=\frac{a+b}{2}$

First we rearrange the central term in the proposed inequality:

$$a\left(\left(\frac{b}{a}\right)^{\sqrt{ab}}+\sqrt{\left(\frac{b}{a}\right)^{a+b}}\right)+b\left(\left(\frac{a}{b}\right)^{\sqrt{ab}}+\sqrt{\left(\frac{a}{b}\right)^{a+b}}\right)=a\left(\frac{b}{a}\right)^{\sqrt{ab}}+a\left(\frac{b}{a}\right)^{\frac{a+b}{2}}+b\left(\frac{a}{b}\right)^{\sqrt{ab}}+b\left(\frac{a}{b}\right)^{\frac{a+b}{2}}$$

$$=\left(a\cdot\frac{b^{\sqrt{ab}}}{a^{\sqrt{ab}}}+b\cdot\frac{a^{\sqrt{ab}}}{b^{\sqrt{ab}}}\right)+\left(a\cdot\frac{b^{\frac{a+b}{2}}}{a^{\frac{a+b}{2}}}+b\cdot\frac{a^{\frac{a+b}{2}}}{b^{\frac{a+b}{2}}}\right)$$

$$=2\left(\frac{a^{1-\sqrt{ab}}b^{\sqrt{ab}}+a^{\sqrt{ab}}b^{1-\sqrt{ab}}}{2}\right)+2\left(\frac{a^{1-\frac{a+b}{2}}b^{\frac{a+b}{2}}+a^{\frac{a+b}{2}}b^{1-\frac{a+b}{2}}}{2}\right)$$

Now the Heinz inequality yields

$$4\sqrt{ab} \le 2\left(\frac{a^{1-\sqrt{ab}}b^{\sqrt{ab}} + a^{\sqrt{ab}}b^{1-\sqrt{ab}}}{2}\right) + 2\left(\frac{a^{1-\frac{a+b}{2}}b^{\frac{a+b}{2}} + a^{\frac{a+b}{2}}b^{1-\frac{a+b}{2}}}{2}\right)$$

$$\le 2\left(\frac{a+b}{2}\right) + 2\left(\frac{a+b}{2}\right) = 2(a+b)$$

Solution 5 by Hatef I. Arshagi – Guilford Technical Community College – Jamestown, NC

First we prove that $a\left(\left(\frac{b}{a}\right)^{\sqrt{ab}} + \sqrt{\left(\frac{b}{a}\right)^{a+b}}\right) + b\left(\left(\frac{a}{b}\right)^{\sqrt{ab}} + \sqrt{\left(\frac{a}{b}\right)^{a+b}}\right) \ge 4\sqrt{ab}$ (1)

To prove this, we will use the well-known inequality that for all $p > 0$ and any real number r

$p^r + \frac{1}{p^r} \ge 2$ (2). For all $a > 0$ and $b > 0$, using (2) in the above step, we can write

$$a\left(\left(\frac{b}{a}\right)^{\sqrt{ab}} + \sqrt{\left(\frac{b}{a}\right)^{a+b}}\right) + b\left(\left(\frac{a}{b}\right)^{\sqrt{ab}} + \sqrt{\left(\frac{a}{b}\right)^{a+b}}\right)$$

$$= \sqrt{ab}\left[\sqrt{\frac{a}{b}}\left(\left(\frac{b}{a}\right)^{\sqrt{ab}} + \sqrt{\left(\frac{b}{a}\right)^{a+b}}\right) + \sqrt{\frac{b}{a}}\left(\left(\frac{a}{b}\right)^{\sqrt{ab}} + \sqrt{\left(\frac{a}{b}\right)^{a+b}}\right)\right]$$

$$= \sqrt{ab}\left[\left(\left(\frac{b}{a}\right)^{-\frac{1}{2}+\sqrt{ab}} + \left(\frac{b}{a}\right)^{-\frac{1}{2}+\frac{a+b}{2}}\right) + \left(\left(\frac{a}{b}\right)^{-\frac{1}{2}+\sqrt{ab}} + \left(\frac{a}{b}\right)^{-\frac{1}{2}+\frac{a+b}{2}}\right)\right]$$

$$\sqrt{ab}\left[\left(\left(\frac{b}{a}\right)^{-\frac{1}{2}+\sqrt{ab}} + \left(\frac{a}{b}\right)^{-\frac{1}{2}+\frac{a+b}{2}}\right) + \left(\left(\frac{b}{a}\right)^{-\frac{1}{2}+\sqrt{ab}} + \left(\frac{a}{b}\right)^{-\frac{1}{2}+\frac{a+b}{2}}\right)\right]$$

$\ge \sqrt{ab}(2+2) = 4\sqrt{ab}$. This completes the proof of (1). Now, we prove that

$a\left(\left(\frac{b}{a}\right)^{\sqrt{ab}} + \sqrt{\left(\frac{b}{a}\right)^{a+b}}\right) + b\left(\left(\frac{a}{b}\right)^{\sqrt{ab}} + \sqrt{\left(\frac{a}{b}\right)^{a+b}}\right) \le 2(a+b)$ (3)

To prove (3), we notice that, for $a \in (0,1]$ and $b \in (0,1]$.

With $a \le b$, we have $\begin{cases} b^{\sqrt{ab}} - a^{\sqrt{ab}} \ge 0 \\ b^{1-\sqrt{ab}} - a^{1-\sqrt{ab}} \ge 0 \end{cases}$ and $\begin{cases} b^{\frac{a+b}{2}} - a^{\frac{a+b}{2}} \ge 0 \\ b^{1-\frac{a+b}{2}} - a^{1-\frac{a+b}{2}} \ge 0 \end{cases}$ (4)

Also: $-2a - 2b = -a^{\sqrt{ab}}a^{1-\sqrt{ab}} - a^{\frac{a+b}{2}}a^{1-\frac{a+b}{2}} - b^{\sqrt{ab}}b^{1-\sqrt{ab}} - b^{\frac{a+b}{2}}b^{1-\frac{a+b}{2}}$ (5)

Now, using (4) and (5), we have

$$a\left(\left(\frac{b}{a}\right)^{\sqrt{ab}} + \sqrt{\left(\frac{b}{a}\right)^{a+b}}\right) + b\left(\left(\frac{a}{b}\right)^{\sqrt{ab}} + \sqrt{\left(\frac{a}{b}\right)^{a+b}}\right) - 2a - 2b$$

$$=2^{1-\sqrt{ab}}b^{\sqrt{ab}}+a^{1-\frac{a+b}{2}}b^{\frac{a+b}{2}}+a^{\sqrt{ab}}b^{1-\sqrt{ab}}+a^{\frac{a+b}{2}}b^{1-\frac{a+b}{2}}$$

$$-a^{\sqrt{ab}}a^{1-\sqrt{ab}}-a^{\frac{a+b}{2}}a^{1-\frac{a+b}{2}}-b^{\sqrt{ab}}b^{1-\sqrt{ab}}-b^{\frac{a+b}{2}}b^{1-\frac{a+b}{2}}$$

$$=\left(a^{1-\sqrt{ab}}b^{\sqrt{ab}}-a^{\sqrt{ab}}a^{1-\sqrt{ab}}\right)+\left(a^{1-\frac{a+b}{2}}b^{\frac{a+b}{2}}-a^{\frac{a+b}{2}}a^{1-\frac{a+b}{2}}\right)$$

$$+\left(a^{\sqrt{ab}}b^{1-\sqrt{ab}}-b^{\sqrt{ab}}b^{1-\sqrt{ab}}\right)+\left(a^{\frac{a+b}{2}}b^{1-\frac{a+b}{2}}-b^{\frac{a+b}{2}}b^{1-\frac{a+b}{2}}\right)$$

$$=a^{1-\sqrt{ab}}\left(b^{\sqrt{ab}}-a^{\sqrt{ab}}\right)+a^{1-\frac{a+b}{2}}\left(b^{\frac{a+b}{2}}-a^{\frac{a+b}{2}}\right)$$

$$-b^{1-\sqrt{ab}}\left(b^{\sqrt{ab}}-a^{\sqrt{ab}}\right)-b^{1-\frac{a+b}{2}}\left(b^{\frac{a+b}{2}}-a^{\frac{a+b}{2}}\right)$$

$$=-\left[\left(b^{\sqrt{ab}}-a^{\sqrt{ab}}\right)\left(b^{1-\sqrt{ab}}-a^{1-\sqrt{ab}}\right)+\left(b^{\frac{a+b}{2}}-a^{\frac{a+b}{2}}\right)\left(b^{1-\frac{a+b}{2}}-a^{1-\frac{a+b}{2}}\right)\right]\le 0$$

This completes the proof of (3). Now, combining the inequalities from (1) and (3), we conclude that

$$4\sqrt{ab}\le a\left(\left(\frac{b}{a}\right)^{\sqrt{ab}}+\sqrt{\left(\frac{b}{a}\right)^{a+b}}\right)+b\left(\left(\frac{a}{b}\right)^{\sqrt{ab}}+\sqrt{\left(\frac{a}{b}\right)^{a+b}}\right)\le 2(a+b)$$

Solution 6 by proposer

Let be $f:[0,1]\to\mathbb{R}; f(x)=a\left(\frac{b}{a}\right)^x+b\left(\frac{a}{b}\right)^x$

$$f'(x)=a\left(\frac{b}{a}\right)^x\ln\frac{b}{a}+b\left(\frac{a}{b}\right)^x\ln\frac{a}{b}=\ln\frac{b}{a}\left[a\left(\frac{b}{a}\right)^x-b\left(\frac{a}{b}\right)^x\right]$$

$$f'(x)=0\Rightarrow a\left(\frac{b}{a}\right)^x=b\left(\frac{a}{b}\right)^x\Rightarrow a\left(\frac{b}{a}\right)^{2x}=b$$

$$\left(\frac{b}{a}\right)^{2x}=\left(\frac{b}{a}\right)^1\Rightarrow 2x=1\Rightarrow x=\frac{1}{2}$$

x	0	$\frac{1}{2}$	1
$f'(x)$	$- - - - - - - - -$	0	$+ + + + + + + +$
$f(x)$	$a+b$ $\searrow$	$2\sqrt{ab}$	$\nearrow$ $a+b$

$$\Rightarrow 2\sqrt{ab}\le a\left(\frac{b}{a}\right)^x+b\left(\frac{a}{b}\right)^x\le a+b \qquad (1)$$

For $x=\sqrt{ab}\in[a,b]\subseteq[0,1]$ in (1):

$$2\sqrt{ab}\le a\left(\frac{b}{a}\right)^{\sqrt{ab}}+b\left(\frac{a}{b}\right)^{\sqrt{ab}}\le a+b \qquad (2)$$

For $x=\frac{a+b}{2}\in[a,b]\subseteq[0,1]$ in (1):

$$2\sqrt{ab} \le a\left(\frac{b}{a}\right)^{\frac{a+b}{2}} + b\left(\frac{a}{b}\right)^{\frac{a+b}{2}} \le a+b \quad (3)$$

By adding (2); (3): $4\sqrt{ab} \le a\left(\left(\frac{b}{a}\right)^{\sqrt{ab}} + \left(\frac{b}{a}\right)^{\frac{a+b}{2}}\right) + b\left(\left(\frac{a}{b}\right)^{\sqrt{ab}} + \left(\frac{a}{b}\right)^{\frac{a+b}{2}}\right) \le 2(a+b)$

$$4\sqrt{ab} \le a\left(\left(\frac{b}{a}\right)^{\sqrt{ab}} + \sqrt{\left(\frac{b}{a}\right)^{a+b}}\right) + b\left(\left(\frac{a}{b}\right)^{\sqrt{ab}} + \sqrt{\left(\frac{a}{b}\right)^{a+b}}\right) \le 2(a+b)$$

PROBLEMS FOR JUNIORS

J.841 If $m, n \ge 0; n \le 4(m+1); x, y, z \in \mathbb{R}_+^* = (0, \infty)$, then in ΔABC with the area F the following inequality holds:

$$\frac{x^{m+2} \cdot a^{4m-n+4}}{(y+z)^m \cdot h_a^n} + \frac{y^{m+2} \cdot b^{4m-n+4}}{(z+x)^m \cdot h_b^n} + \frac{z^{m+2} \cdot c^{4m-n+4}}{(x+y)^m \cdot h_c^n} \ge$$
$$\ge \frac{2^{3m-n+4} \cdot (x-y+yz+zx)}{3^{m+1}} F^{2m-n+2}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

J.842 If $m \ge 0; x, y, z > 0$, then in ΔABC with the area F the following inequality holds:

$$xa^{2m+2} + yb^{2m+2} + z^{2m+2} \ge \frac{4^{m+1}}{3^m}\left((xy)^{\frac{1}{m+1}} + (yz)^{\frac{1}{m+1}} + (zx)^{\frac{1}{m+1}}\right)^{\frac{n+1}{2}} F^{m+1}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

J.843 If $x, y \in \mathbb{R}_+ = [0, \infty), x+y=2$ then in any ΔABC with the area F the following inequality holds:

$$\frac{(a^2+b^2) \cdot m_c^x}{c^y} + \frac{(b^2+c^2) \cdot m_a^x}{a^x} + \frac{(c^2+a^2) \cdot m_b^x}{b^x} \ge 3 \cdot 2^{x+1} F^x$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

J.844 If ABC is a triangle with the area F and $x, y, z \in \mathbb{R}_+^* = (0, \infty)$, then:

$$(x^2a^4 + y^2b^4 + z^2c^4)\left(\frac{1}{(x+y)^2} + \frac{1}{(y+z)^2} + \frac{1}{(z+x)^2}\right) \ge 12F^2$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

J.845 If $x, y, z \in \mathbb{R}_+^* = (0, \infty), t \in \mathbb{R}_+ = [0, \infty)$ and ABC is a triangle with the area F, then:

$$\frac{4x + 3y + z + 2t}{y + 3z + t} a^4 + \frac{x + 4y + 3z + 2t}{z + 3x + t} b^4 + \frac{3x + y + 4z + 2t}{x + 3y + t} c^4 \geq 32F^2$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

J.846 If $x, y, z \in \mathbb{R}_+^* = (0, \infty)$ and $u \in \mathbb{R}_+ = [0, \infty)$, then in any ΔABC the following inequality holds:

$$\frac{y + z + 6u}{x + 3u} \cdot \frac{1}{a} + \frac{z + x + 6u}{y + 3u} \cdot \frac{1}{b} + \frac{x + y + 6u}{z + 3u} \cdot \frac{1}{c} \geq \frac{2\sqrt{3}}{R}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

J.847 Let M an interior point in ΔABC with the area F, x, y, z the distances from point M to the apices A, B, C and d_a, d_b, d_c the distances from M to the sides BC, CA, AB, then:

$$\frac{x^2 a^3}{d_a(d_b + d_c)h_a} + \frac{y^2 b^3}{d_b(d_c + d_a)h_b} + \frac{z^2 c^3}{d_c(d_a + d_b)h_c} \geq 8F$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

J.848 In any ΔABC the following inequality holds:

$$\left(\sum_{cyc} \frac{w_a^3}{h_b + h_c}\right)\left(\sum_{cyc} \frac{1}{(h_a + h_b)^2}\right) \geq \frac{9}{8}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

J.849 If $m, n \in \mathbb{R}_+ = [0, \infty), m + n = 2$ and ABC a triangle with the semiperimeter s and the area F, then:

$$\frac{a^m}{h_a^n} + \frac{b^m}{h_b^n} + \frac{c^m}{h_c^n} \geq \frac{2^m 9}{35F^n}\left(s^2 + \frac{abc}{a + b + c}\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

J.850 If $m, n \in \mathbb{R}_+ = [0, \infty), m + n = 4$ and ΔABC with F area, then prove:

$$\frac{a^m}{h_a^n} + \frac{b^m}{h_b^n} + \frac{c^m}{h_c^n} \geq 2^n F^{2-n}$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania

J.851 If $x, y, z \in \mathbb{R}_+^* = (0, \infty)$ and ΔABC with F area, then prove:

$$\frac{y + z + 6t}{x + 3t} h_a + \frac{z + x + 6t}{y + 3t} h_b + \frac{x + y + 6t}{z + 3t} h_c \geq \frac{36r^2}{R}$$

where $t \in [0, \infty)$. ***Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania***

J.852 If $x, y, z \in \mathbb{R}_+^* = (0, \infty)$ and ΔABC with F area, then prove:

$$\frac{y+z}{x}h_a+\frac{z+x}{y}h_b+\frac{x+y}{z}h_c\geq\frac{36r^2}{R}$$

Proposed by D.M.Bătineţu-Giurgiu, Neculai Stanciu-Romania

J.853 Let be $m\in\mathbb{N}, n,p\in[0,\infty), n+p=4$ and ΔABC with F area, then prove:

$$3m+\left(\frac{x^2a^n}{h_a^p}\right)^{m+1}+\left(\frac{y^2b^n}{h_b^p}\right)^{m+1}+\left(\frac{z^2c^n}{h_c^p}\right)^{m+1}\geq\frac{2^n}{3}(m+1)(xy+yz+zx)F^{n-2}$$

Proposed by D.M.Bătineţu-Giurgiu, Neculai Stanciu-Romania

J.854 If $x,y,z\in\mathbb{R}_+^*=(0,\infty); m,n,p\in[0,\infty), m+n=4$ and ΔABC with F area, then prove:

$$\frac{4x+3y+z+2p}{y+3z+p}\cdot\frac{a^m}{h_a^n}+\frac{x+4y+3z+2p}{z+3x+p}\cdot\frac{b^m}{h_b^n}+\frac{3x+y+4z+2p}{x+3y+p}\cdot\frac{c^m}{h_c^n}\geq 2^{5-n}F^{2-n}$$

Proposed by D.M.Bătineţu-Giurgiu, Neculai Stanciu-Romania

J.855 If $x,y,z\in\mathbb{R}_+^*=(0,\infty)$ and ΔABC with F area, then prove:

$$(ax+by)^4h_ah_b+(bx+cy)^4h_bh_c+(cx+ay)^4h_ch_a\geq 16\sqrt{3}(x+y)^4F^3$$

Proposed by D.M.Bătineţu-Giurgiu, Neculai Stanciu-Romania

J.856 If $x,y,z\in\mathbb{R}_+^*=(0,\infty)$ then prove:

$$\left(\frac{x^3}{y+z}+\frac{y^3}{z+x}+\frac{z^3}{x+y}\right)\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right)\geq\frac{9}{8}$$

Proposed by D.M.Bătineţu-Giurgiu, Neculai Stanciu-Romania

J.857 If $m,n\in\mathbb{R}_+=[0,\infty), m+n=2$ and ΔABC with F area, then prove:

$$\frac{a^m}{h_a^n}+\frac{b^m}{h_b^n}+\frac{c^m}{h_c^n}\geq 2^n\sqrt{3}F^{1-n}+\frac{1}{2^nF^n}\left((a-b)^2+(b-c)^2+(c-a)^2\right)$$

Proposed by D.M.Bătineţu-Giurgiu, Neculai Stanciu-Romania

J.858 $A=\{x/x>0, x^{\sqrt{x}}=4\cdot 2^{\sqrt{x}}\}, B=\{y/y>0, y^{\sqrt{y}}=27\cdot 9^{\sqrt[3]{y}}\}$

$C=\left\{z/z>0, z^{\sqrt{z}}=256\cdot 64^{\sqrt[4]{z}}\right\}$. Find the set Ω such that:

$$A\Delta\Omega\Delta B=C, (X\Delta Y=(X/Y)\cup(Y/X)$$

Proposed by Daniel Sitaru – Romania

J.859 Solve for real numbers:

$$\begin{cases} 0<x,y<\dfrac{\pi}{2}, x+y=\dfrac{5\pi}{6} \\ 4\sin^2 x\sin^2 y(\sin^2 x+\sin^2 y)+4\cos^2 x\cos^2 y(\cos^2 x+\cos^2 y)=\sin^2 2x+\sin^2 2y \end{cases}$$

Proposed by Daniel Sitaru – Romania

J.860 Solve for real numbers:

$$4(\csc^2(x+y)+\csc^2(y+z))=(\csc x+\csc z)(\csc y+\csc(x+y+z))$$

Proposed by Daniel Sitaru – Romania

J.861 If $0<a\le b\le\frac{\sqrt{3}}{3}$ then:

$$a^2b^2(2-a-b)^2(2+a+b)^2\le(1-a^2)(1-b^2)(a+b)^4$$

Proposed by Daniel Sitaru – Romania

J.862 $A=\left\{x|x\in\mathbb{Z},\left[\frac{x^7-14x^5+49x^3-36x}{56}\right]=0,[*]-GIF\right\}$. Find:

$$\Omega=\sum_{x\in A}x$$

Proposed by Daniel Sitaru – Romania

J.863 Solve for real numbers:

$$|\cos x|+|\cos y|=\sqrt{(2+\sin x+\sin y)(2-\sin x-\sin y)}$$

Proposed by Daniel Sitaru – Romania

J.864 In ΔABC the following relationship holds:

$$648\sqrt{3}r^3\le a(a-3b-3c)^2+b(3a-b-c)^2+c(3a-b-c)^2\le 81\sqrt{3}R^3$$

Proposed by Daniel Sitaru – Romania

J.865 Solve for real numbers:

$$\begin{cases} x,y,z>0 \\ \frac{x}{x+1}+\frac{y}{(x+1)(y+1)}+\frac{z}{(x+1)(y+1)(z+1)}+\frac{1}{8\sqrt{xyz}}=1 \end{cases}$$

Proposed by Daniel Sitaru – Romania

J.866 Solve for real numbers:

$$\begin{cases} [x]\cdot\{x\}+1=y,[*]-GIF \\ \sqrt{\frac{xyz}{x^2-xy+y^2}}+\sqrt{\frac{xyz}{y^2-yz+z^2}}+\sqrt{\frac{xyz}{z^2-zx+x^2}}=\sqrt{x}+\sqrt{y}+\sqrt{z} \\ [y]\cdot\{y\}+1=z,\{*\}=*-[*] \end{cases}$$

Proposed by Daniel Sitaru – Romania

J.867 If $1<a<2<b<3<c<4<d<5$ then:

$$\sum_{cyc}\frac{a^5bcd}{(b+c+d)(a-b)(a-c)(a-d)}<\left(\frac{20}{3}\right)^4$$

Proposed by Daniel Sitaru – Romania

J.868 Solve for natural numbers:

$$\frac{(x-2)!!\,(x-3)!!}{(x-4)!!\,(x-5)!!}+\frac{(x-3)!!\,(x-4)!!}{(x-5)!!\,(x-6)!!}+\frac{(x-4)!!\,(x-5)!!}{(x-6)!!\,(x-7)!!}=38$$

Proposed by Daniel Sitaru – Romania

J.869 In $\Delta ABC, G$ – centroid, $E\in(AB), F\in(AC), E, F, G$ – collinears,

$S_1=[AEF], S_2=[EBCF]$. It's possible that: $\frac{S_1}{S_2}<\frac{4}{5}$?

Proposed by Marian Ursărescu – Romania

J.870 In ΔABC the following relationship holds:

$$\frac{2r}{R}+\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}\geq 4$$

Proposed by Marian Ursărescu – Romania

J.871 If in $\Delta ABC, I_a, I_b, I_c$ – excenters then: $S[I_aBC]+S[I_bCA]+S[I_cAB]\geq 3S$

Proposed by Marian Ursărescu – Romania

J.872 If $x,y,z>0$ then:

$$\frac{x^7}{x+2y}+\frac{y^7}{y+2z}+\frac{z^7}{z+2x}\geq x^2y^2z^2$$

Proposed by Marian Ursărescu – Romania

J.873 In any acute ΔABC the following relationship holds:

$$\max\left(\frac{\sin\frac{A}{2}}{\cos\left(\frac{B-C}{2}\right)},\frac{\sin\frac{B}{2}}{\cos\left(\frac{C-A}{2}\right)},\frac{\sin\frac{C}{2}}{\cos\left(\frac{A-B}{2}\right)}\right)\geq\frac{r}{R}$$

Proposed by Marian Ursărescu – Romania

J.874 In ΔABC the following relationship holds:

$$(b+c)m_a+(c+a)m_b+(a+b)m_c\leq 6sR$$

Proposed by Marian Ursărescu – Romania

J.875 If in acute $\Delta ABC, H$ – orthocenter, then:

$$\frac{bc}{AH}+\frac{ca}{BH}+\frac{ab}{CH}\geq\sqrt{3}(a+b+c)$$

Proposed by Marian Ursărescu – Romania

J.876 In acute tetrahedron $ABCD$:

$$\frac{m_A}{\sqrt{h_A\cdot r_A}}+\frac{m_B}{\sqrt{h_B\cdot r_B}}+\frac{m_C}{\sqrt{h_C\cdot r_C}}\leq\frac{4\sqrt{2}}{3}\cdot\frac{R}{r}$$

Proposed by Marian Ursărescu – Romania

J.877 If $a < b < c < d < e < f < g < h, a, b, c, d, e, f, g, h \in \mathbb{R}$ then:

$$(a+b+c+d+e+f+g+h)^2 \geq 16(ah+bg+cf+de)$$

Proposed by Marian Ursărescu – Romania

J.878 If ΔDEF is pedal triangle of I – incentre of ΔABC then:

$$\frac{S[ABC]}{S[DEF]} \leq \frac{R}{r} + \frac{r}{R} + \frac{3}{2}$$

Proposed by Marian Ursărescu – Romania

J.879 In ΔABC the following relationship holds: $(h_a + r_a)(h_b + r_b)(h_c + r_c) \geq 216r^3$

Proposed by Marian Ursărescu – Romania

J.880 In ΔABC the following relationship holds:

$$\sum \frac{m_a^2}{h_a} \leq \sum \frac{m_a^2}{r_a}$$

Proposed by Marin Chirciu – Romania

J.881 In ΔABC the following relationship holds:

$$\left(\frac{2r}{R}\right)^{\frac{3}{2}} \leq \frac{4}{\sqrt{4 + \sum \csc^2 \frac{A}{2}}} \leq 1$$

Proposed by Marin Chirciu – Romania

J.882 In ΔABC the following relationship holds:

$$18r\sqrt{r} \leq \sum \sqrt{bc \cdot h_a} \leq 4(R+r)^2 \sqrt{\frac{1}{2R}}$$

Proposed by Marin Chirciu – Romania

J.883 In ΔABC the following relationship holds:

$$3 \leq \sum \frac{m_a^2}{r_b r_c} \leq \frac{R}{r} + \frac{2r}{R}$$

Proposed by Marin Chirciu – Romania

J.884 In ΔABC the following relationship holds:

$$\frac{8r^2}{3R}(4R+r)^2 \leq \sum bch_a \leq \frac{2R}{3}(4R+r)^2$$

Proposed by Marin Chirciu – Romania

J.885 In ΔABC the following relationship holds:

$$\sum \frac{\tan B + \tan C}{a} \ge 3 \sum \frac{\cot B + \cot C}{a}$$

Proposed by Marin Chirciu – Romania

J.886 In ΔABC the following relationship holds:

$$18r\sqrt{r} \le \sum \sqrt{bc \cdot w_a} \le 6(R+r)\sqrt{\frac{R}{2}}$$

Proposed by Marin Chirciu – Romania

J.887 In ΔABC the following relationship holds:

$$m_a^2 + m_b^2 + m_c^2 \ge p^2 + \lambda r(R - 2r), \lambda \le 2$$

Proposed by Marin Chirciu – Romania

J.888 In $\Delta ABC, I$ – incenter, R_a, R_b, R_c – circumradii of $\Delta BIC, \Delta CIA, \Delta AIB$. Prove that:

$$\frac{3R^2}{p} \le \sum \frac{R_a^2}{h_a(\sin B + \sin C)} \le \frac{3R^5}{8r^3 p}$$

Proposed by Marin Chirciu – Romania

J.889 In $\Delta ABC, I$ – incenter, R_a, R_b, R_c – circumradii of $\Delta BIC, \Delta CIA, \Delta AIB$. Prove that:

$$\sum \frac{R_a^2}{r_a^2} \le \sum \frac{R_a^2}{h_a^2}$$

Proposed by Marin Chirciu – Romania

J.890 In acute ΔABC holds:

$$\sqrt{6(1 + \cos A \cos B \cos C)} \le \frac{1}{2} \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}$$

Proposed by Marin Chirciu – Romania

J.891 In ΔABC the following relationship holds:

$$2 - \frac{2r}{R} \le \frac{\sum r_a^2}{\sum r_b r_c} \le \frac{R}{r} - 1$$

Proposed by Marin Chirciu – Romania

J.892 In ΔABC the following relationship holds:

$$\sum \frac{\cos B + \cos C}{a} \le \frac{p}{9r} \sum \frac{\sin B + \sin C}{a}$$

Proposed by Marin Chirciu – Romania

J.893 In acute ΔABC holds:

$$\sum \frac{a}{b^2 + c^2 - a^2} \ge \frac{9R}{4S}$$

Proposed by Marin Chirciu – Romania

J.894 In ΔABC the following relationship holds:

$$\left(\frac{2r}{R}\right)^{\frac{3}{2}} \leq \frac{5}{\sqrt{16+\sum \cot^2 \frac{A}{2}}} \leq 1$$

Proposed by Marin Chirciu – Romania

J.895 If $x, y, z \geq 1$ then:

$$\left((x+1)(y+1)(z+1)+8xyz\right)\prod_{cyc}(x+3) \geq 16\prod_{cyc}(3x+1)(3y+1)(3z+1)$$

Proposed by Daniel Sitaru – Romania

J.896 $(z^2+z-2)^2+(2z^2+5z+2)^2+(2z^2-z-1)^2=0, z \in \mathbb{C}$.Find: $\Omega=|z|$

Proposed by Daniel Sitaru – Romania

J.897 Solve for natural numbers:

$$34+\prod_{cyc}(2x+1)(2y+1)(2z+1)=2\prod_{cyc}(2x+1)$$

Proposed by Daniel Sitaru – Romania

J.898 Solve for real numbers:

$$\cos 2x+\frac{\sin^3 x-\cos^3 x}{\sin^3 x+\cos^3 x}=\tan\left(x-\frac{\pi}{4}\right)$$

Proposed by Daniel Sitaru – Romania

J.899 Solve for real numbers:

$$\begin{cases} \dfrac{\tan^2 x+\tan^2 y}{\cos^2 y}+\left(\dfrac{\tan y}{\cos z}\right)^2+\left(\dfrac{\tan z}{\cos x}\right)^2=29 \\ \dfrac{\tan^2 y+\tan^2 z}{\cos^2 z}+\left(\dfrac{\tan z}{\cos x}\right)^2+\left(\dfrac{\tan x}{\cos y}\right)^2=19 \\ \dfrac{\tan^2 z+\tan^2 z}{\cos^2 x}+\left(\dfrac{\tan x}{\cos y}\right)^2+\left(\dfrac{\tan y}{\cos z}\right)^2=23 \end{cases}$$

Proposed by Daniel Sitaru – Romania

J.900 Solve for real numbers:

$$\left|\left|\frac{\tan^2 x-1}{\tan^4 x}\right|+\frac{1}{\sin^2 x}-2\cot^3 x\right|+2(\cot x+\cot^3 x)=4$$

Proposed by Daniel Sitaru – Romania

J.901 In ΔABC the following relationship holds:

$$\frac{h_a r_b^2}{m_a}+\frac{h_b r_c^2}{m_b}+\frac{h_c r_a^2}{m_c}\geq\frac{\sqrt{6}}{3}\cdot\frac{r(4R+r)^2}{R+(\sqrt{6}-2)r}$$

Proposed by Daniel Sitaru – Romania

J.902 If $a,b,c>0$ then:

$$\frac{(a^{10}+b^{10})(b^{11}+c^{11})(c^{12}+a^{12})(a^3b^3+b^3c^3+c^3a^3)}{a^9b^9c^9(a^7+b^7)(b^8+c^8)(c^9+a^9)}\geq 27$$

Proposed by Daniel Sitaru – Romania

J.903 In ΔABC the following relationship holds:

$$\frac{a^2-b^2}{\sin C}+\frac{b^2-c^2}{\sin A}+\frac{c^2-a^2}{\sin B}\geq\frac{2(b-c)(a-b)(a-c)}{R}$$

Proposed by Daniel Sitaru – Romania

J.904 In $\Delta ABC, n_a$ –Nagel's cevian, the following relationship holds:

$$\frac{h_a+h_b+h_c}{2r}\geq\sum_{cyc}\frac{n_a\sqrt{r_br_c-r^2}}{m_a^2}$$

Proposed by Bogdan Fuștei-Romania

J.905 In acute ΔABC the following relationship holds:

$$\frac{1}{2}\cdot\sum_{cyc}|b-c|\cdot s_a\leq\sum_{cyc}\sqrt{r_br_c(r_br_c-s_a^2)}$$

Proposed by Bogdan Fuștei-Romania

J.906 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{1}{h_a^2}\geq\frac{\sum m_am_b}{3F^2}$$

Proposed by Bogdan Fuștei-Romania

J.907 In ΔABC the following relationship holds:

$$r_a+r_b+r_c=\frac{R}{2r}\left[2(h_a+h_b+h_c)-\sum_{cyc}\frac{h_bh_c}{h_a}\right]$$

Proposed by Bogdan Fuștei-Romania

J.908 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{m_ah_a}{h_bh_c}\geq\frac{1}{2}\sum_{cyc}\frac{b^2+c^2}{a^2}$$

Proposed by Bogdan Fuștei-Romania

J.909 In $\Delta ABC, n_a$ —Nagel's cevian, g_a —Gergonne cevian, the following relationship holds:

$$\prod_{cyc}(n_a - m_a) \geq \sqrt{g_a g_b g_c}\prod_{cyc}(\sqrt{m_a} - \sqrt{g_a})$$

Proposed by Bogdan Fuştei-Romania

J.910 In $\Delta ABC, n_a$ —Nagel's cevian, the following relationship holds:

$$\frac{m_a}{n_a} + \frac{m_b}{n_b} + \frac{m_c}{n_c} \geq 2\sum_{cyc}\frac{m_b + m_c - m_a}{\sqrt{n_b n_c}}$$

Proposed by Bogdan Fuştei-Romania

J.911 In ΔABC the following relationship holds:

$$s + \sqrt{ab} + \sqrt{bc} + \sqrt{ca} \geq (2b + 2c - a)sinA\sum_{cyc} tan\frac{A}{2}$$

Proposed by Bogdan Fuştei-Romania

J.912 In ΔABC the following relationship holds:

$$\sum_{cyc}\left(\frac{m_a}{w_a} + \sqrt{\frac{m_a}{r_a}} + \sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}}\right) \leq 4\left(1 + \frac{r}{R}\right)$$

Proposed by Bogdan Fuştei-Romania

J.913 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{\sqrt{r_a h_a}}{w_a} \geq \left(1 + \frac{2}{3}\sum_{cyc}\frac{m_a}{h_a}\right)\sqrt{\frac{2r}{R}}$$

Proposed by Bogdan Fuştei-Romania

J.914 If $\begin{cases} x^2 + xy + y^2 = 36, x, y, z > 0 \\ y^2 + yz + z^2 = 100 \\ z^2 + zx + x^2 = 64 \\ a + b + c = 8\sqrt[4]{3}, a, b, c > 0 \end{cases}$ then: $a^2x + b^2y + c^2z \geq 2xyz$

Proposed by Daniel Sitaru – Romania

J.915 If $a, b > 0$ then:

$$\frac{4\sqrt{ab}}{a+b} + \frac{(a+b)^2}{4ab} \geq \frac{a+b}{\sqrt{ab}} + \frac{4ab}{(a+b)^2}$$

Proposed by Daniel Sitaru – Romania

J.916 Solve for real numbers:

$$(1 - 7x)(1 - 9x)(1 - 11x) = \frac{225x^6}{(x-1)(3x-1)(5x-1)}$$

Proposed by Daniel Sitaru – Romania

J.917 Solve for real numbers:

$$\begin{cases} 0 < x, y, z \le 1 \\ (x^2+1)(y^2+1)(z^2+1) = 8 + (x^2-1)(y^2-1)(z^2-1) \end{cases}$$

Proposed by Daniel Sitaru – Romania

J.918 $A(0,0,3), B(4,0,0), C(5,5,0), D(0,6,0)$. Find: $\Omega = m\big(\sphericalangle(AB, CD)\big)$.

Proposed by Daniel Sitaru – Romania

J.919 $ABCD$ – tetrahedron $AB = 5, AC = \sqrt{59}, BC = CD = \sqrt{26}, AD = 3\sqrt{5}, BD = \sqrt{42}$.

Find: $\Omega = m\big(\sphericalangle(AB, CD)\big)$.

Proposed by Daniel Sitaru – Romania

J.920 $ABCD$ – tetrahedron, O – center of circumsphere, G – centroid

$AB = BC = DB = 3, AC = CD = DA = 3\sqrt{2}$. Find: $\Omega = OG$.

Proposed by Daniel Sitaru – Romania

J.921 If $x \in \mathbb{R}$ then: $2\cos^4 x + 3\cos^4(2x) + 6\cos^4(4x) > \frac{1}{4}$

Proposed by Daniel Sitaru – Romania

J.922 $ABCD$ – tetrahedron, $m(\sphericalangle BDC) = m(\sphericalangle CDA) = 60°, DA = DB = DC = 6$.

If Volume $[ABCD] = 9\sqrt{6}$ then find: $\Omega = m(\sphericalangle ADB)$

Proposed by Daniel Sitaru – Romania

J.923 If $0 \le x < \frac{\pi}{4}$ then:

$$\tanh(\sin 2x) \le \tanh(\sin x \cos x) + \frac{\sin x \cos x}{\cosh^2(\sin x \cos x)}$$

Proposed by Daniel Sitaru – Romania

J.924 In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{\cos(B-C) - 2\cos A}{h_a} = 0$$

Proposed by Adil Abdullayev- Azerbaijan

J.925 In ΔABC the following relationship holds:

$$\frac{m_a m_b m_c}{w_a w_b w_c} \le \frac{(a+b)(b+c)(c+a)}{8abc} \cdot \frac{R}{2r}$$

Proposed by Adil Abdullayev- Azerbaijan

J.926 In $\Delta ABC, m(\sphericalangle A) = \frac{m(\sphericalangle B)}{2} = \frac{m(\sphericalangle C)}{4}$, I – incentre, Ω_1 – first Brocard point. Prove that:

$$\Omega_1 I = \frac{R - 2r}{2}$$

Proposed by Adil Abdullayev- Azerbaijan

J.927 In ΔABC the following relationship holds:

$$\frac{R}{r} \geq \frac{4(m_a^2 + m_b^2)}{(m_a + m_b)^2}$$

Proposed by Adil Abdullayev- Azerbaijan

J.928 In ΔABC the following relationship holds:

$$2\left(\frac{R}{r} + \frac{r}{R}\right) + \frac{w_a w_b w_c}{r_a r_b r_c} \geq 6$$

Proposed by Adil Abdullayev- Azerbaijan

J.929 In ΔABC the following relationship holds:

$$\frac{w_a w_b w_c}{h_a h_b h_c} \leq \frac{R}{2r}$$

Proposed by Adil Abdullayev- Azerbaijan

J.930 In ΔABC the following relationship holds:

$$\frac{w_a}{w_b + w_c} + \frac{w_b}{w_c + w_a} + \frac{w_c}{w_a + w_b} \leq \frac{3R}{4r}$$

Proposed by Adil Abdullayev- Azerbaijan

J.931 In ΔABC the following relationship holds:

$$\frac{w_a w_b w_c}{h_a h_b h_c} \geq \frac{9R^2}{a^2 + b^2 + c^2}$$

Proposed by Adil Abdullayev- Azerbaijan

J.932 In ΔABC the following relationship holds:

$$\frac{m_a + m_b}{\sqrt{m_a^2 + m_b^2}} + \frac{m_b + m_c}{\sqrt{m_b^2 + m_c^2}} + \frac{m_c + m_a}{\sqrt{m_c^2 + m_a^2}} \geq 6 \cdot \sqrt{\frac{r}{R}}$$

Proposed by Adil Abdullayev- Azerbaijan

J.933 In ΔABC the following relationship holds:

$$\frac{R}{2r} \geq \sqrt{1 + \frac{27(r_a - r_b)^2(r_a + r_b + 2r_c)^2}{16(4R + r)^2}}$$

Proposed by Adil Abdullayev- Azerbaijan

J.934 In ΔABC the following relationship holds:

$$\frac{h_a}{w_a^2}+\frac{h_b}{w_b^2}+\frac{h_c}{w_c^2}=\frac{1}{2r}+\frac{1}{R}$$

Proposed by Adil Abdullayev- Azerbaijan

J.935 In ΔABC the following relationship holds:

$$\frac{16abc}{(a+b)(b+c)(c+a)}+\sum_{cyc}\frac{n_a^2}{h_a^2}\geq 5+2\sum_{cyc}\frac{(a-b)^2}{ab}$$

Proposed by Adil Abdullayev- Azerbaijan

J.936 In ΔABC the following relationship holds:

$$\sqrt{\frac{1}{m_a^2}+\frac{1}{m_b^2}+\frac{1}{m_c^2}}\leq\frac{m_a+m_b+m_c}{3F}$$

Proposed by Adil Abdullayev- Azerbaijan

J.937 In ΔABC the following relationship holds:

$$(m_a+m_b+m_c)\left(\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}\right)\leq 3\left(\frac{n_a}{r_a}+\frac{n_b}{r_b}+\frac{n_c}{r_c}\right)$$

Proposed by Adil Abdullayev- Azerbaijan

J.938 In ΔABC the following relationship holds:

$$\sqrt{\frac{m_am_bm_c}{h_ah_bh_c}}\leq\frac{R}{2r}$$

Proposed by Adil Abdullayev- Azerbaijan

J.939 In ΔABC the following relationship holds:

$$\frac{m_a^2+m_b^2+m_c^2}{m_am_b+m_bm_c+m_cm_a}+\frac{3h_ah_bh_c}{m_am_bm_c}\leq 4$$

Proposed by Adil Abdullayev- Azerbaijan

J.940 In acute ΔABC the following relationship holds:

$$\left(\frac{r_b}{r_c}+\frac{r_c}{r_a}\right)\cos A+\left(\frac{r_c}{r_a}+\frac{r_a}{r_c}\right)\cos B+\left(\frac{r_a}{r_b}+\frac{r_b}{r_c}\right)\cos C\geq 3$$

Proposed by Adil Abdullayev- Azerbaijan

J.941 In acute $\Delta ABC, o_a, o_b, o_c$ – circumcevians. Prove that:

$$\frac{2(o_a+o_b+o_c)}{o_a+o_b+o_c+h_a+h_b+h_c}\leq\frac{R}{2r}$$

Proposed by Adil Abdullayev- Azerbaijan

J.942 In any triangle ABC the relationship holds:

$$h_a^{\frac{3}{2}}+h_b^{\frac{3}{2}}+h_c^{\frac{3}{2}}\leq h_a\sqrt{w_a}+h_b\sqrt{w_b}+h_c\sqrt{w_c}<(a^2+b^2+c^2)^{\frac{3}{4}}\leq(3R)^{\frac{3}{2}}$$

Proposed by Nguyen Van Canh – BenTre – Vietnam

J.943 In any triangle ABC the relationship holds:

$$h_a^{\frac{3}{2}} + h_b^{\frac{3}{2}} + h_c^{\frac{3}{2}} \le h_a\sqrt{m_a} + h_b\sqrt{m_b} + h_c\sqrt{m_c} < (a^2+b^2+c^2)^{\frac{3}{2}}$$

Proposed by Nguyen Van Canh – BenTre – Vietnam

J.944 In any triangle ABC the following relationship holds: $w_a^2 + w_b^2 + w_c^2 + r(R-2r) \le p^2$

Proposed by Nguyen Van Canh – BenTre – Vietnam

J.945 In triangle ABC the following relationship holds:

$$w_a^2 + w_b^2 + w_c^2 + nr(R-2r) \le p^2, 0 \le n \le 2$$

Proposed by Nguyen Van Canh – BenTre – Vietnam

J.946 In ΔABC, the following relationship holds:

1. $\left(\frac{a+b}{b+c} + \frac{b+c}{c+a} + \frac{c+a}{a+b}\right)^3 \ge \frac{32p^2}{p^2+2Rr+r^2}$
2. $\frac{a+b}{b+c} + \frac{b+c}{c+a} + \frac{c+a}{a+b} \ge \frac{9(3p^2-4Rr-r^2)}{2p^2}$

Proposed by Nguyen Van Canh – BenTre – Vietnam

J.947 If a, b are positive real numbers then:

$$\left(\frac{a+b}{2}\right)^4 \overset{(1)}{\le} \left(\frac{a^2+b^2}{2}\right)^2 \overset{(2)}{\le} \frac{(a^3+b^3)(a+b)}{4} \overset{(3)}{\le} \frac{a^4+b^4}{2}$$

Proposed by Nguyen Van Canh – BenTre – Vietnam

J.948 If a, b, c are positive real numbers such that $abc = 1$ then:

$$\frac{a^4}{b^4+5a^2} + \frac{b^4}{c^4+5a^2} + \frac{c^4}{a^4+5b^2} + \frac{a^3+b^3+c^3}{3} \ge \frac{3}{2}$$

Proposed by Nguyen Van Canh – BenTre – Vietnam

J.949 If a, b, c are positive real numbers and $abc = 1$ then:

$$6\left(\sum_{cyc} ab(a^4+b^4) + \sum a^2b^2(a^2+b^2)\right) + 4\left(\sum a^3\right) \ge 30 + 9\left(\sum ab\,(a+b)\right)$$

Proposed by Nguyen Van Canh – BenTre – Vietnam

J.950 In any triangle ABC the following relationship holds:

$$h_a^{\frac{3}{2}} + h_b^{\frac{3}{2}} + h_a^{\frac{3}{2}} < a\sqrt{m_a} + b\sqrt{m_b} + c\sqrt{m_c} \le 9R\sqrt{\frac{R}{2}}$$

Proposed by Nguyen Van Canh – BenTre – Vietnam

J.951 If a, b, c are positive real numbers such that $ab + bc + ca = 3$, then:

$$1.\ a+b+c\geq abc+2\sqrt{\frac{a^2b^2+b^2c^2+c^2a^2}{3}}$$

$$2.\ \frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}+\frac{81}{4}\sum\frac{a^2b}{(2a+b)^2}\geq\frac{39}{4}$$

Proposed by Nguyen Van Canh – BenTre – Vietnam

J.952 In any triangle ABC holds:

$$\frac{r_a+r_b}{\sqrt{h_a^2+h_b^2}}+\frac{r_b+r_c}{\sqrt{h_b^2+h_c^2}}+\frac{r_c+r_a}{\sqrt{h_c^2+h_a^2}}\geq 6\sqrt{\frac{r}{R}}$$

Proposed by Nguyen Van Canh – BenTre – Vietnam

J.953 If x,y,z are positive real numbers then:

$$\sqrt{xy(x+y)}+\sqrt{yz(y+z)}+\sqrt{zx(z+x)}\geq\sqrt{(x+y)(y+z)(z+x)}+\sqrt{2xyz}$$

Proposed by Nguyen Van Canh – BenTre – Vietnam

J.954 If $a,b,c>0$ and $a+b+c=1$ then:

$$\frac{1}{a^2+b^2+c^2}+\frac{1}{ab(a+b)}+\frac{1}{bc(b+c)}+\frac{1}{ca(c+a)}\geq\frac{87}{2}$$

Proposed by Nguyen Van Canh – BenTre – Vietnam

J.955 In any triangle ABC holds:

$$9\leq\left(\sqrt{\frac{w_a}{m_a}}+\sqrt{\frac{w_b}{m_b}}+\sqrt{\frac{w_c}{m_c}}\right)\left(\sqrt{\frac{m_a}{w_a}}+\sqrt{\frac{m_b}{w_b}}+\sqrt{\frac{m_c}{w_c}}\right)\leq\frac{4R}{r}+1$$

Proposed by Nguyen Van Canh – BenTre – Vietnam

J.956 In any triangle ABC holds:

$$\min\left\{\sin^3\frac{A}{2}+\cos^3\frac{A}{2};\sin^3\frac{B}{2}+\cos^3\frac{B}{2};\sin^3\frac{C}{2}+\cos^3\frac{C}{2}\right\}<\frac{R}{2r}$$

Proposed by Nguyen Van Canh – BenTre – Vietnam

J.957 In ΔABC the following relationship holds:

$$\frac{9}{2R}\leq\sum\frac{r_a}{bc}\cot^2\frac{A}{2}\leq\frac{9}{4r}$$

Proposed by Marin Chirciu – Romania

J.958 In ΔABC the following relationship holds:

$$\left(\frac{2r}{R}\right)^{\frac{3}{2}}\leq\frac{1}{\sqrt{\frac{1}{4}+\sum\sin^2\frac{A}{2}}}\leq 1$$

Proposed by Marin Chirciu – Romania

J.959 In ΔABC the following relationship holds:

$$72pR^3r^2 \leq \sum a^6 \tan\frac{A}{2} \leq 16pR^2 \cdot (4R^3 - 23r^3)$$

Proposed by Marin Chirciu – Romania

J.960 In acute ΔABC holds:

$$\sum \cos^3 A \cos B \geq \frac{1}{2}\left(\frac{r^3}{R^3} + \frac{3r^2}{R^2} + \frac{r}{R} - 1\right)$$

Proposed by Marin Chirciu – Romania

J.961 If $a, b \geq 0$ such that $a + b = 2$ then: $(2 + a^2)(2 + b^2) \geq (2 + a)(2 + b)$

Proposed by Marin Chirciu – Romania

J.962 In ΔABC the following relationship holds:

$$9r \leq \sum \sqrt{\frac{m_b^2 + m_c^2}{2}} \leq \frac{9R}{2}$$

Proposed by Marin Chirciu – Romania

J.963 In ΔABC the following relationship holds:

$$8 \leq \frac{(h_a + r_a)(h_b + r_b)(h_c + r_c)}{w_a w_b w_c} \leq \frac{2R^2}{r^2}$$

Proposed by Marin Chirciu – Romania

J.964 In ΔABC the following relationship holds:

$$\frac{2}{R} \leq \sum \frac{h_a}{bc} \sec^2\frac{A}{2} \leq \frac{1}{r}$$

Proposed by Marin Chirciu – Romania

J.965 In ΔABC the following relationship holds:

$$\sum \frac{h_a}{bc} \tan^2\frac{A}{2} \leq \sum \frac{r_a}{bc} \tan^2\frac{A}{2}$$

Proposed by Marin Chirciu – Romania

J.966 If $a_1, a_2, \dots, a_n$ then:

$$\frac{a_1 + a_2}{a_3^2} + \frac{a_2 + a_3}{a_4^2} \dots + \frac{a_{n-1} + a_n}{a_1^2} + \frac{a_n + a_1}{a_2^2} \geq \frac{n^2}{a_1 + a_2 + \dots + a_n} + \frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}$$

Proposed by Marin Chirciu – Romania

J.967 In ΔABC the following relationship holds:

$$\sum r_a^2 \geq (3r)^4 \sum \frac{1}{r_a^2}$$

Proposed by Marin Chirciu – Romania

J.968 In ΔABC the following relationship holds:

$$(4R+r)^2\cdot\frac{16Rr^2}{3}\le\sum a^5\tan\frac{A}{2}\le(4R+r)^2\cdot\frac{R^7}{12r^4}$$

Proposed by Marin Chirciu – Romania

J.969 In ΔABC the following relationship holds:

$$72pRr^2\le\sum a^4\cot\frac{A}{2}\le 16p(R^3+r^3)$$

Proposed by Marin Chirciu – Romania

J.970 In ΔABC the following relationship holds:

$$4Rr(4R+r)\le\sum a^3\tan\frac{A}{2}\le\frac{R^4}{2r^2}(4R+r)$$

Proposed by Marin Chirciu – Romania

J.971 If $a,b,c>0$ such that $a^2+b^2+c^2=3$ then:

$$\frac{a^3+b^3}{a^2+ab+b^2}+\frac{b^3+c^3}{b^2+bc+c^2}+\frac{c^3+a^3}{c^2+ca+a^2}\ge 2$$

Proposed by Marin Chirciu – Romania

J.972 In ΔABC the following relationship holds:

$$S\left(5-\frac{2r}{R}\right)\le\sum bc\tan\frac{A}{2}\le S\left(2+\frac{R}{r}\right)$$

Proposed by Marin Chirciu – Romania

J.973 In ΔABC the following relationship holds:

$$3\sum\frac{1}{a^3}\tan\frac{A}{2}\le\sum\frac{1}{a^3}\cot\frac{A}{2}$$

Proposed by Marin Chirciu – Romania

J.974 In ΔABC the following relationship holds:

$$\sum\frac{h_a}{bc}\cot^2\frac{A}{2}\ge\sum\frac{r_a}{bc}\cot^2\frac{A}{2}$$

Proposed by Marin Chirciu – Romania

J.975 Let a and b be real numbers such that $ab\ge\frac{1}{n}, n>0$. Prove that:

$$\frac{1}{na^2+1}+\frac{1}{nb^2+1}\ge\frac{2}{nab+1}$$

Proposed by Marin Chirciu – Romania

J.976 In ΔABC the following relationship holds:

$$3\sum b^2c^2\tan\frac{A}{2}\le\sum b^2c^2\cot\frac{A}{2}$$

Proposed by Marin Chirciu – Romania

J.977 In acute ΔABC holds:

$$\sum \tan^3 A \tan B \geq \frac{p^2}{r^2} \geq \frac{16R}{r} - 5$$

Proposed by Marin Chirciu – Romania

J.978 If $a, b > 0$ such that $a + b = 2$ then:

$$(2 + a^4)(2 + b^4) \geq (2 + a^3)(2 + b^3) \geq (2 + a^2)(2 + b^2) \geq (2 + a)(2 + b)$$

Proposed by Marin Chirciu – Romania

J.979 In acute ΔABC holds:

$$\sum \cot^3 A \cot B \geq \frac{4}{3}\left(\frac{r^2}{R^2} + \frac{2r}{R} - 1\right)$$

Proposed by Marin Chirciu – Romania

J.980 In ΔABC the following relationship holds:

$$9S\left(\frac{2r}{R}\right)^2 \leq \sum \frac{h_a(h_b^2 + h_c^2)}{a} \leq 9s$$

Proposed by Marin Chirciu – Romania

J.981 In ΔABC the following relationship holds:

$$\frac{18r^2}{R} \leq \sum \sqrt{\frac{s_b^2 + s_c^2}{2}} \leq \frac{9R}{2}$$

Proposed by Marin Chirciu – Romania

J.982 In ΔABC the following relationship holds:

$$\frac{h_a^3}{h_a^2} + \frac{h_b^3}{h_c^2} + \frac{h_c^3}{h_a^2} \geq 2r\left(5 - \frac{r}{R}\right)$$

Proposed by Marin Chirciu – Romania

J.983 In ΔABC the following relationship holds:

$$\sum \csc^3 A \csc B \geq \frac{16}{3}$$

Proposed by Marin Chirciu – Romania

J.984 In ΔABC the following relationship holds:

$$\frac{16r}{R} \leq \frac{(h_a + r_a)(h_b + r_b)(h_c + r_c)}{s_a s_b s_c} \leq \frac{R^3}{r^3}$$

Proposed by Marin Chirciu – Romania

J.985 In ΔABC the following relationship holds:

$$\frac{6}{R} \leq \sum \frac{h_a}{bc}\csc^2\frac{A}{2} \leq \frac{2(2R - r)}{Rr}$$

Proposed by Marin Chirciu – Romania

J.986 In ΔABC the following relationship holds:

$$\sum \frac{h_a}{bc}\cos^2\frac{A}{2} \le \sum \frac{r_a}{bc}\cos^2\frac{A}{2}$$

Proposed by Marin Chirciu – Romania

J.987 In ΔABC the following relationship holds:

$$6\left(\frac{2r}{R}\right)^2 \le \sum m_a m_b\left(\frac{1}{h_b^2}+\frac{1}{h_c^2}\right) \le 6\left(\frac{R}{2r}\right)^2$$

Proposed by Marin Chirciu – Romania

J.988 In ΔABC the following relationship holds:

$$\sum \frac{1}{a}\cot\frac{A}{2} \ge 3\sum \frac{1}{a}\tan\frac{A}{2}$$

Proposed by Marin Chirciu – Romania

J.989 In $\Delta ABC, H$ – orthocenter, I – incenter, $m(\sphericalangle A) = 90^\circ$. Prove that:

$$\frac{81}{2} < \sum_{cyc} \frac{HI^2 + s^2}{HI^2 - s^2} < \frac{81R^2}{8r^2}$$

Proposed by Radu Diaconu – Romania

J.990 In acute ΔABC the following relationship holds:

$$\frac{16F}{27R^2} \le \sum_{cyc} \frac{a \cdot HA}{s^2 + HI^2} \le \frac{2Rs}{27r^2}$$

Proposed by Radu Diaconu – Romania

J.991 a, b, c, d – sides, r – inradii in a bicentric quadrilateral. Prove that:

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{d}+\frac{d^2}{a} \ge 8r$$

Proposed by Daniel Sitaru – Romania

J.992 If $a, b, c > 0, a + b + c = 2$ then:

$$\frac{a^2+ab+b^2}{a+b}+\frac{b^2+bc+c^2}{b+c}+\frac{c^2+ca+a^2}{c+a} \ge 3$$

Proposed by Daniel Sitaru – Romania

J.993 If e, f – diagonals, R – circumradii, O – circumcenter, I – incenter, s – semiperimeter in a bicentric quadrilateral then:

$$\left(\frac{R-OI}{e}\right)^2 + \left(\frac{R+OI}{f}\right)^2 \ge \frac{s^2}{16R^2}$$

Proposed by Daniel Sitaru – Romania

J.994 In acute ΔABC the following relationship holds:

$$\left(\sqrt{b}+\sqrt{c}\right)\left(\sqrt[3]{c}+\sqrt[3]{a}\right)\left(\sqrt[6]{a}+\sqrt[6]{b}\right)\leq 4\cdot\sqrt[6]{\frac{a^3b^2c}{\sin^3\frac{A}{2}\sin^2\frac{B}{2}\sin\frac{C}{2}}}$$

Proposed by Daniel Sitaru – Romania

J.995 In ΔABC the following relationship holds:

$$\frac{1}{s^2}\sum_{cyc}\frac{(s-a)^2}{m_a+w_a}\geq\frac{1}{2(h_a+h_b+h_c)+9(R-2r)}$$

Proposed by Daniel Sitaru – Romania

J.996 If $x>0, y\geq 0$ then:

$$\frac{(x+2y)^3-(x+y)^3}{2x+3y}+\frac{(x+y)^3-y^3}{2x+y}+\frac{y^3-(x+2y)^3}{2x+2y}\leq\frac{y^2}{2}$$

Proposed by Daniel Sitaru – Romania

J.997 a,b,c,d – sides, e,f – diagonals, r – inradii, s – semiperimeter in a bicentric quadrilateral. Prove that:

$$\frac{a}{s-a}+\frac{b}{s-b}+\frac{c}{s-c}+\frac{d}{s-d}\geq 2+\frac{16r^2}{ef}$$

Proposed by Daniel Sitaru – Romania

J.998 a,b,c,d,e,f – sides, r – inradii in a bicentric hexagon. Prove that:

$$a^2+b^2+c^2+d^2+e^2+f^2\geq 8r^2$$

Proposed by Daniel Sitaru – Romania

J.999 Solve for real numbers:

$$\begin{cases} x,y,z>0 \\ \sqrt{xy}+\sqrt{yz}+\sqrt{zx}+\sqrt{\frac{x^2+y^2}{2}}+\sqrt{\frac{y^2+z^2}{2}}+\sqrt{\frac{z^2+x^2}{2}}=6 \\ x+y+z=3 \end{cases}$$

Proposed by Daniel Sitaru – Romania

J.1000 R – circumradii, r – inradii, s – semiperimeter in a bicentric quadrilateral. Prove that:

$$r+\sqrt{r+4R^2}\geq\frac{32R^2r}{s^2}$$

Proposed by Daniel Sitaru – Romania

J.1001 In ΔABC the following relationship holds:

$$9\cdot\min(a,b,c)\le 2\sum_{cyc} n_a+4r\sum_{cyc}\frac{2r_a+h_a}{s+n_a}$$

Proposed by Bogdan Fuștei – Romania

J.1002 In $\Delta ABC,\omega$ – Brocard angle, holds:

$$\sum_{cyc}\frac{m_a^2+m_b^2}{m_c^2}\le 2\cot^2\omega$$

Proposed by Bogdan Fuștei – Romania

J.1003 In $\Delta ABC, n_a$ – Nagel's cevian, g_a – Gergonne cevian, the following relationship holds:

$$\sum_{cyc}\sqrt{\frac{n_a+g_a}{m_b+m_c-m_a}}\ge 2\sqrt{2}\sum_{cyc}\frac{m_a}{m_b+m_c}$$

Proposed by Bogdan Fuștei – Romania

J.1004 In ΔABC the following relationship holds:

$$6\sqrt{3}\sum_{cyc}\frac{n_a\sin A}{h_a}\le\left(\frac{(2R-r)^2}{r^2}-\frac{R-2r}{R-r}\right)\sum_{cyc}\frac{h_a}{n_a}$$

Proposed by Bogdan Fuștei – Romania

J.1005 Prove that $ag_a(r_b+r_c), bg_b(r_c+r_a), cg_c(r_a+r_b)$ can be sides in a triangle.

Proposed by Bogdan Fuștei – Romania

J.1006 In $\Delta ABC, n_a$ – Nagel's cevian, the following relationship holds:

$$1+\frac{n_a}{h_a}\ge\frac{1}{\sqrt{2}}\left[\frac{1}{\sin\frac{A}{2}}+\sqrt{2\left(\frac{n_a-h_a}{r_a}\right)}\right]$$

Proposed by Bogdan Fuștei – Romania

J.1007 In ΔABC the following relationship holds:

$$2\sum_{cyc} m_a+\sum_{cyc}\left(\sqrt{m_b}-\sqrt{m_c}\right)^2\ge\sum_{cyc}\sqrt{\frac{4a^2+b^2+c^2}{2}}$$

Proposed by Bogdan Fuștei – Romania

J.1008 In ΔABC the following relationship holds:

$$\left(\sum_{cyc} n_a\sqrt{\frac{m_a}{r_a}}\right)^2\ge 2\sqrt{4-\frac{2r}{R}}\cdot\sum_{cyc} n_a n_b\sqrt{\frac{m_a m_b}{r_a r_b}}\sin A$$

Proposed by Bogdan Fuștei – Romania

J.1009 In $\Delta ABC, n_a$ – Nagel's cevian, g_a – Gergonne cevian, the following relationship holds:

$$1+\frac{n_a}{r_a}\geq\frac{1}{\sqrt{2}}\left[\frac{h_a-r}{\cos\frac{B-C}{2}}+\sqrt{\frac{2(2m_a-g_a-h_a)}{r_a}}\right]$$

Proposed by Bogdan Fuștei – Romania

J.1010 In ΔABC the following relationship holds:

$$\frac{3R}{2r}\geq 1+\frac{m_a+m_b+m_c+n_a+n_b+n_c}{h_a+h_b+h_c}$$

Proposed by Bogdan Fuștei – Romania

J.1011 In $\Delta ABC, g_a$ – Gergonne cevian, the following relationship holds:

$$\frac{2R-r}{3r}\geq\sqrt[3]{\frac{(2m_a-g_a)(2m_b-g_b)(2m_c-g_c)}{h_ah_bh_c}}$$

Proposed by Bogdan Fuștei – Romania

J.1012 In $\Delta ABC, n_a$ – Nagel's cevian, the following relationship holds:

$$3(m_aw_a+m_bw_b+m_cw_c)\geq\frac{1}{\sqrt{2}}\left[\sum_{cyc}n_a^2+2\sum_{cyc}r_ah_a+3F\sum_{cyc}\frac{|b-c|}{b+c}\right]$$

Proposed by Bogdan Fuștei – Romania

J.1013 In $\Delta ABC, g_a$ – Gergonne's cevian, the following relationship holds:

$$\frac{2R-r}{3r}\geq\sqrt[3]{\frac{m_am_bm_cw_aw_bw_c}{h_ah_bh_cg_ag_bg_c}}$$

Proposed by Bogdan Fuștei – Romania

J.1014 In $\Delta ABC, g_a$ – Gergonne's cevians, the following relationship holds:

$$\sqrt{\frac{(2R-r)^2}{r^2}-\frac{R-2r}{R-r}}\geq\sum_{cyc}\frac{m_aw_a}{h_ag_a}$$

Proposed by Bogdan Fuștei – Romania

J.1015 In $\Delta ABC, n_a$ – Nagel's cevian, the following relationship holds:

$$\frac{2R-r}{3r}\geq\sqrt[3]{\frac{n_an_bn_c}{h_ah_bh_c}}$$

Proposed by Bogdan Fuștei – Romania

J.1016 In ΔABC the following relationship holds:

$$\frac{1}{6}\sum_{cyc}(n_a+g_a)\geq\sqrt[9]{\frac{m_am_bm_c}{64}\prod_{cyc}(m_a+m_b)^2}$$

Proposed by Bogdan Fuștei – Romania

J.1017 In ΔABC the following relationship holds:

$$\frac{R}{2r}\geq\sqrt{1+\frac{3\left(\frac{a}{c}+\frac{c}{b}+\frac{b}{a}\right)^2(b^2-c^2)^2}{(a+b+c)^4}}$$

Proposed by Bogdan Fuștei – Romania

J.1018 In ΔABC the following relationship holds:

$$\frac{m_a^2}{h_a^2}\geq1+\frac{3\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)^2(b^2-c^2)^2}{(a+b+c)^4}$$

Proposed by Bogdan Fuștei – Romania

J.1019 In $\Delta ABC, n_a$ – Nagel's cevian the following relationship holds:

$$\frac{m_a^3+m_b^3+m_c^3}{r^3}\geq\frac{h_ah_bh_cr}{s}\prod_{cyc}\left(\frac{s}{r}+\frac{n_a^2}{h_a^2}-1\right)$$

Proposed by Bogdan Fuștei – Romania

J.1020 If $z\in\mathbb{C}, t\in[0,2\pi), |z|=1$ then:

$$|z^2+z-\cos t+i\sin t|+|\cos t+i\sin t-z^2+z|+|\cos t+i\sin t+z^2-z|\leq6$$

Proposed by Daniel Sitaru,Dan Nănuți – Romania

J.1021 Solve for real numbers:

$$\begin{cases}\sin x+\sin y=1\\ \cos x+\cos y=\sqrt{3}\\ \sqrt[4]{z+\sin^{-6}x}+\sqrt[4]{z+\sin^{-6}y}=4\sqrt{2}\end{cases}$$

Proposed by Daniel Sitaru, Dan Nănuți – Romania

J.1022 If in $\Delta ABC, m(\sphericalangle A)=90^\circ$ then:

$$\frac{bc}{a(b+c-a)}+\frac{2bc+(a+b+c)^2}{a(a+b+c)}<\frac{3\sqrt{3}+\sqrt{2}+2}{2}$$

Proposed by Daniel Sitaru, Dan Nănuți – Romania

J.1023 If $a,b>0$ then:

$$\left(\frac{2\sqrt{ab}}{a+b}+\frac{a+b}{2\sqrt{ab}}\right)\left(\sqrt{\frac{a^2+b^2}{2ab}}+\sqrt{\frac{2ab}{a^2+b^2}}\right)\left(\frac{\sqrt{2(a^2+b^2)}}{a+b}+\frac{a+b}{\sqrt{2(a^2+b^2)}}\right)\leq\left(\frac{a}{b}+\frac{b}{a}\right)^3$$

Proposed by Daniel Sitaru,Claudia Nănuți – Romania

J.1024 Find $x, y, z \in \left(0, \frac{\pi}{2}\right)$ such that:

$$\begin{cases} \cos 2x + \cos 2y + \cos 2z = 1 \\ 2\sum_{cyc} \sin x \sin y \sin z = 3\prod_{cyc} \cos x \end{cases}$$

Proposed by Daniel Sitaru, Claudia Nănuți – Romania

J.1025 Find $x \in \left(0, \frac{\pi}{2}\right)$ such that: $8\sin x + \sin 2x \cdot \sin 4x = \sqrt{7}$

Proposed by Daniel Sitaru, Claudia Nănuți – Romania

J.1026 Solve for integers: $x^2\sqrt{yz} + y^2\sqrt{zx} + z^2\sqrt{xy} = 3xyz$

Proposed by Mehmet Șahin – Ankara – Turkey

J.1027 Solve for integers: $\sqrt{x^3 + y^3} + \sqrt{y^3 + z^3} + \sqrt{z^3 + x^3} = 2(x + y + z)$

Proposed by Mehmet Șahin – Ankara – Turkey

J.1028 Solve for natural numbers:

$$\varphi\left(\varphi\left(\varphi\left(\varphi\left(\varphi(\varphi(\varphi))\right)\right)\right)\right) = 1, \varphi - \text{Euler's totient function}$$

Proposed by Mehmet Șahin – Ankara – Turkey

J.1029 Solve for natural numbers: $x\sqrt{yz} + y\sqrt{zx} + z\sqrt{xy} = xy + yz + zx$

Proposed by Mehmet Șahin – Ankara – Turkey

J.1030 Solve for natural numbers: $x\sqrt{y} + y\sqrt{z} + z\sqrt{x} = (x + y + z)\sqrt{\frac{x+y+z}{3}}$

Proposed by Mehmet Șahin – Ankara – Turkey

J.1031 Solve for natural numbers:

$$\varphi(n) + \varphi(n+1) \le 4\sqrt{2n+1}, \varphi - \text{Euler's totient function}$$

Proposed by Mehmet Șahin – Ankara – Turkey

J.1032 Solve for integers: $x^3 + y^3 + z^3 = \sqrt[3]{xyz}(x^2 + y^2 + z^2)$

Proposed by Mehmet Șahin – Ankara – Turkey

J.1033 If $x, y, z \ge 1$ then:

$$\frac{x\sqrt[3]{y} + y\sqrt[3]{x}}{x\sqrt{y} + y\sqrt{x}} + \frac{y\sqrt[3]{z} + z\sqrt[3]{y}}{y\sqrt{z} + z\sqrt{y}} + \frac{z\sqrt[3]{x} + x\sqrt[3]{z}}{z\sqrt{x} + x\sqrt{z}} \le 3$$

Proposed by Mehmet Șahin – Ankara – Turkey

J.1034 Solve for integers: $(x^2 + y^2)(x^4 + y^4) = (x + y)^6$

Proposed by Mehmet Șahin – Ankara – Turkey

J.1035 Find all $(x,y,z)\in \mathbb{Z}X\mathbb{Z}X\mathbb{Z}$ such that: $(x+y+z)(x^3+y^3+z^3)=(x^2+y^2+z^2)^2$

Proposed by Mehmet Șahin – Ankara – Turkey

J.1036 Find all $(x,y,z)\in \mathbb{N}\times\mathbb{N}\times\mathbb{N}$ such that: $\sqrt{xy}+\sqrt{yz}+\sqrt{zx}\le 2020$

Proposed by Mehmet Șahin – Ankara – Turkey

J.1037 Solve for integers: $(2a^3+3a^2b+3ab^2+2b^3)(a^2+b^2)=2(a+b)^5$

Proposed by Mehmet Șahin – Ankara – Turkey

J.1038 Solve for natural numbers:

$$\varphi(n)\cdot\varphi(n+1)=288n,\varphi\text{ – Euler's totient function}$$

Proposed by Mehmet Șahin – Ankara – Turkey

J.1039 Find all $(x,y,z,t)\in \mathbb{N}X\mathbb{N}X\mathbb{N}X\mathbb{N}$ such that: $2^x+2^y+2^z=(x+y+z)\cdot 2^t$

Proposed by Mehmet Șahin – Ankara – Turkey

J.1040 Solve for natural numbers:

$$\begin{cases} m+n+p=37 \\ \varphi(mnp)=16\big(\varphi(m)+\varphi(n)+\varphi(p)\big),\varphi-Euler's\ totient\ function \end{cases}$$

Proposed by Mehmet Șahin – Ankara – Turkey

J.1041 If $x,y,z>0$ then:

$$\frac{9\sqrt{2}}{2}\cdot\frac{\sqrt[3]{xyz}}{x^2+y^2+z^2}\le\sum_{cyc}\frac{\sqrt{x(y+z)}}{y^2+z^2}\le\frac{x^2+y^2+z^2}{\sqrt{2}\cdot xyz}$$

Proposed by Mehmet Șahin – Ankara – Turkey

J.1042 Solve for integers: $x(y+z)^2+y(z+x)^2+z(x+y)^2=(x+y+z)^3$

Proposed by Mehmet Șahin – Ankara – Turkey

J.1043 In $\Delta ABC, m(\sphericalangle BAC)=90^\circ, AD\perp BC, D\in(BC), I, I_1, I_2$-incenters in $\Delta ABC, \Delta ABD$ respectively ΔACD. Prove that:

$$[AI_1I_2]=\frac{rh_a}{2}$$

Proposed by Mehmet Șahin – Ankara – Turkey

J.1044 In $\Delta ABC, \Delta A'B'C'$ holds:

$$\frac{\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)\left(\sqrt{a'}+\sqrt{b'}+\sqrt{c'}\right)}{\sqrt[6]{aa'bb'cc'}}\le\frac{2ss'}{\sqrt[3]{2RR'FF'}}$$

Proposed by Daniel Sitaru, Claudia Nănuți – Romania

J.1045 In ΔABC the following relationship holds: $(3a-b)^2(3b-c)^2(3c-a)^2\le 64a^2b^2c^2$

Proposed by Daniel Sitaru, Claudia Nănuți – Romania

J.1046 Solve for real numbers:

$$\begin{cases} x,y,z>0 \\ \sqrt[3]{3a^3+5b^3+7c^3}+\sqrt[3]{3b^3+5c^3+7a^3}+\sqrt[3]{3c^3+5a^3+7b^3}=15 \\ a+b+c=\sqrt[3]{225} \end{cases}$$

Proposed by Daniel Sitaru, Claudia Nănuți – Romania

J.1047 Solve for real numbers:

$$\begin{cases} xyzt=1 \\ \frac{(x-y)^2}{y}+\frac{(y-z)^2}{z}+\frac{(z-t)^2}{t}+\frac{(t-x)^2}{x}=0 \\ \frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{t}+\frac{t^2}{x}=4 \end{cases}$$

Proposed by Daniel Sitaru – Romania

J.1048 If $x,y,z\geq 0$ then:

$$4(x^3+y^3+z^3)+12(x^2z+y^2x+z^2y)\geq 3xyz+15(x^2y+y^2z+z^2x)$$

Proposed by Daniel Sitaru – Romania

J.1049 $a,b,c,d,e\in\mathbb{C}^*$ - different in pairs, $a+b+c+d+e=0, A(a), B(b), C(c),$ $D(d), E(e), a^2+b^2+c^2+d^2+c^2=0, |a|=|b|=|c|=|d|=|e|=4$. Find:

$$\Omega=\frac{[ABCDE]}{AB+BC+CD+DE+EA}$$

Proposed by Daniel Sitaru – Romania

J.1050 If $x,y,z>0, xy+yz+zx=1$ then:

$$\frac{1}{15x^2+1}+\frac{1}{15y^2+1}+\frac{1}{15z^2+1}\geq\frac{1}{2}$$

Proposed by Daniel Sitaru – Romania

J.1051 If $0<x,y,z<1, xy+yz+zx=1$ then:

$$\frac{1}{x^2(x^2+9)}+\frac{1}{y^2(y^2+9)}+\frac{1}{z^2(z^2+9)}\geq\frac{27}{28}$$

Proposed by Daniel Sitaru – Romania

J.1052 In any triangle ABC the relationship holds:

$$\frac{4R+r-\sqrt{(4R+r)^2-3p^2}}{3}\leq\max\{r_a,r_b,r_c\}\leq\frac{4R+r+\sqrt{(4R+r)^2-3p^2}}{3}$$

Proposed by Nguyen Van Canh – Vietnam

J.1053 In ΔABC the following relationship holds:

$$1.\ n_a^2+n_b^2+n_c^2\geq\max\{m_a^2+m_b^2+m_c^2,r_a^2+r_b^2+r_c^2\}+\left(\frac{h_ah_bh_c}{r_ar_br_c}\right)^2(R^2-4r^2)$$

$$2.\ \sqrt{\frac{n_a+m_a+g_a}{3}}+\sqrt{\frac{n_b+m_b+g_b}{3}}+\sqrt{\frac{n_c+m_c+g_c}{3}}\geq\sqrt{m_a}+\sqrt{m_b}+\sqrt{m_c}$$

Proposed by Nguyen Van Canh – Vietnam

J.1054 If a,b,c are positive real numbers then:

$$1.\ \frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq a+b+c+\frac{4(a-c)^2}{a+b+c}$$

$$2.\ a^3+b^3+c^3-3abc\geq 2\left(a-\frac{b+c}{2}\right)^2$$

Proposed by Nguyen Van Canh – Vietnam

J.1055 In any triangle ABC the following relationship holds:

$$\sum\left(\tan\frac{A}{2}\right)+2020\cdot\sum\left(\frac{b-c}{a}\cdot\cos^2\frac{A}{2}\right)+2021\cdot\sum(a\cdot\sin(B-C))=\frac{\sum\left(\cos^2\frac{A}{2}\right)}{2\cdot\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}$$

Proposed by Nguyen Van Canh – Vietnam

J.1056 Let x,y,z be positive real numbers such that $x+y+z=1$. Prove that:

$$3(xy+yz+zx)^4-6xyz(x+y)(y+z)(z+x)(xy+yz+zx)-3x^2y^2z^2$$
$$-xyz(x+y)(y+z)(z+x)\geq 0$$

Proposed by Nguyen Van Canh – Vietnam

J.1057 Solve for real numbers: $x^{[x]}+[x]^x=2.5$, $[*]$ is the greatest integer part of $*$

Proposed by Jalil Hajimir – Toronto – Canada

J.1058 Solve for real numbers: $[x]x\leq\log|x|$, $[*]$ – GIF

Proposed by Jalil Hajimir – Toronto – Canada

J.1059 If $x,y,z>0$ such that $xyz=1$ and $n\in\mathbb{N}$ then:

$$\sum\frac{x^n}{x^n+y^{n+1}+z^{n+1}}\leq 1$$

Proposed by Marin Chirciu – Romania

J.1060 In ΔABC the following relationship holds:

$$24r\leq\sum a\left(\tan\frac{A}{2}+\cot\frac{A}{2}\right)\leq 12R$$

Proposed by Marin Chirciu – Romania

J.1061 In ΔABC the following relationship holds:

$$\frac{1}{s}\left(7-\frac{2r}{R}\right)\leq\sum\left(\frac{1}{b^2}+\frac{1}{c^2}\right)\left(\tan\frac{A}{2}+\cot\frac{A}{2}\right)\leq\frac{1}{S}\left(\frac{2R}{r}+\frac{2r}{R}+1\right)$$

Proposed by Marin Chirciu – Romania

J.1062 In acute ΔABC holds:

$$\sum \frac{a(b+c-a)}{b^2+c^2-a^2} \le \left(\frac{r}{R}\right) \cdot \frac{8+\frac{r}{R}+2\left(\frac{r}{R}\right)^2}{8\frac{r}{R}+3\left(\frac{r}{R}\right)^2-4}$$

Proposed by Marin Chirciu – Romania

J.1063 Solve for real numbers:

$$(0.8)^{[x]}+(0.7)^{[x]} \le 0.1, [*] \text{ is the greatest integer part of } *$$

Proposed by Jalil Hajimir – Toronto – Canada

J.1064 Solve for real numbers: $(x-[x])^{x-[x]}+(x+[x])^{x+[x]}=4$. $[*]$ - GIF

Proposed by Jalil Hajimir – Toronto – Canada

J.1065 In ΔABC the following relationship holds:

$$\frac{a}{\sqrt[3]{a+2b}}+\frac{b}{\sqrt[3]{b+2c}}+\frac{c}{\sqrt[3]{c+2a}} \ge 3\sqrt[3]{2Rr}$$

Proposed by Marian Ursărescu-Romania

J.1066 $z_1, z_2, z_3 \in \mathbb{C}-\{0\}$, different in pairs, $|z_1|=|z_2|=|z_3|=1, A(z_1), B(z_2), C(z_3)$

$$\sum_{cyc} \left|\frac{(z_1-z_2)(z_1-z_3)}{2z_1-z_2-z_3}\right| = 3 \Rightarrow AB=BC=CA$$

Proposed by Marian Ursărescu-Romania

J.1067 $a, b, c \in \mathbb{C}^*$ –different in pairs, $|a|=|b|=|c|, A(a), B(b), C(c)$. Prove that:

$$\sum_{cyc} \left|\frac{b-c}{b+c-2a}\right| = \sqrt{3} \Rightarrow AB=BC=CA$$

Proposed by Marian Ursărescu-Romania

J.1068 $z_A, z_B, z_C \in \mathbb{C}^*$ –different in pairs, $|z_A|=|z_B|=|z_C|=1, a=BC, b=CA, c=AB$.

Prove that:

$$\left|\prod_{cyc} b(z_A-z_B)+c(z_A-z_B)\right| = (a+b+c)^3 \Rightarrow AB=BC=CA$$

Proposed by Marian Ursărescu-Romania

J.1069 In ΔABC the following relationship holds:

$$\frac{ab^2+bc^2+ca^2}{a+b+c} \le 2R(2R-r)$$

Proposed by Marian Ursărescu-Romania

J.1070 $a, b, c \in \mathbb{C}^*$ −different in pairs, $|a| = |b| = |c| = 1, a + b + c \in \{\pm 1, \pm i\}$. Find:

$$\Omega = a^{-2021} + b^{-2021} + c^{-2021}$$

Proposed by Marian Ursărescu-Romania

J.1071 Solve for real numbers: $x^2[x] + \frac{9}{x+1} = 11, [*]$ – great integer function

Proposed by Jalil Hajimir – Toronto – Canada

J.1072 Solve: $\left[\frac{x}{2020}\right] + \left[\frac{x}{2019}\right] + \left[\frac{x}{2018}\right] = x$, $[x]$ is the greatest integer part of x

Proposed by Jalil Hajimir – Toronto – Canada

J.1073 Solve for x: $\left[\frac{x}{2}\right]\left[\frac{x}{3}\right]\left[\frac{x}{4}\right] \leq x^2$, $[x]$ is the greatest integer part of x

Proposed by Jalil Hajimir – Toronto – Canada

J.1074 Prove: $\left[2^{[x]}\right] = [2^x]$, $[x]$ is the greatest integer part of x

Proposed by Jalil Hajimir – Toronto – Canada

J.1075 Generalization for Jalil Hajimir problem

$a \geq 1$ – fixed, $n \in \mathbb{N}, n \geq 2$. Solve for real numbers:

$$\sqrt[n]{(2a+1)x^2 - 3ax + 2a} + \sqrt[n]{(2a-1)x^2 + ax + 2a} = 2\sqrt[n]{2ax^2 - ax + 2a}$$

Proposed by Marin Chirciu – Romania

J.1076 Solve: $\frac{x}{\sqrt{[x]+3}} + \frac{[x]}{\sqrt{5-x}} = 1$, $[*]$ is the greatest integer part of $*$

Proposed by Jalil Hajimir – Toronto – Canada

J.1077 Solve for x: $[x^5] - x[x^4] < 1$, $[*]$ is the greatest integer part of $*$

Proposed by Jalil Hajimir – Toronto – Canada

J.1078 Solve for real numbers: $x[x^2] + x^{2[x]} = (x + [x])^2, [*]$ − GIF

Proposed by Jalil Hajimir – Toronto – Canada

J.1079 $\Omega(n) = n^7 + n^6 + n^5 + n^3 + 2n^2 + 2n + 1$

Find all positive integers n such that $\Omega(n)$ is a prime number.

Proposed by Rajeev Rastogi – India

J.1080 $\begin{cases} 2(x^2 + y^2) + (\sqrt{6} - \sqrt{2})xy = 50 \\ y^2 + z^2 + yz = 144, x, y, z > 0 \\ z^2 + x^2 + \sqrt{2}zx = 169 \end{cases}$. Find: $\Omega = (1 + \sqrt{3})xy + \sqrt{6}yz + 2zx$

Proposed by Rajeev Rastogi – India

J.1081 Let a, b, c be positive real numbers such that $ab + bc + ca = abc$. Prove that:

$$\frac{c^2(a^6+b^6)}{a^4+b^4}+\frac{a^2(b^6+c^6)}{b^4+c^4}+\frac{b^2(c^6+a^6)}{c^4+a^4}\geq\frac{a^2b^2c^2}{3}$$

Proposed by Rajeev Rastogi – India

J.1082 Let $a,b,c\in\mathbb{R}^+$ such that $a+b+c=6$, then prove that:

$$\frac{(a+2)(b+2)^4}{(2ac)^{\frac{2}{3}}+\frac{4}{3}}+\frac{(b+2)(c+2)^4}{(2ab)^{\frac{2}{3}}+\frac{4}{3}}+\frac{(c+2)(a+2)^4}{(2bc)^{\frac{2}{3}}+\frac{4}{3}}\geq 576$$

Proposed by Rajeev Rastogi – India

J.1083 In ΔABC the following relationship holds:

$$\frac{1}{s^2-n_a^2}+\frac{1}{s^2-n_b^2}+\frac{1}{s^2-n_c^2}=\frac{r(4R+r)}{2F^2}$$

Proposed by Ertan Yildirim-Turkiye

J.1084 In ΔABC the following relationship holds:

$$\frac{\sqrt{m_a^4+m_b^4}+\sqrt{m_b^4+m_c^4}+\sqrt{m_c^4+m_a^4}}{ab\cos C+bc\cos A+ca\cos B}\geq\frac{3}{\sqrt{2}}$$

Proposed by Haxverdiyev Taverdi-Afghanistan

J.1085 In ΔABC the following relationship holds:

$$s\geq\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)r\sqrt{3}\geq 3r\sqrt{3}$$

Proposed by Cristian Miu-Romania

J.1086 If $x,y,z>0$ then in acute ΔABC holds:

$$\frac{x(bc+a^2)}{y+z}+\frac{y(ca+b^2)}{z+x}+\frac{z(ab+c^2)}{x+y}\geq 109r^2+4Rr-3s^2$$

Proposed by Mehmet Șahin – Turkiye

J.1087 Find all functions $f:\mathbb{R}\to\mathbb{R},\ f(x)f(y)=f(x+y)+2xy,\forall x,y\in\mathbb{R}$

Proposed by Nguyen Van Canh – Vietnam

J.1088 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{\cos^4 A}{\cos^3\frac{B}{2}\cos^3\frac{C}{2}}\geq\left(\frac{3r}{R}\right)^2$$

Proposed by Florică Anastase – Romania

J.1089 In ΔABC the following relationship holds:

$$\frac{\sin^2 A}{\sin^3 B+2\sin^2 C+8}+\frac{\sin^2 B}{\sin^3 C+2\sin^2 A+8}+\frac{\sin^2 C}{\sin^3 A+2\sin^2 B+8}\le\frac{1}{3}$$

Proposed by Florică Anastase – Romania

J.1090 In ΔABC the following relationship holds:

$$\frac{(b^2+c^2-a^2)^2}{ab\cdot\sin^2 A}+\frac{(a^2+c^2-b^2)^2}{bc\cdot\sin^2 B}+\frac{9(a^2+b^2-c^2)^2}{ca\cdot\sin^2 C}\ge 48r^2$$

Proposed by Florică Anastase – Romania

J.1091 If $a,b,c\in(0,1)$ such that $a+b+c=1$, then prove:

$$\frac{a}{\sqrt{4b^3+c+6}}+\frac{b}{\sqrt{4c^3+a+6}}+\frac{c}{\sqrt{4a^3+b+6}}\le\frac{\sqrt{3}}{3}$$

Proposed by Florică Anastase – Romania

J.1092 In acute ΔABC the following relationship holds:

$$\sum_{cyc}(b+c)\tan\frac{A}{2}\ge\frac{108r^2}{2R-r}$$

Proposed by Marian Ursărescu – Romania

J.1093 In ΔABC the following relationship holds:

$$\sum_{cyc}(b+c)\tan\frac{A}{2}\ge\frac{54R}{2\left(\frac{R}{r}\right)^2+1}$$

Proposed by Marian Ursărescu – Romania

J.1094 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{1}{2r_a^2+r_b^2+r_c^2}\le\frac{1}{2r^2}$$

Proposed by Marian Ursărescu – Romania

J.1095 In ΔABC the following relationship holds:

$$\frac{m_aw_a}{a^2}+\frac{m_bw_b}{b^2}+\frac{m_cw_c}{c^2}\ge\frac{9}{4}$$

Proposed by Marian Ursărescu – Romania

J.1096 In ΔABC the following relationship holds:

$$\frac{m_a}{a^2}+\frac{m_b}{b^2}+\frac{m_c}{c^2}\ge\frac{27}{4(4R+r)}$$

Proposed by Marian Ursărescu – Romania

J.1097 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{m_a(r_b+r_c)}{r_br_c}\ge 7-\frac{2r}{R}$$

Proposed by Marian Ursărescu – Romania

J.1098 In ΔABC the following relationship holds:

$$4R\left(\frac{2R}{r}-1\right)\le\sum_{cyc}\frac{r_b+r_c}{\sin^2 A}\le 4R\left(\left(\frac{R}{r}\right)^2-\frac{R}{r}+1\right)$$

Proposed by Marian Ursărescu – Romania

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

PROBLEMS FOR SENIORS

S.509 Let $m,n\in\mathbb{R}_+=[0,\infty), m+n=2$ and ΔABC with the area F, then:

$$\left(\frac{x\cdot a^m}{h_a^n}+\frac{y\cdot b^m}{h_b^n}+\frac{z\cdot c^m}{h_c^n}\right)^2\ge 4^m(xy+yz+zx)F^{2(1-n)}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.510 Let M an interior point in ΔABC and $x=MA, y=MB, z=MC$, then:

$$\sum_{cyc}\left(\frac{x}{a}\left(\frac{y}{b}+\frac{z}{c}\right)\right)^4=\frac{16}{27}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.511 If $a,b,c,x,y,z \in \mathbb{R}_+^*$, then:

$$\frac{x^4+y^4}{(ax+by)^2+cyz}+\frac{y^4+z^4}{(ay+bz)^2+czx}+\frac{z^4+x^4}{(az+bx)^2+cxy}\geq\frac{2}{3}\cdot\frac{(x+y+z)^2}{(a+b)^2+c}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.512 If $m,n \in \mathbb{R}_+=[0,\infty); x,y,z \in \mathbb{R}_+^*=(0,\infty)$, then in any ΔABC with the area F the following inequality holds:

$$\sum_{cyc}\frac{\big(m(x+y)+n(x+z)\big)a^2}{(ny+mz)h_a^2}\geq 8$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.513 Let M an interior point in ΔABC and $x_A=MA, x_B=MB, x_C=MC$, then:

$$\frac{x_A}{h_a}+\frac{x_B}{h_b}+\frac{x_C}{h_c}\geq 2$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.514 Let $m \in \mathbb{R}_+=[0,\infty)$, an interior point in ΔABC with the area F and $x_A=MB$, $x_B=MB, x_C=MC$. Prove that: $(x_Aa)^{m+1}+(x_Bb)^{m+1}+(x_Cc)^{m+1}\geq\frac{4^{n+1}}{3^m}F^{m+1}$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.515 If M is an interior point in ΔABC and $x=MA, y=MB, z=MC$, then:

$$\sum_{cyc}\left(\frac{\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)^3+27\cdot\frac{x^3}{a^3}}{\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)^2+9\frac{x^2}{a^2}}\right)^2\geq 36$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.516 Find all real roots: $64x^5(x-1)+32x^2(x^2+x+1)-64x+19=0$

Proposed by Daniel Sitaru – Romania

S.517 Find without any software:

$$\Omega=\int\frac{3x^2+x}{1+6x(1+e^{3x})+2e^{3x}+e^{6x}+9x^2}dx$$

Proposed by Daniel Sitaru – Romania

S.518 Find without any software:

$$\Omega=\int_0^{\frac{\pi}{4}}\sin\left(x-\frac{\pi}{4}\right)\sqrt{(1+\sin x)(1+\cos x)}\,dx$$

Proposed by Daniel Sitaru – Romania

S.519 Find:

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{n\cdot 27^n}\sum_{i=1}^{n}\sum_{j=1}^{n}3^{i+j}\binom{3n-i-j}{n}\binom{2n-i-j}{n-i}\right)$$

Proposed by Daniel Sitaru – Romania

S.520 $\Omega(n,r)=\sum_{k=0}^{n}\frac{(-1)^k}{3r+3k-2}\binom{n}{k}, r\in\mathbb{N}, r-$ fixed. Find:

$$\omega(r)=\lim_{n\to\infty}\sqrt[n]{\Omega(n,r)}$$

Proposed by Daniel Sitaru – Romania

S.521 Find without any software:

$$\Omega=\int_{\frac{\pi^5}{1024}}^{\frac{\pi^5}{243}}\frac{\sin(\sqrt[5]{x})\cdot\sin(3\sqrt[5]{x})\cdot\sin(5\sqrt[5]{x})}{\sqrt[5]{x^4}}dx$$

Proposed by Daniel Sitaru – Romania

S.522 Find without any software:

$$\Omega=\int(4\cot^3 x-5\cot^2 x+7\cot x)e^x\,dx$$

Proposed by Daniel Sitaru – Romania

S.523 Let $x_n, n\geq 2$ be the solution of equation: $x+\arctan(x-1)=\sqrt[n]{2018}$. Find:

$$\Omega_1=\lim_{n\to\infty}x_n,\Omega_2=\lim_{n\to\infty}n(x_n-1)$$

Proposed by Marian Ursărescu – Romania

S.524 $f:[a,b]\to\mathbb{R}, f-$continuous, $f-$nonconstant. Prove that:

$\exists c_1,c_2\in[a,b], c_1\neq c_2$ such that $c_1+c_2=a+b, c_1f(c_1)=c_2f(c_2)$.

Proposed by Marian Ursărescu – Romania

S.525 $x_0>0, a>1, x_{n+1}=a^{\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}}$. Find:

$$\Omega=\lim_{n\to\infty}\frac{1}{n^6}\sum_{1\leq i<j<k\leq n}x_ix_jx_k$$

Proposed by Marian Ursărescu – Romania

S.526 Find:

$$\Omega=\lim_{n\to\infty}\prod_{1\leq i<j<k\leq n}\left(1+\frac{ijk}{n^6}\right)$$

Proposed by Marian Ursărescu – Romania

S.527 $x_0 > 0, x_{n+1} = x_n + \frac{1}{x_n^2+x_n+1}$. Find:

$$\Omega = \lim_{n\to\infty} \frac{x_1 + x_2 + \cdots + x_n}{n\sqrt[3]{3}}$$

Proposed by Marian Ursărescu – Romania

S.528 $x_0 > 0, x_{n+1} = x_n + \frac{a}{x_n}, a > 0, n \in \mathbb{N}^*$.Find:

$$\Omega = \lim_{n\to\infty} \frac{\sum_{i,j=1}^{n} x_i x_j}{n^3}$$

Proposed by Marian Ursărescu – Romania

S.529 Find:

$$\Omega = \lim_{n\to\infty} \left(\frac{1}{2} + \sum_{k=1}^{n} \frac{k^p}{k^{4p} + k^{2p} + 1}\right)^{n^{3p-1}}, p \in \mathbb{N}^*$$

Proposed by Marian Ursărescu – Romania

S.530 Find $f: \mathbb{R} \to \mathbb{R}$ such that:

$$(x-y)^2\left(f^2(x) - f^2(y)\right) = (x+y)f^3(x-y)$$

Proposed by Marian Ursărescu – Romania

S.531 If $a, b, c \geq 1$ and $\lambda \geq 1$ then:

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} + 2\lambda(a+b+c) \leq \lambda(a^2+b^2+c^2) + 3(\lambda+1)$$

Proposed by Marin Chirciu – Romania

S.532 If $x, y > 0$ such that $x + y = 2$ and $\lambda \leq 16$ then find:

$$\min P = \left(\lambda x^2 + \frac{1}{y^2}\right)\left(\lambda y^2 + \frac{1}{x^2}\right)$$

Proposed by Marin Chirciu – Romania

S.533 If $a, b, c > 0$ such that $a + b + c + d = 4$ and $\lambda \geq 0$ then:

$$\frac{a^2}{\lambda + ab} + \frac{b^2}{\lambda + bd} + \frac{c^2}{\lambda + cd} + \frac{d^2}{\lambda + da} \geq \frac{4}{\lambda + 1}$$

Proposed by Marin Chirciu – Romania

S.534 If $a, b, c > 0$ such that $a^3 + b^3 + c^3 = 3$ and $0 \leq \lambda \leq 2$ then

$$\frac{1}{1+\lambda ab^2} + \frac{1}{1+\lambda bc^2} + \frac{1}{1+\lambda ca^2} \geq \frac{3}{\lambda+1}$$

Proposed by Marin Chirciu – Romania

S.535 Find:

$$\Omega = \lim_{n\to\infty} \left(\frac{1}{\log(n+1)} \sqrt[n]{\prod_{k=1}^{n} \frac{1}{2k+1}}\right)$$

Proposed by Daniel Sitaru – Romania

S.536 Find without any software:

$$\Omega=\int_0^{\frac{\pi}{4}}\frac{\sin x\left(\sqrt{\frac{1}{\cos x}+\tan x}+\sqrt{\frac{1}{\cos x}-\tan x}\right)}{\sqrt{\cos^5 x+\cos^6 x}}dx$$

Proposed by Daniel Sitaru – Romania

S.537 Find:

$$\Omega=\lim_{n\to\infty}\frac{12^n\cdot(n!)^5}{(n^2+n)^{2n}\cdot(n+1)^n}$$

Proposed by Daniel Sitaru – Romania

S.538 If $n\in\mathbb{N}, n\geq 5$ then:

$$(n-1)^{\frac{1}{n}+\frac{1}{n-1}}>n^{\frac{1}{n+1}+\frac{1}{n}}>(n+1)^{\frac{1}{n+2}+\frac{1}{n+1}}$$

Proposed by Daniel Sitaru – Romania

S.539 If $t\geq 1, x, y>0$ then:

$$\frac{x^{t+1}}{y^t}+\frac{4y^{t+1}}{x^t+y^t}\geq 3\sqrt{\frac{x^{t+1}+2y^{t+1}}{x^{t-1}+2y^{t-1}}}$$

Proposed by Daniel Sitaru – Romania

S.540 Find:

$$\Omega=\lim_{n\to\infty}\left(\left(1+\frac{n^{\frac{1}{n}}}{n}\right)^{\frac{1}{n}}\cdot\sin\frac{1}{n}+\left(1-\frac{n^{\frac{1}{n}}}{n}\right)^{\frac{1}{n}}\cdot\frac{1}{n}\right)$$

Proposed by Daniel Sitaru – Romania

S.541 Find without any software:

$$\Omega=\int\frac{\left(2\tan x+\frac{1}{\cos^2 x}+2\right)^3-1}{\cos^2 x\left(\left(2\tan x+\frac{1}{\cos^2 x}+3\right)^3-1\right)}dx$$

Proposed by Daniel Sitaru – Romania

S.542 If $a, b>0$ then:

$$\frac{a+b}{\sqrt{ab}}+\frac{\sqrt{2(a^2+b^2)}}{a+b}\geq 3+\log\left(\frac{(a+b)\sqrt{a^2+b^2}}{2ab\sqrt{2}}\right)$$

Proposed by Daniel Sitaru – Romania

S.543 If $0 < a \leq b < 1, f: [a,b] \to [0,1], f$ – continuous then:

$$(b-a)^2 \int_a^b f(x)\,dx + \left(\int_a^b \big(1-f(x)\big)dx\right)^3 \leq (b-a)^3$$

Proposed by Daniel Sitaru – Romania

S.544 In $\Delta ABC, n_a$ –Nagel's cevian, g_a –Gergonne cevian, the following relationship holds:

$$|(r_a - r_b)(r_b - r_c)(r_c - r_a)| \geq (n_a - g_a)(n_b - g_b)(n_c - g_c)$$

Proposed by Bogdan Fuştei-Romania

S.545 In $\Delta ABC, n_a$ –Nagel's cevian, the following relationship holds:

$$\sum_{cyc} \frac{n_a^2 - r_b r_c}{r_a} = 2\sum_{cyc} (r_a - h_a)$$

Proposed by Bogdan Fuştei-Romania

S.546 In $\Delta ABC, n_a$ –Nagel's cevian, the following relationship holds:

$$\sum_{cyc} (b+c)cosA \leq 2\sqrt{n_a n_b + n_b n_c + n_c n_a}$$

Proposed by Bogdan Fuştei-Romania

S.547 In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{|m_b - m_c|}{b+c} \geq \frac{3}{4}\left(\sum_{cyc} |a-b|\right)\left(\sum_{cyc} m_a\right)^{-1}$$

Proposed by Bogdan Fuştei-Romania

S.548 In ΔABC the following relationship holds:

$$m_a^2 \geq r_b r_c + \frac{1}{4}(n_a - g_a)^2$$

Proposed by Bogdan Fuştei-Romania

S.549 In ΔABC the following relationship holds:

$$\prod_{cyc} \left(h_a - m_a + \frac{a}{2}\right) < \frac{2r^2 F}{R}$$

Proposed by Bogdan Fuştei-Romania

S.550 In $\Delta ABC, n_a$ –Nagel's cevian, I –incenter, the following relationship holds:

$$\frac{s}{r} \geq \frac{1}{\sqrt{2}}\left(\sum_{cyc} \frac{m_a}{h_a} + \sum_{cyc} \frac{|b-c|}{2r}\sqrt{1 - \frac{AI^2}{w_a^2}}\right) + 2\sum_{cyc} \frac{r_a}{n_a + s}$$

Proposed by Bogdan Fuştei-Romania

S.551 In $\Delta ABC, n_a$ –Nagel's cevian, g_a –Gergonne cevian, the following relationship holds:

$$\sum_{cyc} \frac{\sqrt{n_a r_a g_a}}{h_a} \geq \frac{s}{\sqrt{r}}$$

Proposed by Bogdan Fuștei-Romania

S.552 In $\Delta ABC, n_a$ –Nagel's cevian, g_a –Gergonne cevian, the following relationship holds:

$$\sum_{cyc} \sqrt{r_a}(n_a + g_a) \geq s\sqrt{\frac{2}{R}} \sum_{cyc} \left(\sqrt{\frac{r_b}{r_c}} + \sqrt{\frac{r_b}{r_b}} \right)$$

Proposed by Bogdan Fuștei-Romania

S.553 Find without any software:

$$\Omega = \int_0^1 \frac{2x + 3\sqrt{1-x^2}}{5(1-x^2) + 4x\sqrt{1-x^2}} dx$$

Proposed by Daniel Sitaru – Romania

S.554 Find:

$$\Omega = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{(n-k+1)(\gamma - H_k)}{1+2+\cdots+n}$$

Proposed by Daniel Sitaru – Romania

S.555 Find:

$$\Omega = \lim_{n\to\infty} \prod_{k=1}^{n} \left(1 + \frac{n \tan^{-1}\left(\frac{k}{n}\right)}{k^2 + n^2} \right)$$

Proposed by Daniel Sitaru – Romania

S.556 If $0 < a \leq b$ then:

$$\log\left(\frac{\sinh b}{\sinh a}\right) - \log\frac{b}{a} \leq \int_a^b \frac{1}{x} \log(\cosh x)\, dx \leq \log\left(\frac{\cosh b}{\cosh a}\right)$$

Proposed by Daniel Sitaru – Romania

S.557 If $0 < a \leq b$ then:

$$3\int_a^b \left(\coth\frac{3x}{2} - \tanh\frac{x}{2} \right) dx \leq 2\log\frac{b}{a}$$

Proposed by Daniel Sitaru – Romania

S.558 Find:

$$\Omega = \lim_{n\to\infty}\left|\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}\sin\left(\frac{k}{n}\right)-\sin x\right|, x\in(0,1)$$

Proposed by Daniel Sitaru – Romania

S.559 If $0 < a, b \leq 1$ then:

$$a\left(\sqrt[4]{\left(\frac{b}{a}\right)^{3a+b}}+\sqrt[4]{\left(\frac{b}{a}\right)^{a+3b}}\right)+b\left(\sqrt[4]{\left(\frac{a}{b}\right)^{3a+b}}+\sqrt[4]{\left(\frac{a}{b}\right)^{a+3b}}\right)\leq 2(a+b)$$

Proposed by Daniel Sitaru – Romania

S.560 Find:

$$\Omega = \lim_{x\to\frac{\pi}{6}}\left(x+1-\frac{1}{\sqrt{3}}+\frac{1}{3}\cdot\left(\frac{1}{\sqrt{3}}\right)^3-\frac{1}{5}\cdot\left(\frac{1}{\sqrt{3}}\right)^5+\frac{1}{7}\cdot\left(\frac{1}{\sqrt{3}}\right)^7-\cdots\right)^{\frac{\sqrt{3}}{\sqrt{3}\tan x-1}}$$

Proposed by Adil Abdullayev- Azerbaijan

S.561 Find all positive real numbers α such that: $\ln(\alpha+x)\leq \alpha x, \forall x\geq 0$

Proposed by Nguyen Van Canh – BenTre – Vietnam

S.561 Let $\varphi(x)=x^2+x$. Solve in $\mathbb{R}$: $\varphi(\varphi(x))-2\varphi(x^2)+2020x\leq 0$

Proposed by Nguyen Van Canh – BenTre – Vietnam

S.562 Find all positive real numbers α such that: $e^x\geq \alpha(x+1)+\frac{x^2}{2}+\frac{x^3}{6}, \forall x\geq 0$

Proposed by Nguyen Van Canh – BenTre – Vietnam

S.563 Find all positive real numbers α such that: $e^x\geq \alpha(x+1)+\frac{x^2}{2}, \forall x\in\mathbb{R}$

Proposed by Nguyen Van Canh – BenTre – Vietnam

S.564 If $a,b,c>0$ and $a^3+b^3+c^3=1$ and $\lambda\geq 0$ then:

$$\frac{a^4}{b+\lambda c}+\frac{b^4}{c+\lambda a}+\frac{c^4}{a+\lambda b}\geq\frac{1}{\lambda+1}$$

Proposed by Marin Chirciu – Romania

S.565 If $a,b,c>0$ and $a+b+c=3$ and $\lambda\geq 0, n\in\mathbb{N}, n\geq 2$ then:

$$\frac{a^{3n}}{a^2+\lambda b}+\frac{b^{3n}}{b^2+\lambda c}+\frac{c^{3n}}{c^2+\lambda a}\geq\frac{3}{\lambda+1}$$

Proposed by Marin Chirciu – Romania

S.566 If $a,b,c>0$ and $a^2+b^2+c^2=3$ and $\lambda\geq 0$ then:

$$\frac{a^6}{b+\lambda c}+\frac{b^6}{c+\lambda a}+\frac{c^6}{a+\lambda b}\geq\frac{3}{\lambda+1}$$

Proposed by Marin Chirciu – Romania

S.567 In ΔABC the following relationship holds:

$$\frac{1}{\lambda\left(\frac{r}{h_a}\right)^2}+\frac{1}{\lambda+\left(\frac{r}{h_b}\right)^2}+\frac{1}{\lambda+\left(\frac{r}{h_c}\right)^2}\le\frac{27}{9\lambda+1},\lambda\ge\frac{7}{9}$$

Proposed by Marin Chirciu – Romania

S.568 If $x,y,z>0$ and $x+y+z=1,\lambda\ge\frac{7}{9}$ then:

$$\frac{1}{\lambda+x^2}+\frac{1}{\lambda+y^2}+\frac{1}{\lambda+z^2}\le\frac{27}{9\lambda+1}$$

Proposed by Marin Chirciu – Romania

S.569 If $A=\frac{\sec^3 x}{\sqrt{1+\csc^2 x}}+\frac{\csc^3 x}{\sqrt{1+\sec^2 x}},x\in\left(0,\frac{\pi}{2}\right)$ then find $\min(A)$

Proposed by Marin Chirciu – Romania

S.570 In ΔABC the following relationship holds:

$$\prod_{cyc}\frac{\mu(A)}{r_a-r}\cdot\sum_{cyc}\frac{\mu(A)}{r_a-r}\le\frac{s^2+r^2-8Rr}{32Rr}\cdot\left(\frac{\pi}{3r}\right)^4$$

Proposed by Radu Diaconu – Romania

S.571 In ΔABC the following relationship holds:

$$\prod_{cyc}(r_b+r_c)\mu(A)\cdot\sum_{cyc}(r_b+r_c)\,\mu(A)\le 3(\pi R)^4$$

Proposed by Radu Diaconu – Romania

S.572 In ΔABC the following relationship holds:

$$\sum_{cyc}\sqrt{\frac{\mu(A)\mu(B)r_ar_b}{ab}}\le\frac{\pi\sqrt{3}}{2}$$

Proposed by Radu Diaconu – Romania

S.573 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{w_a+w_b}{\mu(A)+\mu(B)}\cdot\prod_{cyc}\frac{w_a\cdot w_b}{\mu(A)\cdot\mu(B)}\ge 3^{15}\cdot\left(\frac{r}{\pi}\right)^7$$

Proposed by Radu Diaconu – Romania

S.574 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{(\mu(A)+m_a)^2}{\mu(A)\cdot m_a}\ge\frac{2(\pi+9r)^2}{3\pi R}$$

Proposed by Radu Diaconu – Romania

S.575 Find without any software:

$$\Omega = \int_0^{\frac{\pi}{4}} \frac{x + \cos x}{\cos x + \sin\left(\frac{\pi}{4} + x\right)} dx$$

Proposed by Radu Diaconu – Romania

S.576 Find without any software:

$$\Omega = \int_0^1 \frac{e^x(1 - x)}{(x + 2e^x)^2} dx$$

Proposed by Radu Diaconu – Romania

S.577 In nonobtuse ΔABC the following relationship holds:

$$\frac{4\pi s}{27Rr} \le \sum_{cyc} \frac{a \cdot \mu(A)}{m_a \cdot h_a} \le \frac{3\pi R^3}{S^2}$$

Proposed by Radu Diaconu – Romania

S.578 Find:

$$\Omega = \int_0^1 \frac{f(x) - f'(x) + \sin x - \cos x + a}{a + \sin x + f(x) + e^x} dx, a > 0, f \in C^1\big((0, \infty)\big)$$

Proposed by Radu Diaconu – Romania

S.579 In ΔABC holds:

$$\frac{\mu(A) \cdot r_a^2}{r_b^2 + r_c^2} + \frac{\mu(B) \cdot r_b^2}{r_c^2 + r_a^2} + \frac{\mu(C) \cdot r_c^2}{r_a^2 + r_b^2} = \frac{\pi}{2} \Leftrightarrow a = b = c$$

Proposed by Radu Diaconu – Romania

S.580 If in $\Delta ABC, m(\sphericalangle A) = 90^\circ$ then:

$$54r^3 < \begin{vmatrix} r_a & r_b & r_c \\ r_c & r_a & r_b \\ r_b & r_c & r_a \end{vmatrix} < 9R(OI^2 + R^2)$$

Proposed by Radu Diaconu – Romania

S.581 If $e \le a \le b$ then:

$$\int_a^b (\log x)^{\log x} dx \cdot \int_a^b (\log x)^{-\log x} dx \le \log\left(\sqrt{\frac{b}{a}}\right)^{b^2 - a^2}$$

Proposed by Daniel Sitaru – Romania

S.582 If $\frac{\pi}{4} \le x \le y \le z < \frac{\pi}{2}$ then:

$$2\tan x + \left(\sum_{cyc} \tan x\right)\left(\sum_{cyc} \cot x\right) \le 9 + 2\tan z$$

Proposed by Daniel Sitaru – Romania

S.583 Find:

$$\Omega=\lim_{n\to\infty}\left(\frac{3}{4}+\frac{1}{4n}\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k-1}\frac{(n-k+1)^2}{k(k+1)}\right)^n$$

Proposed by Daniel Sitaru – Romania

S.584 Find:

$$\Omega=\int x^{-2}\cdot\tan\left(\frac{1}{2x}\right)\cdot\tan\left(\frac{1}{3x}\right)\cdot\tan\left(\frac{1}{6x}\right)dx$$

Proposed by Daniel Sitaru – Romania

S.585 If $0<a\le b, f:[a,b]\to[0,\infty), f-$continuous, then:

$$2(b-a)\int_a^b f(x)dx\ge\frac{1}{\sqrt{2}}\int_a^b\int_a^b\sqrt{f^2(x)+f^2(y)}\,dxdy+\left(\int_a^b\sqrt{f(x)}\,dx\right)^2$$

Proposed by Daniel Sitaru – Romania

S.586 If $0<a\le b, f:[a,b]\to(0,\infty), f-$continuous, then:

$$\int_a^b\int_a^b\int_a^b\sqrt{f^2(x)f^2(y)+f^2(y)f^2(z)+f^2(z)f^2(x)}\,dxdydz\le\left(\int_a^b f(x)\,dx\right)^3$$

Proposed by Daniel Sitaru – Romania

S.587 If $0<a\le b$ then:

$$\int_a^b\left(\sqrt{e}\right)^{x^2}\cdot\tanh x\,dx\ge\cosh b-\cosh a$$

Proposed by Daniel Sitaru – Romania

S.588 If $a,b>0$ then:

$$\log\left(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a+b}}\right)\cdot\log\left(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{2\sqrt{ab}}}\right)\le\left(\log\sqrt{2}\right)^2$$

Proposed by Daniel Sitaru – Romania

S.589 If $0\le a\le b\le\frac{\pi}{4}$ then:

$$3\log\left(\frac{\cos a}{\cos b}\right)+4\log^3\left(\frac{\cos a}{\cos b}\right)\le 6(b-a)\log^2\left(\frac{\cos b}{\cos a}\right)+(b-a)^3$$

Proposed by Daniel Sitaru – Romania

S.590 $f:\mathbb{R}\to\mathbb{R}, f-$continuous in $x=0, 3f(3x)=3x+f(x), \forall x\in\mathbb{R}$

Prove that $\forall t\in\mathbb{R}, x,y\ge 0$:

$$\frac{3}{8}\cdot\left(\frac{8}{3}f(x)\right)^{\sin^2 t}\cdot\left(\frac{8}{3}f(y)\right)^{\cos^2 t}\leq \sin^2 t\cdot f(x)+\cos^2 t\cdot f(y)$$

Proposed by Daniel Sitaru – Romania

S.591 a,b,c,d,e,f – sides, R – circumradii in a bicentric hexagon. Prove that:

$$abcdef\leq R^6$$

Proposed by Daniel Sitaru – Romania

S.592 Let Δ be area of pedal triangle of first Brocard's point in ΔABC. Prove that:

$$2R\Delta\geq rF$$

Proposed by Daniel Sitaru – Romania

S.593 If AA', BB', CC' - are ninepoint center (O_9) cevians then:

$$2\prod_{cyc}\frac{A'O_9}{AO_9}+\sum_{cyc}\frac{A'O_9\cdot B'O_9}{AO_9\cdot BO_9}\leq\frac{R}{2r}$$

Proposed by Daniel Sitaru – Romania

S.594 In acute ΔABC holds:

$$\sum_{cyc}\sin A+\sum_{cyc}(\sin A)^{-\sin A}<\frac{3R}{r}$$

Proposed by Daniel Sitaru – Romania

S.595 If $x,y,z\in[0,1]$ then in ΔABC holds:

$$(a^x+b^x+c^x)(a^y+b^y+c^y)(a^z+b^z+c^z)\leq\frac{(a+b+c)^3}{\sqrt[3]{(abc)^{3-x-y-z}}}$$

Proposed by Daniel Sitaru – Romania

S.596 If $0<a\leq b<\frac{\pi}{2}$ then:

$$\left(\int_a^b\frac{dx}{\sqrt[3]{\cos^2 x}}\right)^3\leq b-a+\int_a^b\tan^2 x\,dx$$

Proposed by Daniel Sitaru – Romania

S.597 Find:

$$\Omega=\lim_{n\to\infty}\left(n^2\left(\frac{1}{n}\sum_{k=1}^{n}\sin\left(\sqrt{\frac{2k-2n-1}{2n}}\right)-\int_0^1\sin\left(\sqrt{x+1}\right)\right)\right)$$

Proposed by Daniel Sitaru – Romania

S.598 $0 < a \le b < 1, f:[0,1] \to [0,1], f$ - continuous. Prove that:

$$3(b-a)^2 \int_a^b f^2(x)\,dx \le 2(b-a)^3 + \left(\int_a^b f(x)\,dx\right)^3$$

Proposed by Daniel Sitaru – Romania

S.599 If $0 < a \le b$ then:

$$4a^3b^3(b^9-a^9) + 12(b^3-a^3) \ge 9a^3b^3(b^6-a^6) + 36\log\left(\frac{b}{a}\right)$$

Proposed by Daniel Sitaru – Romania

S.600 If $a,b,c > 0, 0 \le x \le \frac{\pi}{4}$ then:

$$\frac{a^{\cos x} + b^{\cos x} + c^{\cos x}}{3\sqrt[3]{(abc)^{\cos x}}} + \frac{3\sqrt[3]{(abc)^{\sin x}}}{a^{\sin x} + b^{\sin x} + c^{\sin x}} \ge 2$$

Proposed by Daniel Sitaru – Romania

S.601 Prove that if $0 < a \le b$ then:

$$\left(\int_a^b \frac{\log x}{x}dx\right)^2 \ge \left(\int_{\frac{a+b}{2}}^b \frac{\log x}{x}dx + \int_{\sqrt{ab}}^b \frac{\log x}{x}dx\right)\left(\int_a^{\frac{a+b}{2}} \frac{\log x}{x}dx + \int_a^{\sqrt{ab}} \frac{\log x}{x}dx\right)$$

Proposed by Daniel Sitaru – Romania

S.602 Let: $u_{n+1} = \alpha u_n^2 + u_n - \frac{1}{3}, u_1 = 0$. Find all real numbers α such that:

$$\lim_{n\to\infty} u_n = \frac{1}{2}$$

Proposed by Nguyen Van Canh – Vietnam

S.603 Let $\varphi_n(x) = \frac{n^2+x}{n-x}, (n \in \mathbb{N}, n \ge 1)$.Find:

a. $\max \varphi_n(x), \forall x \in \left[0, \frac{n}{2}\right]$

b. $\min \varphi_n(x), \forall x \in \left[0; \frac{n}{4}\right]$

Proposed by Nguyen Van Canh – Vietnam

S.604 Let $\varphi_m(x) = \frac{m}{e^x}, \omega_m(x) = \frac{m}{e^{2x}}, \delta_m(x) = \frac{m}{e^{3x}}$. Find all real numbers m such that:

$$\min \varphi_m(x) \cdot \max \omega_m(x) \cdot \min \delta_m(x) = e, \forall x \in [0,1]$$

Proposed by Nguyen Van Canh – Vietnam

S.605 Let $f_\alpha(x) = \sqrt{x^2 + \alpha x + \alpha^2 + 2020}$. Find all real numbers α such that:

$$\max f_\alpha(x) - \min f_\alpha(x) \le 1, \forall x \in [0,1]$$

Proposed by Nguyen Van Canh – Vietnam

S.606 Let $u_{n+1} = u_n^3 - u_n + 1, u_1 = \alpha$. Find all positive real numbers α such that:

$$\lim_{n\to\infty} u_n = 1$$

Proposed by Nguyen Van Canh – Vietnam

S.607 Let $f_\alpha(x) = \sqrt{e^{\alpha x} + e^{-\alpha x}}, (0 \le x \le \ln 2)$. Find all positive real numbers α such that:

$$\max f_a\,(x^2) + \min f_\alpha(x^2) = \frac{1}{2}, \forall x \in [0, \ln 2]$$

Proposed by Nguyen Van Canh – Vietnam

S.608 Let a, b, c be positive real numbers such that $abc = 1$. Find the minimum value of expression:

$$P = \frac{20}{(1+a)^3} + \frac{20}{(1+b)^3} + \frac{20}{(1+c)^3} + \frac{10}{a^{2020}} + \frac{10}{b^{2020}} + \frac{10}{c^{2020}}$$

Proposed by Nguyen Van Canh – Vietnam

S.609 Let: $u_{n+1} = \alpha u_n^2 + u_n + \frac{1}{2}, u_1 = 0$. Find all positive real numbers α such that:

$$\lim_{n\to\infty} u_n = \frac{\sqrt{2}}{2}$$

Proposed by Nguyen Van Canh – Vietnam

S.610 Let $u_n = \sqrt{2021 - \sqrt{2020 + \sqrt{2020 + \sqrt{2020} + \sqrt{2020} + \cdots}}}$ $(n \ge 3, n \in \mathbb{N})$

Find: $\Omega = \lim_{n\to\infty} \frac{u_{n+1}^2}{n+1}$

Proposed by Nguyen Van Canh – Vietnam

S.611 Let $u_{n+1} = u_n^2 - u_n + 1, u_1 = \alpha$.Find all real numbers α such that:

$$\lim_{n\to\infty} u_n = 1$$

Proposed by Nguyen Van Canh – Vietnam

S.612 Let $f_m(x) = \frac{-2020x^2+(2-m)x+m-5}{x+1}, (m \in \mathbb{R})$.Find all real numbers m such that:

$$\max f_m\,(x) + \min f_m\,(x) = 2021, \forall x \in [0,1]$$

Proposed by Nguyen Van Canh – Vietnam

S.613 Let $u_{n+1} = \ln(\alpha^2 + 1 + u_n^2)\,, u_1 = \frac{1}{8}$.Find all real numbers α such that:

$$\lim_{n\to\infty} u_n = 0$$

Proposed by Nguyen Van Canh – Vietnam

S.614 Let: $u_{n+1}=\frac{u_n-3}{2u_n+5}, u_1=\alpha\ (0<\alpha<1)$.Find: $\Omega=\lim_{n\to\infty} u_n^n$

Proposed by Nguyen Van Canh – Vietnam

S.615 Let a,b be real numbers. Prove:

$$\tan^{-1}\left(\frac{2ab}{a+b}\right)+\tan^{-1}(\sqrt{ab})+\tan^{-1}\left(\frac{a+b}{2}\right)\le 3\sqrt{ab}\tan^{-1}\left(\frac{a+b}{2}\right), a,b\ge 1$$

Proposed by Jalil Hajimir – Toronto – Canada

S.616 Find without any software:

$$\int_0^{\frac{\pi}{2}}\frac{\cos^7 x}{\sin x+\cos x}dx$$

Proposed by Jalil Hajimir – Toronto – Canada

S.617 If $x,y,z>0$ then find the minimum of:

$$f(x,y,z)=\sum_{cyc}\frac{x^2}{(x+y)(x+z)}$$

Proposed by Jalil Hajimir – Toronto – Canada

S.618 Prove that: $2\sqrt{e}\le e^{\sin^2 x}+e^{\cos^2 x}\le 1+e, \forall x\in\mathbb{R}$

Proposed by Jalil Hajimir – Toronto – Canada

S.619 If $A,B\in M_n(\mathbb{R}), AB+BA=O_n$ then: $det(A^2+b^2)\ge det\big((A-B)(A+B)\big)$

Proposed by Marian Ursărescu-Romania

S.620 If $A\in M_5(\mathbb{R}), detA\ne 0, Tr((A^*)^{-1})=1$ then: $det(A^2+I_5)\ge(TrA)^2$

Proposed by Marian Ursărescu-Romania

S.621 Let be the sequence $x_0=0, x_1=1, x_{n+2}=3x_{n+1}-x_n, \forall n\in\mathbb{N}$. Find:

$$\lim_{n\to\infty}\sqrt[n]{\sum_{k=0}^{n}\frac{x_{2k+1}}{x_k+x_{k+1}}}$$

Proposed by Marian Ursărescu-Romania

S.622 If $A\in M_5(\mathbb{R}). detA=-1$ then: $det(A^2+I_5)\ge(2TrA-1)^2$

Proposed by Marian Ursărescu-Romania

S.623 If $A\in M_5(\mathbb{R}), detA\ne 0, Tr(A^{-1})^*=1$ then: $det(A+I_5)\ge(TrA)^2$

Proposed by Marian Ursărescu-Romania

S.624 Let be the sequences $x_1,y_1>0, x_{n+1}=x_n-x_n^p$ and $y_{n+1}=\frac{1}{y_1^p}+\frac{1}{y_2^p}+\cdots+\frac{1}{y_n^p}$

$p \in \mathbb{N}, p \geq 2$. Find: $\lim_{n\to\infty}\left(x_n^{p-1} \cdot y_n^{p+1}\right)$

Proposed by Marian Ursărescu-Romania

S.625 $A \in M_5(\mathbb{R}), detA = 1$. Prove that:

$$2det(A^2 + I_5) + TrA \geq Tr(A^{-1}) + Tr(A^{-1})^* + Tr(A^*)$$

Proposed by Marian Ursărescu-Romania

S.626 If $A \in M_4(\mathbb{R}), detA = -1$ then: $det(A^2 + i_4) \geq \frac{1}{2}(TrA)^2$

Proposed by Marian Ursărescu-Romania

S.627 If $A \in M_4(\mathbb{R}), detA \neq 0, Tr(A^{-1}) = -1$ then: $det(A^2 + I_4) \geq (detA + TrA)^{-1}$

Proposed by Marian Ursărescu-Romania

S.628 If $A \in M_4(\mathbb{R}), detA = 1$ then: $det(A^2 + I_4) \geq (TrA - TrA^{-1})^2$

Proposed by Marian Ursărescu-Romania

S.629 Solve for real numbers: $p^{x^4-(p^2-2p+2)x^2+(p-1)^2} = \frac{px}{x^2+p-1}, p > 1$

Proposed by Marian Ursărescu-Romania

S.630 Evaluate:

$$\lim_{x\to 0^-}\frac{\left(3^{\frac{\sin x}{2x}} - \sqrt{3}\right)x}{\sqrt{2^{1-\cos 2x} - 1}}$$

Proposed by Jalil Hajimir – Toronto – Canada

S.631 Let x, y and z be positive real numbers. Find the minimum value of:

$$f(x,y,z) = \frac{e^{4x}}{(e^y - 1)^2} + \frac{e^{4y}}{(e^z - 1)^2} + \frac{e^{4z}}{(e^x - 1)^2}$$

Proposed by Jalil Hajimir – Toronto – Canada

S.632 Find all functions $f: \mathbb{R} \to \mathbb{R}$ such that: $f(x+2y)f(2x+y) - f(x+y) = xy$

Proposed by Jalil Hajimir – Toronto – Canada

S.633 Find without any software:

$$\int_0^a x^2 \log\left(\frac{a - \sqrt{a^2 - x^2}}{a + \sqrt{a^2 - x^2}}\right)dx$$

Proposed by Jalil Hajimir – Toronto – Canada

S.634 Show that:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^k < \frac{\ln 3}{\ln 2}$$

Proposed by Naren Bhandari – Bajura – Nepal

S.635 $\begin{cases}25725-a^2=b^3\\ c^2-25725=d^2\end{cases}$

If a,b,c and d are positive integers. Find the smallest possible value of $a+b+c+d$.

Proposed by Naren Bhandari – Bajura – Nepal

S.636 Prove that:

$$\int_0^1 \frac{\ln\left(x+\sqrt{1+x^2}\right)}{\sqrt{1+x^2}}dx=\frac{\ln^2\left(1+\sqrt{2}\right)}{2}$$

Restriction: It is required to prove the result without the use any sorts of substitution and known primitive of integrand.

Proposed by Naren Bhandari – Bajura – Nepal

S.637 $x_0=2020, x_n=-(x_0+x_1+x_2+\cdots+x_{n-1}), n\ge 1$.Find:

$$\Omega=\lim_{n\to\infty}\left(\frac{x_n}{x_{n+1}}+\frac{x_{n+1}}{x_n}-n\right)$$

Proposed by Rajeev Rastogi – India

S.638 Make an increasing order:

$$\alpha=\sin(\sin 2020^\circ), \beta=\sin(\cos 2020^\circ)$$

$$\gamma=\cos(\sin 2020^\circ), \delta=\cos(\cos 2020^\circ)$$

Proposed by Rajeev Rastogi – India

S.639 If $f:[0,1]\to[0,\infty), f(0)=0, f-$derivable, $0<f'(x)\le 1$ then:

$$\left(\int_0^1 f'(x)dx\right)^{2019}\ge\int_0^1 f^{2020}(x)dx$$

Proposed by Rajeev Rastogi – India

S.640 Prove without any software:

$$\sqrt{2}<\int_2^3 x^{\frac{1}{x}}dx<\sqrt{3}$$

Proposed by Rajeev Rastogi – India

S.641 $f(x)=x^8-9x^7+31x^6+a_0x^5+a_1x^4+a_2x^3+a_3x^2+a_4x+a_5\in\mathbb{R}[x]$

If $f(x)$ has all roots real prove that its lie in $[-2,4]$

Proposed by Rajeev Rastogi – India

S.642 $d: y=ax+b, a,b\in\mathbb{R}, a^2+b^2\neq 0, f(x)=x^4+2x^3-11x^2-13x+35$

$d\cap G_f=\{(\alpha,f(\alpha)),(\beta,f(\beta))\}$. Prove that:

$$\frac{\alpha^2+\beta^2}{a^2+b^2}=\frac{13}{2}$$

Proposed by Rajeev Rastogi – India

S.643 $f'(x)+2f(x)=x+|x|-(x-a)|x-a|, x\neq 0, f(0)=0, a\in\mathbb{R}$ –fixed

Find $f(x)$. ***Proposed by Akerele Olofin-Nigeria***

S.644 Find:

$$\Omega=\lim_{n\to\infty}\sqrt[n]{\left(\sum_{k=1}^{n}2^k\cdot\tan^{-1}\left(\frac{\tan k}{2^k+k+1}\right)\right)\left(\sum_{k=1}^{n}k\cdot k!\right)^{-1}}$$

Proposed by Ruxandra Daniela Tonilă-Romania

S.645 Find:

$$\int\sqrt[3]{x}\cos x\,dx$$

Proposed by Ghuiam Nasery-Afghanistan

S.646 Find:

$$\lim_{x\to 0}\frac{1}{x}\left\{\left(\sum_{k,n\geq 0}\binom{k+n}{n}\frac{x^k y^n}{(k+n)!}-e^y\right)\right\}$$

Proposed by Hussain Reza Zadah-Afghanistan

S.647 Find:

$$\Omega(n)=\lim_{x\to 0}\frac{2^{\tan 2x}\cdot 4^{\tan 4x}\cdot\ldots\cdot(2n)^{\tan(2nx)}-1}{3^{\tan 3x}\cdot 5^{\tan 5x}\cdot\ldots\cdot(2n+1)^{\tan((2n+1)x)}-1}, n\in\mathbb{N}, n\geq 1$$

Proposed by Mohammad Hamed Nasery-Afghanistan

S.648 $z_A, z_B, z_C\in\mathbb{C}, |z_A|=|z_B|=|z_C|$. If $|z_A-z_B|=|z_B-z_C|=|z_C-z_A|$ then find $z\in\mathbb{C}$ such that:

$$\begin{cases}|z-z_A|\leq|z_B+z_C|\\|z-z_B|\leq|z_C+z_A|\\|z-z_C|\leq|z_A+z_B|\end{cases}$$

Proposed by Ionuț Florin Voinea – Romania

S.649 Find:

$$\Omega=\lim_{x\to\infty}\prod_{k=1}^{x-1}\left(1+\frac{k}{x}\right)^{\frac{1}{x}\left(1+\frac{k}{x}\right)^3}$$

Proposed by Mohammad Hamed Nasery-Afghanistan

S.650 Solve for real numbers: $(\sin x)^{\cos x} = (\cos x)^{\sin x}$

Proposed by Mehdi Golafshan-Iran

S.651 If $0 < a < b < 1$, then

$$\int_0^{\frac{\pi}{4}} \frac{a + b\sin x}{b + a\sin x}dx \cdot \int_0^{\frac{\pi}{4}} \frac{1}{b + a\sin x}dx \geq \frac{\pi}{4}\left(\frac{1}{b} - \frac{\sqrt{2}}{2b + a\sqrt{2}}\right)$$

Proposed by Florică Anastase – Romania

S.652 $(b_n)_{n\geq 2}, b_n = \frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}} - \frac{n^2}{\sqrt[n]{n!}}$ be **Bătinețu sequence** and

$$\omega_n = 1 - \frac{\binom{n}{1}}{3} + \frac{\binom{n}{2}}{5} - \cdots + \frac{(-1)^n\binom{n}{n}}{2n+1}$$

Find:

$$\Omega = \lim_{n\to\infty}\left(1 + \frac{\sqrt[n]{\omega_n}}{n!}\right)^{\frac{n!}{b_n^n}}$$

Proposed by Florică Anastase – Romania

S.653 If $0 < a < b \leq \frac{\pi}{2}$, then:

$$\int_a^b \left(\frac{\int_0^x \frac{\sin t}{1+\cos t}dt}{\int_0^x \log\left(\frac{1+\sin t}{1+\cos t}\right)dt}\right)dx \leq \left(a - \frac{a^2}{b}\right)\cdot\frac{\log 2}{\int_0^a \log\left(\frac{1+\sin t}{1+\cos t}\right)dt}$$

Proposed by Florică Anastase – Romania

S.654 $a, b, p, q \in \mathbb{N}$ such that $p(q-b) = a+1$. Find:

$$\Omega = \lim_{m\to\infty}\left(\frac{1}{m}\cdot\lim_{n\to\infty}\sum_{p=1}^{2^m}\sum_{i=1}^{n} i^a \sin^p\left(\frac{i^b}{n^q}\right)\right)$$

Proposed by Florică Anastase – Romania

S.655 Let $F: (0,\infty) \to \mathbb{R}$ be a primitive of function $f: (0,\infty) \to \mathbb{R}$

$f(x) = \frac{2(x - \tan^{-1}x)}{(1+x^2)^2(\tan^{-1}x)^3}$. Find:

$$\Omega = \lim_{x\to\infty}\frac{x}{F(x)}\cdot\int_{\frac{1}{x+1}}^{\frac{1}{x}}\frac{\sin t}{t^2}dt$$

Proposed by Florică Anastase – Romania

S.656 $(x_n)_{n\geq 1}$ – sequence of real numbers such that $x_{n+1} = x_n + \left(\frac{1}{\pi}\right)^{x_n}, x_1 \in \mathbb{R}_+$. Find:

$$\Omega = \lim_{n\to\infty} \frac{x_n}{\sqrt[n]{\log^2 n}} \cdot \int_0^{\pi} \frac{\log(x+n)}{x^2+n\pi} dx$$

Proposed by Florică Anastase – Romania

S.657 If

$$\Omega_n = \frac{1}{2}\left(\prod_{k=1}^{n} \tan^2 \frac{k\pi}{2n+1} - 1\right)$$

then find:

$$\Omega = \lim_{n\to\infty}\left(\frac{\frac{\Omega_1}{\Omega_n} + \frac{\Omega_2}{\Omega_{n-1}} + \cdots + \frac{\Omega_{n-1}}{\Omega_2} + \frac{\Omega_n}{\Omega_1}}{\log\Omega_1 + \log\Omega_2 + \cdots + \log\Omega_n}\right)^{\log \sqrt[n]{\Omega_1\cdot\Omega_2\cdot\ldots\cdot\Omega_n}}$$

Proposed by Florică Anastase – Romania

S.658 If $(a_n)_{n\geq 1}, a_1 = a > 0, a_{n+1} = n\cdot\sqrt{a_n} + 1$ then find:

$$\Omega = \lim_{n\to\infty}\left(\frac{n^3}{\log n}\int_0^a \frac{\log(x+n)}{x^2+na_n}dx\right)$$

Proposed by Florică Anastase – Romania

S.659 $a, b, c \in \mathbb{C}^*$ - different in pairs, $|a| = |b| = |c|, A(a), B(b), C(c)$. Prove that:

$$\sum_{cyc}|(2a-b-c)(2b-c-a)(a-b)| = 9\left|\prod_{cyc}(a-b)\right| \Rightarrow AB = BC = CA$$

Proposed by Marian Ursărescu – Romania

S.660 Find:

$$\Omega = \lim_{n\to\infty}\sqrt[n]{\sum_{k=1}^{n} k^2\binom{n}{k-1}\binom{n}{k}}$$

Proposed by Marian Ursărescu – Romania

S.661

$$x_n = \binom{n}{0}p^n + \binom{n}{3}p^{n-3} + \binom{n}{6}p^{n-6} + \cdots,$$

$$y_n = \binom{n}{1}p^{n-1} + \binom{n}{4}p^{n-4} + \binom{n}{7}p^{n-7} + \cdots,$$

$$z_n = \binom{n}{2}p^{n-2} + \binom{n}{5}p^{n-5} + \binom{n}{8}p^{n-8} + \cdots, p \in \mathbb{N}, p \geq 2$$

Find:

$$\Omega(p) = \lim_{n\to\infty} \sqrt[n]{x_n y_n + y_n x_n + z_n x_n}$$

Proposed by Marian Ursărescu – Romania

S.662 $z_1, z_2, z_3 \in \mathbb{C}^*$, different in pairs, $|z_1| = |z_2| = |z_3|, A(z_1), B(z_2), C(z_3)$

Prove that:

$$\sum_{cyc} \frac{1}{\left|(z_1 - z_2)|z_1 - z_3| + (z_1 - z_3)|z_1 - z_2|\right|^2} = 3\left(\sum_{cyc} |z_1 - z_2|\right)^{-2} \Rightarrow AB = BC = CA$$

Proposed by Marian Ursărescu – Romania

S.663 $z_1, z_2, z_3 \in \mathbb{C} - \{0\}$ - different in pairs, $|z_1| = |z_2| = |z_3| = 1$

$$A(z_1), B(z_2), C(z_3), \sum_{cyc} \frac{2 + \frac{z_1}{z_2} + \frac{z_2}{z_1}}{6 - \frac{z_1}{z_2} - \frac{z_2}{z_1}} = \frac{3}{7}$$

Prove that: $AB = BC = CA$.

Proposed by Marian Ursărescu – Romania

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

UNDERGRADUATE PROBLEMS

U.198 Find:

$$\Omega = \lim_{n\to\infty} \left(\frac{(n!)^{ne}}{e^{1+2^n+3^n+\cdots+n^n}} \cdot (H_n)^{-1}\right)$$

Proposed by Daniel Sitaru – Romania

U.199 If $a, b \geq 0$ then:

$$\int_0^a \left(\int_0^t \log^5(x^2 + x + 2)\, dx \right) dt + \int_0^b \left(\int_0^t \log^5(x^2 + x + 2)\, dx \right) dt \geq ab \log 2$$

Proposed by Daniel Sitaru – Romania

U.200 If $0 < a \leq b < 1, f: [0,1] \to (0, \infty), f$ - continuous then:

$$\int_a^b \int_a^b \int_a^b \int_a^b \big(f(x) + f(z)\big)\big(f(y) + f(t)\big)\, dx\, dy\, dz\, dt \geq 2(b-a)^3 \left(2 \int_a^b f(x)\, dx - 2b + 2a \right)$$

Proposed by Daniel Sitaru – Romania

U.201 Find without softs:

$$\Omega = \lim_{(\mathrm{x,y}) \to (0,0)} \left(\int_{\frac{\pi}{6}+x}^{\frac{\pi}{3}-y} \sqrt{\tan x}\, dx \right) \left(\int_{\frac{\pi}{6}+x}^{\frac{\pi}{3}-y} \sqrt{\cot x}\, dx \right)$$

Proposed by Daniel Sitaru – Romania

U.202

$$\Omega(p) = \sum_{n=1}^{\infty} \frac{n^{p-1} + n^{p-2} + n^{p-3} + \cdots + n}{n^p + n^{p-1} + n^{p-2} + \cdots + n}, p \in \mathbb{N}, p \geq 1$$

Find:

$$\Omega = \lim_{\mathrm{n} \to \infty} \left(\log n - \sum_{p=1}^{n} \Omega(p) \right)$$

Proposed by Daniel Sitaru – Romania

U.203 If $0 < a \leq b < \frac{\pi}{2}$ then:

$$\int_a^b \int_a^b \frac{(\sin^3 x + \tan y)^2 (\cos^3 x + \tan y)}{(\sin^2 x \cdot \cos x + \tan y)^3} dxdy \geq (b-a)^2$$

Proposed by Daniel Sitaru – Romania

U.204 If $0 < a \leq b$ then:

$$8 \int_a^b \int_a^b \frac{x^{12} + y^{12}}{x^5 + y^5} dx\, dy \geq (b-a)(b^8 - a^8)$$

Proposed by Daniel Sitaru – Romania

U.205 If $0 < a \le b < \frac{\pi}{2}$ then:

$$\int_a^b\int_a^b\int_a^b (\tan x \tan y + 1)(\tan y \tan z + 1)(\tan z \tan x + 1)\,dx\,dy\,dz \le (\tan b - \tan a)^3$$

Proposed by Daniel Sitaru – Romania

U.206 If $n \in \mathbb{N}$, then:

$$\int_0^\infty (1 - \tanh(x))^n dx = 2^{n-1}\left(\ln(2) - \sum_{k=1}^{n}\frac{1}{k2^k}\right)$$

Proposed by Angad Singh – India

U.207 If $m > 0$, then prove that:

$$\lim_{x\to 0}\frac{1}{x}\left(\psi\left(\frac{m+x}{2x}\right) - \psi\left(\frac{m}{2x}\right)\right) = \frac{1}{m}$$

Proposed by Angad Singh – India

U.208 Prove that:

$$\int_0^\infty \cos(\pi x^2)\frac{\sinh(\sqrt{2}\pi x) - \sin(\sqrt{2}\pi x)}{\sinh^2\left(\frac{\pi x}{\sqrt{2}}\right) + \sin^2\left(\frac{\pi x}{\sqrt{2}}\right)}dx = \frac{1}{\sqrt{2}}\left(\frac{\pi^{\frac{1}{4}}}{\Gamma\left(\frac{3}{4}\right)} - 1\right)$$

Proposed by Angad Singh – India

U.209 If $a \in \mathbb{R}$, then:

$$\int_0^1 \frac{e^{ax}}{1+x}dx = e^{-1}\left(\ln(2) + \sum_{k=1}^{\infty}\frac{2^k - 1}{k\cdot k!}a^k\right)$$

Proposed by Angad Singh – India

U.210 If $n \in \mathbb{N}$, then:

$$\int_0^\infty (1 - \tanh^{2n}(x))\,dx = \sum_{k=0}^{n-1}\binom{2n}{2k+1}\phi(k,n)$$

where

$$\phi(k,n) = \frac{(2k)!}{(2n-1)_{2k}(2n-2k-1)}\left(1 - \frac{1}{2^{2n-2k-1}}\right) - \sum_{m=1}^{2k}\frac{(2k)_{m-1}}{(2n-1)_m 2^{2n-m}}$$

where $(x)_k$ is the falling factorial.

Proposed by Angad Singh – India

U.211 Prove that:

$$\lim_{x\to 0}\frac{1}{x^2}\left(\psi(x)+\omega\psi(\omega x)+\omega^2\psi(\omega^2 x)+\frac{3}{x}\right)=-3\zeta(3)$$

where $\omega^2+\omega+1=0$ and $\psi(x)$ is the digamma function.

Proposed by Angad Singh – India

U.212 If $n>0$, then

$$\int_0^\infty \frac{\ln(\cosh(x))}{\cosh^n(x)}dx=\int_0^\infty \frac{\psi(n)-\psi\left(\frac{n}{2}\right)-\ln(2)}{\cosh^n(x)}dx$$

where $\psi(n)$ is the digamma function.

Proposed by Angad Singh – India

U.213

$$\int_0^\infty \frac{x-x^3}{(4x-4x^3)^2+(2-6x^2+x^4)^2}dx=\frac{1}{16\sqrt{2}}\left(\pi-\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)\right)$$

Proposed by Angad Singh – India

U.214 If $|a|<1, x^{(0)}=1, x^{(k)}=x(x+1)(x+2)\dots(x+k-1)$ and

$$R(a)=\frac{1}{\Gamma^2\left(\frac{a}{2}\right)\Gamma(1-a)}\sum_{k=0}^{\infty}\frac{a^{(k)}}{k!\left(k+\frac{a}{2}\right)^2}$$

then: $R^2(a)+R^2(1-a)=1$

Proposed by Angad Singh – India

U.215 If

$$\psi(m,a)=\int_0^\infty \frac{e^{-x^2}x^{m-1}}{(1+x^2)^a}$$

and $a\in\mathbb{N}$, then prove that:

$$\psi(2,a)=\frac{1}{2}\sum_{k=1}^{a-1}\frac{(-1)^{k-1}}{(a-1)_k}-\frac{(-1)^a eE_1(1)}{2\Gamma(a)}$$

where $E_1(x)$ is the generalized exponential integral.

Proposed by Angad Singh – India

U.216 Prove that:

$$\sum_{k=1}^{\infty}\frac{1}{k\binom{3k}{k}5202^k}=\frac{1}{53}\left(\frac{6}{5}\tan^{-1}\left(\frac{5}{99}\right)-\ln\left(\frac{18}{17}\right)\right)$$

Proposed by Angad Singh – India

U.217 If $0 < n < \frac{1}{2}$ and

$$K(n) = \frac{\Gamma(n)}{\int_0^\infty x^{n-1} e^x erfc(\sqrt{x}) dx}$$

then

$$K^2(n) + K^2\left(\frac{1}{2} - n\right) = 1$$

Proposed by Angad Singh – India

U.218 Prove that:

$$\int_0^1 \frac{Li_2^2(x)}{x} dx = 2\zeta(2)\zeta(3) - 3\zeta(5)$$

Notation: $Li_s(x)$ is polylogarithmic function and $\zeta(.)$ is Riemann zeta function.

Proposed by Narendra Bhandari – Bajura – Nepal

U.219 Prove that:

$$\sum_{k=1}^{\infty}\sum_{n=1}^{k}\int_0^1 \frac{n}{x^{n(n+1)} - 1}\left((n+1)x^n - \sum_{l=0}^{n} x^{nl}\right)\frac{dx}{(n+1)^2(k-n)!} = \frac{\zeta(2)}{e^{-1}}\ln\left(\frac{A^{12}}{2\pi}\right) - \frac{\zeta(2)}{(e\gamma)^{-1}}$$

Before you approach to solve the integral it is required for this justification to converge of integral.Notation: A is Glashier-Kinkelin constant, $\zeta(.)$ is Riemann zeta function and γ is Euler Mascheroni constant

Proposed by Narendra Bhandari – Bajura – Nepal

U.220 Prove that:

$$\sum_{n}^{\infty} \frac{(-1)^{n+1}}{5n^2}\int_0^{5n}\int_0^{5n}\{y\}^{\lfloor x \rfloor} dx\, dy = \frac{13\pi^2}{60} - \frac{1}{2}\log^2 2 - 2\log^2 \phi$$

where $\{.\}$ is ceiling function, $\lfloor . \rfloor$ is floor function, $\zeta(.)$ is Riemann zeta function and ϕ is Golden ratio.

Proposed by Narendra Bhandari – Bajura – Nepal

U.221 For all $n \geq 1$ being positive integer, prove that:

$$\lim_{m\to\infty}\lim_{n\to\infty}\prod_{k=1}^{m}\cos\left(2\pi\sqrt[2^k]{n^{2^k} + \frac{n^{2^k-1}}{120} + 2^k}\right) = \frac{15}{\phi\pi}\left(\frac{1}{\sqrt{2}} + \sqrt{\frac{3}{2} - \frac{\phi}{\sqrt{2}}(\sqrt{3}-1)\sqrt{1+\phi^2}}\right)$$

where ϕ is Golden ratio.

Proposed by Narendra Bhandari – Bajura – Nepal

U.222 Find:

$$\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\zeta(4)-\zeta(3)-\frac{1}{1^4}-\frac{1}{2^4}-\cdots-\frac{1}{n^4}-\frac{1}{1^3}-\frac{1}{2^3}-\cdots-\frac{1}{n^3}\right)^2$$

Notation: $\zeta(z)$ denotes as Riemann zeta function.

Proposed by Narendra Bhandari – Bajura – Nepal

U.223 Prove without the use hypergeometric identities.

$$\sum_{n=1}^{\infty}\frac{{}_2F_1\left(-n,\frac{1}{2};1;1\right)}{n^3}=2\zeta(3)+\frac{\log^3(4)}{6}-2\zeta(2)\log 2$$

where ${}_2F_1(a,b;c;z)$ is Gauss hypergeometric function and $\zeta(.)$ is Riemann zeta function.

Proposed by Narendra Bhandari – Bajura – Nepal

U.224 Prove that:

$$\sum_{k=1}^{\infty}(-1)^k\prod_{n=1}^{k}\cot\left(\frac{n\pi}{2k+1}\right)=\frac{1}{2}\left(\zeta\left(\frac{1}{2},\frac{5}{4}\right)-\zeta\left(\frac{1}{2},\frac{3}{4}\right)\right)$$

where $\zeta(s,a)$ is Hurwitz zeta function.

Proposed by Narendra Bhandari – Bajura – Nepal

U.225 Prove the following

$$\lim_{n\to\infty}\cot\left(\pi\sqrt{100n^2+n+1}\right)=\frac{16\cos^2(3^\circ)}{\sqrt{36-4\sqrt{5}+4\sqrt{30+6\sqrt{5}}}}\left(1-\frac{8}{4+\sqrt{10-2\sqrt{5}}+\sqrt{15}+\sqrt{3}}\right)$$

Proposed by Narendra Bhandari – Bajura – Nepal

U.226 Find closed form:

$$\int_0^{\infty}\frac{e^{-3x}\cos^3 x\log^2 x\,dx}{\sqrt{x}}$$

Proposed by Kaushik Mahanta – Assam – India

U.227 Generalization and variation of the 77th William Lowel Putam maths competition problem. Prove or disprove

$$\lim_{M\to\infty}\sum_{j=1}^{M}\sum_{p=1}^{j}\left(\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\sum_{n=0}^{\infty}\frac{p}{k2^n+p}-\ln p\right)\frac{1}{M}\frac{\left(\sqrt{2}\right)^{j-1}}{(j-p)!!}=\gamma e\left(\sqrt{\frac{\pi}{2}}\operatorname{erf}(1)+1\right)$$

Notation: γ is Euler – Mascheroni constant, $\operatorname{erf}(.)$ is error function and !! is double factorial.

Proposed by Narendra Bhandari – Bajura – Nepal

U.228 Prove that:

$$\lim_{n\to\infty}\left(\frac{\ln\left(1+\frac{1^2}{n^2}\right)}{\left(1+\frac{1^2}{n^2}\right)^2}+\frac{\ln\left(1+\frac{2^2}{n^2}\right)}{\left(1+\frac{2^2}{n^2}\right)^2}+\cdots+\frac{\ln\left(1+\frac{n^2}{n^2}\right)}{\left(1+\frac{n^2}{n^2}\right)^2}\right)\frac{1}{n}=\frac{1}{4}-\frac{G}{2}-\frac{\pi}{8}+\frac{\pi+1}{4}\ln 2$$

Notation: G is Catalan constant.

Proposed by Narendra Bhandari – Bajura – Nepal

U.229 If $a>0$ then:

$$\int_{-a}^{a}\int_{-a}^{a}|(x+y)(1-xy)|\,dxdy\leq\frac{2}{9}(3a+a^3)^2$$

Proposed by Daniel Sitaru – Romania

U.230 $a,b,c>1, a+b+c=12, \Omega(a)=\lim_{n\to\infty}n^{a-1}\left(\zeta(a)-\zeta_n(a)\right)$

Prove that: $\Omega(a)+\Omega(b)+\Omega(c)\geq 1$

Proposed by Daniel Sitaru – Romania

U.231 Find a closed form:

$$\Omega=\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{1}{2(n+1)^2}\right)\cdot\tan^{-1}\left(n+1-\frac{1}{2(n+1)}\right)$$

Proposed by Daniel Sitaru – Romania

U.232 $0<a\leq b, f:[a,b]\to(0,\infty), f$ – continuous. Prove that:

$$\int_a^b\frac{dx}{f(x)}=\sqrt{3}\Rightarrow\int_a^b\int_a^b\sqrt{\frac{1}{f^2(x)}+\frac{1}{f(x)f(y)}+\frac{1}{f^2(y)}}\,dx\,dy\geq 3(b-a)$$

Proposed by Daniel Sitaru – Romania

U.233 Find a closed form:

$$\Omega=\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)(n+2)(2n+1)(2n+3)}$$

Proposed by Daniel Sitaru – Romania

U.234 Find:

$$\Omega=\lim_{n\to\infty}\left(\frac{3^n\left((2n-1)!!+(2n)!!\right)}{H_{n+1}(2n^2+1)^n+H_n(2n^2+6n+4)^n}\right)$$

Proposed by Daniel Sitaru – Romania

U.235 If $0<a\leq b, f:(0,\infty)\to(0,\infty), f$ – continuous, then:

$$\int_a^b\int_a^b \frac{\sqrt{f(x)f(y)}}{f(x)+f(y)}dx\,dy+\frac{b-a}{2}\int_a^b\sqrt{f(x)}\,dx\le\left(\int_a^b f(x)\,dx\right)\left(\int_a^b\frac{dx}{f(x)}\right)$$

Proposed by Daniel Sitaru – Romania

U.236 If $0<a\le b$ then:

$$\int_a^b\int_a^b\int_a^b \frac{yzdxdydz}{3x^2+2y^2+z^2}\le\frac{(b-a)^2(b+a)}{12}\cdot\log\left(\frac{b}{a}\right)$$

Proposed by Daniel Sitaru – Romania

U.237 If $1<a\le b$ then:

$$\frac{2}{3}\int_a^b\int_a^b\int_a^b(e^{xy}+e^{yz}+e^{zx})\,dx\,dy\,dz\ge 2(b-a)^3+(b^2-a^2)\log\left(\frac{b}{a}\right)$$

Proposed by Daniel Sitaru – Romania

U.238 Find the general solution for:

$$\left(\frac{dy}{dx}\right)^2=y^2+y$$

Proposed by Jalil Hajimir – Toronto – Canada

U.239 Find without softs:

$$\Omega=\int_0^1\log\big(\Gamma(x+2020)\big)\,dx$$

Proposed by Jalil Hajimir – Toronto – Canada

U.240 Let a,b and c be non-negative real numbers. Find the maximum value of:

$$f(x,y,z)=\sum_{cyc}\sqrt{\frac{x+y}{x+y+z}}$$

Proposed by Jalil Hajimir – Toronto – Canada

U.241 Find without any software:

$$\int_0^\infty\frac{\log x}{x^4+4}dx$$

Proposed by Jalil Hajimir – Toronto – Canada

U.242 Find the general solution for:

$$\frac{dX}{dt}=\begin{bmatrix}1 & -2\\ 3 & 1\end{bmatrix}X$$

Proposed by Jalil Hajimir – Toronto – Canada

U.243 Find a closed form:

$$\Omega(n)=\int_{-\infty}^{\infty}\frac{\cos(nx)}{(1+x^2)^2}dx, n\in\mathbb{N}$$

Proposed by Jalil Hajimir – Toronto – Canada

U.244 Evaluate:

$$\int_0^{\infty}x^{\frac{3}{2}}2^{-\frac{x^3}{2}}dx$$

Proposed by Jalil Hajimir – Toronto – Canada

U.245 Find the absolute maximum value of

$f(z)=\left|\frac{3z^2+2}{z^2}\right|$ on the disk $D=\{z\in C:|z+1|\leq 1\}$

Proposed by Jalil Hajimir – Toronto - Canada

U.246 Compute:

$$\oint_{|z|=1}\frac{\sin\pi\, zdz}{(4z-1)^3(2z-1)^2}$$

Proposed by Jalil Hajimir – Toronto – Canada

U.247 Find a closed form:

$$\Omega=\int_0^{\infty}\frac{\cos x}{x^4+4}dx$$

Proposed by Jalil Hajimir – Toronto – Canada

U.248 Find the general solution of:

$$X'=\begin{bmatrix}1 & 4\\ 2 & -1\end{bmatrix}X+\begin{bmatrix}2^t\\ 3\end{bmatrix}$$

Proposed by Jalil Hajimir – Toronto – Canada

U.249 Find the general and particular form of solutions:

$$X'=\begin{bmatrix}1 & -1\\ 2 & 3\end{bmatrix}X+\begin{bmatrix}e^t\\ e^{-t}\end{bmatrix}, x(0,0)=(1,0)$$

Proposed by Jalil Hajimir – Toronto – Canada

U.250 For any $n\geq 1$, let $S(n)=\int_1^{\infty}\frac{tan^{-1}(cot(\pi nx))^2}{x^4}dx$ then show that:

$$\int_{-\infty}^{\infty}\frac{1}{S(n)}dn=\frac{4\sqrt{3}}{\pi}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.251

$$\int_0^\infty \frac{e^{-\pi\sqrt[3]{1+\sqrt[3]{z}}}}{\sqrt[3]{z}}dz = \frac{27e^{-\pi}}{\pi^6}(\pi(\pi(\pi(6+\pi)+20)+40)+40)$$

Proposed by Srinivasa Raghava-AIRMC-India

U.252 Let $y_1(t) = \frac{t-3}{t+1}$ and $y_{m+1}(t) = y_1(y_m(t))$ then prove that:

$$y_{2018}\left(\frac{2017}{2021}\right) = 2020$$

Proposed by Srinivasa Raghava-AIRMC-India

U.253 If $\frac{1+\sqrt{x^3-x-1}-x}{1+\sqrt{x^2-x-1}-x} = \frac{1+x}{1-x}$ then prove that:

$$x^8 - 14x^7 + 47x^6 - 84x^5 + 119x^4 - 102x^3 + 9x^2 + 8x + 32 = 0$$

Proposed by Srinivasa Raghava-AIRMC-India

U.254 Show that:

$$\sum_{n=2}^{\infty} \frac{999F_n + 22}{F_{n-1}F_{n+1}} = 2020$$

Proposed by Srinivasa Raghava-AIRMC-India

U.255 Compute the real part:

$$Re\left(\int_{e^{-\frac{i\pi}{3}}}^{e^{\frac{i\pi}{3}}} Li_3\left(e^{\frac{i\pi}{3}}\right)dx\right)$$

Proposed by Srinivasa Raghava-AIRMC-India

U.256 Prove the relation:

$$1 + \int_0^\infty \left(tanh\left(\frac{x}{2}\right) + tanh(x)\right)^2 \frac{e^{-x}}{x}dx = log\left(\frac{A^{12}}{\sqrt[3]{2}\pi^2\Gamma\left(\frac{5}{4}\right)^4}\right) + \frac{4G}{\pi}$$

A –Glasier constant, G –Catalan constant.

Proposed by Srinivasa Raghava-AIRMC-India

U.257 If we define the function $F(m) = \sum_{n=1}^{\infty} \frac{\sqrt{5^{n+1}}(\phi^{mn}+1)}{n^2\binom{2n}{n}\phi^{2n}}$ then find the value of $\frac{\partial F}{\partial m}$ at the point $m = 1$.

Proposed by Srinivasa Raghava-AIRMC-India

U.258 If, for $y \geq 1, V(y) = \int_0^{\pi y} cos\left(\frac{x}{y}\right) cos(xy)dx$ then prove that:

$$\int_0^\infty V(y)dy \int_0^\infty \frac{V(y)}{y^4}dy = \int_0^\infty \frac{V(y)^2}{y}dy$$

Proposed by Srinivasa Raghava-AIRMC-India

U.259 Prove the relation:

$$\int_0^\infty \left(\int_0^\infty \frac{\big(x+tanh(x)\big)sin(xy)}{cosh(x)}dx\right)\frac{dy}{y} = \pi\int_0^\infty \frac{x+tanh(x)}{cosh(x)}dx = 2\pi G + \pi$$

$G -$ is Catalan constant.

Proposed by Srinivasa Raghava-AIRMC-India

U.260 If $a,b>0, f(a,b)=\int_0^\infty \frac{(1-e^{-ax})^2 cosbx}{x}dx$ then for $a,b>\frac{3}{2}$, prove inequality:

$$f\left(\sqrt{ab},\frac{a+b}{2}\right)+log(a+b)>1$$

Proposed by Srinivasa Raghava-AIRMC-India

U.261 Prove the relation:

$$\prod_{k=1}^{\infty} exp\left(\frac{F_k}{k\sqrt{5^{k-1}}}\right) = \phi^2+2$$

F_k –Fibonacci number, ϕ –Golden Ratio.

Proposed by Srinivasa Raghava-AIRMC-India

U.262 If $a,b,c\in\mathbb{R}, b^2>4ac$ the prove the relation:

$$\sum_{k=-\infty}^{\infty}\frac{(-1)^k}{ak^2+bk+c} = \frac{cos\left(\frac{\pi b}{2a}\right)}{cos\left(\frac{\pi\sqrt{b^2-4abc}}{2a}\right)}\sum_{k=-\infty}^{\infty}\frac{1}{ak^2+bk+c}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.262 For $n\ge 1$, we have:

$$\int_0^{\frac{\pi}{2}}\frac{tanx}{nlog^2(tanx)+\pi^2}dx = \frac{1}{2\sqrt{n}}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.263 Evaluate the integral for $(x^2+y^2+z^2+w^2)\le 1$

$$\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\frac{\sqrt{1-(x^2+y^2+z^2+w^2)}}{1+x^2+y^2+z^2+w^2}dw\,dz\,dy\,dx$$

Proposed by Srinivasa Raghava-AIRMC-India

U.264 Let the recurrence relation:

$K(n+1)+k(n)=\phi^n+\phi^{2n}+\phi^{3n}$ with,$K(1)=\sqrt{5}$, then compute the sum

$\sum_{n=1}^{\infty}\frac{K(n)}{\phi^{4n}}, \phi$ –Golden Ratio.

Proposed by Srinivasa Raghava-AIRMC-India

U.265 If for $Re(n) > 0, U(n) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1+x^2}{1+y^2}e^{-ny^2-x^2}dydx$ then prove the relation:

$$\int U(n)dn = U(n) + 3\pi\sqrt{n}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.266 Find:

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{n=0}^{\infty}\frac{\binom{2n}{m+n}}{\phi^{k+3n}} = \phi^7, \phi - \text{Golden ratio.}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.267 For $x \in (0,1)$, if $f(x) = n - k^x$ and $g(x) = 1 + k^x$ for $n > 0, k \geq 0$ some consecutive integers $n > k$, then show that:

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{2n^2}\int_0^1(\ln f(x) + 2\ln g(x))\,dx = -\frac{\eta(2)}{\zeta(2)}\ln\left(\frac{1}{2^{\zeta(2)}}\prod_{m\geq 1}\sqrt[m^2]{m}\right) = \eta(2)\ln\left(\frac{4e^{\gamma}\pi}{A^{12}}\right)$$

where $\zeta(.)$ is Riemann zeta function, $\eta(.)$ Is Dirichlet eta function, A is Glashier-Kinkelin constant, γ is Euler-Mascheroni constant and e is Euler – number.

Proposed by Naren Bhandari – Bajura – Nepal

U.268 Show that the quartic equation:

$$F(X) = X^4 - 144X^3 + 6656X^2 - 98304X + 65536 = 0$$

has 4 real zero's $\alpha_1, \alpha_2, \alpha_3, \alpha_4$. Let α_3 be the smallest zero such that

$$\alpha_3 = a - b\sqrt{5} - 4\sqrt{c + d\sqrt{5}}, a, b, c, d \in \mathbb{Z}^+$$

then show that for $k = a + b + c + d + 1$ and $j \in \mathbb{Z}^+ \setminus \{1\}$

$$\sum_{\substack{n_1,n_2,\dots,n_k=0\\ n_1\neq n_2\neq\dots\neq n_k}}\prod_{i=1}^{k}\frac{\alpha_3}{j^{n_i}} > \frac{\ln^{77}(2)\cdot 77!\cdot j^{77}}{(j-1)(j^2-1)\dots(j^{76}-1)(j^{77}-1)}$$

Proposed by Naren Bhandari – Bajura – Nepal

U.269 For all $j, i \geq 0$, prove that:

$$\lim_{j\to\infty}\lim_{n\to\infty}\sum_{i=1}^{\infty}\sum_{k=0}^{n}\sum_{l=1}^{k+1}\sin^{-1}\left(\frac{\pi^{n^{2020}}}{n+(j+1)k}\right)\tan^{-1}\left(\frac{\pi^{-\Gamma\left(\frac{1}{n^{2020}}\right)}}{n+(i+1)(k-l+1)}\right)\frac{(i+1)^{-2}}{\ln(1+j)^{\sqrt[j]{1+j}}}$$

$$=\frac{\pi^\gamma}{e}\left(\ln\left(\frac{A^{12}}{2\pi}\right)-\gamma\right)\zeta(2)$$

where γ, A, e are Euler-Mascheroni, Glashier-Kinkelin constant and Euler number

Proposed by Naren Bhandari – Bajura – Nepal

U.270 Show that the integral

$$\int_1^\infty \frac{\ln(x^4-2x^2+1)}{x\sqrt{x^2-1}}dx=0$$

Proposed by Naren Bhandari – Bajura – Nepal

U.271 Prove that the following equality

$$\lim_{m\to\infty}\prod_{n=2}^{m}\left(\sum_{j=0}^{m}\sum_{i=1}^{3}\frac{i+1}{j+i}\left(\prod_{i=1}^{3}(j+i)\right)^{-1}-\frac{1}{n}\prod_{k=1}^{n-1}\sin\left(\frac{k\pi}{n}\right)\right)=\frac{\sqrt[24]{2^{13}}}{e^{\zeta(2)\ln^{-1}2}}\sqrt{\frac{\pi}{\ln 2}}$$

Proposed by Naren Bhandari – Bajura – Nepal

U.272 For all $n\geq 1$, prove:

$$I(n)=\int_0^1\frac{x^n}{1+x}dx=\frac{1}{2}\left(\psi^0\left(\frac{n+1}{2}\right)-\psi^0\left(\frac{2n+1}{n}\right)\right)=\frac{1}{2}\left(H_{\frac{n}{2}}-H_{\frac{n-1}{2}}\right)$$

and hence $\lim_{n\to\infty} I(n)=0$.

Notation: H_n denotes n[th] Harmonic number and $\psi^0(x)$ is digamma function.

Proposed by Naren Bhandari – Bajura – Nepal

U.273 For $n\geq 0$, prove that:

$$\int_0^{\frac{1}{4}\pi}\left(\frac{1-\tan x}{1+\tan x}\right)^n dx=\frac{1}{4}\left(\psi^0\left(\frac{n+3}{4}\right)-\psi^0\left(\frac{n+1}{4}\right)\right)=\frac{1}{4}\left(H_{\frac{n-1}{4}}-H_{\frac{n-3}{4}}\right)$$

Notation: $\psi^0(x)$ and H_n denotes digamma function and n[th] haromonic number respectively.

Proposed by Naren Bhandari – Bajura – Nepal

U.274 Let the sequence of the function $f_n(x)=(1-x^k)^n$ where $n,k\in\mathbb{N}$, then prove

$$\lim_{M\to\infty}\lim_{n\to\infty}\sum_{N=1}^{M}\left(\sum_{m=1}^{N}\sum_{k=1}^{m}\sqrt[n]{\int_0^1 f_n(x)\,dx}\,\frac{1}{m^2}+\frac{1}{\ln(N+1)}\right)\frac{1}{M}=\gamma$$

γ denotes Euler-Mascheroni constant.

Proposed by Naren Bhandari – Bajura – Nepal

U.275 For positive integers $1\leq j\leq n$, prove or disprove that:

$$\sum_{k=1}^{N}\sum_{j=1}^{k}\binom{2N}{2k}\binom{k}{j}=\frac{1}{\left(1+\sqrt{2}\right)^{2n}}+\frac{2P_{2n}}{\sqrt{2}}-\frac{4^n}{2}$$

where P_n is n^th Pell number.

Proposed by Naren Bhandari – Bajura – Nepal

U.276 For $x \in \mathbb{R}$, find the general solution for x such that the following is always finite and if

$$4\sin x\sum_{k=1}^{n}\sin\left((4k-1)x\right)=1-\cos\left((pn+q)x\right)\sec(x)$$

with p, q are positive co-prime integers, then show that for $k>0$ and

$$\alpha_n=\left\{\left\lfloor\frac{pn-q}{2}\right\rfloor : n\in\mathbb{N}\right\}$$

$$\alpha_1+\cfrac{1}{\alpha_2+\cfrac{1}{\alpha_3+\cfrac{1}{\alpha_4+\cfrac{1}{\dots}}}}\cdot\sum_{N=\alpha_1,\alpha_2,\alpha_3\dots}^{\infty}\frac{4k}{N^2}\sin^2\left(\frac{N}{k}\right)=\pi\coth(1)=\pi\left(1+\frac{2}{e^2-1}\right)$$

Proposed by Naren Bhandari – Bajura – Nepal

U.277 For $M \le N, p$ positive integers, prove or disprove

$$\lim_{n\to\infty}\sum_{j=M}^{N}\left(\sum_{k=1}^{j}\left(\sum_{l=1}^{k}\left(1+F_n(l)\right)^{\frac{1}{nl}}\frac{1}{k}\right)^{n}\right)^{p}\binom{N-M+1}{1}^{-1}=\frac{1}{2}\sum_{q=1}^{p}\frac{N^p(M-1)^q}{N^q(M-1)}$$

$$\frac{1}{p+1}\sum_{q=1}^{p+1}N^{p+1-q}(M-1)^{q-1}+\sum_{j=2}^{p}\sum_{q=1}^{p-j+1}\frac{B_j}{j}\binom{p}{j-1}N^{p-j+1-q}(M-1)^{q-1}$$

where $F_n(l)=(-1)^{l+1}\frac{\sqrt[nl]{nl}}{nl}$ and B_n is n^th Bernoulli number.

Proposed by Naren Bhandari – Bajura – Nepal

U.278 Prove:

$$\int_0^{\infty}\frac{e^{\pi x}}{(e^{2\pi x}+1)(n^2+4x^2)}dx=\frac{1}{8n}\left(\psi^0\left(\frac{n+3}{4}\right)-\psi^0\left(\frac{n+1}{4}\right)\right)$$

where $\psi^0(.)$ Is Digamma function.

Proposed by Naren Bhandari – Bajura – Nepal

U.279 For all $k, m, n \in \mathbb{N}$, prove that:

$$\int_0^1\frac{\ln^m(x)}{1+x^n}\frac{dx}{\sqrt[k]{\frac{1}{x}}}=(-1)^m\frac{m!}{(2n)^{m+1}}\left(\zeta\left(m+1,\frac{k+1}{2kn}\right)-\zeta\left(m+1,\frac{1}{2}+\frac{k+1}{2nk}\right)\right)$$

$$=\frac{1}{(2n)^{m+1}}\left(\psi^m\left(\frac{1}{2}+\frac{k+1}{2kn}\right)-\psi^m\left(\frac{k+1}{2kn}\right)\right)$$

Notation: $\psi^n(x)$ is polygamma function and $\zeta(s,a)$ is Hurwitz zeta function.

Proposed by Naren Bhandari – Bajura – Nepal

U.280 Evaluate the following product:

$$\prod_{n=1}^{\infty}\prod_{k=0}^{\infty}\log\left(2+\frac{(-1)^k}{n+k}\right)$$

Proposed by Naren Bhandari – Bajura – Nepal

U.281 For $0\le j\le n$, prove:

$$\sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{k=j}^{n}\binom{n}{k}\binom{k}{j}\binom{n}{j}\frac{2^j}{8^n(2n+1)^3}=\frac{\pi}{4}\left(\frac{\zeta(2)}{2}+\ln^2 2\right)$$

where $\zeta(.)$ is Riemann zeta function.

Proposed by Naren Bhandari – Bajura – Nepal

U.282 Prove that:

$$\lim_{n\to\infty}\left(\frac{\ln\left(1+\frac{1^2}{n^2}\right)}{\left(1+\frac{1^2}{n^2}\right)^2}+\frac{\ln\left(1+\frac{2^2}{n^2}\right)}{\left(1+\frac{2^2}{n^2}\right)^2}+\cdots+\frac{\ln\left(1+\frac{n^2}{n^2}\right)}{\left(1+\frac{n^2}{n^2}\right)^2}\right)\frac{1}{n}=\frac{1}{4}-2G-3\pi+6(\pi+1)\ln 2$$

Notation: G is Catalan constant

Proposed by Naren Bhandari – Bajura – Nepal

U.283 If $(ax^2+1)\frac{d^2y}{dx^2}+(bx^2+cx+d)\frac{dy}{dx}+ey=0$, then show that:

$$(ax^2+1)\frac{d^{n+2}y}{dx^{n+2}}+(bx^2+cx+2axn+d)\frac{d^{n+1}y}{dx^{n+1}}+$$

$$+(an^2-an+2bnx+cn+e)\frac{d^ny}{dx^n}+(n^2-n)\frac{d^{n-1}y}{dx^{n-1}}=0$$

Proposed by Akerele Olofin-Nigeria

U.284 If $\Omega_1=9\int_0^\infty\frac{tanh\left(\frac{\pi x}{2}\right)}{x(x^2+9)}dx$ and $\Omega_2=\int_0^\infty\left(\frac{\sqrt{log(2t)}}{cosh(2t)}\right)^2dt$. Then show that:

$$\Omega_1+\Omega_2=\psi_0(2)+\frac{log2}{2}+\frac{1}{2}log\left(\frac{\pi}{8}\right)-\left(\psi_0\left(\frac{1}{2}\right)+\frac{\gamma}{2}\right)$$

where $\psi(*)-$ and $\gamma-$denotes the digamma function and Euler-Mascheroni constant

Proposed by Akerele Olofin-Nigeria

U.285 Show that:

$$\psi^{(0)}\left(\frac{1}{7}\right)+\psi^{(0)}\left(\frac{2}{7}\right)+\psi^{(0)}\left(\frac{3}{7}\right)+\cdots+\psi^{(0)}\left(\frac{6}{7}\right)=-(7log7+6\gamma)$$

where $\psi(*)-$ and $\gamma-$denotes the digamma function and Euler-Mascheroni constant

Proposed by Akerele Olofin-Nigeria

U.286 Show that:

$$\int_0^1\frac{(1-\phi^5)(1-\phi^6)(1-\phi^7)}{(\phi-1)\big(log(\phi)\big)}d\phi=log\left\{\frac{\Gamma(14)\Gamma(13)\Gamma(12)}{\Gamma(6)\Gamma(7)\Gamma(8)\Gamma(19)}\right\}$$

Proposed by Akerele Olofin-Nigeria

U.287 Show that:

$$\int_0^{\frac{1}{2}}\frac{log^2(1-\omega)}{\omega}d\omega=\frac{\zeta(3)}{4}-\frac{log^3(2)}{3}$$

Proposed by Akerele Olofin-Nigeria

U.288 If $\phi_0=\sum_{\lambda\ge1}\frac{\tau(\lambda)}{\lambda^{16}}+(\sum_{1024\le k\le1}[log_2 k])(mod\ 100)$, then find the closed form for the sum where $\tau(\lambda)$ and $[*]$ denotes the number of divisors of λ and greatest integer function

Proposed by Akerele Olofin-Nigeria

U.289 If $\sigma(x)=\Gamma(x)^{\Gamma(x)^{\Gamma(x)^{\cdot^{\cdot}}}}$, then prove that:

$$\sigma(3)=\frac{-w(-log(2))}{log(2)}$$

where $w(*)-$ denotes the Lambert $W-$function. ***Proposed by Akerele Olofin-Nigeria***

U.290 $\eta(x)=\sum_{n=1}^{\infty}\frac{d^n\eta}{dx^n}$, when $x=2,\eta(x)=e$. Show that $\eta(\pi)=e^{\frac{\pi}{2}}$

Proposed by Akerele Olofin-Nigeria

U.291 If $\psi(k)=\sum_{\beta\ge1}\frac{1}{\beta(\beta+1)^k}, Re(k)>0$. Show that: $\psi(k)=k-\varphi$, where $\varphi=\sum_{k\le i\le2}\zeta(i)$

Proposed by Akerele Olofin-Nigeria

U.292 Solve the system:

$$\begin{cases}6\frac{d^2x}{dt^2}+13\frac{d^2y}{dt^2}+7x=0\\6\frac{d^2x}{dt^2}+17\frac{d^2x}{dt^2}+7y=0\end{cases}$$, given that, at $t=0,x=\frac{7}{4},y=1,\frac{dx}{dt}=0,\frac{dy}{dt}=0$

Proposed by Akerele Olofin-Nigeria

U.293 Prove that:

$$\lim_{z\to 3}\frac{2^{\omega}-4}{3(\sqrt{\omega-1}-1)}=\frac{8}{3}log2,\text{ where }\frac{\omega 1}{z}\prod_{n=1}^{\infty}\frac{\left(1+\frac{1}{n}\right)^{z}}{\left(1+\frac{z}{n}\right)}$$

Proposed by Akerele Olofin-Nigeria

U.294 If $a<b,-\xi=(a-b)$ and $0<\sqrt{\frac{\xi}{b}}<1$. Prove that:

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{d\omega}{\sqrt{asin^2\omega+bcos^2\omega}}=\frac{1}{\sqrt{b}}\left(K\left(\sqrt{\frac{\xi}{b}}\right)-F\left(\sqrt{\frac{\xi}{b}},\frac{\pi}{4}\right)\right)$$

where $K(k)$ –denotes the complete elliptical integral of the first kind and $F(k,\psi)$ denotes the incomplete elliptical integral of the first kind. ***Proposed by Akerele Olofin-Nigeria***

U.295 Prove that:

$$\frac{1}{192}\left(\psi_{(1)}\left(\frac{11}{12}\right)-\psi_{(1)}\left(\frac{5}{12}\right)-\psi_{(1)}\left(\frac{7}{12}\right)+\psi_{(1)}\left(\frac{1}{12}\right)\right)=\frac{\pi^2}{8\sqrt{3}}$$

where $\psi_{(1)}(*)$ –denotes the trigamma function. ***Proposed by Akerele Olofin-Nigeria***

U.296 If $\Lambda_{\mathrm{N}}=\sum_{N\le k\le 1}\left(\frac{(-1)^k C(N,k)}{k}\right)$. Prove that: $\Lambda_{10}=-H_{10}$

where H and $C(N,k)$ denotes the nth harmonic number and the binomial coefficients

Proposed by Akerele Olofin-Nigeria

U.297 Let $a_1,a_2,\dots,a_n$ be $n>1$ distinct real numbers. Prove that:

$$\underbrace{min}_{1\le j<k\le n}\left(a_j-a_k\right)^2\le\frac{12}{n(n^2-1)}\left(\sum_{j=1}^{n}a_j^2-\frac{1}{n}\left(\sum_{j=1}^{n}a_j\right)^2\right)$$

Proposed by Akerele Olofin-Nigeria

U.298 Find a closed form:

$$\Omega=\int_{e^e}^{\infty}\frac{log\left(log(logx)\right)}{x^{101}}dx$$

Proposed by Akerele Olofin-Nigeria

U.299 Prove the relation:

$$\int_{-1}^{1}\int_{-1}^{1}\left(\int_{-\infty}^{\infty}\frac{t^2-x}{t^2+y}\frac{\sin(\pi t)}{t}dt\right)dydx=\frac{e^{\pi}\left(-\frac{4}{\pi}-4i\right)-\frac{4}{\pi}-4}{e^{\pi}}$$

Proposed by K. Srinivasa Raghava-AIRMC-India

U.300 Prove:

$$\sum_{n=0}^{\infty}\sum_{m=1}^{z}(-1)^n \ln\left(1+\frac{1}{n+m}\right)=\ln\left(\frac{\pi}{2^z}\right)+\ln\left(\frac{\Gamma(z+1)}{\Gamma^2\left(\frac{z+1}{2}\right)}\right)$$

Proposed by Asmat Qatea-Afghanistan

U.301 Let $u_1=a\geq 0, u_{n+1}=\ln\left(2020-a+\sqrt{u_n}\right)$. Find all numbers $a\geq 0$ such that

$$\lim_{n\to\infty}u_n=0$$

Proposed by Nguyen Van Canh-Vietnam

U.302 Prove that:

$$\int_0^{\frac{\pi}{2}}\frac{\left(\tan^{-1}\left(\frac{\sin x}{2}\right)\right)^2}{\sin x}dx=$$

$$=\left(\frac{\pi^2}{4}+Li_2\left(\frac{1}{\varphi^2}\right)-Li_2\left(-\frac{1}{\varphi^2}\right)\right)\ln\varphi+\left(Li_3\left(\frac{1}{\varphi^2}\right)-Li_3\left(-\frac{1}{\varphi^2}\right)\right)-\frac{7}{4}\zeta(3)$$

Proposed by Ghazaly Abiodun-Nigeria

U.303 Show that:

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}\left(\frac{\sin n}{n}\right)^2=\frac{\pi(\cosh(2)-1)}{4\sinh(\pi)}-\frac{1}{2}$$

Proposed by Johnathan Haw-USA

U.304 If a,b are real numbers and p,m,k are positive integers such that $p>m$ then prove that:

$$\lim_{n\to\infty}\frac{\sum_{v=1}^{n}(an^p+bn^m+v)^k}{\sum_{v=1}^{n+1}(an^p+bn^m-v+1)^k}=1$$

It is required to prove the result at least in 2 different ways.

Proposed by Naren Bhandari-Nepal

U.305 Prove that:

$$\int_0^{\infty}e^{-2x}\left(\sinh(x)+\tanh(x)\right)\frac{dx}{x}=\log\left(\frac{\pi}{2}\right)+\frac{\log(2)}{2}$$

Proposed by Abdul Mukhtar-Nigeria

U.306 Show that:

$$\int_0^1\frac{\ln(5x^2+4x+1)}{(x+1)(3x+1)}dx=\frac{-\ln^2 2}{2}+\ln 2\ln\left(\frac{2}{3}\right)-\frac{\pi^2}{48}-\frac{1}{4}Li_2(-4)+Li_2\left(\frac{2}{3}\right)-Li_2\left(\frac{1}{3}\right)$$

Proposed by Hamza Djahel-Afghanistan

U.307 Prove that: $\forall k,s,n\in\mathbb{N}, k=2s+1$

$$\sum_{n=0}^{\infty}\frac{1}{n+3k}\cos\left(\frac{n\pi}{3}\right)=\frac{1}{2}\left(1-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\cdots-\cdots-\cdots-\frac{1}{3k-1}\right)$$

$$-\frac{1}{6}\left(1-\frac{1}{2}+\frac{1}{3}-\cdots+\cdots-\frac{1}{k-1}\right)$$

Proposed by Amrit Awasthi-India

U.308 Evaluate:

$$\lim_{n\to\infty}\frac{\sqrt{n}}{e^n}e^{\int_0^{\infty}\left\lfloor\frac{n}{e^x}\right\rfloor dx}$$

Notation $\lfloor . \rfloor$ denotes the floor function.

Proposed by Arghyadeep Chaterjee-India

U.309 When $a_0=r$ and $a_{n+1}=(a_n)^2$. Define: $P_{n(r)}=\prod_{k=0}^{n}(1+a_k)$. Evaluate:

$$\lim_{n\to\infty}\int_0^1 P_n(r)-\left(\frac{n+1}{r+1}\right)dr$$

Proposed by Jibran Iqbal-Algerie

U.310 Find:

$$\Omega(1,2)=\int\frac{\log x}{(1+\log x)^2}dx, \Omega(2,3)=\int\frac{\log^2 x}{(1+\log x)^3}dx$$

$$\Omega(m,n)=\int\frac{\log^m x}{(1+\log x)^n}dx, m,n\in\mathbb{N}, m,n\in\mathbb{N}$$

Proposed by Durmuș Ogmen-Turkiye

U.311 Prove that:

$$1-\frac{1}{4}\left(\frac{1}{2}\right)\frac{1}{3^2}+\frac{1}{16}\left(\frac{1\cdot 3}{2\cdot 4}\right)\frac{1}{5^2}-\frac{1}{64}\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)\frac{1}{7^2}+\cdots=\frac{\pi^2}{10}$$

Proposed by Lunjapao Baite – India

U.312 Let the recurrence: $S(n-3)+S(n-2)+S(n-1)+S(n)=1+(-1)^n$

$S(0)=-1; S(1)=0; S(2)=1$ then show that:

$$\sum_{n=1}^{\infty}\frac{S(n)}{n^3}=\frac{55\zeta(3)}{64}-\frac{\pi^2(8+3\pi)}{192}$$

Proposed by K. Srinivasa Raghava – AIRMC-India

U.313 Prove that:

$$\sum_{k=1}^{\infty}\left(\frac{1}{(6k-6)!}+\frac{1}{(6k-5)!}-\frac{1}{(6k-3)!}-\frac{1}{(6k-2)!}\right)=\sqrt{\frac{4e}{3}}\sin\left(\frac{2\pi+3\sqrt{3}}{6}\right)$$

Proposed by Asmat Qatea-Afghanistan

U.314 Find:

$$\int_0^{\infty}\sin^{-1}\left(\frac{1}{1+x^2}\right)dx$$

Proposed by Ajetunmobi Abdulqoyyum – Nigeria

U.315 Prove that:

$$\sum_{n=0}^{\infty}\frac{1}{n+3}\cos\left(\frac{n\pi}{3}\right)=\frac{1}{4}$$

Proposed by Amrit Awasthi-India

U.316 Prove that:

$$\int_0^{\pi}\arctan^2\left(\frac{\sin(x)}{\pi+\cos(x)}\right)dx=\frac{\pi^3}{30}-\frac{\pi}{2}(\ln(\phi))^2$$

where ϕ is the golden ratio.

Proposed by Ty Halpen-USA

U.317 Evaluate:

$$\int_0^{\infty}x^n\ln(x)\cos(x)e^{-x}\,dx, n>0$$

Proposed by Abdul Mukhtar-Nigeria

U.318 Evaluate the value of I

$$I=\int_{-\pi}^{+\pi}\frac{(2\pi)^{\frac{x}{\pi}}\Gamma\left(n+1-\frac{x}{\pi}\right)\sin\left[\left(n+\frac{1}{2}\right)x\right]\zeta\left(1-\frac{x}{\pi}\right)}{\left(n-\frac{x}{\pi}\right)\zeta\left(\frac{x}{\pi}\right)\left(1-2^{\frac{x}{\pi}}\right)\left(1-\frac{x}{\pi}\right)_{n-1}}dx$$

Where, $\Gamma(x)\to$ Gamma function, $\zeta(x)\to$ Riemann Zeta function, $\eta(x)\to$ Dirichlet's Eta function, $(\alpha)_k\to$ Pochhammer's Symbol ***Proposed by Ankush Kumar Parcha-India***

U.319 Let $a\geq 0$ be positive number. And

$$\sum_{n=1}^{\infty}\frac{a^{\frac{1}{p}}}{n^{\frac{1}{p}}(a+n)}=\frac{\pi}{\sin\frac{\pi}{p}}$$

For which p this equality is correct? ***Proposed by Javad Asadzadeh-Azerbaijan***

U.320 Evaluate in a closed form:

$$\Omega = \sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty}\sum_{k=1}^{m+n} \frac{(-1)^{m+k}}{(n+1)^2(m+k)^2(n+m)^2}$$

Proposed by Mokhtar Khassani-Algerie

U.321 Find:

$$I(c) = \int_{-c}^{c} \left(\frac{\sec(x)}{a^{\frac{1}{x^{2n+1}}}+1} + \frac{\csc(x)}{b^{\frac{1}{x^{2m}}}+1} \right) dx$$

$$a, b, c \in \mathbb{R}, n, m \in \mathbb{N}$$

Proposed by Precious Itsuokor-Nigeria

U.322 Calculate

$$2\prod_{k=1}^{k=\infty} \left[\frac{\sum_{n=0}^{n=\infty} \left(\frac{1}{2k}\right)^n}{\sum_{n=0}^{n=\infty} \left(\frac{1}{2k+1}\right)^n} \right]$$

Proposed by Timson Azeez Folorunsho-Nigeria

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

ROMANIAN MATHEMATICAL MAGAZINE-R.M.M.-WINTER 2022

PROBLEMS FOR JUNIORS

JP.391 In $\Delta ABC, P$ −inner point, $M, L \in [AB], D, E \in [BC], F, K \in [CA], AM = AF, BL = BE,$

$CK = CD, |DE| = a_1, |FK| = b_1, |LM| = c_1,$ $(M, P, F), (C, P, L), (D, P, K)$ −are collinear.Prove that:

$$F = \frac{1}{2}(a_1 r_a + b_1 r_b + c_1 r_c)$$

Proposed by Mehmet Şahin-Ankara-Turkiye

JP.392 $z_1, z_2, z_3 \in \mathbb{C}^*$ different in pairs such that $|z_1| = |z_2| = |z_3| = 1, A(z_1), B(z_2), C(z_3)$. Prove that:

$$\text{If } \sum_{cyc} \frac{z_2 z_3}{14 z_2 z_3 - z_2^2 - z_3^2} = \frac{1}{5} \text{ then } AB = BC = CA.$$

Proposed by Marian Ursărescu-Romania

JP.393 In ΔABC the following relationship holds: $h_a^3 + h_b^3 + h_c^3 \leq \frac{81}{8}(9R^3 - 64r^3)$

Proposed by Marian Ursărescu-Romania

JP.394 If $x, y, z, t > 0$ then:

$$\frac{x^4+1}{y\sqrt{x^4-x^2+1}} + \frac{y^4+1}{z\sqrt{y^4-y^2+1}} + \frac{z^4+1}{z\sqrt{z^4-z^2+1}} + \frac{t^4+1}{x\sqrt{t^4-t^2+1}} \geq 8$$

Proposed by Daniel Sitaru-Romania

JP.395. Let R and r be the circumradius and inradius, respectively, of triangle ABC. Let D, E and F be chosen on sides BC, CA and AB, so that AD, BE and CF bisect the angles of ABC. Prove:

$$\frac{DE}{AB} + \frac{EF}{BC} + \frac{FD}{CA} \leq \frac{3}{4}\left(1 + \frac{R}{2r}\right)$$

Proposed by George Apostolopoulos-Greece

JP.396. Let h_a, h_b, h_c be the altitudes from the vertices A, B, C respectively, R the circumradius and r the inradius of a triangle ABC. Let A_1, B_1 and C_1 be chose on the sides BC, CA and AB so that AA_1, BB_1 and CC_1 bisect the angles of ABC. Let h_A, h_b and h_C denote the altitudes of triangles AB_1C_1, BC_1A_1 and CA_1B_1 from the vertices A, B, C, respectively.Prove that:

$$\sqrt[3]{\frac{h_a}{h_A}} + \sqrt[3]{\frac{h_b}{h_B}} + \sqrt[3]{\frac{h_c}{h_C}} \leq 3\sqrt[3]{2} \cdot \frac{R}{2r}$$

Proposed by George Apostolopoulos-Greece

JP. 397. Solve for real numbers: $3^{\log_2(3^x-1)} = 2^{\log_3(2^x+1)} + 1$

Proposed by Ionuț Florin Voinea-Romania

JP.398 If $a, b, c > 0$; $ab\sqrt{ab} + bc\sqrt{bc} + ca\sqrt{ca} = 3$ then:

$$\sum_{cyc} \left(a\sqrt{a} - b\sqrt{b}\right)^2 + 24 \leq \sum_{cyc} (a+b)^3$$

Proposed by Daniel Sitaru-Romania

JP.399 In ΔABC the following relationship holds:

$$a^4 + b^4 + c^4 \geq 16F^2 + \frac{1}{2}\left((a^2-b^2)^2 + (b^2-c^2)^2 + (c^2-a^2)^2\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru

JP.400 If $x,y,z>0$; $\sqrt[3]{xy}+\sqrt[3]{yz}+\sqrt[3]{zx}=3$ then: $\sqrt[3]{x(y+z)}+\sqrt[3]{y(z+x)}+\sqrt[3]{z(x+y)}\geq 3\sqrt[3]{2}$

Proposed by Daniel Sitaru-Romania

JP.401 Find all sets x,y,z of positive integers such that : $\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}=\frac{1}{2}$

Proposed by George Apostolopoulos-Greece

JP.402. In acute ΔABC the following relationship holds:

$$R-2r\leq 3(\max\{h_a,h_b,h_c\}-\min\{h_a,h_b,h_c\})$$

Proposed by Cristian Miu-Romania

JP.403. If $x,y,z\in(0,1)$ then in ΔABC the following relationship holds:

$$\frac{1}{(y+z)(1-x^2)h_a^4}+\frac{1}{(z+x)(1-y^2)h_b^4}+\frac{1}{(x+y)(1-z^2)h_c^4}\geq\frac{3\sqrt{3}}{4F^2}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

JP.404 If $x,y>0$; $z\in\left[0,\frac{\pi}{2}\right]$; $\frac{1}{x+\sin z}+\frac{1}{y+\sin z}\geq 1$ then: $\frac{1}{x}+\frac{1}{y}\geq 2\sin z$

Proposed by D.M.Bătinețu-Giurgiu, Daniel Sitaru-Romania

JP.405 In ΔABC the following relationship holds:

$$\frac{a^3}{2b+2c-a}+\frac{b^3}{2c+2a-b}+\frac{c^3}{2a+2b-c}\geq\frac{4\sqrt{3}F}{3}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

PROBLEMS FOR SENIORS

SP.391 Find:

$$\Omega=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\left(1+\left(\sum_{i=1}^{k}i^2\right)^{-1}\right)^{\sum_{i=1}^{k}i^2}-ne\right)$$

Proposed by Florică Anastase-Romania

SP.392 Let $(x_n)_{n\geq1},(y_n)_{n\geq1}$ be sequences of real numbers such that $x_1=1,y_1=a^2+a+1$,

$$a>1,\forall n\in\mathbb{N}^* \text{ and } x_{n+1}=\frac{a^2+ax_n+x_ny_n}{y_n},y_{n+1}=\frac{a^2+ay_n+x_ny_n}{x_n}.\ \text{Find:}$$

$$\lim_{n\to\infty}x_n,\lim_{n\to\infty}y_n.$$

Proposed by Florică Anastase-Romania

SP.393 If $S_n = e^{\frac{1}{n}+\frac{1}{n+1}+\cdots+\frac{1}{n+k\cdot n}}, n, k \in \mathbb{N}, k \geq 1$ then find:

$$\Omega = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\left(1+\frac{1}{n}\right)^{\frac{k}{S_n}} - n\right)$$

Proposed by Florică Anastase-Romania

SP.394 If $(F_n)_{n\geq 0}, F_0 = F_1 = 1, F_{n+2} = F_{n+1} + F_n, \forall n \in \mathbb{N}$ and $a, b, c \in \mathbb{R}_+^*$ with

$a + b + c \leq 24$, then prove:

$$\frac{F_n}{\sqrt{F_n^2 + aF_{n+1}F_{n+2}}} + \frac{F_{n+1}}{\sqrt{F_{n+1}^2 + bF_{n+2}F_n}} + \frac{F_{n+2}}{\sqrt{F_{n+2}^2 + cF_nF_{n+1}}} \geq 1, \forall n \in \mathbb{N}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

SP.395 $A \in M_2(\mathbb{Q})$ such that $\det(A^2 - 2I_2) = 0$. Find: $\Omega = \det(A^2 - 3A + 3I_2)$

Proposed by Marian Ursărescu-Romania

SP.396 Find all continuous functions $f: \mathbb{R} \to \mathbb{R}$ such that:

$$f(x) - f(ax) + f(a^2x) - f(a^3x) + f(a^4x) = x, \forall x \in \mathbb{R}, a \in (0,1)$$

Proposed by Marian Ursărescu-Romania

SP.397 Find:

$$\Omega = \lim_{p\to\infty}\frac{1}{p^a}\cdot\sum_{m=1}^{p}\sum_{n=1}^{m}\sum_{k=1}^{n}\frac{k^2}{2k^2 - 2nk + n^2}$$

Proposed by Florică Anastase – Romania

SP.398 If $a, b > 0, b \neq 1$ solve for real numbers:

$$\sum_{k=1}^{n}\frac{x + \log_b(a+k)}{2x + \log_b(a+k)(a-k+n)} - \frac{n+1}{2} = 0$$

Proposed by Florică Anastase – Romania

SP.399 If $m, n > 0; x, y, z \in (0,1)$ then:

$$\frac{1}{(1-x^2)(my+nz)} + \frac{1}{(1-y^2)(mz+nx)} + \frac{1}{(1-z^2)(mx+ny)} \geq \frac{9\sqrt{3}}{2}$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

SP.400 In $\Delta ABC, \lambda > 0$ the following relationship holds:

$$\prod_{cyc}\left(\frac{\sin A}{\sin B \sin C}+\lambda^2\right)\geq 9\lambda^2\cdot\sqrt{\frac{3}{4}}$$

Proposed by Florică Anastase – Romania

SP.401 Let I be the incentre of triangle ABC and let A', B' and C' be the intersections of the rays AI, BI, and CI with the circumcircle of the triangle. Prove that $[A'B'C'] \geq [ABC]$, where $[*]$ −represent the area.

Proposed by George Apostolopoulos- Greece

SP.402 Let $m_a, m_b m_c$ be the lengths of the medians of a triangle ABC with inradius r and circumradius R. Prove that:

$$\frac{8r^2}{R^4}\leq\sum_{cyc}\frac{\sin^2 B+\sin^2 C}{m_a^2}\leq\frac{1}{2r^2}$$

Proposed by George Apostolopoulos- Greece

SP.403 In $\Delta ABC, \lambda > 0, n \in \mathbb{N}$ the following relationship holds:

$$\prod_{cyc}\left(\frac{\sin^{2n} A}{\sin^{2n+4} B}+\lambda^2\right)\geq 12\lambda^2$$

Proposed by Florică Anastase – Romania

SP.404 In ΔABC the following relationship holds: $3(a^2+b^2+c^2)+4\left(h_a^2+h_b^2+h_c^2\right)\geq 24\sqrt{3}F$

Proposed by Daniel Sitaru-Romania

SP.405 If $a, b, c > 0$ then:

$$\sqrt{a^2+5ab+7b^2}+\sqrt{b^2+5bc+7c^2}+\sqrt{c^2+5ca+7a^2}\geq\sqrt{13}(a+b+c)$$

Proposed by Daniel Sitaru-Romania

UNDERGRADUATE PROBLEMS

UP. 391

$$x_n=\sum_{i=1}^{n}\sin\frac{(2i-1)x}{n^2}, x>0$$

Find:

$$\Omega=\lim_{x\to 0}\left(\lim_{n\to\infty}\left(\left(\sum_{i=1}^{n}a_i^{1+x_n}\right)\cdot\left(\sum_{i=1}^{n}a_i\right)^{-1}\right)^{\frac{1}{x_n}}\right), a_i>0, i=\overline{1,n}$$

Proposed by Florică Anastase - Romania

UP.392 Find

$$\lim_{n\to\infty}\frac{1}{\sqrt[n]{(n!)((2n-1)!!)}}\cdot\sum_{k=1}^{n}\left[\left(\sqrt{k+1}+\sqrt{k}\right)^2\right]$$

where $[*]$ great integer part.

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

UP.393 Calculate the integral:

$$\Omega=\int_0^\infty\frac{\tan^{-1}x}{x^4+1}dx$$

It is required to express the integral value with the usual mathematical constants and $\psi_1\left(\frac{3}{8}\right)$, where $\psi_1(x)$ is the trigamma function.

Proposed by Vasile Mircea Popa-Romania

UP.394

$$\Omega(a)=\int_1^a\frac{x}{\log(1+x^2)}dx, a>2$$

Prove that:

$$\frac{(a+b+c)^2}{3\sqrt{2}}+\frac{1}{\sqrt{2}}\log\left(\frac{a^2b^2c^2}{e^3}\right)\le\sum_{cyc}\Omega(a)\le(a+b+c)^2+\log\left(\frac{a^2b^2c^2}{e\sqrt{e}}\right)$$

Proposed by Florică Anastase-Romania

UP.395 For $a>0,b>1$ find:

$$\Omega=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{\left(a+b^k\right)\tan^{-1}\left(\frac{n^2}{n^2-kn+k^2}\right)}{b^k+2a+b^{n-k}}$$

Proposed by Florică Anastase-Romania

UP.396 Find:

$$\Omega(a)=\int_{\frac{1}{a}}^{a}\frac{\sin^2x\cdot\tan\left(\frac{1}{x^2}\right)}{(1+x^2)\left(\sin^2\left(\frac{1}{x}\right)\tan\left(\frac{1}{x^2}\right)+\sin^2x\tan\left(\frac{1}{x^2}\right)\right)}dx$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

UP.397 Find:

$$\Omega=\lim_{n\to\infty}\left(\sqrt[n]{(2n-1)!!}\right)^2\left(e^{H_{2n+2}-H_{n+1}}-e^{H_{2n}-H_n}\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

UP.398 If $0 < a \leq b$ then:

$$\int_0^{\sqrt{ab}} e^{-x^2}\,dx - \int_0^{\frac{a+b}{2}} e^{-x^2}\,dx \geq \tan^{-1}(\sqrt{ab}) - \tan^{-1}\left(\frac{a+b}{2}\right)$$

Proposed by Daniel Sitaru-Romania

UP.399 Find:

$$\Omega(t) = \lim_{x\to\infty}\left(\left(\Gamma(x+2)\right)^{\frac{t}{x+1}} - \left(\Gamma(x+1)\right)^{\frac{t}{x}}\right)\cdot x^{1-t}\,; t > 0$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

UP.400 If $0 < a < b < \frac{\pi}{2}$ then:

$$\int_a^b \frac{(3-\tan x)^2}{\cos^2(\sqrt{\tan x})}dx + \int_a^b \frac{(3-\cot x)^2}{\cos^2(\sqrt{\cot x})}dx \geq 18(b-a) + 3\log(\cot a\cdot\tan b)$$

Proposed by Florică Anastase – Romania

UP.401 If $0 < a < b < \frac{\pi}{2}$ then:

$$\int_a^b \left(\frac{\cos x}{\tan(\sin x)} + \frac{\sin x}{\tan(\cos x)} + \frac{\sin x\cdot\tan x}{\tan(\sin x)} + \frac{\cos x\cdot\cot x}{\tan(\cos x)}\right)dx \leq \frac{5}{3}\log(\cot a\cdot\tan b)$$

Proposed by Florică Anastase – Romania

UP.402 If $\frac{1}{e} < a \leq b < e$ then:

$$\sin^{-1}(\log(\sqrt{ab})) - \sin^{-1}\left(\log\left(\frac{a+b}{2}\right)\right) \leq \sqrt{1-\log^2\left(\frac{a+b}{2}\right)} - \sqrt{1-\log^2(\sqrt{ab})}$$

Proposed by Daniel Sitaru-Romania

UP.403 If $0 < a \leq b$ then:

$$\int_a^b\int_a^b \sqrt{x^2+5xy+8y^2}\,dxdy \geq \frac{\sqrt{14}}{2}(b+a)(b-a)^2$$

Proposed by Daniel Sitaru-Romania

UP.404 If $0 < a \leq b < 1$ then:

$$\log\left(\frac{1-a+b-ab}{1+a-b-ab}\right) \geq \frac{3\sqrt{3}}{2}(b^2-a^2)$$

Proposed by Daniel Sitaru-Romania

UP.405 Find:

$$\Omega(m)=\int_0^{\frac{\pi}{m}}\sin(mx)\cdot\log\left(1+\tan\left(\frac{\pi}{m}\right)\tan x\right)dx\,;m>2$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

INDEX OF AUTHORS RMM-35

NOTĂ: Pentru a publica probleme propuse, articole și note matematice în RMM puteți trimite materialele pe mailul: dansitaru63@yahoo.com