Partition Function Inequality

Introduction

In this article, we will obtain an upper bound for the partition function $p(n)$ using elementary methods of combinatorics.

Prerequisite

Definition 1

Let $p(n)$ be the number of ways of writing n as the sum of positive integers (order is not important).

Definition 2

Let $p(m, n)$ be the number of solutions of the equation

$$\sum_{i=1}^{m} a_i = n$$

where, $a_i \in \mathbb{N} \ \forall \ 1 \leq i \leq m \leq n$ (order is not important).

Definition 3

Let $\phi(m, n)$ be the number of solutions of the equation

$$\sum_{i=1}^{m} a_i = n$$

where, $a_i \in \mathbb{N} \ \forall \ 1 \leq i \leq m \leq n$ (order is important).

Lemma 1

For all $m, n \in \mathbb{N}$ and $m \leq n$, we have,

$$p(m, n) \leq \phi(m, n)$$

Proof: Since, $p(m,n)$ and $\phi(m,n)$ differ only in the order of the summands, therefore, the number of ways of writing n as the sum of m positive integers where order is important is greater than or equal to the number of ways of writing n as the sum of m positive integers where order is not important. This completes the proof of lemma 1.

Lemma 2

For all $m, n \in \mathbb{N}$ and $m \leq n$, we have,

$$\phi(m,n) = \binom{n-1}{m-1}$$

Proof: From definition 3 and elementary combinatorics, it can be shown that, $\phi(m,n)$ is the coefficient of x^n in the series expansion of $(x + x^2 + x^3 + ...)^m = x^m(1-x)^{-m}$, that is, the coefficient of x^{n-m} in the series expansion of $(1-x)^{-m}$. We know that if $|x| < 1$, then,

$$(1-x)^{-m} = 1 + \sum_{k=1}^{\infty} \frac{m(m+1)...(m+k-1)}{k!} x^k = \sum_{k=0}^{\infty} \frac{(m+k-1)!}{(m-1)!k!} x^k$$

which implies, the coefficient of x^{n-m} in the series expansion of $(1-x)^{-m}$ is $\frac{(n-1)!}{(m-1)!(n-m)!}$, which completes the proof of lemma 2.

Lemma 3

For all $n \in \mathbb{N}$, we have,

$$\sum_{m=1}^{n} p(m,n) = p(n)$$

Proof: From definition 2, it can be shown that $p(m,n)$ is the number of ways of writing n as the sum of m positive integers where order is not important. Thus the sum,

$$\sum_{m=1}^{n} p(m,n)$$

represents the total number of ways of writing n as the sum of m positive integers where $m = 1, 2, 3..., n$. This is the same as definition 1, which completes the proof of lemma 3.

Lemma 4

For all $n \in \mathbb{N}$, we have,

$$\sum_{m=1}^{n} \binom{n-1}{m-1} = 2^{n-1}$$

Proof: From the binomial theorem it is known that if $n \in \mathbb{N}$ and $x \in \mathbb{C}$, then,

$$(1+x)^n = \sum_{m=0}^{n} \binom{n}{m} x^m$$

substituting $x = 1$, replacing n by $n-1$ and replacing m by $m-1$ completes the proof of lemma 4.

Inequality

For all $n \in \mathbb{N}$, we have,

$$p(n) \leq 2^{n-1}$$

Proof: From lemma 1, we have,

$$p(m,n) \leq \phi(m,n)$$

summing up both the sides from $m = 1$ to $m = n$, we obtain,

$$\sum_{m=1}^{n} p(m,n) \leq \sum_{m=1}^{n} \phi(m,n)$$

using lemma 2, lemma 3 and lemma 4, we obtain,

$$p(n) \leq \sum_{m=1}^{n} \binom{n-1}{m-1} = 2^{n-1}$$

which completes the proof of the inequality.

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