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INEQUALITIES RELATED TO GENERALIZED HYPERBOLIC FUNCTIONS AND LOGARITHMIC MEAN

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Abstract: This article introduces some new inequalities related to generalized hyperbolic functions and logarithmic mean.

Keywords: Hiperbolic functions, logarithmic mean

1. INTRODUCTION

HUYGENS' INEQUALITY

$$2\frac{\sin x}{x}+\frac{\tan x}{x}>3, \qquad (1)$$

CUSA- HUYGENS INEQUALITY

$$\frac{\sin x}{x}<\frac{\cos x+2}{3} \qquad (2)$$

These inequalities are true for every $x\in\left(0,\frac{\pi}{2}\right)$.[1–6]

$$\sinh(x)=\frac{e^x-e^{-x}}{2},\cosh(x)=\frac{e^x+e^{-x}}{2},\tanh(x)=\frac{e^x-e^{-x}}{e^x+e^{-x}}$$

These functions are called hyperbolic sine, cosine and tangent functions, respectively.Logarithmic mean has applications in mathematics and physics. In this article we will present a generalization of the logarithmic mean. The

logarithmic mean of two positive numbers a and b is the number $L(a,b)$ defined as[7]

$$L(a,b)=\frac{a-b}{loga-logb}\ ,a\neq b\ ,$$

with the convention that:

$$L(a,a)=lim_{b\to a}L(a,b)=a\ .$$

2. PRELIMINARIES

DEFINITION. $sinh_{\varphi}(x)=\frac{\varphi^{x}-\varphi^{-x}}{2}$, $cosh_{\varphi}(x)=\frac{\varphi^{x}+\varphi^{-x}}{2}$, $\quad(3)$
$tanh_{\varphi}(x)=\frac{\varphi^{x}-\varphi^{-x}}{\varphi^{x}+\varphi^{-x}}$, $x\in R$, $\varphi>1$

These functions are called generalized hyperbolic sine, cosine, and tangent functions, respectively.[8]

LEMMA 1.[3] If $\forall\varphi>1$ and $x\geq 0$ then the following inequality is satisfied:

$$sinh_{\varphi}(x)\geq xln\varphi \qquad (4)$$

Proof. **Let $f: R^{+}\to R$ be a function defined by $f(x)=sinh_{\varphi}(x)-xln\varphi$**

The derivative of $f(x)$ is $\quad f'(x)=ln\varphi\left(cosh_{\varphi}(x)-1\right)$

If we apply the AM-GM inequality, we get that:

$cosh_{\varphi}(x)\geq 1$, for all $x\in R$.

Then we obtain : $\quad f'(x)\geq 0$, for all $x\in R$.

This show that: $f(x), x\in R$ is an increasing function. Then we obtain

For all $x\geq 0$, $f(x)\geq f(0)=0$. □

LEMMA 2.[3] ***If $\forall\varphi>1$ and $x\geq 0$ then the following inequality is satisfied:***

$$tanh_{\varphi}(x) \leq xln\varphi \qquad (5)$$

Proof: Let $f: R^{+} \to R$ be a function defined by $f(x) = tanh_{\varphi}(x) - xln\varphi$

The derivative of f(x) is $f'(x) = ln\varphi\left(\frac{1}{cosh_{\varphi}^{2}} - 1\right) \leq 0.$

This show that $f(x), x \geq 0$ is an decreasing function. Then we obtain :

For all $x \geq 0$, $f(x) \leq f(0) = 0$. □

MAIN RESULT

THEOREM 1 ***If $x \geq 0$ and $s > f > 1$ then the following inequality is satisfied:***

$$sinh_{s}(x) \geq sinh_{f}(x) \qquad (6)$$

Proof: Let $\mathrm{f}: \mathrm{R}^{+} \to \mathrm{R}$ be a function defined by

$$f(x) = sinh_{s}(x) - sinh_{f}(x)$$

$$f'(x) = ln\varphi\left(cosh_{s}(x) - cosh_{f}(x)\right) = \frac{s^{x}+s^{-x}}{2} - \frac{f^{x}+f^{-x}}{2}$$

Obviously for all $x \geq 0$, $(s^{x}-f^{x})\left(1 - \frac{1}{s^{x}f^{x}}\right) \geq 0$.This show that:

$\frac{s^{x}+s^{-x}}{2} - \frac{f^{x}+f^{-x}}{2} \geq 0$. Then we obtain $f'(x) \geq 0$, for all $x \geq 0$.This show $f(x), x \geq 0$ is an increasing function. Then we obtain For all $x \geq 0$, $f(x) \geq f(0) = 0$. □

THEOREM 2: ***If $x \in R$ and $s > f > 1$ then the following inequality is satisfied:***

$$cosh_{s}(x) \geq cosh_{f}(x) \qquad (7)$$

Proof: Let $\mathrm{f}: \mathrm{R}^{+} \to \mathrm{R}$ be a function defined by $f(x) = cosh_{s}(x) - cosh_{f}(x)$

$f(x)$ is an even function then enough to show that theorem is true for $x \geq 0$.

The derivative of $f(x)$ is $f'(x) = ln\varphi\left(sinh_{s}(x) - sinh_{f}(x)\right)$

if we use *Theorem 1*, then we obtain $f'(x) \geq 0$. This show $f(x), x \geq 0$ is an increasing function. Then we obtain for all $x \geq 0$, $f(x) \geq f(0) = 0$.

THEOREM 3: a and b are two positive numbers and $a \neq b$ then the following inequality is satisfied:

$$\sqrt{ab} < \frac{a-b}{(\log_\varphi a - \log_\varphi b)\ln\varphi} < \frac{a+b}{2}, \varphi > 1 \qquad (8)$$

Proof: **From *Lemma* 2 we get that for all $0 \neq z \in R$ the following inequality is true:**

$$\frac{\tanh_\varphi(z)}{z\ln\varphi} < 1,$$

If we take : $z = \frac{x-y}{2}$ then $\frac{\tanh_\varphi\left(\frac{x-y}{2}\right)}{\frac{x-y}{2}\ln\varphi} < 1$, this means that:

$$\frac{\varphi^{\frac{x-y}{2}} - \varphi^{\frac{y-x}{2}}}{\frac{x-y}{2}\left(\varphi^{\frac{x-y}{2}} + \varphi^{\frac{y-x}{2}}\right)\ln\varphi} < 1$$

$$\frac{\varphi^{\frac{x-y}{2}} - \varphi^{\frac{y-x}{2}}}{\frac{x-y}{2}\left(\varphi^{\frac{x-y}{2}} + \varphi^{\frac{y-x}{2}}\right)\ln\varphi} = \frac{\varphi^x - \varphi^y}{\frac{x-y}{2}(\varphi^x + \varphi^y)\ln\varphi} \cdot \frac{\varphi^{-\frac{y+x}{2}}}{\varphi^{-\frac{y+x}{2}}} < 1$$

This show that $\frac{\varphi^x - \varphi^y}{\frac{x-y}{2}\ln\varphi} < \varphi^x + \varphi^y$

If we take $a = \varphi^x, b = \varphi^y$ then we obtain: $\frac{a-b}{(\log_\varphi a - \log_\varphi b)\ln\varphi} < \frac{a+b}{2}$.

From *Lemma 1* we get that for all $0 \neq z \in R$ the following inequality is true:

$$\frac{\sinh_\varphi(z)}{z\ln\varphi} > 1,$$

If we take : $z = \frac{x-y}{2}$ then $\frac{\sinh_\varphi\left(\frac{x-y}{2}\right)}{\frac{x-y}{2}\ln\varphi} > 1$, this means that:

$$\frac{\varphi^{\frac{x-y}{2}} - \varphi^{\frac{y-x}{2}}}{\frac{x-y}{2}\ln\varphi} = \frac{\varphi^x - \varphi^y}{\frac{x-y}{2}\ln\varphi}\varphi^{-\frac{y+x}{2}} > 1, \text{ we obtain}$$

$\frac{\varphi^x-\varphi^y}{\frac{x-y}{2}ln\varphi} > \varphi^{\frac{y+x}{2}}$, Now we take $a = \varphi^x, b = \varphi^y$ then we obtain:

$$\frac{a-b}{(\log_\varphi a - \log_\varphi b) ln\,\varphi} > \sqrt{ab}\,.$$

The proof of theorem is complete. □

Theorem 4 and theorem 5 are proved in [8]. We will give a few different proofs:

THEOREM 4 (HUYGENS' INEQUALITY)*If $x \neq 0$ and $x \in R$ then the following inequality is satisfied:*

$$2\frac{sinh_\varphi(x)}{x} + \frac{tanh_\varphi(x)}{x} > 3ln\varphi, \forall\varphi > 1 \qquad (9)$$

Proof: Let $f: R^+ \to R$ be a function defined by

$$f(x) = 2sinh_\varphi(x) + tanh_\varphi(x) - 3xln\varphi, x \geq 0$$

$$f'(x) = \left(2cosh_\varphi(x) + \frac{1}{cosh^2_\varphi(x)} - 3\right) ln\varphi$$

If we apply the AM-GM inequality, we get that: $f'(x) \geq 0$, for all $x \in R$.

Then we obtain $f(x), x \geq 0$ is an increasing function. This show

$\forall x \geq 0, f(x) \geq f(0)$. We obtain: $2sinh_\varphi(x) + tanh_\varphi(x) \geq 3xln\varphi,\ x \geq 0$

Now if we divide both sides of the inequality by $x > 0$ then we obtain

$$2\frac{sinh_\varphi(x)}{x} + \frac{tanh_\varphi(x)}{x} > 3ln\varphi.$$

The proof of theorem is complete. □

THEOREM 5(CUSA-HUYGENS INEQUALITY) *If $x \neq 0$ and $x \in R$ then the following inequality is satisfied:*

$$\frac{sinh_\varphi(x)}{x} < \left[\frac{cosh_\varphi(x)+2}{3}\right] ln\varphi, \forall\varphi > 1 \qquad (10)$$

Proof: Let $f: R^+ \to R$ be a function defined by

$$f(x) = x\left[cosh_\varphi(x) + 2\right]ln\varphi - 3sinh_\varphi(x), \qquad x \geq 0$$

$$f'(x) = 2\left(1 - cosh_\varphi(x)\right)ln\varphi + xsinh_\varphi(x)(ln\varphi)^2$$

$$f''(x) = (ln\varphi)^2\left(xcosh_\varphi(x)ln\varphi - sinh_\varphi(x)\right)$$

From *Lemma 2* we obtain that: $f''(x) \geq 0, x \geq 0$

This show that $f'(x)$ and $f(x)$ are increasing and positive functions.

$$x\left[cosh_\varphi(x) + 2\right]ln\varphi \geq 3sinh_\varphi(x)$$

Now if we divide both sides of the inequality by $x > 0$ then we obtain

$$\frac{sinh_\varphi(x)}{x} < \left[\frac{cosh_\varphi(x) + 2}{3}\right]ln\varphi$$

The proof of theorem is complete. □

THEOREM 6: If $x \geq 0$ and $s > f > 1$ then the following inequality is satisfied:

a) $sinh_s(x)\ln f \geq sinh_f(x)\ln s$ (11)

b) $tanh_s(x)\ln f \leq tanh_f(x)\ln s$ (12)

c) $cosh_s(x)\ln f \geq cosh_f(x)\ln s$ (13)

Proof: a) $f: R^+ \to R$ be a function defined by

$$f(x) = sinh_s(x)\ln f - sinh_f(x)\ln s$$

$$f'(x) = \ln f \ln s\left(cosh_s(x) - cosh_f(x)\right)$$

From *Theorem 2* we obtain $f'(x) \geq 0$, for all $x \in R$. Then we obtain $f(x), x \geq 0$ is an increasing function. This show $f(x) \geq f(0)$ if $x \geq 0$.That means

$sinh_s(x)\ln f - sinh_f(x)\ln s \geq 0$.The proof of *part a)* is complete.

b) $f: R^+ \to R$ be a function defined by

$$f(x) = tanh_s(x)\ln f - tanh_f(x)\ln s$$

$$f'(x) = \ln f \ln s\left(\frac{1}{cosh_s^2(x)} - \frac{1}{cosh_f^2(x)}\right)$$

From *Theorem 2* we obtain $f'(x) \leq 0$, for all $x \in R$. This show that $f(x), x \geq 0$ is an decreasing function. Then we obtain :

For all $x \geq 0$, $f(x) \leq f(0) = 0$.

$tanh_s(x)\ln f - tanh_f(x)\ln s \leq 0$.The proof of *part b)* is complete.

c) $f: R^+ \to R$ be a function defined by $f(x) = cosh_s(x)\ln f - cosh_f(x)\ln s$

$$f'(x) = \ln f \ln s\left(sinh_s(x) - sinh_f(x)\right)$$

From *Theorem 1* we obtain $f'(x) \geq 0$ for all $x \geq 0$. That means $f(x), x \geq 0$ is an increasing function. This show $f(x) \geq f(0)$ if $x \geq 0$. This means that

$cosh_s(x)\ln f - cosh_f(x)\ln s \geq 0$.The proof of theorem is complete.

REFERENCES

1. Chen C-P, Cheung W-S (2012) Sharpness of Wilker and Huygens type inequalities. J Inequal Appl 2012:72 . https://doi.org/10.1186/1029-242X-2012-72

2. Ibrahimov S. Huygens' and Wilker'S inequalities revisited(2022)

3. Nantomah K, Prempeh E (2020) Some Inequalities for Generalized Hyperbolic Functions. Moroccan Journal of Pure and Applied Analysis 6:76–92 . https://doi.org/10.2478/mjpaa-2020-0007

4. Neuman E (2012) On Wilker and Huygens type inequalities. Mathematical Inequalities & Applications 271–279 . https://doi.org/10.7153/mia-15-22

5. **Neuman E, Sándor J (2010) On some inequalities involving trigonometric and hyperbolic functions with emphasis on the Cusa-Huygens, Wilker, and Huygens inequalities. Mathematical Inequalities & Applications 715–723 . https://doi.org/10.7153/mia-13-50**

6. **Sandor J, Olah-Gal R**

On Cusa-Huygens type trigonometric and hyperbolic inequalities. J S 9

7. **Carlson B.C some inequalities for hypergeometric functions'. Proc. Amer. Math. Soc.** 17 **(1966), 32-39**

8. **Nantomah K (2020) Cusa-Huygens, Wilker and Huygens Type Inequalities for Generalized Hyperbolic Functions. EJMS 277–289 . https://doi.org/10.34198/ejms.5221.277289**

9. Daniel Sitaru-Romanian Mathematical Magazine:

http://www.ssmrmh.ro/2020/12/13/a-simple-proof-for-huygens-inequality/