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ROMANIAN MATHEMATICAL MAGAZINE

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Jensen + Nesbitt

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This paper presents two possibilities for generating new inequalities obtained especially by the successive application of the Jensen and Nesbitt inequalities in certain conditions of monotony. By choosing specific convex / concave functions, it is get various applications.

Keywords: *Jensen's inequality, Nesbitt's inequality, convex / concave function*

2000 *Mathematics Subject Classification*: 26D15

The inequalities of *Nesbitt* and *Jensen*, - two already classical inequalities - are well known in mathematical literature and practice :

1. Proposition (*Nesbitt's inequality*, [5])

If $a, b, c > 0$, then,

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}, \qquad (N)$$

having equality iff $a=b=c$.

2. Proposition (*Jensen's inequality*, [1])

Let $f: \mathbf{I}\subset\mathbb{R}\longrightarrow\mathbb{R}$ a *convex function* on the interval I. Then for any $x_k\in\mathbf{I}$, we have

$$\frac{1}{n}\cdot\sum_{k=1}^{n}f(x_k)\geq f\left(\frac{1}{n}\cdot\sum_{k=1}^{n}x_k\right), \qquad (J)$$

If f is a *concave function* on I, the inequality sign in (J) is reversed.

Equality in (J) occurs if and only if $x_1=x_2=\cdots=x_n$, or when the function f is a function *linear* (*affine*).

In the following we will highlight some inequalities that result from the successive application of of the *inequalities of Jensen* (for case $n=3$) and *Nesbitt*, together with certain properties of monotony of the functions considered. Here is a first result of this kind.

3. *Proposition*

If the function $f : I \subset (0, \infty) \longrightarrow \mathbb{R}_+$ is a *convex* and *increasing* function on I, then :

$$f\left(\frac{a}{b+c}\right)+f\left(\frac{b}{c+a}\right)+f\left(\frac{c}{a+b}\right) \ge 3\cdot f\left(\frac{1}{2}\right) , \qquad (1)$$

for any $a, b, c > 0$.

Proof

Indeed, using *Jensen's inequality* for *convex functions* in the first instance, then by *Nesbitt's inequality* and also taking into account the fact that the function is *increasing* , we obtain successively :

$$\sum_{cyc} f\left(\frac{a}{b+c}\right) \overset{(J)}{\ge} 3f\left(\frac{1}{3}\sum_{cyc}\frac{a}{b+c}\right) \overset{(N)}{\ge} 3f\left(\frac{3}{2}\cdot\frac{1}{3}\right) = 3f\left(\frac{1}{2}\right)$$

Equality occurs if $a = b = c$.

4. *Corollary* *(a generalization of Nesbitt's inequality)*

For any $p > 1$ and for any $a, b, c > 0$, the following inequality occurs ,

$$\left(\frac{a}{b+c}\right)^p + \left(\frac{b}{c+a}\right)^p + \left(\frac{c}{a+b}\right)^p \ge \frac{3}{2^p} , \qquad (2)$$

Proof

Consider the function, $f : (0, \infty) \longrightarrow \mathbb{R}_+$, $f(x) = x^p$, $p > 1$, which is obviously *convex* and *ascending*, so with *Proposition* 3, we have : $\left(\frac{a}{b+c}\right)^p + \left(\frac{b}{c+a}\right)^p + \left(\frac{c}{a+b}\right)^p \ge 3\cdot\left(\frac{1}{2}\right)^p$, with equally if $a = b = c$. For $p = 1$, inequality (N) is obtained .

5. *Application*

For any $q > 1$ and for any $a, b, c > 0$, the following inequality occurs ,

$$q^{\frac{a}{b+c}} + q^{\frac{b}{c+a}} + q^{\frac{c}{a+b}} \ge 3\cdot\sqrt{q} , \qquad (3)$$

Proof

|Let the function , $f : (0, \infty) \longrightarrow \mathbb{R}_+$, $f(x) = q^x$, $q > 1$, which is obviously *convex* and *ascending*, so with *Proposition* 3, we have : $q^{\frac{a}{b+c}} + q^{\frac{b}{c+a}} + q^{\frac{c}{a+b}} \ge 3\cdot q^{\frac{1}{2}}$, with equally if $a = b = c$.

6. *Application* , [2]

For $a, b, c > 0$, there is the inequality ,

$$\frac{\sqrt{a^2+(b+c)^2}}{b+c}+\frac{\sqrt{b^2+(c+a)^2}}{c+a}+\frac{\sqrt{c^2+(a+b)^2}}{a+b}\geq\frac{3}{2}\cdot\sqrt{5} \quad . \tag{4}$$

Proof

Let the function , $f: I\subset(0,\infty)\longrightarrow\mathbb{R}_+$, $f(x)=\sqrt{x^2+1}$, for which we have:

$f'(x)=\frac{x}{\sqrt{x^2+1}}>0$, $f''(x)=\frac{1}{(x^2+1)\cdot\sqrt{x^2+1}}>0$, so the function f is convex and increasing on $(0,\infty)$. With Proposition 3, we have ,

$$\sqrt{\left(\frac{a}{b+c}\right)^2+1}+\sqrt{\left(\frac{b}{c+a}\right)^2+1}+\sqrt{\left(\frac{c}{a+b}\right)^2+1}\overset{(Prop.3)}{\geq}3\cdot f\left(\frac{1}{2}\right)=3\cdot\sqrt{\left(\frac{1}{2}\right)^2+1}=\frac{3}{2}\cdot\sqrt{5} ,$$

hence the inequality (2) , with equally if $a=b=c$.

7. *Remark*

If a, b, c are the lengths of the sides of a triangle , then ,

$$\frac{a}{b+c},\frac{b}{c+a},\frac{c}{a+b}\in(0,1), \qquad (5)$$

Indeed, from $b+c>a$, result , $\frac{a}{b+c}<1$. Analogous : $\frac{b}{c+a}<1$, $\frac{c}{a+b}<1$ ×

8. *Application*

For a real number $p>1$ and a, b, c – sides of a triangle , we have the following inequality ,

$$\frac{a^p}{(-a+b+c)\cdot(b+c)^{p-1}}+\frac{b^p}{(a-b+c)\cdot(c+a)^{p-1}}+\frac{c^p}{(a+b-c)\cdot(a+b)^{p-1}}\geq\frac{3}{2^{p-1}} \quad . \tag{6}$$

Proof

Consider the function , $f:(0,1)\longrightarrow\mathbb{R}_+$, $f(x)=\frac{x^p}{1-x}$, for which we have :

$$f'(x)=\frac{x^{p-1}[p-(p-1)x]}{(1-x)^2}>0 \text{ in intervalul } \left(0,1+\frac{1}{p-1}\right)\supset(0,1),$$

$$f''(x)=\frac{x^{p-2}[p(p-1)(1-x)^2+2x(p-(p-1)x)]}{(1-x)^3}>0 , \ (\forall)\, x\in(0,1) .$$ It turns out that

the function f is *convex* and *increasing* on (0, 1), so with *Proposition* 3, and *Remark* 7, we have ,

$$\frac{\left(\frac{a}{b+c}\right)^p}{1-\left(\frac{a}{b+c}\right)}+\frac{\left(\frac{b}{c+a}\right)^p}{1-\left(\frac{b}{c+a}\right)}+\frac{\left(\frac{c}{a+b}\right)^p}{1-\left(\frac{c}{a+b}\right)}\geq 3\cdot\frac{\left(\frac{1}{2}\right)^p}{1-\frac{1}{2}} \Leftrightarrow$$

$$\Leftrightarrow \frac{a^p}{(-a+b+c)\cdot(b+c)^{p-1}}+\frac{b^p}{(a-b+c)\cdot(c+a)^{p-1}}+\frac{c^p}{(a+b-c)\cdot(a+b)^{p-1}}\geq\frac{3}{2^{p-1}} .$$

Equality occurs if $a=b=c$.

9. *Application*

In the triangle **ABC** , on the sides a, b, c, we have the inequality ,

$$arcsin\left(\frac{a}{b+c}\right)+arcsin\left(\frac{b}{c+a}\right)+arcsin\left(\frac{c}{a+b}\right)\geq\frac{\pi}{2} , \qquad (7)$$

Proof

How ,

$$\frac{a}{b+c},\frac{b}{c+a},\frac{c}{a+b}\in(0,1)$$

(*Remark* 7) , we consider the function ,

$f:(0,1)\longrightarrow(0,\pi/2)$, $f(x)=arcsin\,x$, which is *convex* and *ascending* on $(0,1)$.

Then with *Proposition* 3, we have : $arcsin\left(\frac{a}{b+c}\right)+arcsin\left(\frac{b}{c+a}\right)+arcsin\left(\frac{c}{a+b}\right)\geq 3\,arcsin\,\frac{1}{2}$,

that is, the inequality in the statement . Equality occurs in the case of the equilateral triangle .

10. *Proposition*

If the function $f:I\subset(0,\infty)\longrightarrow\mathbb{R}_+$ is a *concave* and *descending* function on interval I ,

then : $$f\left(\frac{a}{b+c}\right)+f\left(\frac{b}{c+a}\right)+f\left(\frac{c}{a+b}\right)\leq 3\cdot f\left(\frac{1}{2}\right) , \qquad (8)$$

Proof

First , using *Jensen inequality* for *concave* functions , then *Nesbitt inequality* and taking into account and the fact that the function is decreasing , we obtain successively :

$$\sum_{cyc} f\left(\frac{a}{b+c}\right)\overset{(J)}{\leq}3f\left(\frac{1}{3}\sum_{cyc}\frac{a}{b+c}\right)\overset{(N)}{\leq}3f\left(\frac{3}{2}\cdot\frac{1}{3}\right)=3f\left(\frac{1}{2}\right)$$

Equality occurs if $a=b=c$.

11. *Application* , [3]

In triangle **ABC** , with sides a, b, c, we have inequality ,

$$\frac{\sqrt{a+b-c}}{a+b}+\frac{\sqrt{-a+b+c}}{b+c}+\frac{\sqrt{a-b+c}}{c+a}\leq\frac{3\cdot\sqrt{3}}{2\cdot\sqrt{a+b+c}}\ , \qquad (9)$$

Proof

The inequality in the statement can be written in the equivalent forms :

$$\frac{\sqrt{(a+b-c)(a+b+c)}}{a+b}+\frac{\sqrt{(-a+b+c)(a+b+c)}}{b+c}+\frac{\sqrt{(a-b+c)(a+b+c)}}{c+a}\leq\frac{3\cdot\sqrt{3}}{2}\Leftrightarrow$$

$$\Leftrightarrow\frac{\sqrt{(a+b)^2-c^2}}{a+b}+\frac{\sqrt{(b+c)^2-a^2}}{b+c}+\frac{\sqrt{(c+a)^2-b^2}}{c+a}\leq\frac{3\cdot\sqrt{3}}{2}\Leftrightarrow$$

$$\Leftrightarrow\sqrt{1-\left(\frac{a}{b+c}\right)^2}+\sqrt{1-\left(\frac{b}{c+a}\right)^2}+\sqrt{1-\left(\frac{c}{a+b}\right)^2}\leq\frac{3\cdot\sqrt{3}}{2}\ . \qquad (10)$$

Consider the function , $f:(0,1)\longrightarrow\mathbb{R}_+$, $f(x)=\sqrt{1-x^2}$ (*the circle function- in the first dial*), which is obviously *concave* and *decreasing* on $(0,1)$.

How ,

$$\frac{a}{b+c},\frac{b}{c+a},\frac{c}{a+b}\in(0,1)$$

(*Remark* 7) , then with *Proposition* 10, we have :

$$\sqrt{1-\left(\frac{a}{b+c}\right)^2}+\sqrt{1-\left(\frac{b}{c+a}\right)^2}+\sqrt{1-\left(\frac{c}{a+b}\right)^2}\leq 3\cdot f\left(\frac{1}{2}\right)=3\cdot\sqrt{1-\left(\frac{1}{2}\right)^2}=\frac{3\cdot\sqrt{3}}{2}\ ,$$

so there is inequality (10) .

12. *Application* , [4]

In triangle **ABC** , with sides a, b, c, we have inequality ,

$$arccos\left(\frac{a}{b+c}\right)+arccos\left(\frac{b}{c+a}\right)+arccos\left(\frac{c}{a+b}\right)\leq\pi\ , \qquad (11)$$

with equally if $a=b=c$.

Proof

With *Remark* 7 , we have ,

$$\frac{a}{b+c},\frac{b}{c+a},\frac{c}{a+b}\in(0,1)$$

To solve we consider the function ,

$f:(0,1)\longrightarrow(0,\pi/2)$, $f(x)=\arccos x$, which is *concave* and *decreasing* on $(0,1)$.
Then with *Proposition* 10, we have :

$$\arccos\left(\frac{a}{b+c}\right)+\arccos\left(\frac{b}{c+a}\right)+\arccos\left(\frac{c}{a+b}\right)\le 3\arccos\frac{1}{2}=\pi$$

that is, the inequality in the statement. Equality occurs in the case of the equilateral triangle .

13. *Remark*

Inequality (11) can also be obtained from inequality (7) , using identity ,

$$\arccos x=\pi/2-\arcsin x \quad (12)$$

14. *Remark*

Note that only the possibilities of association : (f – *convex* , f – *ascending*) - from *Proposition* 3, and (f – *concave* , f – *descending*) - from *Proposition* 10 can be considered. The other two possibilities of association do not ensure the transitivity of the inequality relationship .

For the above applications - demonstrated by the successive application of *Jensen* and *Nesbitt* inequalities - there are also other ways to demonstrate - as happened in the group posts : [2] , [3] , [4] .
Obviously, many other applications of Sentences 3 and 10 can be obtained and demonstrated, respecting the above scenarios .

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