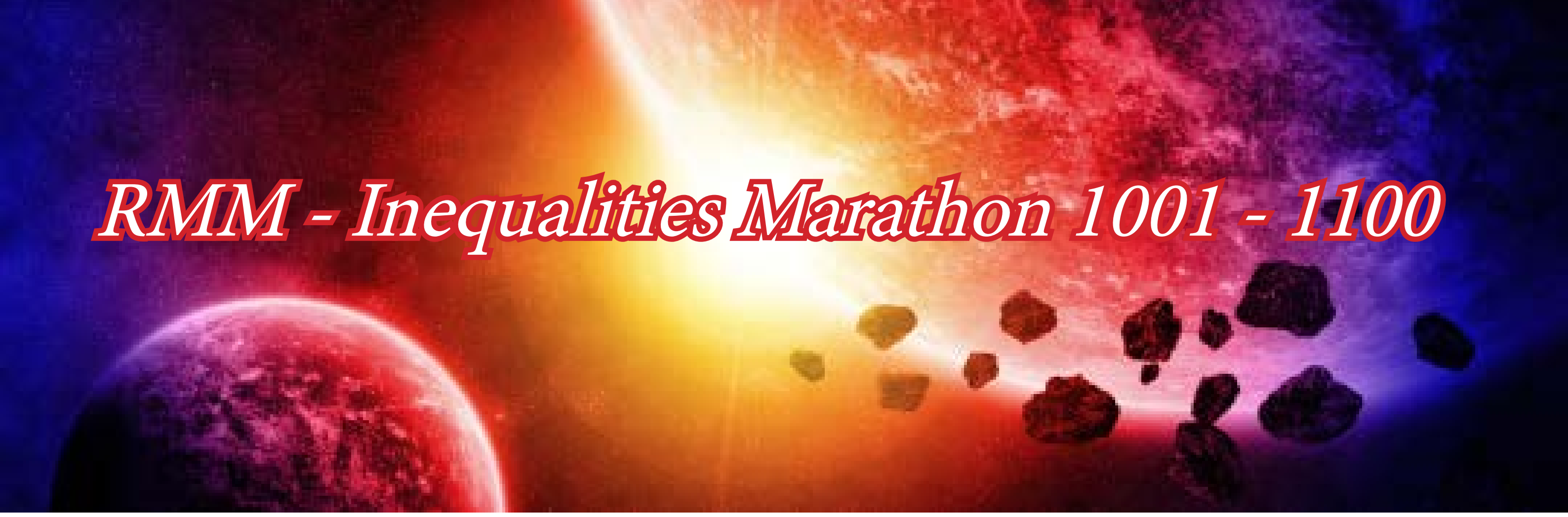
RMM - Inequalities Marathon 1001 - 1100



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1001. If $0 < a \leq 1 \leq b$ then: $(a+b-1)^{a+b-1}+1 \leq a^a+b^b$.

Proposed by Daniel Sitaru-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$If\ a=b=1\ we\ have:\ (a+b-1)^{a+b-1}+1=a^a+b^b.\ Assume\ now\ that\ a\neq b.$$

$$Let\ f(x)=x^x,\ \ x>0.\ We\ have:$$

$$f''(x)=x^{x-1}+(\log x+1)^2.x^x>0\ then\ f\ is\ convex\ on\ (0,\infty).$$

$$Let\ \ t=\frac{b-1}{b-a}\in[0,1].\ We\ have:$$

$$1=t.a+(1-t).b\ \ and\ \ a+b-1=(1-t).a+t.b.$$

$$Then\ by\ Jensen's\ inequality\ we\ have:$$

$$1=f(1)=f(t.a+(1-t).b)\leq t.f(a)+(1-t).f(b)=t.a^a+(1-t).b^b$$

$$And:\ \ (a+b-1)^{a+b-1}=f(a+b-1)=f\big((1-t).a+t.b\big)\leq(1-t).a^a+t.b^b$$

$$Summing\ up\ the\ two\ inequalities, we\ obtain:$$

$$(a+b-1)^{a+b-1}+1\leq a^a+b^b,\ \ the\ desired\ result.$$

Solution 2 by Daoudi Abdessattar-Tunisia

$f:(0,\infty)\to(0,\infty), f(x)=x^x$ convex function and (b,a) majorizes $(b+a-1,1)$ if $a+b-1$ is greater than 1, (b,a) majorizes $1, a+b-1)$ if $a+b-1$ is smaller than 1, then by Karamata's inequality

$(a+b-1)^{a+b-1}+1^1\leq a^a+b^b$. Equality holds for $a=b=1$.

1002. If $x,y,z\in\mathbb{R}$ then prove that:

$$(x^2+y^2+z^2)^2-(xy+yz+zx)^2\geq\sqrt{6}(x-y)(y-z)(z-x)(x+y+z).$$

Proposed by Hikmat Mammadov-Azerbaijan

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Let\ a=x-z\ and\ b=y-z.\ We\ have:$$

$$(x^2+y^2+z^2)^2-(xy+yz+zx)^2$$

$$=(x^2+y^2+z^2-xy-yz-zx)(x^2+y^2+z^2+xy+yz+zx)=$$

$$=\frac{1}{2}[(x-y)^2+(y-z)^2+(z-x)^2][(x+y)^2-(x-z)(y-z)+2z^2]=$$

$$=\frac{1}{2}[(a-b)^2+b^2+a^2][(a+b+2z)^2-ab+2z^2]$$

$$=(a^2-ab+b^2)(a^2+ab+b^2+4(a+b)z+6z^2).$$

Then : $(x^2+y^2+z^2)^2-(xy+yz+zx)^2$

$$=6(a^2-ab+b^2)z^2+4(a^3+b^3)z+a^4+a^2b^2+b^4$$

And : $(x-y)(y-z)(z-x)(x+y+z)=(a-b).b.(-a).(a+b+3z)$

$$=-ab(a^2-b^2)-3ab(a-b)z.$$

So the problem becomes to prove that :

$$6(a^2-ab+b^2)z^2+\left[4(a^3+b^3)+3\sqrt{6}ab(a-b)\right]z+a^4+a^2b^2+b^4$$
$$+\sqrt{6}ab(a^2-b^2)\geq 0$$

Which is a quadratic on z.

Since $a^2-ab+b^2\geq 0,\forall a,b\in R$ *so it suffices to prove :*

$$\Delta=\left[4(a^3+b^3)+3\sqrt{6}ab(a-b)\right]^2$$
$$-24(a^2-ab+b^2)\left(a^4+a^2b^2+b^4+\sqrt{6}ab(a^2-b^2)\right)\leq 0$$

$$\Leftrightarrow -8a^6+24a^5b+6a^4b^2-52a^3b^3+6a^2b^4+24ab^5-8b^6\leq 0$$

$\Leftrightarrow -(a+b)^2(a-2b)^2(2a-b)^2\leq 0$ *which is true. So the proof is completed.*

Equality holds iff $a=b=0$ *or* $a+b=0$ *or* $a-2b=0$ *or* $2a-b=0$.

$$\Leftrightarrow\ x=y=z \text{ or } (x,y,z)=\left(\frac{\sqrt{6}}{6}a,\frac{6+\sqrt{6}}{6}a,-\frac{6-\sqrt{6}}{6}a\right),$$

$a\in R,$ *and their permutation.*

1003. If $a,b,c>0,p,q,r>1,pq+qr+rp=pqr$ then :

$$abcpqr\leq qra^p+rpb^q+pqc^r.$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

$f:(0,\infty)\to\mathbb{R},f(x)=e^x,f''(x)=e^x>0,f$ *−convex. By Jensen:*

$$f(\lambda_1 x+\lambda_2 y+\lambda_3 z)\le \lambda_1 f(x)+\lambda_2 f(y)+\lambda_3 f(z), x,y,z>0$$

$$\lambda_1,\lambda_2,\lambda_3>0, \lambda_1+\lambda_2+\lambda_3=1$$

$$e^{\lambda_1 x+\lambda_2 y+\lambda_3 z}\le \lambda_1 e^x+\lambda_2 e^y+\lambda_3 e^z$$

$$\lambda_1=\frac{1}{p}, \lambda_2=\frac{1}{q}, \lambda_3=\frac{1}{r}, \lambda_1+\lambda_2+\lambda_3=\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=\frac{pq+qr+rp}{pqr}=1$$

$$e^{\frac{1}{p}x+\frac{1}{q}y+\frac{1}{r}z}\le \frac{1}{p}e^x+\frac{1}{q}e^y+\frac{1}{r}e^z \Rightarrow e^{\frac{1}{p}x}\cdot e^{\frac{1}{q}y}\cdot e^{\frac{1}{r}z}\le \frac{1}{p}e^x+\frac{1}{q}e^y+\frac{1}{r}e^z$$

$$\begin{cases} e^{\frac{1}{p}x}=a \\ e^{\frac{1}{q}y}=b \\ e^{\frac{1}{r}z}=c \end{cases} \Rightarrow \begin{cases} \frac{1}{p}x=lna \\ \frac{1}{q}y=lnb \\ \frac{1}{r}z=lnc \end{cases} \Rightarrow \begin{cases} x=plna \\ y=qlnb \\ z=rlnc \end{cases}$$

$$abc\le \frac{1}{p}e^{plna}+\frac{1}{q}e^{qlnb}+\frac{1}{r}e^{rlnc} \Rightarrow abc\le \frac{1}{p}\left(e^{lna}\right)^p+\frac{1}{q}\left(e^{lnb}\right)^q+\frac{1}{r}\left(e^{lnc}\right)^r$$

$$abc\le \frac{1}{p}a^p+\frac{1}{q}b^q+\frac{1}{r}c^r$$

$$abcpqr\le qra^p+rpb^q+pqc^r.$$

Equality holds for: $a^p=b^q=c^r$.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Since\ \frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1\ then\ by\ Weighted\ AM-GM\ we\ have:$$

$$\frac{1}{p}.a^p+\frac{1}{q}.b^q+\frac{1}{r}.c^r\ge (a^p)^{\frac{1}{p}}.(b^q)^{\frac{1}{q}}.(c^r)^{\frac{1}{r}}=abc.$$

$$Multiplying\ the\ both\ sides\ by\ pqr\ we\ get: qra^p+rpb^q+pqc^r\ge abcpqr.$$

Solution 3 by Tapas Das-India

Let (a^p, b^q, c^r) with associated weight (qr, pr, pq).By AM-GM, we get:

$$\frac{qr\cdot a^p+pr\cdot b^q+pq\cdot c^r}{pq+qr+rp}\ge\left(a^{pqr}\cdot b^{pqr}\cdot c^{pqr}\right)^{\frac{1}{pq+qr+rp}}$$

$$\frac{qr\cdot a^p+pr\cdot b^q+pq\cdot c^r}{pqr}\ge(abc)^{\frac{prq}{pq+qr+rp}}=abc$$

$$pq+qr+rp=pqr$$

Hence,

$$qra^p+rpb^q+pqc^r\ge abcpqr.$$

1004. **If $a,b,c>0$ and $\lambda\ge 0$ then**

$$\frac{a^3}{a^2+\lambda ab+b^2}+\frac{b^3}{b^2+\lambda bc+c^2}+\frac{c^3}{c^2+\lambda ca+a^2}+\frac{9}{(\lambda+2)(a+b+c)}\ge\frac{6}{\lambda+2}.$$

Proposed by Marin Chirciu-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:$$

$$\frac{a^3}{a^2+\lambda ab+b^2}=a-\frac{ab(\lambda a+b)}{a^2+b^2+\lambda ab}\overset{AM-GM}{\ge}a-\frac{ab(\lambda a+b)}{2ab+\lambda ab}=a-\frac{\lambda a+b}{\lambda+2}=\frac{2a-b}{\lambda+2}.$$

$$Similarly\ we\ have:\ \frac{b^3}{b^2+\lambda bc+c^2}\ge\frac{2b-c}{\lambda+2}\quad and\quad\frac{c^3}{c^2+\lambda ca+a^2}\ge\frac{2c-a}{\lambda+2}.$$

$$Summing\ up\ these\ inequalities\ we\ get$$

$$\frac{a^3}{a^2+\lambda ab+b^2}+\frac{b^3}{b^2+\lambda bc+c^2}+\frac{c^3}{c^2+\lambda ca+a^2}\ge\frac{a+b+c}{\lambda+2}.$$

$$Then:\ \frac{a^3}{a^2+\lambda ab+b^2}+\frac{b^3}{b^2+\lambda bc+c^2}+\frac{c^3}{c^2+\lambda ca+a^2}+\frac{9}{(\lambda+2)(a+b+c)}\ge$$

$$\ge\frac{1}{\lambda+2}\left[(a+b+c)+\frac{9}{a+b+c}\right]\overset{AM-GM}{\ge}\frac{6}{\lambda+2}.$$

$$Equality\ holds\ iff\ a=b=c.$$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

For $a,b,c>0$ and $\lambda\ge 0$ we give $a+b+c=d\Rightarrow\frac{a+b+c}{d}=1$

$$\frac{a^3}{a^2+\lambda ab+b^2}+\frac{b^3}{b^2+\lambda bc+c^2}+\frac{c^3}{c^2+\lambda ca+a^2}\ge\frac{a+b+c}{\lambda+2}$$

$$\frac{(dx)^3}{(dx)^2+\lambda dxdy+(dy)^2}+\frac{(dy)^3}{(dy)^2+\lambda dydz+(dz)^2}+\frac{(dz)^3}{(dz)^2+\lambda dzdx+(dx)^2}\ge\frac{d}{\lambda+2}$$

$$\frac{dx^3}{x^2+\lambda xy+y^2}+\frac{dy^3}{y^2+\lambda yz+z^2}+\frac{dz^3}{z^2+\lambda zx+x^2}\geq\frac{d}{\lambda+2}$$

$$\frac{x^3}{x^2+\lambda xy+y^2}+\frac{y^3}{y^2+\lambda yz+z^2}+\frac{z^3}{z^2+\lambda zx+x^2}\geq\frac{1}{\lambda+2}$$

$$\frac{(x^2+y^2+z^2)^2(\lambda+2)}{x^3+y^3+z^3+\lambda(x^2y+y^2z+z^2x)+xy^2+yz^2+zx^2}\geq 1$$

$$\left(x^4+y^4+z^4+2(x^2y^2+y^2z^2+z^2x^2)\right)(\lambda+2)\geq$$

$\geq x^3+y^3+z^3+\lambda(x^2y+y^2z+z^2x)+xy^2+yz^2+zx^2$ **which is true, because**

$$x^2y+y^2z+z^2x\leq\frac{1}{9}\text{ and }(x^2+y^2+z^2)^2\geq\frac{1}{9}$$

$$2(x^2+y^2+z^2)\geq(x^3+y^3+z^3+xy^2+yz^2+zx^2)(x+y+z)$$

$$\frac{a^3}{a^2+\lambda ab+b^2}+\frac{b^3}{b^2+\lambda bc+c^2}+\frac{c^3}{c^2+\lambda ca+a^2}\geq\frac{a+b+c}{\lambda+2}$$

$$\frac{a+b+c}{\lambda+2}+\frac{9}{(\lambda+2)(a+b+c)}\geq\frac{6}{\lambda+2}$$

1005. **Let $a,b,c>0, a+b+c=1$. Prove that :**

$$\sqrt{\frac{ab}{c}}+\sqrt{\frac{bc}{a}}+\sqrt{\frac{ca}{b}}\geq\sum_{cyc}\sqrt{a^2+b^2+3abc}.$$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

The given inequality is equivalent to :

$$ab+bc+ca\geq\sum_{cyc}\sqrt{abc(a^2+b^2+3abc)}$$

By CBS inequality we have :

$$\sum_{cyc}\sqrt{abc(a^2+b^2+3abc)}\leq\sqrt{\left(\sum\nolimits_{cyc}ab\right)\left[\sum\nolimits_{cyc}c(a^2+b^2+3abc)\right]}$$

$$=\sqrt{\left(\sum\nolimits_{cyc}ab\right)\left[\sum\nolimits_{cyc}c(a^2+b^2)+3abc\right]}=$$

$$=\sqrt{\left(\sum_{cyc}ab\right)\cdot\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a\right)}=ab+bc+ca.$$

$$So\ the\ proof\ is\ completed.\ Equality\ holds\ iff\ a=b=c=\frac{1}{3}.$$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

For $a,b,c>0$ and $a+b+c=1$, we will obtain that

$$\sqrt{\frac{ba}{c}}+\sqrt{\frac{bc}{a}}+\sqrt{\frac{ca}{b}}\geq\sqrt{a^2+b^2+3abc}+\sqrt{b^2+c^2+3abc}+\sqrt{c^2+a^2+3abc}$$

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq\sqrt{\frac{a}{bc}+\frac{b}{ca}+3}+\sqrt{\frac{b}{ca}+\frac{c}{ab}+3}+\sqrt{\frac{c}{ab}+\frac{a}{bc}+3}$$

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq\sqrt{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(\frac{a}{b}+\frac{b}{a}+\frac{b}{c}+\frac{c}{b}+\frac{c}{a}+\frac{a}{c}+3(a+b+c)\right)}$$

$$\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2\geq\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(\frac{a}{b}+\frac{b}{a}+\frac{b}{c}+\frac{c}{b}+\frac{c}{a}+\frac{a}{c}+3\right)$$

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{a}{c}+\frac{c}{b}+\frac{b}{a}+3$$

Solution 3 by Nguyen Van Canh-Ben Tre-Vietnam

$$\sqrt{\frac{ab}{c}}+\sqrt{\frac{bc}{a}}+\sqrt{\frac{ca}{b}}\geq\sqrt{a^2+b^2+3abc}+\sqrt{b^2+c^2+3abc}+\sqrt{c^2+a^2+3abc}$$

$$\Leftrightarrow ab+bc+ca\geq\sqrt{abc}\left(\sqrt{a^2+b^2+3abc}+\sqrt{b^2+c^2+3abc}+\sqrt{c^2+a^2+3abc}\right);\ (*)$$

By AM − GM Inequality we have:

$$\sqrt{abc(a^2+b^2+3abc)}=\sqrt{ab(ca^2+cb^2+3abc^2)}\leq\frac{ab+ca^2+cb^2+3abc^2}{2};$$

Similary:

$$\sqrt{abc(b^2+c^2+3abc)}\leq\frac{bc+ab^2+ac^2+3bca^2}{2};$$

$$\sqrt{abc(c^2+a^2+3abc)}\leq\frac{ac+bc^2+ba^2+3acb^2}{2};$$

$$\Rightarrow \sqrt{abc}\left(\sqrt{a^2+b^2+3abc}+\sqrt{b^2+c^2+3abc}+\sqrt{c^2+a^2+3abc}\right)$$

$$\le \frac{\sum ab+\sum ab(a+b)+3abc\sum a}{2}=\frac{\sum ab+(\sum a)(\sum ab)-3abc+3abc\sum a}{2}$$

$$=\frac{2\sum ab-3abc+3abc}{2}=\sum ab;$$

$\Rightarrow (*)$true. Proved. Equality $\Leftrightarrow a=b=c=\frac{1}{3}$.

1006. Let $a,b,c>0, a+b+c=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$. Prove that :

$$\sum_{cyc}\frac{1}{a^3(a^2+ab+b^2)}\ge\frac{\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2}{9}.$$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By AM − GM inequality we have :

$$\frac{1}{a^3(a^2+ab+b^2)}+\sqrt{\frac{a^2+ab+b^2}{27}}+\sqrt{\frac{a^2+ab+b^2}{27}}$$

$$\ge 3\sqrt[3]{\frac{1}{a^3(a^2+ab+b^2)}\cdot\sqrt{\frac{a^2+ab+b^2}{27}}^{\,2}}=\frac{1}{a}$$

$$Then:\ \frac{1}{a^3(a^2+ab+b^2)}\ge\frac{1}{a}-\frac{2}{3}\sqrt{\frac{a^2+ab+b^2}{3}}\quad(1)$$

By CBS inequality we have :

$$\sqrt{\frac{a^2+ab+b^2}{3}}+\frac{\sqrt{ab}}{3}\le\sqrt{\left(1+\frac{1}{3}\right)\left(\frac{a^2+ab+b^2}{3}+\frac{ab}{3}\right)}=\frac{2}{3}(a+b)$$

$$Then:\ \sqrt{\frac{a^2+ab+b^2}{3}}\le\frac{2}{3}(a+b)-\frac{\sqrt{ab}}{3}\quad(2)$$

$$\textit{From } (1)\ \&\ (2) \textit{ we get}: \frac{1}{a^3(a^2+ab+b^2)} \ge \frac{1}{a} - \frac{2}{3}\cdot\left(\frac{2}{3}(a+b) - \frac{\sqrt{ab}}{3}\right)$$

$$= \frac{1}{a} - \frac{4(a+b)}{9} + \frac{2\sqrt{ab}}{9} \quad (\textit{and analogs})$$

$$\textit{Thus, } \sum_{cyc} \frac{1}{a^3(a^2+ab+b^2)} \ge \sum_{cyc}\frac{1}{a} - \frac{4}{9}\sum_{cyc}(a+b) + \frac{2}{9}\sum_{cyc}\sqrt{ab}$$

$$= \sum_{cyc} a - \frac{8}{9}\sum_{cyc} a + \frac{2}{9}\sum_{cyc}\sqrt{ab} = \frac{\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2}{9}.$$

$$\textit{Equality holds iff } a = b = c = 1.$$

Solution 2 by Nguyen Van Canh-Ben Tre-Vietnam

By AM − GM Inequality we have:

$$2\sqrt{3(a^2+ab+b^2)} = 2\sqrt{\left(a+\sqrt{ab}+b\right).3\left(a-\sqrt{ab}+b\right)}$$

$$\le a+\sqrt{ab}+b+3\left(a-\sqrt{ab}+b\right) = 2\left(2a-\sqrt{ab}+2b\right);$$

$$\Rightarrow a^2+ab+b^2 \le \frac{\left(2a-\sqrt{ab}+2b\right)^2}{3};$$

Similary:

$$b^2+bc+c^2 \le \frac{\left(2b-\sqrt{bc}+2\sqrt{c}\right)^2}{3}; c^2+ca+a^2 \le \frac{\left(2c-\sqrt{ca}+2a\right)^2}{3}$$

Thus,

$$\sum \frac{1}{a^3(a^2+ab+b^2)} \ge \sum \frac{3}{a^3\left(2a-\sqrt{ab}+2b\right)^2}$$

$$= 3\sum \frac{\left(\frac{1}{a}\right)^3}{\left(2a-\sqrt{ab}+2b\right)^2} \overset{\text{Radon Inequality}}{\ge} \frac{3\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^3}{\left(4\sum a - \sum\sqrt{ab}\right)^2}$$

$$= \frac{3(a+b+c)^3}{\left(4\sum a - \sum\sqrt{ab}\right)^2} \overset{(*)}{\ge} \frac{\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2}{9};$$

$$(*) \Leftrightarrow \frac{3p^3}{(4p-q)^2} \ge \frac{p+2q}{9}; \left(p = \sum a \ge q = \sum\sqrt{ab}\right)$$

$$\Leftrightarrow 27p^3 \geq (p+2q)(4p-q)^2;$$

$$\Leftrightarrow 11\,p^3 - 24\,p^2q + 15\,p\,q^2 - 2\,q^3 \geq 0 \Leftrightarrow (11\,p - 2\,q)(p - q)^2 \geq 0;$$

Which is true since $p \geq q$. Proved. Equality $\Leftrightarrow a = b = c = 1$.

1007. For $a, b, c > 0, \frac{1}{a}+\frac{1}{b}+\frac{1}{c} = 3$, prove:

$$\frac{a}{b^2+c^2}+\frac{b}{c^2+a^2}+\frac{c}{a^2+b^2} \geq \frac{3}{2}\left(\frac{a+b+c}{ab+bc+ca}\right)^2$$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Proof}: \frac{1}{a}+\frac{1}{b}+\frac{1}{c} = 3 \Rightarrow 1 = \frac{\sum_{cyc} ab}{3abc} \Rightarrow \sum_{cyc}\frac{a}{b^2+c^2} \geq \frac{3}{2}\left(\frac{\sum_{cyc} a}{\sum_{cyc} ab}\right)^2$$

$$\Leftrightarrow \frac{\sum_{cyc} ab}{3abc}.\sum_{cyc}\frac{a}{b^2+c^2} \overset{(*)}{\geq} \frac{3}{2}\left(\frac{\sum_{cyc} a}{\sum_{cyc} ab}\right)^2$$

Assigning $b+c = x, c+a = y, a+b = z \Rightarrow x+y-z = 2c > 0, y+z-x = 2a > 0$ and $z+x-y = 2b > 0 \Rightarrow x+y > z, y+z > x, z+x > y$
$\Rightarrow x, y, z$ form sides of a
triangle with semiperimeter, circumradius and inradius $=$ s, R, r (say) yielding $2\sum_{cyc} a$

$$= \sum_{cyc} x = 2s \Rightarrow \sum_{cyc} a = s \Rightarrow a = s-x, b = s-y, c = s-z$$

Via aforementioned substitutions, $\sum_{cyc} a^2 = \left(\sum_{cyc} a\right)^2 - 2\sum_{cyc} ab = s^2 - 2\sum_{cyc}(s-x)(s-y)$

$$= s^2 - 2(4Rr+r^2) \Rightarrow \sum_{cyc} a^2 \overset{(\bullet)}{=} s^2 - 8Rr - 2r^2$$

$$\text{Also}, \sum_{cyc} a^3 = \left(\sum_{cyc} a\right)^3 - 3(a+b)(b+c)(c+a) = s^3 - 3xyz = s^3 - 12Rrs$$

$$\Rightarrow \sum_{cyc} a^3 \overset{(\bullet\bullet)}{=} s^3 - 12Rrs \text{ and } \sum_{cyc} a^2b^2 = \left(\sum_{cyc} ab\right)^2 - 2abc\sum_{cyc} a$$

$$= (4Rr+r^2)^2 - 2r^2s.s$$

$$\sum_{cyc} a^2b^2 \overset{(\bullet\bullet\bullet)}{=} (4Rr+r^2)^2 - 2r^2s^2$$

$$\text{Again, }\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a^3\right) = \sum_{cyc} a^5 + \left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a\right) - abc\left(\sum_{cyc} ab\right)$$

$$\Rightarrow \sum_{cyc} a^5 + \left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a\right) = \left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a^3\right) + abc\left(\sum_{cyc} ab\right)$$

$$\overset{via\ (\bullet),(\bullet\bullet)}{=} (s^2 - 8Rr - 2r^2)(s^3 - 12Rrs) + r^2s(4Rr + r^2)$$

$$\Rightarrow \sum_{cyc} a^5 + \left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a\right) \overset{(\bullet\bullet\bullet\bullet)}{=} (s^2 - 8Rr - 2r^2)(s^3 - 12Rrs) + r^2s(4Rr + r^2)$$

$$\text{Now, } \sum_{cyc} \frac{a}{b^2 + c^2} = \frac{\sum_{cyc}\left(a(c^2 + a^2)(a^2 + b^2)\right)}{\prod_{cyc}(b^2 + c^2)}$$

$$= \frac{(\sum_{cyc} a^2b^2)(\sum_{cyc} a) + \sum_{cyc} a^5}{(\sum_{cyc} a^2)(\sum_{cyc} a^2b^2) - a^2b^2c^2} \overset{via\ (\bullet),(\bullet\bullet\bullet),(\bullet\bullet\bullet\bullet)}{=} \frac{(s^2 - 8Rr - 2r^2)(s^3 - 12Rrs) + r^2s(4Rr + r^2)}{(s^2 - 8Rr - 2r^2)((4Rr + r^2)^2 - 2r^2s^2) - r^4s^2} \Rightarrow (*) \Leftrightarrow$$

$$\frac{4Rr + r^2}{3r^2s} \cdot \frac{(s^2 - 8Rr - 2r^2)(s^3 - 12Rrs) + r^2s(4Rr + r^2)}{(s^2 - 8Rr - 2r^2)((4Rr + r^2)^2 - 2r^2s^2) - r^4s^2} \ge \frac{3}{2} \cdot \frac{s^2}{(4Rr + r^2)^2}$$

$$\Leftrightarrow 2(4Rr + r^2)^3 . \left((s^2 - 8Rr - 2r^2)(s^2 - 12Rr) + r^2(4Rr + r^2)\right)$$
$$\ge 9r^2s^2\left((s^2 - 8Rr - 2r^2)((4Rr + r^2)^2 - 2r^2s^2) - r^4s^2\right)$$

$$\Leftrightarrow 18rs^6 + (128R^3 - 48R^2r - 192Rr^2 - 34r^3)s^4 - rs^2(2560R^4 + 1024R^3r - 192R^2r^2 - 128Rr^3 - 14r^4)$$
$$+ r^2\left(12288R^5 + 12800R^4r + 5120R^3r^2 + 960R^2r^3 + 80Rr^4 + 2r^5\right) \overset{(**)}{\ge} 0$$

and $\because 18r(s^2 - 16Rr + 5r^2)^3 \overset{Gerretsen}{\ge} 0 \therefore$ in order to prove $(**)$, it suffices to prove : LHS of $(**)$ $\ge 18r(s^2 - 16Rr + 5r^2)^3$

$$\Leftrightarrow (128R^3 - 48R^2r + 672Rr^2 - 304r^3)s^4 - rs^2(2560R^4 + 1024R^3r - 13632R^2r^2 + 8768Rr^3 - 1336r^4)$$
$$+r^2\left(12288R^5 + 12800R^4r + 78848R^3r^2 - 68160R^2r^3 + 21680Rr^4 - 2248r^5\right) \ge 0 \overset{(***)}{\ge} 0$$ *and*

$\because (128R^3 - 48R^2r + 672Rr^2 - 304r^3)(s^2 - 16Rr + 5r^2)^2 \overset{Gerretsen}{\ge} 0 \therefore$ in order to prove $(***)$, it suffices to prove : LHS of $(***)$ $\ge (128R^3 - 48R^2r + 672Rr^2 - 304r^3)(s^2 - 16Rr + 5r^2)^2$

$$\Leftrightarrow (1536R^4 - 3840R^3r + 8352R^2r^2 - 7680Rr^3 + 1704r^4)s^2 \overset{(****)}{\ge} r\left(20480R^5 - 45568R^4r + 104064R^3r^2 - 118384R^2r^3 + 43760Rr^4 - 5352r^5\right)$$

Now, $\because 1536R^4 - 3840R^3r + 8352R^2r^2 - 7680Rr^3 + 1704r^4$
$= (R - 2r)\left(1152R^3 + 384R^2(R - 2r) + 6816Rr^2 + 5952r^3\right) + 13608r^4 \overset{Euler}{\ge} 13608r^4 > 0 \therefore$ *LHS of* $(****) \overset{Rouche}{\ge}$

$$\left(1536R^4 - 3840R^3r + 8352R^2r^2 - 7680Rr^3 + 1704r^4\right)\left(2R^2 + 10Rr - r^2 - 2(R - 2r)\sqrt{R^2 - 2Rr}\right) \overset{?}{\ge} r\left(20480R^5 - 45568R^4r + 104064R^3r^2 - 118384R^2r^3 + 43760Rr^4 - 5352r^5\right)$$

$$\Leftrightarrow 3072R^6-12800R^5r+22336R^4r^2-32064R^3r^3+36640R^2r^4-19040Rr^5+3648r^6\overset{?}{\geq}2(R-2r)(1536R^4-3840R^3r+8352R^2r^2-7680Rr^3+1704r^4).\sqrt{R^2-2Rr}$$

$$\Leftrightarrow (R-2r)(3072R^5-6656R^4r+9024R^3r^2-14016R^2r^3+8608Rr^4-1824r^5)r^6\overset{?}{\geq}2(R-2r)(1536R^4-3840R^3r+8352R^2r^2-7680Rr^3+1704r^4).\sqrt{R^2-2Rr}$$

$$\overset{\because R-2r\geq 0\text{ via Euler}}{\Leftrightarrow} 3072R^5-6656R^4r+9024R^3r^2-14016R^2r^3+8608Rr^4-1824r^5\underbrace{\overset{?}{\geq}}_{(*****)}2(1536R^4-3840R^3r+8352R^2r^2-7680Rr^3+1704r^4).\sqrt{R^2-2Rr}$$

Again, $\because 3072R^5-6656R^4r+9024R^3r^2-14016R^2r^3+8608Rr^4-1824r^5=(R-2r)(2816R^4+256R^3(R-2r)+8000R^2r^2+1984Rr^3+12576r^4)+23328r^5\overset{\text{Euler}}{\geq}23328r^5>0\ \therefore (*****)\Leftrightarrow\left(3072R^5-6656R^4r+9024R^3r^2-14016R^2r^3+8608Rr^4-1824r^5\right)^2\overset{?}{\geq}4(R^2-2Rr)(1536R^4-3840R^3r+8352R^2r^2-7680Rr^3+1704r^4)^2$

$$\Leftrightarrow 25165824t^9-156237824t^8+467927040t^7-916881408t^6+1258520576t^5-1104685056t^4+530042880t^3-95773952t^2-8173056t+3326976\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\Big((t-2)\big((t-2).P(t)+7635057408\big)+2885580288\Big)+544195584\overset{?}{\geq}0\rightarrow\text{true}$$

$\because t-2\overset{\text{Euler}}{\geq}0$ and

$P(t)=22544384t^6+2621440t^5(t-2)+134479872t^4+154238976t^3+528252928t^2+1289803776t+3163742208\overset{\text{Euler}}{>}0\Rightarrow(*****)\Rightarrow(****)\Rightarrow(***)\Rightarrow(**)$

$\Rightarrow (*)$ is true

$$\Rightarrow \forall\, a,b,c>0\ \Big|\ \frac{1}{a}+\frac{1}{b}+\frac{1}{c}=3,\ \frac{a}{b^2+c^2}+\frac{b}{c^2+a^2}+\frac{c}{a^2+b^2}\geq\frac{3}{2}\left(\frac{a+b+c}{ab+bc+ca}\right)^2\ (\text{QED})$$

1008. Prove that :

$$\frac{(x^2+y^2+z^2-xy-yz-zx)^3}{(x-y)^2(y-z)^2(z-x)^2}\geq\frac{27}{4},\quad \forall x\neq y\neq z\neq x\in\mathbb{R}.$$

Proposed by Neculai Stanciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

The inequality is symmetrical, WLOG, we may assume that $x>y>z$.

Let $a=x-y>0,\ b=y-z>0.$ *We have :* $x-z=a+b$.

We have : $2(x^2+y^2+z^2-xy-yz-zx)=(x-y)^2+(y-z)^2+(x-z)^2=$

$$=a^2+b^2+(a+b)^2\overset{CBS}{\geq}\frac{(a+b)^2}{2}+(a+b)^2=\frac{3(a+b)^2}{2}$$

Then :

$$(x^2+y^2+z^2-xy-yz-zx)^3=\frac{27(a+b)^6}{64}\overset{AM-GM}{\geq}\frac{27(a+b)^2.(2\sqrt{ab})^4}{64}$$
$$=\frac{27}{4}.a^2b^2(a+b)^2.$$

$$\textit{Therefore},\ \frac{(x^2+y^2+z^2-xy-yz-zx)^3}{(x-y)^2(y-z)^2(z-x)^2}\geq\frac{27}{4}.$$

1009. Let $a,b,c\geq 0: a+b+c=3$. Prove that :

$$\frac{a+1}{\sqrt{a^2+3}+4bc}+\frac{b+1}{\sqrt{b^2+3}+4ca}+\frac{c+1}{\sqrt{c^2+3}+3ab}\geq 1.$$

Proposed by Phan Ngoc Chau-Vietnam

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\frac{a+1}{\sqrt{a^2+3}+4bc}=\frac{(a+1)^2}{(a+1)\sqrt{a^2+3}+4bc(a+1)}\overset{AM-GM}{\geq}\frac{(a+1)^2}{\frac{(a+1)^2+(a^2+3)}{2}+4bc(a+1)}$$

$$\textit{Then}:\ \frac{a+1}{\sqrt{a^2+3}+4bc}\geq\frac{(a+1)^2}{a^2+a+2+4bc(a+1)}\quad(\textit{and analogs})$$

Thus,

$$\sum_{cyc}\frac{a+1}{\sqrt{a^2+3}+4bc}\geq\sum_{cyc}\frac{(a+1)^2}{a^2+a+2+4bc(a+1)}\overset{CBS}{\geq}\frac{\left(\sum_{cyc}a+3\right)^2}{\left(\sum_{cyc}a\right)^2+\sum_{cyc}a+6+12abc+2\sum_{cyc}bc}$$

$$\textit{Since}:\ ab+bc+ca\leq\frac{(a+b+c)^2}{3}=3\ \textit{and}$$

$$abc\leq\left(\frac{a+b+c}{3}\right)^3=1\ \textit{then we get}:$$

$$\sum_{cyc}\frac{a+1}{\sqrt{a^2+3}+4bc}\geq\frac{(3+3)^2}{3^2+3+6+12.1+2.3}=1.$$

$$\textit{Equality holds iff } a=b=c=1.$$

Solution 2 by Michael Serghiou-Greece

$$\sum_{cyc}\frac{a+1}{\sqrt{a^2+3}+4bc}\geq 1;(1)$$

If one, or two of $a,b,c=0$ it is easy to prove (1).

Let $(p,q,r)=(\sum a,\sum ab,\prod a), p=3$ and $q\le 3, r\le 1$

First we have: $\sqrt{a^2+3}=\frac{1}{2}\sqrt{(a^2+3)4}\le\frac{1}{4}(a^2+7)$, so (1) becomes the stronger

$$\sum_{cyc}\frac{a+1}{\frac{1}{4}(a^2+7)+4bc}\ge 1;(2)$$

Now, we exploit the convexity of $f(t)=\frac{1}{t}$ on $(0,\infty)$ $\left(f''(t)=\frac{2}{t^3}>0\right)$ and the generalized Jensen's inequality, we have:

$$\sum_{i=1}^{3}f(\lambda_i x_i)\ge\left(\sum_{i=1}^{3}\lambda_i\right)f\left(\frac{\sum_{i=1}^{3}\lambda_i x_i}{\sum_{i=1}^{3}\lambda_i x_i}\right)\text{ with }\lambda_i\in\{a+1,b+1,c+1\}, i=1,2,3$$

We have the stronger inequality:

$$LHS_{(2)}\ge\frac{36}{\sum(a+1)\left[\frac{1}{4}(a^2+7)+4bc\right]}\ge 1$$

After some computation we get:

$$36\ge\left[\frac{1}{4}(78-11q+3r)+12r+4q\right]\text{ which reduces to:}$$

$\frac{1}{4}[5(3-q)+51(1-r)]\ge 0$ and obviously holds. We are done!

Equality holds for $q=3, r=1, p=3\Leftrightarrow a=b=c=1$.

1010. For $a_i>0, i\in\overline{1,n}$, prove that:

$$\sum_{i=1}^{n}\frac{a_i}{(a_1+a_2+\dots+a_n)-a_i}\le\sum_{i=1}^{n}\frac{a_i^2}{(a_1^2+a_2^2+\dots+a_n^2)-a_i^2}$$

Proposed by Hikmat Mammadov-Azerbaijan

Solution by Soumitra Mandal-Chandar Nagore-India

Let's prove that:

$$\sum_{cyc}\frac{x^2}{y^2+z^2}\ge\sum_{cyc}\frac{x}{y+z}\Leftrightarrow\sum_{cyc}x\left(\frac{x}{y^2+z^2}-\frac{1}{y+z}\right)\ge 0\Leftrightarrow$$

$$\sum_{cyc}x\cdot\frac{y(x-y)+z(x-z)}{(y+z)(y^2+z^2)}\ge 0\Leftrightarrow$$

$$\sum_{cyc}\frac{xy(x-y)}{(y+z)(y^2+z^2)}+\sum_{cyc}\frac{z(x-z)}{(y+z)(y^2+z^2)}\ge 0 \Leftrightarrow$$

$$\sum_{cyc}\frac{xy(x-y)}{(y+z)(y^2+z^2)}+\sum_{cyc}\frac{yx(y-x)}{(y+z)(x^2+z^2)}\ge 0 \Leftrightarrow$$

$$\sum_{cyc}xy(x-y)\left\{\frac{1}{(y+z)(y^2+z^2)}-\frac{1}{(x+z)(x^2+z^2)}\right\}\ge 0 \Leftrightarrow$$

$$\sum_{cyc}xy(x-y)\cdot\frac{(z+x)(z^2+x^2)-(y+z)(y^2+z^2)}{(y+z)(z+x)(y^2+z^2)(x^2+z^2)}\ge 0 \Leftrightarrow$$

$$\sum_{cyc}xy(x-y)\cdot\frac{x^3-y^3+z(x^2-y^2)+z^2(x-y)}{(y+z)(z+x)(y^2+z^2)(z^2+x^2)}\ge 0 \Leftrightarrow$$

$$(x^2+y^2+z^2+xy+yz+zx)\sum_{cyc}\frac{xy(x-y)^2}{(y+z)(z+x)(y^2+z^2)(z^2+x^2)}\ge 0$$

which is true. So, is being established:

$$\sum_{cyc}\frac{x^2}{y^2+z^2}\ge\sum_{cyc}\frac{x}{y+z}$$

Similarly, applying the same logic, we have:

$$\sum_{i=1}^{n}\frac{a_1^2}{a_2^2+a_3^2+\cdots+a_n^2}\ge\sum_{i=1}^{n}\frac{a_1}{a_2+a_3+\cdots+a_n}$$

Therefore,

$$\sum_{i=1}^{n}\frac{a_i}{(a_1+a_2+\cdots+a_n)-a_i}\le\sum_{i=1}^{n}\frac{a_i^2}{(a_1^2+a_2^2+\cdots+a_n^2)-a_i^2}$$

1011. **If $0\le x,y,z\le\frac{\pi}{4}$ then :**

$$2\cos^2 x.\cos^2 y.\cos^2 z\le 1+\cos 2x.\cos 2y.\cos 2z\le 8\cos^2 x.\cos^2 y.\cos^2 z$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$2\cos^2 x.\cos^2 y.\cos^2 z\overset{(1)}{\lesssim}1+\cos 2x.\cos 2y.\cos 2z\overset{(2)}{\lesssim}8\cos^2 x.\cos^2 y.\cos^2 z.$$

$$Since\ \ 0\le x,y,z\le\frac{\pi}{4}\ then\ we\ have:\ \frac{1}{2}\le\cos^2 x,\cos^2 y,\cos^2 z\le 1.$$

$$We\ have:\ \ 1+\cos 2x.\cos 2y.\cos 2z$$
$$=1+(2\cos^2 x-1)(2\cos^2 y-1)(2\cos^2 z-1)=$$

$$=8\cos^2 x.\cos^2 y.\cos^2 z-4(\cos^2 x.\cos^2 y+\cos^2 y.\cos^2 z+\cos^2 z.\cos^2 x)$$
$$+2(\cos^2 x+\cos^2 y+\cos^2 z).$$

$$Now\ we\ have:$$

$$(1)\Leftrightarrow 3\cos^2 x.\cos^2 y.\cos^2 z-2(\cos^2 x.\cos^2 y+\cos^2 y.\cos^2 z+\cos^2 z.\cos^2 x)$$
$$+(\cos^2 x+\cos^2 y+\cos^2 z)\ge 0$$

$$\Leftrightarrow \cos^2 x\,(1-\cos^2 y)(1-\cos^2 z)+\cos^2 y\,(1-\cos^2 z)(1-\cos^2 x)$$
$$+\cos^2 z\,(1-\cos^2 x)(1-\cos^2 y)\ge 0$$

$$Which\ is\ true.\ \ Equality\ holds\ iff\ (x,y,z)=(0,0,z)\ \ and\ their\ permutation.$$

$$Also\ we\ have:$$

$$(2)\Leftrightarrow -2(\cos^2 x.\cos^2 y+\cos^2 y.\cos^2 z+\cos^2 z.\cos^2 x)$$
$$+(\cos^2 x+\cos^2 y+\cos^2 z)\le 0$$

$$\Leftrightarrow -2\cos^2 x.\left(\cos^2 y-\frac{1}{2}\right)-2\cos^2 y.\left(\cos^2 z-\frac{1}{2}\right)-2\cos^2 z.\left(\cos^2 x-\frac{1}{2}\right)\le 0$$

$$Which\ is\ true\ because\ \ \cos^2 x,\cos^2 y,\cos^2 z\ge\frac{1}{2}.$$

$$Equality\ holds\ iff\ x=y=z=\frac{\pi}{4}.$$

$$Therefore,\ \ 2\cos^2 x.\cos^2 y.\cos^2 z\le 1+\cos 2x.\cos 2y.\cos 2z$$
$$\le 8\cos^2 x.\cos^2 y.\cos^2 z.$$

Solution 2 by Tapas Das-India

$$2\cos^2 x\cos^2 y\cos^2 z=\frac{1}{4}(2\cos^2 x\cdot 2\cos^2 y\cdot 2\cos^2 z)=$$

$$=\frac{1}{4}(1+\cos 2x)(1+\cos 2y)(1+\cos 2z)=\frac{1}{4}(1+a)(1+b)(1+c)$$

$$a=\cos 2x\,,b=\cos 2y\,,c=\cos 2z\Rightarrow a,b,c\in(0,1)$$

$$0 \le x \le \frac{\pi}{4} \Rightarrow 0 \le 2x \le \frac{\pi}{2}$$

$$(a-1)(b-1) \ge 0 \Rightarrow ab+1 \ge a+b$$

$$2(ab+1) \ge a+b+ab+1 = (a+1)(b+1); (1)$$

$$2(ab+1)\cdot 2(c+1) \ge 2(a+1)(b+1)(c+1)$$

$$4(ab+1)(c+1) \ge 2(a+1)(b+1)(c+1)$$

$$2(ab+1)(c+1) \ge (a+1)(b+1)(c+1)$$

$$(ab+1)(c+1) \ge \frac{1}{2}(a+1)(b+1)(c+1); (2)$$

From (1) and (2), we get :

$$2(abc+1) \ge \frac{1}{2}(a+1)(b+1)(c+1)$$

$$abc+1 \ge \frac{1}{4}(a+1)(b+1)(c+1)$$

$$1+\cos 2x \cos 2y \cos 2z \ge \frac{1}{2}(1+\cos 2x)(1+\cos 2y)(1+\cos 2z)$$

$$1+\cos 2x \cos 2y \cos 2z \ge \frac{1}{4}\cdot 2\cos 2x \cos 2y \cos 2z =$$

$$= 2\cos^2 x \cos^2 y \cos^2 z$$

$$8\cos^2 x \cos^2 y \cos^2 z = (1+\cos 2x)(1+\cos 2y)(1+\cos 2z) =$$

$$= 1+a+b+c+ab+bc+ca \ge 1+abc =$$

$$= 1+\cos 2x \cos 2y \cos 2z$$

Solution 3 by Fayssal Abdelli-Bejaia-Algerie

$$0 \le x, y, z \le \frac{\pi}{4}$$

$$2\cos^2 x \cos^2 y \cos^2 z \le 1+\cos 2x \cos 2y \cos 2z \le 8\cos^2 x \cos^2 y \cos^2 z$$

$$2\cos^2 x \cos^2 y \cos^2 z \le 1+\cos 2x \cos 2y \cos 2z\,; (A)$$

$$2\cos^2 x\cos^2 y\cos^2 z \le 1+(1-2\sin^2 x)(1-2\sin^2 y)(1-2\sin^2 z)$$

$$2(1-\sin^2 x)(1-\sin^2 y)(1-\sin^2 z) \le$$

$$\le 1+(1-2\sin^2 x-2\sin^2 y+4\sin^2 x\sin^2 y)(1-2\sin^2 z)$$

$$\sin^2 x\sin^2 y+\sin^2 y\sin^2 z+\sin^2 z\sin^2 x-3\sin^2 x\sin^2 y\sin^2 z\ge 0; (B)$$

But: $\sin^2 x\sin^2 y\ge\sin^2 x\sin^2 y\sin^2 z$, $\sin^2 z\sin^2 y\ge\sin^2 x\sin^2 y\sin^2 z$,

$\sin^2 x\sin^2 z\ge\sin^2 x\sin^2 y\sin^2 z$, because $\sin^2 x, \sin^2 y, \sin^2 z\in(0,1)\Rightarrow$

$\Rightarrow (B)$ is true$\Rightarrow (A)$ is true.

$$1+\cos 2x\cos 2y\cos 2z\le 8\cos^2 x\cos^2 y\cos^2 z; (C)$$

$$1+(2\cos^2 x-1)(2\cos^2 y-1)(2\cos^2 z-1)\le 8\cos^2 x\cos^2 y\cos^2 z$$

$$2\cos^2 x+2\cos^2 y+2\cos^2 z-4\cos^2 x\cos^2 y-4\cos^2 y\cos^2 z-\cos^2 z\cos^2 x\le 0$$

$$2\cos^2 y(1-2\cos^2 z)+2\cos^2 x(1-2\cos^2 y)+2\cos^2 z(1-\cos^2 x)$$

But $0\le x\le\frac{\pi}{4}\Rightarrow\frac{\sqrt{2}}{2}\le\cos x\le 1$

$\frac{1}{2}\le\cos^2 x\le 1\Leftrightarrow -1\le 1-2\cos^2 x\le 0$ (and analogs) $\Rightarrow (C)$ is true.

1012. If $a,b,c>0, abc=1$, then:

$$\sum_{cyc}\frac{(a+b+1)^4}{a^5+b^5+1}\le 81$$

Proposed by Marin Chirciu-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

Via repeated Chebyshev, $a^5+b^5+1\ge\frac{1}{3^4}(a+b+1)^5\Rightarrow\frac{(a+b+1)^4}{a^5+b^5+1}$

$$\le\frac{81}{a+b+1}\text{ and analogs}\Rightarrow\sum_{cyc}\frac{(a+b+1)^4}{a^5+b^5+1}\le 81\sum_{cyc}\frac{1}{a+b+1}\overset{?}{\le}81$$

$$\Leftrightarrow (a+b+1)(b+c+1)(c+a+1)\overset{?}{\ge}(a+b+1)(b+c+1)+(b+c+1)(c+a+1)$$

$$+(c+a+1)(a+b+1)\Leftrightarrow\sum_{cyc}a^2b+\sum_{cyc}ab^2+2abc-2-2(a+b+c)\overset{?}{\ge}0$$

$$\overset{\because abc=1}{\Leftrightarrow}\sum_{cyc}\left(ab\left(\sum_{cyc}a-c\right)\right)\overset{?}{\geq}2\sum_{cyc}a$$

$$\Leftrightarrow\frac{2}{3}\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)+\frac{1}{3}\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)\underset{(*)}{\overset{?}{\geq}}2\sum_{cyc}a+3abc$$

$$\text{Now, via } A-G, \frac{2}{3}\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)\geq\frac{2}{3}\left(\sum_{cyc}a\right).3\sqrt[3]{a^2b^2c^2}\overset{\because abc=1}{=}2\sum_{cyc}a$$

$$\therefore\frac{2}{3}\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)\overset{(i)}{\geq}2\sum_{cyc}a$$

$$\text{Also, via } A-G, \frac{1}{3}\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)\geq\frac{1}{3}.3\sqrt[3]{abc}.3\sqrt[3]{a^2b^2c^2}=3abc$$

$$\therefore\frac{1}{3}\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)\overset{(ii)}{\geq}3abc\ \therefore (i)+(ii)\Rightarrow(*)\text{ is true}$$

$$\Rightarrow\forall a,b,c>0\ \middle|\ abc=1,\sum_{cyc}\frac{(a+b+1)^4}{a^5+b^5+1}\leq 81\text{ with equality iff } a=b=c=1$$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\sum_{cyc}\frac{(a+b+1)^4}{a^5+b^5+1}\leq 81,\qquad 3^3\sum_{cyc}\frac{a^4+b^4+1}{a^5+b^5+1}\leq 81$$

$$3\cdot 3^3\sum_{cyc}\frac{a^4+b^4+1}{(a^4+b^4+1)(a+b+1)}\leq 81,\qquad \sum_{cyc}\frac{1}{a+b+1}\leq 1$$

$$(a+b+1)(b+c+1)+(b+c+1)(c+a+1)+(c+a+1)(a+b+1)\leq$$
$$\leq(a+b+1)(b+c+1)(c+a+1)$$
$$a^2+b^2+c^2+3(ab+bc+ca)+4(a+b+c)+3\leq$$
$$\leq a^2+b^2+c^2+a^2b+b^2c+c^2a+a^2c+c^2b+b^2a+3(ab+bc+ca)+$$
$$+2(a+b+c)+2abc+1$$

$$2(a+b+c)\leq\frac{a}{c}+\frac{c}{b}+\frac{b}{a}+\frac{b}{c}+\frac{c}{a}\text{ which is true, because}$$

$$2\left(\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}\right)\leq\frac{x^4}{yz}+\frac{y^4}{zx}+\frac{z^4}{xy}+\frac{x^2y^2}{z^2}+\frac{y^2z^2}{x^2}+\frac{z^2x^2}{y^2}$$

$$\frac{x^2}{y^2}+\frac{y^2}{z^2}+\frac{z^2}{x^2}\leq\frac{x^3}{y^3}+\frac{y^3}{z^3}+\frac{z^3}{x^3}$$

1013. If $n\in\mathbb{N}, n\ge 3$ and $m, x_k\in[1,\infty), k=\overline{1,n}, m=\frac{x_1+x_2+\dots+x_n}{n}$, then:

$$\sum_{k=1}^{n}\left(x_1^{x_k}+x_2^{x_k}+\dots+x_n^{x_k}\right)\ge n^2\cdot m^m$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

Solution by Tapas Das-India

$$\sum_{k=1}^{n}\left(x_1^{x_k}+x_2^{x_k}+\dots+x_n^{x_k}\right)\ge n\sum_{k=1}^{n}\sqrt[n]{x_1^{x_k}\cdot x_2^{x_k}\cdot\ldots\cdot x_n^{x_k}}=$$

$$=n\sum_{k=1}^{n}x_k^{\frac{x_1+x_2+\dots+x_n}{n}}=n\sum_{k=1}^{n}x_k^m\ge n\cdot n\left(\frac{x_1+x_2+\dots+x_n}{n}\right)^m=n^2\cdot m^m$$

1014. If $x,y,z>0$ then:

$$\frac{(x+y)^4}{x^4+14x^2y^2+y^4}+\frac{(y+z)^4}{y^4+14y^2z^2+z^4}+\frac{(z+x)^4}{z^4+14z^2x^2+x^4}\ge 3$$

Proposed by Marin Chirciu-Romania

Solution 1 by Fayssal Abdelli-Bejaia-Algerie

$$\frac{(x+y)^4}{x^4+14x^2y^2+y^4}\overset{(?)}{\ge}1\Leftrightarrow x^4+4x^3y+6x^2y^2+4xy^3+y^4\ge x^4+14x^2y^2+y^4$$

$$4x^3y+4xy^3-8x^2y^2\ge 0\Leftrightarrow 4xy(x^2-2xy+y^2)\ge 0\Leftrightarrow$$

$$4xy(x-y)^2\ge 0 \text{ true.}$$

Similarly, we have:

$$\frac{(y+z)^4}{y^4+14y^2z^2+z^4}\ge 1 \text{ and } \frac{(z+x)^4}{z^4+14z^2x^2+x^4}\ge 1$$

Hence,

$$\frac{(x+y)^4}{x^4+14x^2y^2+y^4}+\frac{(y+z)^4}{y^4+14y^2z^2+z^4}+\frac{(z+x)^4}{z^4+14z^2x^2+x^4}\ge 3$$

Solution 2 by Tapas Das-India

$$(x+y)^4=x^4+4x^3y+6x^2y^2+4xy^3+y^4\ge$$

$$\ge x^4+y^4+6x^2y^2+4xy\cdot 2\sqrt{x^2y^2}=x^4+14x^2y^2+y^4$$

Hence,

$$\frac{(x+y)^4}{x^4+14x^2y^2+y^4} \geq 1$$

Similarly, we have:

$$\frac{(y+z)^4}{y^4+14y^2z^2+z^4} \geq 1 \text{ and } \frac{(z+x)^4}{z^4+14z^2x^2+x^4} \geq 1$$

Hence,

$$\frac{(x+y)^4}{x^4+14x^2y^2+y^4}+\frac{(y+z)^4}{y^4+14y^2z^2+z^4}+\frac{(z+x)^4}{z^4+14z^2x^2+x^4} \geq 3$$

1015. If $a, b > e$ then:

$$(a+b)^{2\sqrt{ab}} \leq 4^{\sqrt{ab}} \cdot \left(\sqrt{ab}\right)^{a+b}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Florentin Vișescu-Romania

$$(a+b)^{2\sqrt{ab}} \leq 4^{\sqrt{ab}} \cdot \left(\sqrt{ab}\right)^{a+b}, \quad \left(\frac{a+b}{2}\right)^{2\sqrt{ab}} \leq \left(\sqrt{ab}\right)^{a+b}$$

$$\log\left(\frac{a+b}{2}\right)^{2\sqrt{ab}} \leq \log\left(\sqrt{ab}\right)^{a+b}, \quad 2\sqrt{ab}\log\left(\frac{a+b}{2}\right) \leq (a+b)\log\left(\sqrt{ab}\right)$$

$$\frac{\log\left(\frac{a+b}{2}\right)}{\frac{a+b}{2}} \leq \frac{\log\left(\sqrt{ab}\right)}{\sqrt{ab}}$$

Let $f: [(0,\infty) \to \mathbb{R}, f(x) = \frac{\log x}{x}, f'(x) = \frac{1-\log x}{x^2}$

$$f'(x) = 0 \Leftrightarrow x = e$$

x	0 $\qquad$ e $\qquad$ ∞
$f'(x)$	$+ + + + + + +0 - - - - - - - - - -$
$f(x)$	$\nearrow$ $\frac{1}{e}$ $\searrow$

If $e < \sqrt{ab} \leq \frac{a+b}{2}, f \searrow$ **then** $\frac{\log\left(\sqrt{ab}\right)}{\sqrt{ab}} \geq \frac{\log\left(\frac{a+b}{2}\right)}{\frac{a+b}{2}}$

If $\sqrt{ab} \leq \frac{a+b}{2} \leq e, f \nearrow$ **then** $\frac{\log\left(\sqrt{ab}\right)}{\sqrt{ab}} \leq \frac{\log\left(\frac{a+b}{2}\right)}{\frac{a+b}{2}}$

Solution 2 by Ravi Prakash-New Delhi-India

$$(a+b)^{2\sqrt{ab}} \le 4^{\sqrt{ab}}\left(\sqrt{ab}\right)^{a+b}, \qquad (a+b)^{2\sqrt{ab}} \le 2^{2\sqrt{ab}}(ab)^{\frac{a+b}{2}}$$

$$\left(\frac{a+b}{2}\right)^{2\sqrt{ab}} \le (ab)^{\frac{a+b}{2}}, \qquad \left(\frac{a+b}{2}\right)^{\frac{2}{a+b}} \le \left(\sqrt{ab}\right)^{\frac{1}{\sqrt{ab}}}; (1)$$

$$\text{Let } f(x) = x^{\frac{1}{x}}, x \ge e \Rightarrow \log f(x) = \frac{1}{x}\log x$$

$$\frac{f'(x)}{f(x)} = \frac{1}{x^2}(1-\log x) < 0, \forall x > e$$

$$f \text{ -is a decreasing function on } [e,\infty)$$

$$\text{As } a,b \ge e, \frac{a+b}{2} \ge \sqrt{ab} \ge e \Rightarrow f\left(\frac{a+b}{2}\right) \le f\left(\sqrt{ab}\right)$$

$$(a+b)^{2\sqrt{ab}} \le 4^{\sqrt{ab}} \cdot \left(\sqrt{ab}\right)^{a+b}$$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$(a+b)^{2\sqrt{ab}} \le 4^{\sqrt{ab}}.\left(\sqrt{ab}\right)^{a+b} \Leftrightarrow 2\sqrt{ab}.\ln(a+b) \le \sqrt{ab}.\ln 4 + (a+b).\ln\left(\sqrt{ab}\right)$$

$$\Leftrightarrow \sqrt{ab}(2\ln(a+b) - \ln 4) \le (a+b).\ln\left(\sqrt{ab}\right)$$

$$\Leftrightarrow \sqrt{ab}(2\ln(a+b) - 2\ln 2) \le (a+b).\ln\left(\sqrt{ab}\right) \Leftrightarrow 2.\sqrt{ab}.\ln\frac{a+b}{2} \le (a+b).\ln\left(\sqrt{ab}\right)$$

$$\Leftrightarrow \frac{\ln\frac{a+b}{2}}{\frac{a+b}{2}} \le \frac{\ln\left(\sqrt{ab}\right)}{\sqrt{ab}} \Leftrightarrow \frac{\ln\frac{a+b}{2}}{\frac{a+b}{2}} - \frac{\ln\left(\sqrt{ab}\right)}{\sqrt{ab}} \le 0$$

$$\Leftrightarrow f\left(\frac{a+b}{2}\right) - f\left(\sqrt{ab}\right) \overset{(*)}{\le} 0, \text{ where } f(x) = \frac{\ln x}{x}$$

$$\text{Now, via MVT and} \because \frac{a+b}{2} \overset{A-G}{\ge} \sqrt{ab} \therefore f\left(\frac{a+b}{2}\right) - f\left(\sqrt{ab}\right)$$

$$= \left(\frac{a+b}{2} - \sqrt{ab}\right).f'(\xi) \left(\text{where } \sqrt{ab} < \xi < \frac{a+b}{2}\right)$$

$$\Rightarrow f\left(\frac{a+b}{2}\right) - f\left(\sqrt{ab}\right) \overset{(i)}{=} \frac{\left(\sqrt{a}-\sqrt{b}\right)^2}{2}.\frac{1-\ln\xi}{\xi^2}$$

$$\text{Now, } \xi > \sqrt{ab} \overset{a,b>e}{>} \sqrt{e.e} \Rightarrow \xi > e \Rightarrow \ln\xi > 1 \Rightarrow 1 - \ln\xi < 0 \Rightarrow \frac{1-\ln\xi}{\xi^2} < 0$$

$$\Rightarrow \frac{\left(\sqrt{a}-\sqrt{b}\right)^2}{2}.\frac{1-\ln\xi}{\xi^2} \le 0 \overset{\text{via (i)}}{\Rightarrow} f\left(\frac{a+b}{2}\right) - f\left(\sqrt{ab}\right) \le 0 \Rightarrow (*) \text{ is true}$$

$$\Rightarrow \forall\ a,b > e, (a+b)^{2\sqrt{ab}} \le 4^{\sqrt{ab}}.\left(\sqrt{ab}\right)^{a+b}, \text{equality iff } a = b \ (\text{QED})$$

Solution 4 by Sanong Huayrerai-Nakon Pathom-Thailand

For $x \ge y \ge 2e$ let $x = yk, k \ge 1$. Hence,

$$(a+b)^{2\sqrt{ab}} \le 4^{\sqrt{ab}} \cdot \left(\sqrt{ab}\right)^{a+b}$$

$$(a+b)^{2\sqrt{ab}} \le 2^{2\sqrt{ab}}\left(\frac{2\sqrt{ab}}{2}\right)^{a+b}, \quad x^y \le 2^y\left(\frac{y}{2}\right)^x$$

$$(yk)^y \le 2^y\left(\frac{y}{2}\right)^x, \quad y^y k^y \cdot 2^{yk} \le 2^y y^{yk}$$

$$k^y \le \left(\frac{y}{2}\right)^{y(k-1)}$$

$k \le e^{k-1}$ which is true from $e^{k-1} \ge (k-1)+1 = k$.

Solution 5 by Said Cerbach-Algiers-Algerie

Let $f: [(0,\infty) \to \mathbb{R}, f(x) = \frac{\log x}{x}, f'(x) = \frac{1-\log x}{x^2}$

If $e < \sqrt{ab} \le \frac{a+b}{2}, f \searrow$ then $\frac{\log(\sqrt{ab})}{\sqrt{ab}} \ge \frac{\log\left(\frac{a+b}{2}\right)}{\frac{a+b}{2}}$

$$2\sqrt{ab}\log\left(\frac{a+b}{2}\right) \le (a+b)\log\left(\sqrt{ab}\right), \qquad \log\left(\frac{a+b}{2}\right)^{2\sqrt{ab}} \le \log\left(\sqrt{ab}\right)^{a+b}$$

$$\left(\frac{a+b}{2}\right)^{2\sqrt{ab}} \le \left(\sqrt{ab}\right)^{a+b}, \qquad (a+b)^{2\sqrt{ab}} \le 4^{\sqrt{ab}} \cdot \left(\sqrt{ab}\right)^{a+b}$$

1016. If $x, y, z > 0$ and $\frac{1}{xy} + \frac{1}{yz} + \frac{1}{zx} = 1$ then prove that:

$$(x-1)(y-1)(z-1) \le 6\sqrt{3} - 10$$

Proposed by Hikmat Mammadov-Azerbaijan

Solution 1 by Tapas Das-India

$$\frac{1}{xy} + \frac{1}{yz} + \frac{1}{zx} = 1 \Leftrightarrow x + y + z = xyz; (1)$$

Since $x < xyz$ we have $yz > 1$ and similarly, $zx > 1, xy > 1$.

Let $x \le 1, y \ge 1, z \ge 1$ then $(x-1)(y-1)(z-1) \le 0$.

Let $x \ge 1, y \ge 1, z \ge 1$ and $x - 1 = a; y - 1 = b; z - 1 = c; a, b, c > 0$ then:

$a+1+b+1+c+1=(a+1)(b+1)(c+1)$ and using (1), we obtain:

$$abc+ab+bc+ca=2;\quad (2)$$

Let $t=\sqrt[3]{abc}\Rightarrow ab+bc+ca\geq 3t^2;(3)$

From (2) and (3): $t^3+3t^2\leq abc+ab+bc+ca=2$

$(x+1)(x^2+2x-2)\leq 0$. So, we must have: $x^2+2x-2\leq 0\Rightarrow$

$$(x+1)^2\leq 3\Rightarrow x\leq\sqrt{3}-1$$

$$x^3\leq\left(\sqrt{3}-1\right)^3=6\sqrt{3}-10$$

Solution 2 by Soumitra Mandal-Chandar-Nagore-India

Let $x-1=a; y-1=b; z-1=c\Rightarrow x=a+1; y=b+1; z=c+1$

$$\because x,y,z>0\Rightarrow a+1,b+1,c+1>0$$

$$\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}=1\Rightarrow\sum_{cyc}\frac{1}{(a+1)(b+1)}=1\Rightarrow$$

$$3+a+b+c=(1+a)(1+b)(1+c)$$

$$2=ab+bc+ca+abc\Rightarrow 2\geq 3\sqrt[3]{(abc)^2}+abc$$

$2-3m^2-m^3\geq 0$, where $m^3=abc$

$$3(1-m^2)-(1+m^3)\geq 0\Rightarrow(1+m)(2-2m-m^2)\geq 0$$

$2-2m-m^2\geq 0$, where $1+m\neq 0$ and $1+m>0$

$$\Rightarrow 3\geq(1+m)^2\Rightarrow\sqrt{3}-1\geq m\Rightarrow\sqrt{3}-1\geq\sqrt[3]{abc}\Rightarrow\left(\sqrt{3}-1\right)^3\geq abc$$

$$3\sqrt{3}-1-3\cdot\left(\sqrt{3}\right)^2\cdot 1+3\cdot\sqrt{3}\cdot 1\geq abc\Rightarrow 6\sqrt{3}-10\geq abc$$

$$(x-1)(y-1)(z-1)\leq 6\sqrt{3}-10$$

Equality holds for $x=y=z=\sqrt{3}$.

Solution 3 by Soumava Chakraborty-Kolkata-India

$$\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}=1\Rightarrow\sum_{cyc}x\overset{(i)}{=}xyz\text{ and }(x-1)(y-1)(z-1)\leq 6\sqrt{3}-10$$

$$\Leftrightarrow xyz-1-\sum_{cyc}xy+\sum_{cyc}x\leq 6\sqrt{3}-10\overset{\text{via (i)}}{\Leftrightarrow}2\sum_{cyc}x-\sum_{cyc}xy\leq 6\sqrt{3}-9$$

$$\Leftrightarrow\left(\sum_{cyc}xy\right)\sqrt{\frac{\sum_{cyc}x}{xyz}}-2\sum_{cyc}x+(6\sqrt{3}-9)\sqrt{\frac{xyz}{\sum_{cyc}x}}\ge 0\left(\because (i)\Rightarrow 1=\sqrt{\frac{\sum_{cyc}x}{xyz}}=\sqrt{\frac{xyz}{\sum_{cyc}x}}\right)$$

$$\Leftrightarrow\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x\right)-2\left(\sum_{cyc}x\right)\sqrt{xyz\left(\sum_{cyc}x\right)}+(6\sqrt{3}-9)xyz\overset{(\bullet)}{\ge}0$$

Assigning $y+z=a, z+x=b, x+y=c\Rightarrow a+b-c=2z>0, b+c-a=2x>0$ and $c+a-b=2y>0\Rightarrow a+b>c, b+c>a, c+a>b$ $\Rightarrow a,b,c$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say) yielding $2\sum_{cyc}x=\sum a=2s\Rightarrow\sum_{cyc}x=s\Rightarrow x=s-a, y=s-b, z=s-c$

Via aforementioned substitutions, $\sum_{cyc}xy=\sum_{cyc}(s-a)(s-b)\overset{(ii)}{=}4Rr+r^2$ and $\sqrt{xyz\left(\sum_{cyc}x\right)}=\sqrt{(s-a)(s-b)(s-c).s}$

$$\Rightarrow\sqrt{xyz\left(\sum_{cyc}x\right)}\overset{(iii)}{=}rs$$

$\therefore$ via (ii), (iii), $(\bullet)\Leftrightarrow(4Rr+r^2)s-2s.rs+(6\sqrt{3}-9)r^2s\ge 0\Leftrightarrow 4R+r-2s+(6\sqrt{3}-9)r\ge 0\Leftrightarrow 2s\le 4R-8r+6\sqrt{3}r\Leftrightarrow s\le(3\sqrt{3}-4)r+2R\to$ true via Blundon

$$\Rightarrow(\bullet)\text{ is true}\therefore(x-1)(y-1)(z-1)\le 6\sqrt{3}-10\ \forall\, x,y,z>0\ \Big|\ \frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}=1\ (QED)$$

1017. If $a,b,c>0, a^3+b^3+c^3+d^3=1$ then:

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+d}+\frac{1}{d+a}\le\frac{1}{2abcd}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Tapas Das-India

$$\frac{a^3+b^3+c^3+d^3}{4}\ge\frac{a^2+b^2+c^2+d^2}{4}\cdot\frac{a+b+c+d}{4}$$

$$\frac{1}{4}\ge\frac{a+b+c+d}{4}\cdot\frac{4(a^2b^2c^2d^2)^{\frac{1}{4}}}{4}$$

$$\frac{1}{\sqrt{abcd}}\ge(a+b+c+d);\ \ (1)$$

$$\frac{1}{a+b}+\frac{1}{b+c}=\frac{a+b+c+d}{(a+b)(c+d)}\le\frac{(a+b+c+d)}{2\sqrt{ab}\cdot 2\sqrt{cd}}=\frac{a+b+c+d}{4\sqrt{abcd}}$$

$$\frac{1}{c+d}+\frac{1}{d+a}=\frac{a+b+c+d}{(c+d)(d+a)}\le\frac{a+b+c+d}{4\sqrt{abcd}}$$

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+d}+\frac{1}{d+a}\le 2\frac{a+b+c+d}{4\sqrt{abcd}}=\frac{1}{2}\frac{a+b+c+d}{\sqrt{abcd}}\le$$

$$\le\frac{1}{2}\cdot\frac{1}{\sqrt{abcd}}\cdot\frac{1}{\sqrt{abcd}}=\frac{1}{2abcd}$$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+d}+\frac{1}{d+a}\le\frac{1}{2abcd}$$

$$\frac{1}{4}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{b}+\frac{1}{c}+\frac{1}{c}+\frac{1}{d}+\frac{1}{d}+\frac{1}{a}\right)\le\frac{1}{2}\cdot\frac{a^3+b^3+c^3+d^3}{abcd}$$

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\le\frac{a^3+b^3+c^3+d^3}{abcd}$$

$abc+bcd+cda+dab\le a^3+b^3+c^3+d^3$ true, because:

$a^3+b^3+c^3\ge 3abc$(and analogs). Hence,

$$3(a^3+b^3+c^3+d^3)\ge 3(abc+bcd+cda+dab)\Leftrightarrow$$

$$abc+bcd+cda+dab\le a^3+b^3+c^3+d^3$$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$2abcd\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+d}+\frac{1}{d+a}\right)$$
$$=\frac{2ab}{a+b}.cd+\frac{2bc}{b+c}.da+\frac{2cd}{c+d}.ab+\frac{2da}{d+a}.bc\overset{H\le G}{\le}\sqrt{ab}.cd+\sqrt{bc}.da$$
$$+\sqrt{cd}.ab+\sqrt{da}.bc$$
$$=\sqrt{abcd}\left(\sqrt{cd}+\sqrt{da}+\sqrt{ab}+\sqrt{bc}\right)\overset{CBS}{\le}\sqrt{abcd}.\sqrt{c+d+a+b}.\sqrt{d+a+b+c}$$
$$\therefore 2abcd\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+d}+\frac{1}{d+a}\right)\overset{(*)}{\le}\sqrt{abcd}\left(\sum_{cyc}a\right)$$
$$\text{Again, }1=\sum_{cyc}a^3\overset{Chebyshev}{\ge}\frac{1}{4}\left(\sum_{cyc}a\right)\left(\sum_{cyc}a^2\right)\overset{Chebyshev}{\ge}\frac{1}{4}\left(\sum_{cyc}a\right).4\sqrt[4]{abcd}\Rightarrow 1$$
$$\ge\sqrt{abcd}\left(\sum_{cyc}a\right)\overset{via\ (*)}{\ge}2abcd\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+d}+\frac{1}{d+a}\right)$$

$$\Rightarrow \frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+d}+\frac{1}{d+a}\le\frac{1}{2abcd}\ \forall\, a,b,c,d>0\,\big|\, a^3+b^3+c^3+d^3=1\ (QED)$$

1018. Let $a,b,c>1$ be real numbers such that

$abc(a-1)(b-1)(c-1)=1$. Prove that :

$$a^6+b^6+c^6+9(a^4+b^4+c^4)+18(a^2+b^2+c^2)\ge$$
$$\ge 3(a^5+b^5+c^5)+13(a^3+b^3+c^3)+12(a+b+c)+57$$

Proposed by Kunihiko Chikaya-Tokyo-Japan

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

The given inequality is equivalent to :

$$\sum_{cyc}(a^6-3a^5+9a^4-13a^3+18a^2-12a+8)\ge 81 \Leftrightarrow \sum_{cyc}(a^2-a+2)^3\ge 81.$$

By AM – GM inequality we have :

$$\sum_{cyc}(a^2-a+2)^3=\sum_{cyc}[a(a-1)+1+1]^3\ge\sum_{cyc}27.a(a-1).1.1$$

$$\ge 27.3\sqrt[3]{abc(a-1)(b-1)(c-1)}=81.$$

So the proof is completed. Equality holds iff $a=b=c=\frac{1+\sqrt{5}}{2}$.

1019. Let $a,b,c>0:ab+bc+ca=3.$ Prove that:

$$\frac{1}{abc}-1\ge\frac{1}{24}.\sum_{cyc}\frac{(a-b)^2}{a^2-ab+b^2}$$

Proposed by Phan Ngoc Chau-Vietnam

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $p:=a+b+c,\ \ q:=ab+bc+ca=3,\ \ r:=abc\le 1.$

Since $(a-b)^2=(a^2-ab+b^2)-ab,$

then the desired inequality can be written as

$$\sum_{cyc}\frac{ab}{a^2-ab+b^2}+\frac{24}{abc}\ge 27.$$

By AM – GM inequality we have : $\frac{ab}{a^2-ab+b^2}+c(a^2-ab+b^2)+\frac{1}{abc}\ge 3.$

Summing up this inequality with similar ones we get :

$$\sum_{cyc}\frac{ab}{a^2-ab+b^2}+\frac{3}{abc}\geq 9-\sum_{cyc}c(a^2-ab+b^2)=9-pq+6r=9-3p+6r$$

$$So\ it\ suffices\ to\ prove\ that:\ \frac{21}{r}\geq 18+3p-6r.$$

$$From\ the\ known\ inequality\ (ab+bc+ca)^2\geq 3abc(a+b+c)\ we\ get:\ \frac{9}{r}\geq 3p$$

$$Therefore,\ \frac{21}{r}-(18+3p-6r)\geq\frac{21}{r}-\left(18+\frac{9}{r}-6r\right)=\frac{6(1-r)(2-r)}{r}\overset{r\leq 1}{\geq}0.$$

$$So\ the\ proof\ is\ completed.\ Equality\ holds\ iff\ a=b=c=1.$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\frac{1}{abc}-1\geq\frac{1}{24}\left(\frac{(a-b)^2}{a^2-ab+b^2}+\frac{(b-c)^2}{b^2-bc+c^2}+\frac{(c-a)^2}{c^2-ca+a^2}\right)\Leftrightarrow\frac{24}{abc}-24$$
$$\geq\sum_{cyc}\frac{a^2-ab+b^2-ab}{a^2-ab+b^2}=3-\sum_{cyc}\frac{ab}{a^2-ab+b^2}$$
$$\Leftrightarrow\frac{24}{abc}-27+\sum_{cyc}\frac{ab}{a^2-ab+b^2}\overset{(*)}{\geq}0$$

$$\text{Now,}\sum_{cyc}\frac{ab}{a^2-ab+b^2}=abc\sum_{cyc}\frac{1}{c(a^2-ab+b^2)}\overset{\text{Bergstrom}}{\geq}\frac{9abc}{\sum_{cyc}\left(ab(\sum_{cyc}a-c)\right)-3abc}$$
$$=\frac{9abc}{(\sum_{cyc}a)(\sum_{cyc}ab)-6abc}\ \therefore\ \text{LHS of }(*)$$
$$\geq\frac{24}{abc}-27+\frac{9abc}{(\sum_{cyc}a)(\sum_{cyc}ab)-6abc}$$
$$\overset{?}{\geq}0\Leftrightarrow\frac{8}{abc}\overset{?}{\geq}9-\frac{3abc}{(\sum_{cyc}a)(\sum_{cyc}ab)-6abc}=\frac{9(\sum_{cyc}a)(\sum_{cyc}ab)-57abc}{(\sum_{cyc}a)(\sum_{cyc}ab)-6abc}$$
$$\Leftrightarrow\frac{8(\sum_{cyc}ab).\sqrt{\sum_{cyc}ab}}{3\sqrt{3}abc}\overset{?}{\geq}\frac{9(\sum_{cyc}a)(\sum_{cyc}ab)-57abc}{(\sum_{cyc}a)(\sum_{cyc}ab)-6abc}$$
$$\Leftrightarrow\boxed{\frac{64}{27a^2b^2c^2}\cdot\left(\sum_{cyc}ab\right)^3\underset{(**)}{\overset{?}{\geq}}\left(\frac{9(\sum_{cyc}a)(\sum_{cyc}ab)-57abc}{(\sum_{cyc}a)(\sum_{cyc}ab)-6abc}\right)^2}$$

Assigning $b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0, y+z-x=2a>0$ **and** $z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y$
$\Rightarrow x,y,z$ **form sides of a**

$\text{triangle with semiperimeter, circumradius and inradius} = s, R, r$ (say) yielding $2\sum_{cyc} a$

$$= \sum_{cyc} x = 2s \Rightarrow \boxed{\sum_{cyc} a \overset{(\bullet)}{=} s} \Rightarrow a = s - x, b = s - y, c = s - z \Rightarrow \boxed{abc \overset{(\bullet\bullet)}{=} r^2 s}$$

Via aforementioned substitutions, $\sum_{cyc} ab = \sum_{cyc} (s - x)(s - y) = 4Rr + r^2$

$$\Rightarrow \boxed{\sum_{cyc} ab \overset{(\bullet\bullet\bullet)}{=} 4Rr + r^2}$$

$\therefore$ via $(\bullet), (\bullet\bullet), (\bullet\bullet\bullet), (**)$ transforms into : $\dfrac{64r^3(4R+r)^3}{27r^4s^2} \geq \left(\dfrac{9s(4Rr+r^2) - 57r^2s}{s(4Rr+r^2) - 6r^2s}\right)^2$

$$= \frac{16(9R - 12r)^2}{(4R - 5r)^2} \Leftrightarrow \boxed{4(4R - 5r)^2(4R + r)^3 \overset{(***)}{\geq} 27r(9R - 12r)^2 s^2}$$

Now, LHS of $(***) \overset{\text{Gerretsen}}{\leq} 27r(9R - 12r)^2(4R^2 + 4Rr + 3r^2) \overset{?}{\leq} 4(4R - 5r)^2(4R + r)^3$

$$\Leftrightarrow 4096t^5 - 15916t^4 + 14068t^3 + 4159t^2 + 2984t - 11564 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t - 2)\Big((t - 2)\big(4096t^3 + 246t^2 + 222t(t - 2) + 511\big) + 6804\Big) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$\Rightarrow (***) \Rightarrow (**) \Rightarrow (*)$ is true

$$\Rightarrow \frac{1}{abc} - 1 \geq \frac{1}{24}\left(\frac{(a - b)^2}{a^2 - ab + b^2} + \frac{(b - c)^2}{b^2 - bc + c^2} + \frac{(c - a)^2}{c^2 - ca + a^2}\right) \forall a, b, c > 0 \mid ab + bc + ca = 3 \text{ (QED)}$$

1020. If $a, b, c > 0, a + b + c = 1$ and $0 \leq \lambda \leq \frac{1}{4}$ then:

$$\frac{a^5}{b} + \frac{b^5}{c} + \frac{c^5}{a} + \lambda abc \geq \frac{\lambda + 1}{27}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Michael Sterghiou-Greece

$$a + b + c = 1, \lambda \in \left[0, \frac{1}{4}\right] \Rightarrow \frac{a^5}{b} + \frac{b^5}{c} + \frac{c^5}{a} + \lambda abc \geq \frac{\lambda + 1}{27}; (1)$$

Let $(p, q, r) = (\sum a, \sum ab, \prod a), p = 1, q \leq \frac{1}{3}, r \leq \frac{1}{27}$

$$\left(\sum_{cyc} \frac{a^5}{b}\right)\left(\sum_{cyc} ab\right) \overset{CBS}{\geq} \left(\sum_{cyc} a^3\right)^2 \Rightarrow \sum_{cyc} \frac{a^5}{b} \geq \frac{(1 - 3q + 3r)^2}{q}$$

$$\sum_{cyc} a^3 = p^3 - 3pq + 3r$$

It suffices to prove that:

$$f(\lambda)=\frac{(1-3q+3r)^2}{q}+\lambda r-\frac{\lambda+1}{27}\geq 0 \text{ as } \lambda\left(r-\frac{1}{27}\right)\leq 0\Rightarrow f\searrow$$

So, we need to show $f\left(\frac{1}{4}\right)\geq 0$ as $\lambda\leq\frac{1}{4}$ or $\frac{1}{108q}f(q,r)\geq 0$, where

$$f(q,r)=972q^2-1917qr-653q+972r^2+648r+108$$

$$f'(r)=27(-71q+72r+24)>0 \text{ as } 71q\leq\frac{71}{3}, r>0\Rightarrow f(q,r)\nearrow$$

As $r\geq\frac{4q-1}{9}$ (3rd degree Schur) it suffices that

$$f\left(q,\frac{4q-1}{9}\right)\geq 0 \text{ or } 8(3q-1)(13q-6)\geq 0 \text{ which holds as } q\leq\frac{1}{3}, 13q<6$$

Equality holds for $a=b=c=\frac{1}{3}$.

Solution 2 by Soumava Chakraborty-Kolkata-India

Assigning $b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y\Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say) yielding $2\sum a=\sum x=2s\Rightarrow \boxed{\sum a=s}\Rightarrow a=s-x, b=s-y, c=s-z$

Via aforementioned substitutions, $\sum_{cyc}ab=\sum_{cyc}(s-x)(s-y)=4Rr+r^2\Rightarrow \boxed{\sum_{cyc}ab\overset{(i)}{=}4Rr+r^2}$

$$\text{and }\sum_{cyc}a^2=\left(\sum_{cyc}a\right)^2-2\sum_{cyc}ab\overset{\text{via (i)}}{=}s^2-2(4Rr+r^2)\Rightarrow\sum_{cyc}a^2\overset{(ii)}{=}s^2-8Rr-2r^2$$

$$\text{Now, }\frac{a^5}{b}+\frac{b^5}{c}+\frac{c^5}{a}$$

$$=\frac{a^6}{ab}+\frac{b^6}{bc}+\frac{c^6}{ca}\overset{\text{Bergstrom}}{\geq}\frac{\left(\sum_{cyc}a^3\right)^2}{\sum_{cyc}ab}\overset{\text{Holder}}{\geq}\frac{\left(\sum_{cyc}a^3\right).\frac{1}{9}\left(\sum_{cyc}a\right)^3}{\sum_{cyc}ab}\overset{\text{Chebyshev}}{\geq}\frac{1}{27\sum_{cyc}ab}\left(\sum_{cyc}a\right)^4\left(\sum_{cyc}a^2\right)$$

$$\Rightarrow\boxed{\frac{a^5}{b}+\frac{b^5}{c}+\frac{c^5}{a}+\lambda abc-\frac{\lambda+1}{27}}$$

$$\geq\frac{1}{27\sum_{cyc}ab}\left(\sum_{cyc}a\right)^4\left(\sum_{cyc}a^2\right)+\lambda abc\left(\sum_{cyc}a\right)-\left(\frac{\lambda+1}{27}\right)\left(\sum_{cyc}a\right)^4\left(\because 1=\sum_{cyc}a=\left(\sum_{cyc}a\right)^4\right)$$

$$=\frac{1}{27}\left(\sum_{cyc}a\right)^4\left(\frac{\sum_{cyc}a^2}{\sum_{cyc}ab}-1\right)-\lambda\left(\sum_{cyc}a\right)\left(\frac{\left(\sum_{cyc}a\right)^3}{27}-abc\right)\boxed{\overset{?}{\geq}0}$$

$$\Leftrightarrow\frac{1}{27}\left(\sum_{cyc}a\right)^4\left(\frac{\sum_{cyc}a^2-\sum_{cyc}ab}{\sum_{cyc}ab}\right)\overset{?}{\geq}\lambda\left(\sum_{cyc}a\right).\frac{\left(\sum_{cyc}a\right)^3-27abc}{27}$$

$$\Leftrightarrow\left(\sum_{cyc}a\right)^3\left(\frac{\sum_{cyc}a^2-\sum_{cyc}ab}{\sum_{cyc}ab}\right)\underset{(*)}{\overset{?}{\geq}}\lambda\left(\left(\sum_{cyc}a\right)^3-27abc\right)$$

$$\text{Now,}\left(\sum_{cyc}a\right)^3\left(\frac{\sum_{cyc}a^2-\sum_{cyc}ab}{\sum_{cyc}ab}\right)$$

$$\geq\frac{1}{4}\left(\left(\sum_{cyc}a\right)^3-27abc\right)\overset{\text{via (i),(ii)}}{\Leftrightarrow}\frac{4s^3}{4Rr+r^2}(s^2-8Rr-2r^2-4Rr-r^2)\geq s^3-27r^2s$$

$$\Leftrightarrow 4s^2(s^2-12Rr-3r^2)\overset{(\bullet)}{\geq}(4Rr+r^2)(s^2-27r^2)$$

$$\text{Now, }2s^2-27Rr\overset{\text{Geretsen}}{\geq}2(16Rr-5r^2)-27Rr=5r(R-2r)\overset{\text{Euler}}{\geq}0\Rightarrow 2s^2\geq 27Rr\Rightarrow \text{LHS of }(\bullet)$$

$$\geq 54Rr(s^2-12Rr-3r^2)\overset{?}{\geq}(4Rr+r^2)(s^2-27r^2)$$

$$\Leftrightarrow(50R-r)s^2+27r^2(4R+r)\underset{(\bullet\bullet)}{\overset{?}{\geq}}54R(12Rr+3r^2)$$

$$\text{Again, }2s^2\geq 27Rr\Rightarrow \text{LHS of }(\bullet\bullet)\geq\frac{27Rr}{2}(50R-r)+27r^2(4R+r)\overset{?}{\geq}54R(12Rr+3r^2)$$

$$\Leftrightarrow 50R^2-Rr+8Rr+2r^2\overset{?}{\geq}48R^2+12Rr\Leftrightarrow 2R^2-5Rr+2r^2\overset{?}{\geq}0$$

$$\Leftrightarrow(2R-r)(R-2r)\overset{?}{\geq}0\to\text{true}\because R\overset{\text{Euler}}{\geq}2r\Rightarrow(\bullet\bullet)\Rightarrow(\bullet)\text{ is true}\Rightarrow\left(\sum_{cyc}a\right)^3\left(\frac{\sum_{cyc}a^2-\sum_{cyc}ab}{\sum_{cyc}ab}\right)$$

$$\geq\frac{1}{4}\left(\left(\sum_{cyc}a\right)^3-27abc\right)\overset{\frac{1}{4}\geq\lambda}{\geq}\lambda\left(\left(\sum_{cyc}a\right)^3-27abc\right)$$

$$\left(\because\left(\sum_{cyc}a\right)^3-27abc\overset{A-G}{\geq}0\right)\Rightarrow(*)\text{ is true}\Rightarrow\frac{a^5}{b}+\frac{b^5}{c}+\frac{c^5}{a}+\lambda abc-\frac{\lambda+1}{27}\geq 0\Rightarrow\frac{a^5}{b}+\frac{b^5}{c}+\frac{c^5}{a}+\lambda abc$$

$$\geq\frac{\lambda+1}{27}\ \forall a,b,c>0\,|\,a+b+c=1\text{ and }0\leq\lambda\leq\frac{1}{4}\ \text{(QED)}$$

Solution 3 by Daoudi Abdessatar-Tunisia

$$a+b+c=1,\lambda\in\left[0,\frac{1}{4}\right]\Rightarrow\frac{a^5}{b}+\frac{b^5}{c}+\frac{c^5}{a}+\lambda abc\geq\frac{\lambda+1}{27};(I)$$

It suffices to prove that:

$$\sum_{cyc}\frac{a^5}{b}-\frac{5}{108}+\frac{1}{4}abc\geq 0$$

Since $abc\leq\frac{1}{27}$ and $\lambda\leq\frac{1}{4}$, by AM-GM inequality, we have:

$$\frac{a^5}{b}+\frac{a^5}{b}+\frac{1}{9}(ab+ab)\geq\frac{4}{3}a^3(\text{and similarly})$$

Let $(p,q,r)=\sum a,\sum ab,\prod a)$ then:

$$\sum_{cyc}\frac{a^5}{b}\ge\frac{2}{3}\sum_{cyc}a^3-\frac{1}{9}q=2r+\frac{2}{3}-\frac{19}{9}q;$$

$f(q)=-\frac{19}{9}q+\frac{9}{4}r+\frac{67}{108}$ is a decreasing function, $q=\frac{1}{3},r=\frac{1}{27}$

$$\min\{f(q)\}=-\frac{19}{9}\cdot\frac{1}{3}\cdot\frac{9}{4}\cdot\frac{1}{27}+\frac{67}{108}=0$$

1021. Let $a,b,c>0, ab+bc+ca=a+b+c$. Prove that:

$$\frac{\sqrt[3]{4(ab+ac)}}{a^2}+\frac{\sqrt[3]{4(bc+ba)}}{b^2}+\frac{\sqrt[3]{4(ca+cb)}}{c^2}\ge3\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-1\right)$$

Proposed by Phan Ngoc Chau-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$(ab+bc+ca)^2\ge3abc(a+b+c)\overset{a+b+c=ab+bc+ca}{=}3abc(ab+bc+ca)\Rightarrow ab+bc+ca\ge3abc\Rightarrow\sum_{cyc}\frac{1}{a}\overset{(i)}{\ge}3$$

$$\text{Now, }\frac{\sqrt[3]{4(ab+ac)}}{a^2}+\frac{\sqrt[3]{4(bc+ba)}}{b^2}+\frac{\sqrt[3]{4(ca+cb)}}{c^2}$$

$$=\sum_{cyc}\frac{\left(\frac{1}{a}\right)^2}{\sqrt[3]{\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{ab+ac}}}\overset{A-G}{\ge}3\sum_{cyc}\frac{\left(\frac{1}{a}\right)^2}{\frac{1}{2}+\frac{1}{2}+\frac{1}{ab+ac}}\overset{\text{Bergstrom}}{\ge}\frac{3\left(\sum_{cyc}\frac{1}{a}\right)^2}{3+\sum_{cyc}\frac{1}{ab+ac}}\overset{A-G}{\ge}\frac{3\left(\sum_{cyc}\frac{1}{a}\right)^2}{3+\frac{1}{2}\cdot\sum_{cyc}\frac{1}{a\sqrt{bc}}}$$

$$=\frac{3\left(\sum_{cyc}\frac{1}{a}\right)^2}{3+\frac{1}{2}\cdot\sum_{cyc}\left(\sqrt{\frac{1}{ab}}\cdot\sqrt{\frac{1}{ac}}\right)}$$

$$\overset{CBS}{\ge}\frac{3\left(\sum_{cyc}\frac{1}{a}\right)^2}{3+\frac{1}{2}\cdot\sqrt{\sum_{cyc}\frac{1}{ab}}\cdot\sqrt{\sum_{cyc}\frac{1}{ac}}}=\frac{3\left(\sum_{cyc}\frac{1}{a}\right)^2}{3+\frac{1}{2}\sum_{cyc}\frac{1}{ab}}$$

$$=\frac{3\left(\sum_{cyc}\frac{1}{a}\right)^2}{3+\frac{1}{2}\sum_{cyc}\frac{1}{a}}\left(\because\frac{ab+bc+ca}{abc}=\frac{a+b+c}{abc}\Rightarrow\sum_{cyc}\frac{1}{ab}=\sum_{cyc}\frac{1}{a}\right)\overset{?}{\ge}3\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-1\right)$$

$$=3\left(\sum_{cyc}\frac{1}{a}-1\right)\Leftrightarrow\frac{x^2}{3+\frac{x}{2}}\overset{?}{\ge}x-1\quad\left(x=\sum_{cyc}\frac{1}{a}\right)$$

$$\Leftrightarrow\frac{2x^2}{x+6}-x+1\overset{?}{\ge}0\Leftrightarrow\frac{2x^2-x^2-6x+x+6}{x+6}\overset{?}{\ge}0\Leftrightarrow x^2-5x+6\overset{?}{\ge}0\Leftrightarrow(x-3)(x-2)\overset{?}{\ge}0$$

$$\to\text{true}\because x=\sum_{cyc}\frac{1}{a}\ge3\text{ via (i)}$$

$$\therefore \frac{\sqrt[3]{4(ab+ac)}}{a^2}+\frac{\sqrt[3]{4(bc+ba)}}{b^2}+\frac{\sqrt[3]{4(ca+cb)}}{c^2}\ge 3\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-1\right)\ \forall\, a,b,c > 0\ \middle|\ ab+bc+ca=a+b+c\ (QED)$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By AM − GM inequality we have :

$$3.\frac{\sqrt[3]{4(ab+ac)}}{3a^2}+\frac{4}{3(ab+ac)}+2.\frac{2}{3}\ge 6\sqrt[6]{\left(\frac{\sqrt[3]{4(ab+ac)}}{3a^2}\right)^3.\frac{4}{3(ab+ac)}.\left(\frac{2}{3}\right)^2}=\frac{4}{a}$$

$$Then:\ \frac{\sqrt[3]{4(ab+ac)}}{a^2}\ge\frac{4}{a}-\frac{4}{3}-\frac{4}{3(ab+ac)}\overset{CBS}{\ge}\frac{4}{a}-\frac{4}{3}-\frac{1}{3}\left(\frac{1}{ab}+\frac{1}{ac}\right)$$

Summing up this inequality with similar ones obtained by permutation we get :

$$\frac{\sqrt[3]{4(ab+ac)}}{a^2}+\frac{\sqrt[3]{4(ab+ac)}}{a^2}+\frac{\sqrt[3]{4(ab+ac)}}{a^2}\ge 4\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-4-\frac{2}{3}\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ac}\right)$$

$$Since\ \ ab+bc+ca=a+b+c\ \ we\ have:\ \frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ac}$$

And from the inequality

$$\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2\ge 3\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ac}\right)\ we\ get:\ \frac{1}{a}+\frac{1}{b}+\frac{1}{c}\ge 3.$$

$$Then:\ \frac{\sqrt[3]{4(ab+ac)}}{a^2}+\frac{\sqrt[3]{4(ab+ac)}}{a^2}+\frac{\sqrt[3]{4(ab+ac)}}{a^2}\ge\frac{10}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-4$$
$$\ge 3\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-1\right).$$

Equality holds iff $a=b=c=1$.

1022. **If $a,b,c,d>0$ and $a^3+b^3+c^3+d^3=1$ then:**

$$\frac{1}{2}\ge\frac{a^2b^2}{a+b}+\frac{b^2c^2}{b+c}+\frac{c^2d^2}{c+d}+\frac{d^2a^2}{d+a}$$

Proposed by Alex Szoros-Romania

Solution 1 by Adrian Popa-Romania

$$\frac{a^2b^2}{a+b}+\frac{b^2c^2}{b+c}+\frac{c^2d^2}{c+d}+\frac{d^2a^2}{d+a}\overset{AGM}{\le}\frac{a^2b^2}{2\sqrt{ab}}+\frac{b^2c^2}{2\sqrt{bc}}+\frac{c^2d^2}{2\sqrt{cd}}+\frac{d^2a^2}{2\sqrt{da}}=$$

$$=\frac{1}{2}\left((ab)^{\frac{3}{2}}+(bc)^{\frac{3}{2}}+(cd)^{\frac{3}{2}}+(da)^{\frac{3}{2}}\right)\overset{?}{\le}\frac{1}{2}$$

$$1=a^3++b^3+c^3+d^3=\frac{a^3+b^3}{2}+\frac{b^3+c^3}{2}+\frac{c^3+d^3}{2}+\frac{d^3+a^3}{2}\overset{AGM}{\ge}$$

$$\geq (ab)^{\frac{3}{2}} + (bc)^{\frac{3}{2}} + (cd)^{\frac{3}{2}} + (da)^{\frac{3}{2}}$$

$$1 \geq (ab)^{\frac{3}{2}} + (bc)^{\frac{3}{2}} + (cd)^{\frac{3}{2}} + (da)^{\frac{3}{2}} \text{ true!}$$

Solution 2 by George Titakis-Greece

$$\frac{a^2b^2}{a+b} + \frac{b^2c^2}{b+c} + \frac{c^2d^2}{c+d} + \frac{d^2a^2}{d+a} \leq \frac{(a+b)^4}{16(a+b)} + \frac{(b+c)^4}{16(b+c)} + \frac{(c+d)^4}{16(c+d)} + \frac{(d+a)^4}{16(d+a)} =$$

$$= \frac{1}{16}[(a+b)^3 + (b+c)^3 + (c+d)^3 + (d+a)^3] =$$

$$= \frac{1}{16}[2(a^3+b^3+c^3+d^3) + 3ab(a+b) + 3bc(b+c) + 3cd(c+d) + 3da(d+a)] \leq$$

$$\leq \frac{1}{16}[2(a^3+b^3+c^3+d^3) + 3 \cdot 2(a^3+b^3+c^3+d^3)] =$$

$$= \frac{8(a^3+b^3+c^3+d^3)}{16} = \frac{1}{2}$$

Equality holds for $a = b = c = \frac{1}{\sqrt[3]{4}}$.

Solution 3 by Tapas Das-India

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2) \geq (a+b)(2ab - ab) = ab(a+b)$$

$$\frac{a^2b^2}{a+b} + \frac{b^2c^2}{b+c} + \frac{c^2d^2}{c+d} + \frac{d^2a^2}{d+a} =$$

$$= \frac{ab \cdot ab(a+b)}{(a+b)^2} + \frac{bc \cdot bc(b+c)}{(b+c)^2} + \frac{cd \cdot cd(c+d)}{(c+d)^2} + \frac{da \cdot da(d+a)}{(d+a)^2} \leq$$

$$\leq \frac{ab}{(a+b)^2}(a^3+b^3) + \frac{bc}{(b+c)^2}(b^3+c^3) + \frac{cd}{(c+d)^2}(c^3+d^3) + \frac{da}{(d+a)^2}(d^3+a^3) \leq$$

$$\leq \frac{ab}{4ab}(a^3+b^3) + \frac{bc}{4bc}(b^3+c^3) + \frac{cd}{4cd}(c^3+d^3) + \frac{da}{4da}(d^3+a^3) =$$

$$= \frac{a^3+b^3}{4} + \frac{b^3+c^3}{4} + \frac{c^3+d^3}{4} + \frac{d^3+a^3}{4} = \frac{2(a^3+b^3+c^3+d^3)}{4} = \frac{1}{2}$$

Solution 4 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\frac{a^3+b^3+c^3+d^3}{2} \geq \frac{(ab)^2}{a+b} + \frac{(bc)^2}{b+c} + \frac{(cd)^2}{c+d} + \frac{(da)^2}{d+a}$$

$$\frac{a^3+b^3}{4} \geq \frac{(ab)^2}{a+b} \ (\text{and analogs})$$

$$\Leftrightarrow 2\sqrt{ab}\cdot 2\sqrt{(ab)^3}\geq 4(ab)^2 \Leftrightarrow 4(ab)^2\geq 4(ab)^2 \text{ true.}$$

Thus,

$$\frac{a^3+b^3}{4}+\frac{b^3+c^3}{4}+\frac{c^3+d^3}{4}+\frac{d^3+a^3}{4}=\frac{2(a^3+b^3+c^3+d^3)}{4}=\frac{1}{2}$$

Solution 5 by Soumava Chakraborty-Kolkata-India

$$\frac{a^2b^2}{a+b}+\frac{b^2c^2}{b+c}+\frac{c^2d^2}{c+d}+\frac{d^2a^2}{d+a}$$

$$=\frac{2ab}{a+b}\cdot\frac{ab}{2}+\frac{2bc}{b+c}\cdot\frac{bc}{2}+\frac{2cd}{c+d}\cdot\frac{cd}{2}$$

$$+\frac{2da}{d+a}\cdot\frac{da}{2}\overset{H\leq G}{\leq}\frac{1}{2}\left(\sqrt{ab}.ab+\sqrt{bc}.bc+\sqrt{cd}.cd+\sqrt{da}.da\right)$$

$$=\frac{1}{2}\left(a\sqrt{a}.b\sqrt{b}+b\sqrt{b}.c\sqrt{c}+c\sqrt{c}.d\sqrt{d}\right.$$

$$\left.+d\sqrt{d}.a\sqrt{a}\right)\overset{CBS}{\leq}\frac{1}{2}\sqrt{a^3+b^3+c^3+d^3}.\sqrt{b^3+c^3+d^3+a^3}$$

$$=\frac{1}{2}(a^3+b^3+c^3+d^3)\overset{a^3+b^3+c^3+d^3=1}{=}\frac{1}{2}$$

$$\Rightarrow\frac{1}{2}\geq\frac{a^2b^2}{a+b}+\frac{b^2c^2}{b+c}+\frac{c^2d^2}{c+d}+\frac{d^2a^2}{d+a}\ \text{(QED)}$$

1023. If $x,y,z>0,\frac{1}{1+x}+\frac{1}{1+y}+\frac{1}{1+z}=1$ then:

$$x+y+z\geq\frac{3}{4}xyz$$

Proposed by Marin Chirciu-Romania

Solution 1 by Tapas Das-India

$$\frac{1}{1+x}+\frac{1}{1+y}+\frac{1}{1+z}=1\Leftrightarrow$$

$$(1+y)(1+z)+(1+x)(1+z)+(1+y)(1+z)=(1+x)(1+y)(1+z)$$

$$xyz=2+x+y+z\Rightarrow x+y+z=xyz-2;\ (1)$$

Now, $x+y+z\geq 3(xyz)^{\frac{1}{3}}$

$$xyz-2\geq 3(xyz)^{\frac{1}{3}}$$

Let $(xyz)^{\frac{1}{3}}=a$, then $a^3-2\geq 3a\Rightarrow a^3-3a-2\geq 0$

$$(a-2)(a+1)^2 \geq 0 \Rightarrow a \geq 2 \Rightarrow (xyz)^{\frac{1}{3}} \geq 2 \Rightarrow xyz \geq 8$$

$$\text{Now, } x+y+z-\frac{3}{4}xyz = xyz-2-\frac{3}{4}xyz = \frac{1}{4}xyz-2 \geq 0 \Rightarrow$$

$$x+y+z \geq \frac{3}{4}xyz$$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\frac{1}{1+x}+\frac{1}{1+y}+\frac{1}{1+z}=1 \Leftrightarrow$$

$$(1+y)(1+z)+(1+x)(1+z)+(1+y)(1+z)=(1+x)(1+y)(1+z)$$

$$xyz = 2+x+y+z \Rightarrow x+y+z = xyz-2$$

$$4xyz-8 \geq 3xyz \Rightarrow xyz \geq 8 \Rightarrow x+y+z \geq \frac{3}{4}xyz$$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$\frac{1}{1+x}+\frac{1}{1+y}+\frac{1}{1+z}=1=2+\sum_{cyc}\frac{1}{1+x}=3 \Rightarrow 2=\sum_{cyc}\left(1-\frac{1}{1+x}\right) \Rightarrow 2$$

$$=\sum_{cyc}\frac{x}{1+x} \overset{\text{Jensen}}{\leq} 3.\frac{\frac{\sum_{cyc}x}{3}}{1+\frac{\sum_{cyc}x}{3}} \left(\because f(t)=\frac{t}{1+t} \text{ is concave } \forall\, t>0 \text{ as } f''(t)\right.$$

$$\left.=\frac{-2}{(1+t)^3}<0\right)$$

$$\Rightarrow 2 \leq \frac{3\sum_{cyc}x}{3+\sum_{cyc}x} \Rightarrow 3\sum_{cyc}x \geq 6+2\sum_{cyc}x \Rightarrow \sum_{cyc}x \overset{(*)}{\geq} 6$$

$$\text{Now, } \frac{1}{1+x}+\frac{1}{1+y}+\frac{1}{1+z}=1 \Rightarrow \sum_{cyc}((1+y)(1+z))=\prod_{cyc}(1+x) \Rightarrow \sum_{cyc}(1+y+z+yz)$$

$$=1+xyz+\sum_{cyc}xy+\sum_{cyc}x \Rightarrow 3+2\sum_{cyc}x+\sum_{cyc}xy=1+xyz+\sum_{cyc}xy+\sum_{cyc}x$$

$$\Rightarrow xyz=\sum_{cyc}x+2 \Rightarrow \frac{3}{4}xyz=\frac{3}{4}\sum_{cyc}x+\frac{3}{2}=\sum_{cyc}x+\frac{3}{2}-\frac{1}{4}\sum_{cyc}x \overset{\text{via }(*)}{\geq} \sum_{cyc}x+\frac{3}{2}-\frac{6}{4}=\sum_{cyc}x$$

$$\Rightarrow x+y+z \geq \frac{3}{4}xyz \;\forall\, x,y,z>0 \,\Big|\, \frac{1}{1+x}+\frac{1}{1+y}+\frac{1}{1+z}=1 \text{ (QED)}$$

Solution 4 by Hikmat Mammadov-Azerbaijan

$$\frac{1}{1+x}+\frac{1}{1+y}+\frac{1}{1+z}=1 \Leftrightarrow$$

$$(1+y)(1+z)+(1+x)(1+z)+(1+y)(1+z)=(1+x)(1+y)(1+z)$$

$$xyz = 2 + x + y + z \Rightarrow x + y + z = xyz - 2$$

$$x + y + z \geq 3(x + y + z + 2) \Leftrightarrow x + y + z \geq 6; (1)$$

$$x + y + z + 2 = xyz \leq \left(\frac{x + y + z}{3}\right)^3$$

$$a := a + y + z \overset{?}{\geq} 6 \Rightarrow a^3 - 27a - 54 \geq 0 \Rightarrow (a - 6)(a^2 + 6a + 9) \geq 0$$

$$(a - 6)(a + 3)^2 \geq 0 \Rightarrow a \geq 6; (2)$$

From (1) and (2), it follows that: $x + y + z \geq \frac{3}{4}xyz$

Equality holds for $x = y = z = 2$.

1024. If $a, b, c > 0, abc = a + b + c$ and $\lambda \geq 0$ then:

$$\sqrt{ab + bc + ca}\left(\lambda + \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) \geq 3(\lambda + 1)$$

Proposed by Marin Chirciu-Romania

Solution 1 by Tapas Das-India

$$a + b + c = abc; (1)$$

$$\frac{a + b + c}{3} \geq (abc)^{\frac{1}{3}} + b + c \geq 3(abc)^{\frac{1}{3}} \Rightarrow abc \geq 3(abc)^{\frac{1}{3}}$$

$$(abc)^{\frac{2}{3}} \geq 3 \Rightarrow abc \geq 3^{\frac{3}{2}}; (2)$$

$$ab + bc + ca \geq 3(abc)^{\frac{2}{3}} = 9 \Rightarrow \sqrt{ab + bc + ca} \geq 3$$

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} = \frac{a + b + c}{abc} = \frac{abc}{abc} = 1$$

$$\sqrt{ab + bc + ca}\left(\lambda + \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) \geq 3\lambda + \sqrt{ab + bc + ca} = 3(\lambda + 1)$$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

$$1 = \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} \Rightarrow 9 \leq ab + bc + ca \text{ and } 3 \leq \sqrt{ab + bc + ca}$$

$$3 \leq 3\left(\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}\right) \leq \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^2 \leq 3\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)$$

$$\Rightarrow 1 \leq \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$$

$$\sqrt{ab + bc + ca}\left(\lambda + \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) \geq \sqrt{9}\left(\lambda + \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) =$$

$$= 3\left(\lambda + \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) = 3(\lambda + 1)$$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$\left(\sum_{cyc} ab\right)^2 \geq 3abc\sum_{cyc} a = 3abc.\sqrt{\left(\sum_{cyc} a\right)^2} \geq 3abc.\sqrt{3\sum_{cyc} ab}$$

$$\Rightarrow \left(\sum_{cyc} ab\right).\sqrt{\sum_{cyc} ab} \overset{(i)}{\geq} 3\sqrt{3}abc$$

$$\text{Again, } abc = \sum_{cyc} a \overset{A-G}{\geq} 3.\sqrt[3]{abc} \Rightarrow \sqrt[3]{a^2b^2c^2} \geq 3 \Rightarrow a^2b^2c^2 \geq 27 \Rightarrow abc \overset{(ii)}{\geq} 3\sqrt{3} \therefore (i)\bullet(ii)$$

$$\Rightarrow \left(\sum_{cyc} ab\right).\sqrt{\sum_{cyc} ab} \geq 3\sqrt{3}.3\sqrt{3} \Rightarrow \left(\sqrt{\sum_{cyc} ab}\right)^3 \geq 27 \Rightarrow \sqrt{\sum_{cyc} ab} \overset{(*)}{\geq} 3$$

$$\Rightarrow \lambda(\sqrt{ab+bc+ca} - 3) \geq 0 \ (\because \lambda \geq 0) \Rightarrow \sqrt{ab+bc+ca}.\lambda \overset{(**)}{\geq} 3\lambda$$

$$\text{Further, } \sum_{cyc}\frac{1}{a^2} \geq \frac{1}{3}\left(\sum_{cyc}\frac{1}{a}\right)^2 \geq \frac{1}{3}.3\sum_{cyc}\frac{1}{ab} = \frac{a+b+c}{abc} = 1 \Rightarrow \sqrt{ab+bc+ca}\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)$$

$$\geq \sqrt{ab+bc+ca} \overset{\text{via } (*)}{\geq} 3 \therefore \sqrt{ab+bc+ca}\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right) \overset{(***)}{\geq} 3$$

$$\therefore (**) + (***) \Rightarrow \sqrt{ab+bc+ca}.\lambda + \sqrt{ab+bc+ca}\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right) \geq 3\lambda + 3$$

$$\Rightarrow \sqrt{ab+bc+ca}\left(\lambda + \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right) \geq 3(\lambda+1) \ \forall \ a,b,c > 0 \mid abc$$

$$= a+b+c \text{ (QED)}$$

1025. Let $a,b,c \geq 0, a+b+c = a^2+b^2+c^2$. Prove that:

$$2\sqrt{2}\left(\sqrt{ab+ac} + \sqrt{bc+ba} + \sqrt{ca+cb}\right) + 9 \geq 7(a^2+b^2+c^2)$$

Proposed by Phan Ngoc Chau-Vietnam

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

If $a = b = c = 0$ *the inequality is true. Assume that* $a+b+c > 0$.

Homogenizing the given inequality we get the equivalent expression :

$$2\sqrt{2}(a+b+c)\left(\sqrt{ab+ac}+\sqrt{bc+ba}+\sqrt{ca+cb}\right)+9(a^2+b^2+c^2)$$
$$\geq 7(a+b+c)^2$$
$$\Leftrightarrow 2(a+b+c)\sum_{cyc}\sqrt{2a(b+c)}+2\sum_{cyc}a^2\geq 7\sum_{cyc}a(b+c)$$

By AM − GM inequality we have :

$$2(a+b+c)\sqrt{2a(b+c)}+2a^2=2.\sqrt{2a^3(b+c)}+4.\sqrt{\frac{a(b+c)^3}{2}}+2a^2\geq$$
$$\geq 7\sqrt[7]{\left(\sqrt{2a^3(b+c)}\right)^2.\left(\sqrt{\frac{a(b+c)^3}{2}}\right)^4.2a^2}=7a(b+c)$$

Then : $2(a+b+c)\sqrt{2a(b+c)}+2a^2\geq 7a(b+c)$ *(and analogs)*

Summing up this inequality with similar ones yields the desired inequality.

Equality holds iff $(a,b,c)=(1,1,1)$.

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\boxed{\text{Case 1}: a,b,c>0}\; 2\sqrt{2a(b+c)}\overset{G-H}{\geq}\frac{8a(b+c)}{2a+b+c}=\frac{8a(b+c+2a-2a)}{2a+b+c}$$
$$=8a-\frac{16a^2}{2a+b+c}\text{ and analogs}$$
$$\Rightarrow 2\sqrt{2}\left(\sqrt{ab+ac}+\sqrt{bc+ba}+\sqrt{ca+cb}\right)+9$$
$$\geq 8\sum_{cyc}a-16\sum_{cyc}\frac{a^2}{2a+b+c}+9\overset{?}{\geq}7(a^2+b^2+c^2)$$
$$\Leftrightarrow 8\sum_{cyc}a-16\sum_{cyc}\frac{a^2}{2a+b+c}$$
$$+\frac{9\sum_{cyc}a^2}{\sum_{cyc}a}\overset{?}{\geq}7\sum_{cyc}a\left(\because\sum_{cyc}a=\sum_{cyc}a^2\Rightarrow 1=\frac{\sum_{cyc}a^2}{\sum_{cyc}a}\right)$$
$$\Leftrightarrow\sum_{cyc}a+\frac{9\sum_{cyc}a^2}{\sum_{cyc}a}\underset{(*)}{\overset{?}{\geq}}16\sum_{cyc}\frac{a^2}{2a+b+c}$$

Assigning $b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y$
$\Rightarrow x,y,z$ form sides of a

triangle with semiperimeter, circumradius and inradius = s, R, r (say) yielding $2\sum_{cyc} a$

$$= \sum_{cyc} x = 2s \Rightarrow \sum_{cyc} a = s \Rightarrow a = s - x, b = s - y, c = s - z$$

Via aforementioned substitutions, $\sum_{cyc} a^2 = \left(\sum_{cyc} a\right)^2 - 2\sum_{cyc} ab = s^2 - 2\sum_{cyc}(s-x)(s-y)$

$$= s^2 - 2(4Rr + r^2) \Rightarrow \sum_{cyc} a^2 \overset{(\bullet)}{=} s^2 - 8Rr - 2r^2$$

Now, $\sum_{cyc} \frac{a^2}{2a+b+c} = \sum_{cyc} \frac{a^2}{a+b+c+a} = \sum_{cyc} \frac{(s-x)^2}{y+z} = \sum_{cyc} \frac{(2s-x-s)^2}{2s-x}$

$$= \sum_{cyc} \frac{(2s-x)^2 - 2s(2s-x) + s^2}{2s-x} = \sum_{cyc}(2s - x - 2s) + s^2 \sum_{cyc} \frac{1}{y+z}$$

$$= -2s + s^2 \frac{\sum_{cyc}((z+x)(x+y))}{\prod_{cyc}(y+z)}$$

$$= -2s + s^2 \frac{\left(\sum_{cyc} x^2 + 2\sum_{cyc} xy\right) + \sum_{cyc} xy}{2s(s^2 + 2Rr + r^2)}$$

$$= -2s + \frac{s}{2(s^2 + 2Rr + r^2)}\left(\left(\sum_{cyc} x\right)^2 + s^2 + 4Rr + r^2\right)$$

$$= -2s + \frac{s(5s^2 + 4Rr + r^2)}{2(s^2 + 2Rr + r^2)} = \frac{s(s^2 - 4Rr - 3r^2)}{2(s^2 + 2Rr + r^2)}$$

$$\Rightarrow 16\sum_{cyc} \frac{a^2}{2a+b+c} = \frac{8s(s^2 - 4Rr - 3r^2)}{s^2 + 2Rr + r^2} \overset{?}{\leq} \sum_{cyc} a + \frac{9\sum_{cyc} a^2}{\sum_{cyc} a} \overset{\text{via } (\bullet)}{=} s + \frac{9(s^2 - 8Rr - 2r^2)}{s}$$

$$= \frac{10s^2 - 72Rr - 18r^2}{s}$$

$$\Leftrightarrow (5s^2 - 36Rr - 9r^2)(s^2 + 2Rr + r^2) \overset{?}{\geq} 4s^2(s^2 - 4Rr - 3r^2)$$

$$\Leftrightarrow s^4 - (10Rr - 8r^2)s^2 - r^2(72R^2 + 54Rr + 9r^2) \underset{(**)}{\overset{?}{\geq}} 0$$

$$\text{Now, LHS of } (**) \overset{\text{Gerretsen}}{\geq} (6Rr + 3r^2)s^2$$

$$- r^2(72R^2 + 54Rr + 9r^2) \overset{\text{Gerretsen}}{\geq} (6Rr + 3r^2)(16Rr - 5r^2)$$

$$- r^2(72R^2 + 54Rr + 9r^2) = 24r^2(2R^2 - 3Rr - 2r^2) = 24r^2(R - 2r)(2R + r)$$

$\overset{\text{Euler}}{\geq} 0 \Rightarrow (**) \Rightarrow (*)$ is true $\therefore 2\sqrt{2}\left(\sqrt{ab+ac} + \sqrt{bc+ba} + \sqrt{ca+cb}\right) + 9$

$\geq 7(a^2 + b^2 + c^2)$, equality iff $a = b = c = 1$

$\boxed{\text{Case 2 : Exactly one among } a, b, c = 0}$ WLOG we may assume $a = 0$ and b, c

> 0 and then : $2\sqrt{2}\left(\sqrt{ab+ac} + \sqrt{bc+ba} + \sqrt{ca+cb}\right) + 9$

$\geq 7(a^2 + b^2 + c^2)$

$$\Leftrightarrow 4.\sqrt{2bc}+\frac{9(b^2+c^2)}{b+c}\geq 7(b+c)\ \left(\because b^2+c^2=b+c\Rightarrow 1=\frac{b^2+c^2}{b+c}\right)$$

$$\Leftrightarrow 4\sqrt{2}.\sqrt{bc}.(b+c)+9(b^2+c^2)\overset{(***)}{\geq}7(b^2+c^2+2bc)$$

$$\text{Now, LHS of }(***)\overset{G-H}{\geq}4\sqrt{2}.\frac{2bc}{b+c}.(b+c)+9(b^2+c^2)\overset{?}{\geq}7(b^2+c^2+2bc)$$

$$\Leftrightarrow 8\sqrt{2}bc+2b^2+2c^2-14bc\underbrace{\overset{?}{\geq}}_{(****)}0$$

$$\text{Now, via } A-G,\text{ LHS of }(****)\geq(8\sqrt{2}+4-14)bc=2(4\sqrt{2}-5)bc=\frac{2(32-25)}{4\sqrt{2}+5}bc>0$$

$$\Rightarrow(****)\Rightarrow(***)\text{ is true}$$

$$\therefore 2\sqrt{2}\left(\sqrt{ab+ac}+\sqrt{bc+ba}+\sqrt{ca+cb}\right)+9\geq 7(a^2+b^2+c^2)\ (\text{strict inequality})$$

$\boxed{\text{Case 3 : Exactly 2 among } a,b,c=0}$ WLOG we may assume $b=c=0$ and $a>0$ and then : $2\sqrt{2}\left(\sqrt{ab+ac}+\sqrt{bc+ba}+\sqrt{ca+cb}\right)+9\geq 7(a^2+b^2+c^2)$

$$\Leftrightarrow 9>0\rightarrow\text{true}$$

$$\therefore 2\sqrt{2}\left(\sqrt{ab+ac}+\sqrt{bc+ba}+\sqrt{ca+cb}\right)+9\geq 7(a^2+b^2+c^2)\ (\text{strict inequality})$$

$\therefore$ combining all cases, $2\sqrt{2}\left(\sqrt{ab+ac}+\sqrt{bc+ba}+\sqrt{ca+cb}\right)+9\geq 7(a^2+b^2+c^2)\ \forall a,b,c\geq 0\ \middle|\ a+b+c=a^2+b^2+c^2$, equality iff $a=b=c=1$ (QED)

Solution 3 by Michael Sterghiou-Greece

$$2\sqrt{2}\left(\sqrt{ab+ac}+\sqrt{bc+ba}+\sqrt{ca+cb}\right)+9\geq 7(a^2+b^2+c^2);\ (1)$$

Let $(p,q,r)=\left(\sum a,\sum ab,\prod a\right)$. We homogenise (1) as

$$2\sqrt{2}\left(\sum_{cyc}a\right)\left(\sum_{cyc}\sqrt{ab+ac}\right)+9\sum_{cyc}a^2\geq 7\left(\sum_{cyc}a\right)^2$$

Multiplying by $a+b+c$ and considering that $\sum a=\sum a^2$.

Now, we assume $p=3$. By AM-GM:

$$\sqrt{ab+ac}=\sqrt{\frac{1}{2}\cdot 2a(b+c)}\geq\frac{1}{\sqrt{2}}\cdot\frac{2}{\frac{1}{2a}+\frac{1}{b+c}}=\frac{4}{\sqrt{2}}\cdot\frac{a(b+c)}{a+3}$$

$$(1)\Rightarrow 24\left(\sum_{cyc}\frac{a(b+c)}{a+3}\right)+9(9-2q)-63\geq 0;\ (2)$$

This reduces to: $4\cdot\frac{5r+9q}{3q+r+54}-(q-1)\geq 0;\ (3)$

If $q\leq 1$ we are done, so let $q>1$, $(3)\Rightarrow$

$-3q^2-15q+54+r(21-q)\geq 0$ or the stronger as

$r \ge \frac{4q-9}{3}$ (3rd degree Schur's inequality) true as $q \le \frac{p^2}{3} = 3$ and $q > 1$ by assumption.

Equality holds for $a = b = c = 1$.

1026. If $x, y, z > 0$ then:

$$\sum_{cyc} \frac{x}{\sqrt[3]{y^3 + 25xyz + z^3}} \ge 1$$

Proposed by Marin Chirciu-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \frac{x.x^{\frac{1}{3}}}{\sqrt[3]{xy^3 + 25x^2yz + xz^3}}$$

$$= \sum_{cyc} \frac{x^{\frac{4}{3}}}{\sqrt[3]{xy^3 + 25x^2yz + xz^3}} \overset{\text{Radon}}{\ge} \frac{\left(\sum_{cyc} x\right)^{\frac{4}{3}}}{\left(\sum_{cyc} x^3y + \sum_{cyc} xy^3 + 25xyz\left(\sum_{cyc} x\right)\right)^{\frac{1}{3}}} \overset{?}{\ge} 1$$

$$\Leftrightarrow \frac{\left(\sum_{cyc} x\right)^4}{\sum_{cyc} x^3y + \sum_{cyc} xy^3 + 25xyz\left(\sum_{cyc} x\right)} \overset{?}{\ge} 1$$

$$\Leftrightarrow \sum_{cyc} x^4 + 4\sum_{cyc} x^3y + 4\sum_{cyc} xy^3 + 6\sum_{cyc} x^2y^2 + 12xyz\left(\sum_{cyc} x\right) \overset{?}{\ge} \sum_{cyc} x^3y + \sum_{cyc} xy^3 + 25xyz\left(\sum_{cyc} x\right)$$

$$\Leftrightarrow \sum_{cyc} x^4 + 3\sum_{cyc} x^3y + 3\sum_{cyc} xy^3 + 6\sum_{cyc} x^2y^2 \underset{(*)}{\overset{?}{\ge}} 13xyz\left(\sum_{cyc} x\right)$$

$$\text{Now, } \sum_{cyc} x^4 + 3\sum_{cyc} x^3y + 3\sum_{cyc} xy^3 + 6\sum_{cyc} x^2y^2$$

$$\ge \sum_{cyc} x^2y^2 + 3\sum_{cyc}(x^3y + xy^3) + 6\sum_{cyc} x^2y^2 \overset{A-G}{\ge} 7\sum_{cyc} x^2y^2 + 6\sum_{cyc} x^2y^2$$

$$= 13\sum_{cyc} x^2y^2 \ge 13xyz\left(\sum_{cyc} x\right) \Rightarrow (*) \text{ is true}$$

$$\therefore \sum_{cyc} \frac{x}{\sqrt[3]{y^3 + 25xyz + z^3}} \ge 1 \ \forall\, x, y, z > 0 \ (QED)$$

Solution 2 by Biswajt Ghosh-India

Let us prove that:

$$\frac{x}{\sqrt[3]{y^3+25xyz+z^3}}<\frac{1}{3}\Leftrightarrow 3x<\sqrt[3]{y^3+25xyz+z^3}$$

$$27x^3<y^3+25xyz+z^3=y^3+\frac{25}{3}(3xyz)+z^3$$

$$27x^3<(y^3+z^3)+\frac{25}{3}(x^3+y^3+z^3)$$

$$81x^3<28(y^3+z^3)+25x^3$$

$$56x^3<28(y^3+z^3)\Rightarrow 2x^3<y^3+z^3;(1)$$

Similarly,

$2y^3<x^3+z^3;(2)$ and $2z^3<x^3+y^3;(3)$

By adding (1), (2) and (3):

$$2(x^3+y^3)<(x^3+y^3)+2z^3\Rightarrow x^3+y^3<2z^3;(4)$$

From (3) and (4): $2z^3<2z^3$ contradiction!

Hence,

$$\frac{x}{\sqrt[3]{y^3+25xyz+z^3}}>\frac{1}{3}\Rightarrow\sum_{cyc}\frac{x}{\sqrt[3]{y^3+25xyz+z^3}}\geq 1$$

Equality holds for $x=y=z=1$.

Solution 3 by Michael Sterghiou-Greece

$$\sum_{cyc}\frac{x}{\sqrt[3]{y^3+25xyz+z^3}}\geq 1;(1)$$

Let $(p,q,r)=(\sum x,\sum xy,\prod x)$. WLOG, let $p=3$. From the convexity of $f(t)=\frac{1}{\sqrt{t}}$, weights

x,y,z:

$(2);\ \sum_{cyc}\frac{x}{\sqrt[3]{y^3+25xyz+z^3}}\geq\left(\sum_{cyc}x\right)\cdot\frac{1}{\sqrt[3]{\frac{\sum x(y^3+25xyz+z^3)}{x+y+z}}}=\frac{3}{\left[\frac{1}{3}\sum(xy^3+xz^3+25r)\right]^{\frac{1}{3}}}$ which must be

greater than 1. As:

$$\sum(xy^3+xz^3)=(\sum x^2)(\sum xy)-(\sum x)xyz=(9-2q)q-3r$$

$$\sum x(xyz)=3r$$

$$(2)\Rightarrow\frac{81}{(9-2q)q-3r+75r}\geq 1\Leftrightarrow 2q^2-9q-72r+81\geq 0,\text{as } r\leq\frac{q^2}{9}$$

It suffices that $3(3-q)(2q+9)\ge 0$ which is true as $q\le \frac{p^2}{3}=3$.

Equality holds for $x=y=z$.

1027. Let $a,b,c>0, a+b+c=3$. Prove that:

$$3+2\left(\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}\right)\ge\sum_{cyc}\sqrt{1+4\left(\frac{1}{a^2}+\frac{1}{b^2}\right)}$$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\sqrt{1+4\left(\frac{1}{a^2}+\frac{1}{b^2}\right)}=\sum_{cyc}\left(\sqrt{\frac{4a^2+4b^2+a^2b^2}{ab}}\cdot\sqrt{\frac{1}{ab}}\right)\overset{CBS}{\le}\sqrt{\sum_{cyc}\frac{4a^2+4b^2+a^2b^2}{ab}}\cdot\sqrt{\sum_{cyc}\frac{1}{ab}}$$

$$=\sqrt{\frac{1}{a^2b^2c^2}}\cdot\sqrt{4\sum_{cyc}c(a^2+b^2)+\sum_{cyc}ca^2b^2}\cdot\sqrt{\sum_{cyc}a}$$

$$=\frac{1}{abc}\sqrt{\sum_{cyc}a}\cdot\sqrt{4\sum_{cyc}(ab(a+b))+abc\sum_{cyc}ab}$$

$$=\frac{1}{abc}\sqrt{\sum_{cyc}a}\cdot\sqrt{4\sum_{cyc}\left(ab\left(\sum_{cyc}a-c\right)\right)+abc\sum_{cyc}ab}$$

$$=\frac{1}{abc}\sqrt{\sum_{cyc}a}\cdot\sqrt{4\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)-12abc+abc\sum_{cyc}ab}$$

$$\overset{\because 3=a+b+c}{=}\frac{1}{abc}\sqrt{\frac{4}{9}\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a\right)^4-\frac{12abc}{9}\left(\sum_{cyc}a\right)^3+abc\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a\right)}\overset{?}{\le}3$$

$$+2\left(\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}\right)=\frac{1}{abc}\left(3abc+2\sum_{cyc}a^2\right)$$

$$\Leftrightarrow\left(3abc+2\sum_{cyc}a^2\right)^2\overset{?}{\ge}\frac{4}{9}\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a\right)^4-\frac{12abc}{9}\left(\sum_{cyc}a\right)^3+abc\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a\right)$$

$$\Leftrightarrow 9a^2b^2c^2 + 4\left(\sum_{cyc} a^2\right)^2 + 12abc\left(\sum_{cyc} a^2\right) \overset{?}{\geq} \frac{4}{9}\left(\sum_{cyc} ab\right)\left(\sum_{cyc} a\right)^4 - \frac{12abc}{9}\left(\sum_{cyc} a\right)^3 + abc\left(\sum_{cyc} ab\right)\left(\sum_{cyc} a\right)$$

$$\overset{\because 3 = a+b+c}{\Leftrightarrow} 9a^2b^2c^2 + \frac{4}{9}\left(\sum_{cyc} a^2\right)^2\left(\sum_{cyc} a\right)^2 + \frac{12abc}{3}\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a\right) \overset{?}{\geq} \frac{4}{9}\left(\sum_{cyc} ab\right)\left(\sum_{cyc} a\right)^4 - \frac{12abc}{9}\left(\sum_{cyc} a\right)^3 + abc\left(\sum_{cyc} ab\right)\left(\sum_{cyc} a\right)$$

$$\Leftrightarrow 81a^2b^2c^2 + 36abc\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a\right) + 4\left(\sum_{cyc} a^2\right)^2\left(\sum_{cyc} a\right)^2 \overset{?}{\geq} 4\left(\sum_{cyc} ab\right)\left(\sum_{cyc} a\right)^4 - 12abc\left(\sum_{cyc} a\right)^3 + 9abc\left(\sum_{cyc} ab\right)\left(\sum_{cyc} a\right)$$

$$\overset{expanding\ and\ re-arranging}{\Leftrightarrow} \boxed{4\sum_{cyc} a^6 + 4\sum_{cyc} a^5b + 4\sum_{cyc} ab^5 + 20abc\left(\sum_{cyc} a^3\right) + 6a^2b^2c^2 \underbrace{\overset{?}{\geq}}_{(*)} 4\sum_{cyc} a^4b^2 + 4\sum_{cyc} a^2b^4 + 8\sum_{cyc} a^3b^3 + 9abc\left(\sum_{cyc} a^2b + \sum_{cyc} ab^2\right)}$$

$$\text{Now, } 4\sum_{cyc} a^5b + 4\sum_{cyc} ab^5 \underbrace{\overset{A-G}{\geq}}_{(1)} 8\sum_{cyc} a^3b^3$$

$$\text{Also, } 4\sum_{cyc} a^6 + 12a^2b^2c^2 \underbrace{\overset{Schur}{\geq}}_{(2)} 4\sum_{cyc} a^4b^2 + 4\sum_{cyc} a^2b^4$$

$$\text{Moreover, } \sum_{cyc} a^3 + 3abc \underbrace{\overset{Schur}{\geq}}_{(i)} \sum_{cyc} a^2b + \sum_{cyc} ab^2 \text{ and } \sum_{cyc} a^3 \underbrace{\overset{A-G}{\geq}}_{(ii)} 3abc \therefore (i) + (ii) \Rightarrow 2\sum_{cyc} a^3$$

$$\geq \sum_{cyc} a^2b + \sum_{cyc} ab^2 \Rightarrow 18abc\sum_{cyc} a^3 \overset{(3)}{\geq} 9abc\left(\sum_{cyc} a^2b + \sum_{cyc} ab^2\right) \text{ and}$$

$$lastly,\ 2abc\left(\sum_{cyc} a^3\right) \underbrace{\overset{A-G}{\geq}}_{(4)} 6a^2b^2c^2 \therefore (1) + (2) + (3) + (4) \Rightarrow (*) \text{ is true}$$

$$\Rightarrow \sum_{cyc} \sqrt{1 + 4\left(\frac{1}{a^2} + \frac{1}{b^2}\right)} \leq 3 + 2\left(\frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab}\right) \forall a, b, c > 0 \mid a + b + c = 3 \text{ (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$By\ AM - GM\ inequality\ we\ have:$$

$$2\sqrt{1+4\left(\frac{1}{a^2}+\frac{1}{b^2}\right)}\le\frac{1+4\left(\frac{1}{a^2}+\frac{1}{b^2}\right)}{1+\frac{2c}{ab}}+\left(1+\frac{2c}{ab}\right)$$

$$=\left(\frac{ab+4c\left(\frac{b}{ca}+\frac{a}{bc}\right)}{ab+2c}+\left(\frac{2c}{ab}-1\right)\right)+2=$$

$$=\frac{4c\left(\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}\right)}{ab+2c}+2=2\left(\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}\right)\left(1-\frac{ab}{ab+2c}\right)+2.$$

$$\textit{Then}:\ \sqrt{1+4\left(\frac{1}{a^2}+\frac{1}{b^2}\right)}\le 1+\left(\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}\right)\left(1-\frac{ab}{ab+2c}\right)\ (\textit{and analogs})$$

Summing up this inequality with similar ones we get:

$$\sum_{cyc}\sqrt{1+4\left(\frac{1}{a^2}+\frac{1}{b^2}\right)}$$

$$\le 3+\left(\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}\right)\left(3-\sum_{cyc}\frac{ab}{ab+2c}\right)\overset{CBS}{\le}3$$

$$+\left(\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}\right)\left(3-\frac{\left(\sum_{cyc}ab\right)^2}{\sum_{cyc}ab(ab+2c)}\right)=$$

$$=3+\left(\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}\right)\left(3-\frac{\left(\sum_{cyc}ab\right)^2}{\sum_{cyc}(ab)^2+6abc}\right)$$

$$=3+\left(\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}\right)\left(3-\frac{\left(\sum_{cyc}ab\right)^2}{\sum_{cyc}(ab)^2+2abc\sum_{cyc}a}\right)=$$

$$=3+\left(\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}\right)(3-1)=3+2\left(\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}\right),\ \textit{the desired result.}$$

Equality holds for $a=b=c=1$.

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

$$3+2\left(\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}\right)\ge\sum_{cyc}\sqrt{1+4\left(\frac{1}{a^2}+\frac{1}{b^2}\right)}$$

$$3+2\left(\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}\right)\ge\frac{1}{3}\sum_{cyc}\sqrt{\frac{9}{ab}\left(ab+4\left(\frac{a}{b}+\frac{b}{a}\right)\right)}$$

$$3+2\left(\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}\right)\ge\frac{1}{3}\left(9\sum_{cyc}\frac{1}{ab}+\sum_{cyc}ab+4\sum_{cyc}\left(\frac{a}{b}+\frac{b}{a}\right)\right)$$

$$18+12\left(\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}\right)\ge 9\sum_{cyc}\frac{1}{ab}+\sum_{cyc}ab+4\sum_{cyc}\left(\frac{a}{b}+\frac{b}{a}\right)$$

$$18abc+12(a^2+b^2+c^2)\ge$$

$$\ge 9(a+b+c)+(ab)^2c+(bc)^2a+(ca)^2b^2+4\sum_{cyc}(a^2b+ab^2)$$

$$18abc+4(a^3+b^3+c^3)+\sum_{cyc}(ab^2+a^2b)\ge$$

$$\ge 27+\sum_{cyc}a^2b^2c+4\sum_{cyc}a^2b+ab^2)$$

$$12abc+3(a^3+b^3+c^3)\ge 3\sum_{cyc}a^2b+\sum_{cyc}a^2b^2c \text{ true, because:}$$

$$3[(a^3+b^3+c^3)+3abc]\ge 3\sum_{cyc}(a^2b+ab^2) \text{ and } 3abc\ge\sum_{cyc}a^2b^2c$$

1028. Let $a,b,c>0, a+b+c=1$. Prove that:

$$\frac{a(a^2-bc)}{abc(a^2-bc)+3(bc)^2}+\frac{b(b^2-ca)}{abc(b^2-ca)+3(ca)^2}+\frac{c(c^2-ab)}{abc(c^2-ab)+3(ba)^2}\ge 0$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a(a^2-bc)}{abc(a^2-bc)+3(bc)^2}+\frac{b(b^2-ca)}{abc(b^2-ca)+3(ca)^2}+\frac{c(c^2-ab)}{abc(c^2-ab)+3(ab)^2}\ge 0$$

$$\Leftrightarrow\sum_{cyc}\frac{(a^3-abc+3bc)-3bc}{bc(a^3-abc+3bc)}\ge 0\Leftrightarrow\sum_{cyc}\frac{1}{bc}\ge 3\sum_{cyc}\frac{1}{a^3-abc+3bc}$$

$$\overset{\because 1=a+b+c}{\Leftrightarrow}\boxed{\sum_{cyc}\frac{1}{bc}\overset{(*)}{\ge}3\sum_{cyc}\frac{1}{a^3-abc+3bc(a+b+c)}}$$

$$\text{Now, } a^3-abc+3bc(a+b+c)=a^3+2abc+3b^2c+3bc^2$$

$$=a^3+b^2c+bc^2+2bc(b+c)+2abc\overset{A-G}{\ge}3\sqrt[3]{a^3.b^2c.bc^2}$$

$$+2bc(a+b+c)\overset{a+b+c=1}{=}3abc+2bc$$

$$\Rightarrow \frac{1}{a^3-abc+3bc(a+b+c)} \le \frac{1}{3abc+2bc} \text{ and analogs} \Rightarrow 3\sum_{cyc}\frac{1}{a^3-abc+3bc(a+b+c)}$$

$$\le 3\sum_{cyc}\frac{1}{3abc+2bc} \overset{?}{\le} \sum_{cyc}\frac{1}{bc} = \frac{a+b+c}{abc} \overset{a+b+c=1}{=} \frac{1}{abc}$$

$$\Leftrightarrow \frac{1}{3abc} \overset{?}{\ge} \frac{1}{bc(3a+2)} + \frac{1}{ca(3b+2)} + \frac{1}{ab(3c+2)}$$

$$= \frac{a(3b+2)(3c+2)+b(3c+2)(3a+2)+c(3a+2)(3b+2)}{abc(3a+2)(3b+2)(3c+2)}$$

$$\Leftrightarrow (3a+2)(3b+2)(3c+2) \overset{?}{\ge} 3a(3b+2)(3c+2)+3b(3c+2)(3a+2)$$

$$+3c(3a+2)(3b+2) \overset{\text{expanding and re-arranging}}{\Leftrightarrow} \boxed{4 \underbrace{\overset{?}{\ge}}_{(**)} 27abc+9\sum_{cyc}ab}$$

$$\text{Now, } 1 = a+b+c \overset{A-G}{\ge} 3.\sqrt[3]{abc} \Rightarrow 3.\sqrt[3]{abc} \le 1 \Rightarrow 27abc \overset{(i)}{\le} 1 \text{ and } 3\sum_{cyc}ab$$

$$\le (a+b+c)^2 \overset{a+b+c=1}{=} 1 \Rightarrow 9\sum_{cyc}ab \overset{(ii)}{\le} 3 \therefore (i)+(ii) \Rightarrow 27abc+9\sum_{cyc}ab \le 4$$

$$\Rightarrow (**) \Rightarrow (*) \text{ is true}$$

$$\therefore \frac{a(a^2-bc)}{abc(a^2-bc)+3(bc)^2} + \frac{b(b^2-ca)}{abc(b^2-ca)+3(ca)^2} + \frac{c(c^2-ab)}{abc(c^2-ab)+3(ab)^2}$$

$$\ge 0 \ \forall\, a,b,c>0 \ \middle|\ a+b+c=1 \text{ (QED)}$$

1029. If $a,b,c>0, a+b+c=\lambda>0$ then:

$$\frac{a}{(b+c)^3}+\frac{b}{(c+a)^3}+\frac{c}{(a+b)^3} \ge \frac{27}{8\lambda^2}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Tapas Das-India

$$\frac{a}{(b+c)^3}+\frac{b}{(c+a)^3}+\frac{c}{(a+b)^3} = \frac{a^4}{(ab+ac)^3}+\frac{b^4}{(bc+ba)^3}+\frac{c^4}{(ca+cb)^3} \overset{Radon}{\ge}$$

$$\ge \frac{(a+b+c)^4}{(2ab+2bc+2ca)^3} = \frac{\lambda^4}{8(ab+bc+ca)^3} \ge$$

$$\ge \frac{\lambda^4}{8\cdot\frac{(a+b+c)^6}{27}} = \frac{27\lambda^4}{8\lambda^6} = \frac{27}{8\lambda^2}$$

Solution 2 by Daoudi Abdessattar-Tunisia

$$\left(\sum_{cyc}\frac{a}{(b+c)^3}\right)\left(\sum_{cyc}a\right)\left(\sum_{cyc}a\right)\overset{Holder}{\geq}\left(\sum_{cyc}\frac{a}{b+c}\right)^3\overset{Nesbitt}{\geq}\left(\frac{3}{2}\right)^3$$

$$\frac{a}{(b+c)^3}+\frac{b}{(c+a)^3}+\frac{c}{(a+b)^3}\geq\frac{27}{8\lambda^2}$$

Solution 3 by Marian Dincă-Romania

Let $f(x)=\frac{1}{x^3}$ convex function, decreasing and use weighted Jensen's inequality:

$$\frac{a}{(b+c)^3}+\frac{b}{(c+a)^3}+\frac{c}{(a+b)^3}=af(b+c)+bf(c+a)+cf(a+b)\geq$$

$$\geq(a+b+c)f\left(\frac{a(b+c)+b(c+a)+c(a+b)}{a+b+c}\right)\geq(a+b+c)f\left(\frac{2(a+b+c)}{3}\right)=\frac{27}{8\lambda^2}$$

Solution 4 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\frac{a}{(b+c)^3}+\frac{b}{(c+a)^3}+\frac{c}{(a+b)^3}=\frac{\left(\frac{a}{b+c}\right)^2}{a(b+c)}+\frac{\left(\frac{b}{c+a}\right)^2}{b(c+a)}+\frac{\left(\frac{c}{a+b}\right)^2}{c(a+b)}\geq$$

$$\geq\frac{\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)^2}{2(ab+bc+ca)}\geq\frac{\left(\frac{(a+b+c)^2}{2(ab+bc+ca)}\right)^2}{2(ab+bc+ca)}\geq\frac{27}{8\lambda^2}$$

$$\frac{(a+b+c)^4}{8(ab+bc+ca)^3}\geq\frac{27}{8\lambda^2}\Leftrightarrow(a+b+c)^4\lambda^2\geq27(ab+bc+ca)^3$$

$$(a+b+c)^6(\geq27(ab+bc+ca)^3$$

$(a+b+c)^2(a+b+c)^2(a+b+c)^2\geq27(ab+bc+ca)^3$ which is true because:

$$(a+b+c)^2\geq3(ab+bc+ca)$$

1030. If $a,b,c>0$ then:

$$\frac{\frac{a(b^3+c^3)}{b+c}+\frac{b(c^3+a^3)}{c+a}+\frac{c(a^3+b^3)}{a+b}}{\frac{a(b^2+c^2)}{b+c}+\frac{b(c^2+a^2)}{c+a}+\frac{c(a^2+b^2)}{a+b}}\leq\frac{a^3+b^3+c^3}{a^2+b^2+c^2}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\textit{The desired inequality is succesively equivalent to}:$$

$$(a^3+b^3+c^3)\left(\frac{a(b^2+c^2)}{b+c}+\frac{b(c^2+a^2)}{c+a}+\frac{c(a^2+b^2)}{a+b}\right)$$

$$\geq (a^2+b^2+c^2)\left(\frac{a(b^3+c^3)}{b+c}+\frac{b(c^3+a^3)}{c+a}+\frac{c(a^3+b^3)}{a+b}\right)$$

$$\sum_{cyc}\frac{a^4(b^2+c^2)}{b+c}+\sum_{cyc}\frac{a(b^2+c^2)(b^3+c^3)}{b+c}\geq\sum_{cyc}\frac{a^3(b^3+c^3)}{b+c}+\sum_{cyc}\frac{a(b^2+c^2)(b^3+c^3)}{b+c}$$

$$\sum_{cyc}\frac{a^3[a(b^2+c^2)-(b^3+c^3)]}{b+c}\geq 0 \;\Leftrightarrow\; \sum_{cyc}\left(\frac{a^3b^2(a-b)}{b+c}-\frac{a^3c^2(c-a)}{b+c}\right)\geq 0$$

$$\Leftrightarrow \sum_{cyc}\left(\frac{a^3b^2(a-b)}{b+c}-\frac{b^3a^2(a-b)}{c+a}\right)\geq 0 \;\Leftrightarrow$$

$$\sum_{cyc}\frac{a^2b^2(a-b)[a(c+a)-b(b+c)]}{(b+c)(c+a)}\geq 0\Leftrightarrow\sum_{cyc}\frac{a^2b^2(a-b)^2(a+b+c)}{(b+c)(c+a)}\geq 0$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$\text{Assigning } b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0, y+z-x=2a$
$>0 \text{ and } z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y$
$\Rightarrow x,y,z \text{ form sides}$
$\text{of a triangle with semiperimeter, circumradius and inradius}$

$$=s,R,r\ (say)\text{ yielding } 2\sum_{cyc}a=\sum_{cyc}x=2s\Rightarrow \boxed{\sum_{cyc}a\overset{(\bullet)}{=}s}\Rightarrow a=s-x, b$$

$$=s-y, c=s-z\Rightarrow \boxed{abc\overset{(\bullet\bullet)}{=}r^2s}$$

$$\text{Via aforementioned substitutions, }\sum_{cyc}ab=\sum_{cyc}(s-x)(s-y)=4Rr+r^2$$

$$\Rightarrow\boxed{\sum_{cyc}ab\overset{(\bullet\bullet\bullet)}{=}4Rr+r^2}\text{ and }\sum_{cyc}a^2$$

$$=\left(\sum_{cyc}a\right)^2-2\sum_{cyc}ab\overset{\text{via }(\bullet),(\bullet\bullet\bullet)}{=}s^2-2\sum_{cyc}(s-x)(s-y)$$

$$= s^2 - 2(4Rr + r^2) \Rightarrow \sum_{cyc} a^2 \overset{(\bullet\bullet\bullet\bullet)}{=} s^2 - 8Rr - 2r^2 \text{ and also}, \sum_{cyc} a^3$$

$$= \left(\sum_{cyc} a\right)^3 - 3(a+b)(b+c)(c+a) \overset{\text{via }(\bullet)}{=} s^3 - 3xyz = s^3 - 12Rrs$$

$$\Rightarrow \sum_{cyc} a^3 \overset{(\bullet\bullet\bullet\bullet\bullet)}{=} s^3 - 12Rrs$$

$$\text{Now}, \sum_{cyc} \frac{a(b^2+c^2)}{b+c} = \sum_{cyc} \frac{a(b+c)^2 - 2abc}{b+c} = \sum_{cyc}\big(a(b+c)\big) - 2abc.\frac{\sum_{cyc}\big((c+a)(a+b)\big)}{\prod_{cyc}(b+c)}$$

$$= 2\sum_{cyc} ab - 2abc.\frac{\left(\sum_{cyc} a^2 + 2\sum_{cyc} ab\right) + \sum_{cyc} ab}{\prod_{cyc}(b+c)}$$

$$= 2\sum_{cyc} ab - 2abc.\frac{\left(\sum_{cyc} a\right)^2 + \sum_{cyc} ab}{\prod_{cyc}(b+c)} \overset{\text{via }(\bullet),(\bullet\bullet),(\bullet\bullet\bullet)}{=} 2(4Rr + r^2) - \frac{2r^2 s(s^2 + 4Rr + r^2)}{4Rrs}$$

$$= \frac{4R(4Rr + r^2) - r(s^2 + 4Rr + r^2)}{2R} \Rightarrow \boxed{\sum_{cyc} \frac{a(b^2+c^2)}{b+c} \overset{(i)}{=} \frac{(4R - r)(4Rr + r^2) - rs^2}{2R}}$$

$$\text{Again}, \sum_{cyc} \frac{a(b^3+c^3)}{b+c} = \sum_{cyc} a(b^2 - bc + c^2) = \sum_{cyc} ab(a+b) - 3abc \overset{\text{via }(\bullet\bullet)}{=} \sum_{cyc}(s-x)(s-y)z - 3r^2 s$$

$$= \sum_{cyc}\Big((s^2 - s(2s - z) + xy)z\Big) - 3r^2 s$$

$$= -s^2 \sum_{cyc} x + s\sum_{cyc} x^2 + 3xyz - 3r^2 s = -2s^3 + 2s(s^2 - 4Rr - r^2) + 12Rrs - 3r^2 s$$

$$\Rightarrow \boxed{\sum_{cyc} \frac{a(b^3+c^3)}{b+c} \overset{(ii)}{=} 4Rrs - 5r^2 s}$$

$$\text{Now}, \frac{\frac{a(b^3+c^3)}{b+c} + \frac{b(c^3+a^3)}{c+a} + \frac{c(a^3+b^3)}{a+b}}{\frac{a(b^2+c^2)}{b+c} + \frac{b(c^2+a^2)}{c+a} + \frac{c(a^2+b^2)}{a+b}} \le \frac{a^3+b^3+c^3}{a^2+b^2+c^2} \Leftrightarrow \left(\sum_{cyc} a^3\right)\left(\sum_{cyc} \frac{a(b^2+c^2)}{b+c}\right)$$

$$\ge \left(\sum_{cyc} a^2\right)\left(\sum_{cyc} \frac{a(b^3+c^3)}{b+c}\right)$$

$$\overset{\text{via }(\bullet\bullet\bullet\bullet),(\bullet\bullet\bullet\bullet\bullet),(i),(ii)}{\Leftrightarrow} (s^3 - 12Rrs).\frac{(4R - r)(4Rr + r^2) - rs^2}{2R} \ge (s^2 - 8Rr - 2r^2)(4Rrs - 5r^2 s)$$

$$\Leftrightarrow (s^2 - 12Rr)(16R^2 - r^2 - s^2) \ge 2R(4R - 5r)(s^2 - 8Rr - 2r^2)$$

$$\Leftrightarrow \boxed{\boxed{s^4 - (8R^2 + 22Rr - r^2)s^2 + 8Rr(4R + r)^2 \overset{(*)}{\le} 0}}$$

$$\text{Now, Rouche} \Rightarrow s^2 - (m - n) \ge 0 \text{ and } s^2 - (m + n) \le 0, \text{where } m = 2R^2 + 10Rr - r^2 \text{ and } n$$

$$= 2(R - 2r).\sqrt{R^2 - 2Rr} \therefore \big(s^2 - (m + n)\big)\big(s^2 - (m - n)\big) \le 0$$

$$\Rightarrow s^4 - s^2(2m) + m^2 - n^2 \le 0 \Rightarrow s^4 - s^2(4R^2 + 20Rr - 2r^2) + r(4R + r)^3 \le 0$$

$$\Rightarrow \text{in order to prove } (*), \text{it suffices to prove :}$$

$$s^4 - (8R^2 + 22Rr - r^2)s^2 + 8Rr(4R + r)^2 \le s^4 - s^2(4R^2 + 20Rr - 2r^2) + r(4R + r)^3$$

$$\Leftrightarrow \boxed{\boxed{(4R^2 + 2Rr + r^2)s^2 \overset{(**)}{\ge} r(4R - r)(4R + r)^2}}$$

$$\text{Now, LHS of } (**) \overset{\text{Rouche}}{\geq} (4R^2+2Rr+r^2)\left(2R^2+10Rr-r^2-2(R-2r).\sqrt{R^2-2Rr}\right) \overset{?}{\geq} r(4R-r)(4R+r)^2$$

$$\Leftrightarrow 4R^4-10R^3r+R^2r^2+6Rr^3 \overset{?}{\geq} (R-2r)(4R^2+2Rr+r^2).\sqrt{R^2-2Rr}$$

$$\Leftrightarrow R(R-2r)(4R^2-2Rr-3r^2) \overset{?}{\geq} (R-2r)(4R^2+2Rr+r^2).\sqrt{R^2-2Rr}$$

$$\Leftrightarrow R(4R^2-2Rr-3r^2) \overset{?}{\geq} (4R^2+2Rr+r^2).\sqrt{R^2-2Rr}\left(\because R-2r \overset{\text{Euler}}{\geq} 0\right)$$

$$\Leftrightarrow R^2(4R^2-2Rr-3r^2)^2 \overset{?}{\geq} (R^2-2Rr)(4R^2+2Rr+r^2)^2$$

$$\left(\because 4R^2-2Rr-3r^2=(R-2r)(4R+6r)+9r^2 \overset{\text{Euler}}{\geq} 9r^2>0\right) \Leftrightarrow 2r^3(4R+r)^2 \overset{?}{\geq} 0 \to \text{true} \Rightarrow (**)$$

$$\Rightarrow (*) \text{ is true } \therefore \frac{\frac{a(b^3+c^3)}{b+c}+\frac{b(c^3+a^3)}{c+a}+\frac{c(a^3+b^3)}{a+b}}{\frac{a(b^2+c^2)}{b+c}+\frac{b(c^2+a^2)}{c+a}+\frac{c(a^2+b^2)}{a+b}} \leq \frac{a^3+b^3+c^3}{a^2+b^2+c^2}$$

1031. If $a,b,c,x,y>0$ then:

$$\sum_{cyc(a,b,c)}\left(\frac{\sqrt{a^{x+y}}}{\sqrt{b^{x+y}}+\sqrt{c^{x+y}}}-\frac{a^{\sqrt{xy}}}{b^{\sqrt{xy}}+c^{\sqrt{xy}}}\right)\geq 0.$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\text{Let } f(t)=\sum_{cyc}\frac{a^t}{b^t+c^t},\quad t>0.$$

We have :

$$f'(t)=\sum_{cyc}\frac{ln(a).a^t(b^t+c^t)-a^t(ln(b).b^t+ln(c).c^t)}{(b^t+c^t)^2}=$$

$$=\sum_{cyc}\left(\frac{a^tb^t(ln(a)-ln(b))}{(b^t+c^t)^2}-\frac{c^ta^t(ln(c)-ln(a))}{(b^t+c^t)^2}\right)$$

$$=\sum_{cyc}\left(\frac{a^tb^t(ln(a)-ln(b))}{(b^t+c^t)^2}-\frac{a^tb^t(ln(a)-ln(b))}{(c^t+a^t)^2}\right)=$$

$$=\sum_{cyc}\frac{a^tb^t(ln(a)-ln(b))(a^t-b^t)(a^t+b^t+2c^t)}{(b^t+c^t)^2(c^t+a^t)^2}\geq 0,$$

because $ln(a)-ln(b)$ and a^t-b^t have the same sign for any $a,b,t>0$.

Then f is increasing on $(0,\infty)$.

$$\text{Since } \frac{x+y}{2}\geq\sqrt{xy} \text{ then we have : } f\left(\frac{x+y}{2}\right)\geq f(\sqrt{xy}) \Leftrightarrow$$

$$\sum_{cyc}\frac{\sqrt{a^{x+y}}}{\sqrt{b^{x+y}}+\sqrt{c^{x+y}}}\ge\sum_{cyc}\frac{a^{\sqrt{xy}}}{b^{\sqrt{xy}}+c^{\sqrt{xy}}}.$$

Therefore,

$$\sum_{cyc}\left(\frac{\sqrt{a^{x+y}}}{\sqrt{b^{x+y}}+\sqrt{c^{x+y}}}-\frac{a^{\sqrt{xy}}}{b^{\sqrt{xy}}+c^{\sqrt{xy}}}\right)\ge 0.$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\text{Let } z=\frac{x+y}{2},\ \ t=\sqrt{xy},\ \ p=z-t\overset{AM-GM}{\ge}0.$$

The desired inequality is succesively equivalent to :

$$\sum_{cyc}\left(\frac{a^z}{b^z+c^z}-\frac{a^t}{b^t+c^t}\right)\ge 0\Leftrightarrow\sum_{cyc}\frac{a^z(b^t+c^t)-a^t(b^z+c^z)}{(b^z+c^z)(b^t+c^t)}\ge 0$$

$$\Leftrightarrow\sum_{cyc}\left(\frac{a^tb^t(a^p-b^p)}{(b^z+c^z)(b^t+c^t)}-\frac{c^ta^t(c^p-a^p)}{(b^z+c^z)(b^t+c^t)}\right)\ge 0$$

$$\Leftrightarrow\sum_{cyc}\left(\frac{a^tb^t(a^p-b^p)}{(b^z+c^z)(b^t+c^t)}-\frac{a^tb^t(a^p-b^p)}{(c^z+a^z)(c^t+a^t)}\right)\ge 0$$

$$\Leftrightarrow\sum_{cyc}\frac{a^tb^t(a^p-b^p)[(c^z+a^z)(c^t+a^t)-(b^z+c^z)(b^t+c^t)]}{(b^z+c^z)(b^t+c^t)(c^z+a^z)(c^t+a^t)}\ge 0$$

$$\Leftrightarrow\sum_{cyc}\frac{a^tb^t(a^p-b^p)[(a^{z+t}-b^{z+t})+c^z(a^t-b^t)+c^t(a^z-b^z)]}{(b^z+c^z)(b^t+c^t)(c^z+a^z)(c^t+a^t)}\ge 0$$

Which is true because $a^p-b^p, a^{z+t}-b^{z+t}, a^t-b^t$ and a^z-b^z have the same sign. So the proof is complete.

1032. **If $a,b,c>0$ then:**

$$8\left(\sum_{cyc}\frac{a^5}{b^2+c^2}\right)\left(\sum_{cyc}\frac{a^6}{b^3+c^3}\right)\left(\sum_{cyc}\frac{a^7}{b^4+c^4}\right)\ge(a^3+b^3+c^3)^3$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Sanong Huayrerai-Nakon Pathom-Thailand

$$8\left(\sum_{cyc}\frac{a^5}{b^2+c^2}\right)\left(\sum_{cyc}\frac{a^6}{b^3+c^3}\right)\left(\sum_{cyc}\frac{a^7}{b^4+c^4}\right)\ge$$

$$=8\cdot\frac{(\sum a^3)^3}{3^3}\cdot\frac{(\sum a^2)^2}{2(\sum a^2b^2)}\cdot\frac{(\sum a^3)^3}{2(\sum a^3b^3)}\cdot\frac{(\sum a^4)^2}{2(\sum a^4b^4)}\ge$$

$$\ge\frac{(\sum a^3)^3}{3^3}\cdot\frac{3(\sum a^2b^2)}{\sum a^2b^2}\cdot\frac{3(\sum a^3b^3)}{\sum a^3b^3}\cdot\frac{3(\sum a^4b^4)}{\sum a^4b^4}=(a^3+b^3+c^3)^3$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\text{Schur}\Rightarrow\forall\, u,v,w>0,\ u^3+v^3+w^3+3uvw\ge\sum_{cyc}u^2v+\sum_{cyc}uv^2\ \text{ and } u^3+v^3+w^3\ge 3uvw$$

$$\therefore\ \text{summing up},\ 2\sum_{cyc}u^3\overset{(*)}{\ge}\sum_{cyc}u^2v+\sum_{cyc}uv^2$$

$$\sum_{cyc}\frac{a^7}{b^4+c^4}=\sum_{cyc}\frac{a^9}{a^2b^4+a^2c^4}\overset{\text{Holder}}{\ge}\frac{\left(\sum_{cyc}a^3\right)^3}{3\left(\sum_{cyc}a^4b^2+\sum_{cyc}a^2b^4\right)}\overset{\text{via }(*)}{\ge}\frac{\left(\sum_{cyc}a^3\right)^3}{6\sum_{cyc}a^6}$$

$$\therefore\sum_{cyc}\frac{a^7}{b^4+c^4}\overset{(i)}{\ge}\frac{\left(\sum_{cyc}a^3\right)^3}{6\sum_{cyc}a^6}$$

$$\text{WLOG assuming } a\ge b\ge c\Rightarrow a^6\ge b^6\ge c^6 \text{ and } \frac{1}{b^3+c^3}\ge\frac{1}{c^3+a^3}\ge\frac{1}{a^3+b^3}$$

$$\therefore\sum_{cyc}\frac{a^6}{b^3+c^3}\overset{\text{Chebyshev}}{\ge}\frac{1}{3}\left(\sum_{cyc}a^6\right)\left(\sum_{cyc}\frac{1}{b^3+c^3}\right)\overset{\text{Bergstrom}}{\ge}\frac{1}{3}\left(\sum_{cyc}a^6\right)\left(\frac{9}{2\sum_{cyc}a^3}\right)$$

$$\therefore\sum_{cyc}\frac{a^6}{b^3+c^3}\overset{(ii)}{\ge}\frac{3\sum_{cyc}a^6}{2\sum_{cyc}a^3}$$

$$\sum_{cyc}\frac{a^5}{b^2+c^2}=\sum_{cyc}\frac{a^6}{ab^2+ac^2}\overset{\text{Bergstrom}}{\ge}\frac{\left(\sum_{cyc}a^3\right)^2}{\sum_{cyc}a^2b+\sum_{cyc}ab^2}\overset{\text{via }(*)}{\ge}\frac{\left(\sum_{cyc}a^3\right)^2}{2\left(\sum_{cyc}a^3\right)}$$

$$\therefore\sum_{cyc}\frac{a^5}{b^2+c^2}\overset{(iii)}{\ge}\frac{\sum_{cyc}a^3}{2}$$

$$\therefore\ (i)\bullet(ii)\bullet(iii)\Rightarrow 8\left(\sum_{cyc}\frac{a^5}{b^2+c^2}\right)\left(\sum_{cyc}\frac{a^6}{b^3+c^3}\right)\left(\sum_{cyc}\frac{a^7}{b^4+c^4}\right)$$

$$\ge 8\bullet\frac{\sum_{cyc}a^3}{2}\bullet\frac{3\sum_{cyc}a^6}{2\sum_{cyc}a^3}\bullet\frac{\left(\sum_{cyc}a^3\right)^3}{6\sum_{cyc}a^6}=\left(a^3+b^3+c^3\right)^3\ \text{(QED)}$$

1033. Let $a,b,c>0$. Prove that:

$$\frac{\frac{1}{a}+b}{\sqrt{\frac{1}{a}+a}}+\frac{\frac{1}{b}+c}{\sqrt{\frac{1}{b}+b}}+\frac{\frac{1}{c}+a}{\sqrt{\frac{1}{c}+c}}\ge 3\sqrt{2}$$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:\ \frac{\frac{1}{a}+b}{\sqrt{\frac{1}{a}+a}}=\frac{\sqrt{a}.\left(\frac{1}{a}+b\right)}{\sqrt{1+a^2}}=$$

$$=\frac{\sqrt{a}.\left(\frac{1}{a}+b\right)}{\left(\sqrt{1+a^2}+\sqrt{2a}\right)-\sqrt{2a}}\overset{CBS}{\ge}\frac{\sqrt{a}.\left(\frac{1}{a}+b\right)}{\sqrt{(1+1)[(1+a^2)+2a]}-\sqrt{2a}}=$$

$$=\frac{\sqrt{a}.\left(\frac{1}{a}+b\right)}{\sqrt{2}(a+1-\sqrt{a})}=\left(\frac{\sqrt{a}.\left(\frac{1}{a}+b\right)}{\sqrt{2}(a-\sqrt{a}+1)}+\frac{\sqrt{2}(a-\sqrt{a}+1)}{\sqrt{a}}\right)-\frac{\sqrt{2}(a-\sqrt{a}+1)}{\sqrt{a}}\ge$$

$$\overset{AM-GM}{\ge}2\sqrt{\frac{1}{a}+b}-\sqrt{2}\left(\sqrt{a}-1+\frac{1}{\sqrt{a}}\right)\overset{CBS}{\ge}\sqrt{2}\left(\sqrt{\frac{1}{a}}+\sqrt{b}\right)-\sqrt{2}\left(\sqrt{a}-1+\frac{1}{\sqrt{a}}\right)$$

$$Then:\ \frac{\frac{1}{a}+b}{\sqrt{\frac{1}{a}+a}}\ge\sqrt{2}(\sqrt{b}-\sqrt{a})+\sqrt{2}.$$

$$Similarly\ we\ have:\ \frac{\frac{1}{b}+c}{\sqrt{\frac{1}{b}+b}}\ge\sqrt{2}(\sqrt{c}-\sqrt{b})+\sqrt{2}\ \ and$$

$$\frac{\frac{1}{c}+a}{\sqrt{\frac{1}{c}+c}}\ge\sqrt{2}(\sqrt{a}-\sqrt{c})+\sqrt{2}.$$

Summing up these inequalities yields the desired result.

Equality holds iff $a=b=c=1$.

Solution 2 by Said Cerbach-Algiers-Algerie

$$\frac{\frac{1}{a}+b}{\sqrt{\frac{1}{a}+a}}+\frac{\frac{1}{b}+c}{\sqrt{\frac{1}{b}+b}}+\frac{\frac{1}{c}+a}{\sqrt{\frac{1}{c}+c}}\ge 3\sqrt{2}$$

$$\frac{\frac{1}{a}+b}{\sqrt{\frac{1}{a}+a}}=\frac{\frac{1}{a}+a+b-a}{\sqrt{\frac{1}{a}+a}}=\sqrt{\frac{1}{a}+a}+\frac{b-a}{\sqrt{\frac{1}{a}+a}}$$

Hence,

$$\frac{\frac{1}{a}+b}{\sqrt{\frac{1}{a}+a}}+\frac{\frac{1}{b}+c}{\sqrt{\frac{1}{b}+b}}+\frac{\frac{1}{c}+a}{\sqrt{\frac{1}{c}+c}}=\sqrt{\frac{1}{a}+a}+\sqrt{\frac{1}{b}+b}+\sqrt{\frac{1}{c}+c}+$$

$$+\frac{b-a}{\sqrt{\frac{1}{a}+a}}+\frac{c-b}{\sqrt{\frac{1}{b}+b}}+\frac{a-c}{\sqrt{\frac{1}{c}+c}}$$

$$\frac{\frac{1}{a}+b}{\sqrt{\frac{1}{a}+a}}+\frac{\frac{1}{b}+c}{\sqrt{\frac{1}{b}+b}}+\frac{\frac{1}{c}+a}{\sqrt{\frac{1}{c}+c}}\geq 3\sqrt{2}+f(a,b,c)$$

$$\sqrt{a+\frac{1}{a}}\overset{AGM}{\geq}\sqrt{2}$$

We will to prove that $f(a,b,c)\geq 0$

$$\frac{df}{da}=-\frac{1}{\sqrt{a+\frac{1}{a}}}+\frac{1}{\sqrt{c+\frac{1}{c}}}-\frac{1}{2}\cdot\frac{b-a-\frac{b}{a^2}+\frac{1}{a}}{\left(a+\frac{1}{a}\right)^{\frac{3}{2}}}$$

$$\frac{df}{db}=-\frac{1}{\sqrt{\frac{1}{b}+b}}+\frac{1}{\sqrt{\frac{1}{a}+a}}-\frac{1}{2}\cdot\frac{c-b-\frac{c}{b^2}+\frac{1}{b}}{\left(b+\frac{1}{b}\right)^{2}}$$

$$\frac{df}{dc}=-\frac{1}{\sqrt{\frac{1}{c}+c}}+\frac{1}{\sqrt{\frac{1}{b}+b}}-\frac{1}{2}\cdot\frac{a-c-\frac{a}{c^2}+\frac{1}{c}}{\left(c+\frac{1}{c}\right)^{\frac{3}{2}}}$$

$$\vec{\Delta}f(1,1,1)=\vec{0},$$

$$\text{Hess}f(1,1,1)=\begin{pmatrix}\frac{1}{\sqrt{2}} & 0 & 0\\ 0 & \frac{1}{\sqrt{2}} & 0\\ 0 & 0 & \frac{1}{\sqrt{2}}\end{pmatrix}$$

Hence, $f(a,b,c)\geq f(1,1,1)$.

1034. Let $a,b,c>0$, $a+b+c=3$. Prove that:

$$\frac{a^2-b}{b(b+c)}+\frac{b^2-c}{c(c+a)}+\frac{c^2-a}{a(a+b)}\geq 0$$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:\ \frac{3(a^2-b)}{b(b+c)}=\frac{3a^2-(a+b+c)b}{b(b+c)}=\frac{3a^2-ab}{b(b+c)}-1$$

$$=\frac{3a^2-ab+2b(b+c)}{b(b+c)}-3=$$

$$=\frac{a(a+b)}{b(b+c)}+\frac{2(a^2+b^2-ab+bc)}{b(b+c)}-3\overset{AM-GM}{\geq}\frac{a(a+b)}{b(b+c)}+\frac{2(ab+bc)}{b(b+c)}-3$$

$$=\frac{a(a+b)}{b(b+c)}+\frac{2(c+a)}{(b+c)}-3$$

$$Then:\ 3.\frac{a^2-b}{b(b+c)}\geq\frac{a(a+b)}{b(b+c)}+\frac{2(c+a)}{(b+c)}-3\ \ (and\ analogs)$$

Summing up this inequality with similar ones we get:

$$3.LHS\geq\left(\frac{a(a+b)}{b(b+c)}+\frac{b(b+c)}{c(c+a)}+\frac{c(c+a)}{a(a+b)}\right)+2\left(\frac{c+a}{b+c}+\frac{a+b}{c+a}+\frac{b+c}{a+b}\right)-9\overset{AM-GM}{\geq}3$$

$$+2.3-9=0.$$

$$Therefore,\ \frac{a^2-b}{b(b+c)}+\frac{b^2-c}{c(c+a)}+\frac{c^2-a}{a(a+b)}\geq 0.\ \ Equality\ holds\ iff\ a=b=c.$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\frac{a^2-b}{b(b+c)}+\frac{b^2-c}{c(c+a)}+\frac{c^2-a}{a(a+b)}\geq 0\Leftrightarrow\frac{a^2}{b(b+c)}+\frac{b^2}{c(c+a)}+\frac{c^2}{a(a+b)}$$

$$\geq\sum_{cyc}\frac{1}{b+c}\overset{\because 1=\frac{a+b+c}{3}}{=}\frac{\sum_{cyc}\big((c+a)(a+b)\big)}{\prod_{cyc}(b+c)}.\frac{a+b+c}{3}$$

$$\Leftrightarrow 3\left(\prod_{cyc}(b+c)\right)\left(\frac{a^2}{b(b+c)}+\frac{b^2}{c(c+a)}+\frac{c^2}{a(a+b)}\right)\overset{(*)}{\geq}\left(\sum_{cyc}a^2+3\sum_{cyc}ab\right)\left(\sum_{cyc}a\right)$$

$$\text{Now, }\left(\frac{a^2}{b(b+c)}+\frac{b^2}{c(c+a)}+\frac{c^2}{a(a+b)}\right)\left(\sum_{cyc}(a(b+c))\right)\left(\sum_{cyc}b\right)\overset{\text{Holder}}{\geq}\left(\sum_{cyc}a\right)^3$$

$$\Rightarrow \frac{a^2}{b(b+c)}+\frac{b^2}{c(c+a)}+\frac{c^2}{a(a+b)}\overset{(i)}{\geq}\frac{\left(\sum_{cyc}a\right)^2}{2\sum_{cyc}ab}$$

$$\text{Also, }\prod_{cyc}(b+c)\overset{(ii)}{\geq}\frac{8}{9}\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)\therefore (i)\bullet(ii)\Rightarrow \text{LHS of }(*)$$

$$\geq 3.\frac{8}{9}\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right).\frac{\left(\sum_{cyc}a\right)^2}{2\sum_{cyc}ab}\overset{?}{\geq}\left(\sum_{cyc}a^2+3\sum_{cyc}ab\right)\left(\sum_{cyc}a\right)$$

$$\Leftrightarrow 4\left(\sum_{cyc}a^2+2\sum_{cyc}ab\right)\overset{?}{\geq}3\left(\sum_{cyc}a^2+3\sum_{cyc}ab\right)$$

$$\Leftrightarrow \sum_{cyc}a^2\overset{?}{\geq}\sum_{cyc}ab\rightarrow \text{true}\Rightarrow (*)\text{ is true}\therefore \frac{a^2-b}{b(b+c)}+\frac{b^2-c}{c(c+a)}+\frac{c^2-a}{a(a+b)}\geq 0\ \forall a,b,c$$

$$>0\ \big|\ a+b+c=3\ (QED)$$

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\frac{a^2-b}{b(b+c)}+\frac{b^2-c}{c(c+a)}+\frac{c^2-a}{a(a+b)}\geq 0$$

$$\frac{a^2}{b(b+c)}+\frac{b^2}{c(c+a)}+\frac{c^2}{a(a+b)}\geq\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}$$

$$\frac{a^2}{b}(c+a)(a+b)+\frac{b^2}{c}(a+b)(b+c)+\frac{c^2}{a}(b+c)(c+a)\geq$$

$$\geq (a+b)(b+c)+(b+c)(c+a)+(c+a)(a+b)$$

$$\frac{a^4}{b}+\frac{b^4}{c}+\frac{c^4}{a}+a^3+b^3+c^3+a^2c+b^2a+c^2b+\frac{a^3c}{b}+\frac{c^3b}{a}+\frac{b^3a}{c}\geq$$

$$\geq (a+b+c)^2+ab+bc+ca=9+ab+bc+ca$$

$$a^3+b^3+c^3+a^2c+c^2b+b^2c\geq 6$$

$$\frac{a^4}{b}+\frac{b^4}{c}+\frac{c^4}{a}+\frac{a^3c}{b}+\frac{c^3b}{a}+\frac{b^3a}{c}\geq 3+ab+bc+ca$$

$$\frac{a^3}{b}(a+c)+\frac{b^3}{c}(b+a)+\frac{c^3}{a}(c+b)\geq 3+ab+bc+ca$$

$$\frac{a^4(a+c)^2}{ab(a+c)}+\frac{b^4(b+a)^2}{bc(a+b)}+\frac{c^4(c+b)^2}{ca(c+b)}\geq 3+ab+bc+ca$$

$$\frac{\left(a^2(a+c)+b^2(b+a)+c^2(c+b)\right)^2}{a^2b+b^2c+c^2a+3abc}\geq 3+ab+bc+ca$$

$$(a^3+b^3+c^3+a^2c+b^2a+c^2b)^2\geq$$

$$\geq(3+ab+bc+ca)(a^2b+b^2c+c^2a+3abc)$$

$$\left(\frac{2(a+b+c)^3}{9}\right)^2\geq(3+ab+bc+ca)(a^2b+b^2c+c^2a+3abc)$$

$$\left(\frac{2(a+b+c)^3}{9}\right)^2\geq\frac{2}{9}(3+ab+bc+ca)(a+b+c)^3$$

$$\frac{2}{9}(a+b+c)^3\geq 9(3+ab+bc+ca)\text{ true from } a+b+c=3$$

$$ab+bc+ca\leq 3.$$

1035. Let $a,b,c\geq 0: ab+bc+ca>0$. Prove that:

$$\frac{2(a^2b+b^2c+c^2a)-3abc}{a^2b+b^2c+c^2a-abc}\geq\frac{bc}{a^2+bc}+\frac{ca}{b^2+ca}+\frac{ab}{c^2+ab}\geq\frac{a^2b+b^2c+c^2a}{a^2b+b^2c+c^2a-abc}$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Lemma: \ If\ x,y,z\geq 0: xy+yz+zx>0\ then:$$

$$\frac{x}{x+y}+\frac{y}{y+z}+\frac{z}{z+x}\geq\frac{x+y+z}{x+y+z-\sqrt[3]{xyz}}.$$

$$Proof: If\ xy+yz+zx\geq\sqrt[3]{xyz}(x+y+z).\ \ By\ CBS\ inequality\ we\ have:$$

$$\sum_{cyc}\frac{x}{x+y}\geq\frac{\left(\sum_{cyc}x\right)^2}{\sum_{cyc}x(x+y)}=\frac{\left(\sum_{cyc}x\right)^2}{\left(\sum_{cyc}x\right)^2-\sum_{cyc}xy}\geq\frac{\left(\sum_{cyc}x\right)^2}{\left(\sum_{cyc}x\right)^2-\sqrt[3]{xyz}.\sum_{cyc}x}$$

$$=\frac{x+y+z}{x+y+z-\sqrt[3]{xyz}}.$$

$$If\ xy+yz+zx\leq\sqrt[3]{xyz}(x+y+z).\ \ By\ CBS\ inequality\ we\ have:$$

$$\sum_{cyc}\frac{x}{x+y}\geq\frac{\left(\sum_{cyc}zx\right)^2}{\sum_{cyc}z^2x(x+y)}=\frac{\left(\sum_{cyc}zx\right)^2}{\left(\sum_{cyc}zx\right)^2-xyz\sum_{cyc}x}\geq\frac{1}{1-\frac{xyz\sum_{cyc}x}{\left(\sum_{cyc}zx\right)^2}}$$

$$\geq\frac{1}{1-\frac{xyz\sum_{cyc}x}{\left(\sqrt[3]{xyz}.\sum_{cyc}x\right)^2}}=\frac{x+y+z}{x+y+z-\sqrt[3]{xyz}}.$$

So the poof of the lemma is completed.

For $x=b^2c,\ \ y=a^2b,\ \ z=c^2a,$ *we have :*

$$\sum_{cyc}\frac{b^2c}{b^2c+a^2b}\geq\frac{a^2b+b^2c+c^2a}{a^2b+b^2c+c^2a-abc}.$$

$$Then:\ \frac{bc}{a^2+bc}+\frac{ca}{b^2+ca}+\frac{ab}{c^2+ab}\geq\frac{a^2b+b^2c+c^2a}{a^2b+b^2c+c^2a-abc}.$$

For $x=a^2b,\ \ y=b^2c,\ \ z=c^2a,$ *we have :* $\sum_{cyc}\frac{a^2b}{a^2b+b^2c}\geq\frac{a^2b+b^2c+c^2a}{a^2b+b^2c+c^2a-abc}.$

Then :

$$\sum_{cyc}\frac{bc}{a^2+bc}=\sum_{cyc}\left(1-\frac{a^2b}{a^2b+b^2c}\right)\leq 3-\frac{a^2b+b^2c+c^2a}{a^2b+b^2c+c^2a-abc}$$

$$=\frac{2(a^2b+b^2c+c^2a)-3abc}{a^2b+b^2c+c^2a-abc}.$$

So the proof is completed. Equality holds for $a=b=c.$

1036. Let $a,b,c>0:a+b+c=3.$ Prove that:

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{36}{a^2b+b^2c+c^2a}\geq 15$$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Lemma : *If* $a,b,c>0:a+b+c=3$ *then* $a^2b+b^2c+c^2a\leq 4-\frac{3abc}{ab+bc+ca}.$

Proof of Lemma : *We have :* $27[4-(a^2b+b^2c+c^2a)]$

$$=4(a+b+c)^3-27(a^2b+b^2c+c^2a)=$$

$$=\sum_{cyc} a(b+4c-2a)^2 \overset{CBS}{\geq} \frac{\left[\sum_{cyc}\sqrt{abc}(b+4c-2a)\right]^2}{\sum_{cyc} bc}$$

$$=\frac{81abc}{ab+bc+ca}, \textit{the proof is complete.}$$

Also by Schur's inequality we have : $(a+b+c)^3+9abc$

$$\geq 4(a+b+c)(ab+bc+ca)$$

$$\Leftrightarrow\ 27+9abc\geq 12(ab+bc+ca)\ \text{ then : }\ 4(ab+bc+ca)-3abc\leq 9\ \ (1)$$

Back to main problem, we have :

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{36}{a^2b+b^2c+c^2a} \overset{AM-GM\ \&\ Lemma}{\geq}$$

$$\geq \frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}+\frac{36(ab+bc+ca)}{4(ab+bc+ca)-3abc}=$$

$$=\frac{3}{abc}+\left(9+\frac{27abc}{4(ab+bc+ca)-3abc}\right)\overset{(1)}{\geq}3\left(\frac{1}{abc}+abc\right)+9\overset{AM-GM}{\geq}3.2+9=15,$$

the desired result. Equality holds iff $a=b=c$.

1037. Let $a,b,c\geq 0: a+b+c=3$ and $n\geq 2$. Prove that:

$$2\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}-3\right)\geq\ 3n\left(\sqrt[n]{\frac{5ab+4ca}{9}}+\sqrt[n]{\frac{5bc+4ab}{9}}+\sqrt[n]{\frac{5ca+4bc}{9}}-3\right)$$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$2\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}-3\right)\overset{(*)}{\geq}\ 3n\left(\sqrt[n]{\frac{5ab+4ca}{9}}+\sqrt[n]{\frac{5bc+4ab}{9}}+\sqrt[n]{\frac{5ca+4bc}{9}}-3\right)$$

By AM − GM inequality we have :

$$n\sqrt[n]{\frac{5ab+4ca}{9}}\leq\sqrt{\frac{5ab+4ca}{9}}+\sqrt{\frac{5ab+4ca}{9}}+(n-2).1=\frac{2}{3}\sqrt{5ab+4ca}+n-2\leq$$

$$\leq\frac{1}{3}\left(\frac{5ab+4ca}{2a+\sqrt{ab}}+\left(2a+\sqrt{ab}\right)\right)+n-2=\frac{1}{3}\left(\frac{4a(a+b+c)}{2a+\sqrt{ab}}+2\sqrt{ab}\right)+n-2$$

$$Then: \quad 3n\left(\sqrt[n]{\frac{5ab+4ca}{9}}-1\right) \le 2\sqrt{ab}+\frac{12a}{2a+\sqrt{ab}}-6$$

$$= 2\sqrt{ab}-\frac{6\sqrt{b}}{2\sqrt{a}+\sqrt{b}} \quad (and\ analogs)$$

Summing up this inequality with similar ones we get:

$$RHS_{(*)} \le 2\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)-6\sum_{cyc}\frac{\sqrt{b}}{2\sqrt{a}+\sqrt{b}} \overset{CBS}{\le} 2\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)$$

$$-\frac{6\left(\sum_{cyc}\sqrt{b}\right)^2}{\sum_{cyc}\sqrt{b}\left(2\sqrt{a}+\sqrt{b}\right)} =$$

$$= 2\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)-6 = 2\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}-3\right), \quad as\ desired.$$

Equality holds iff $a=b=c=1$.

1038. If $a,b>0$ then: $(3a+3b)^5+(4a)^4a \ge 15(4a)^4b$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By AM − GM inequality we have:

$$(3a+3b)^5+(4a)^4a = 4.\frac{(3a+3b)^5}{4}+(4a)^4a \ge 5\sqrt[5]{\left(\frac{(3a+3b)^5}{4}\right)^4.(4a)^4a} =$$

$$= 5a(3a+3b)^4 = 5a(a+a+a+3b)^4 \ge 5a.4^4.a^3.3b = 15(4a)^4b.$$

Equality holds iff $a=3b$.

1039. If $x_i>0\ \left(i=\overline{1,n}\right)$, such that $\sum_{i=1}^n x_i = 1$, and f is convex, then prove the following inequalities

$$i)\ \sum_{i=1}^n f(x_i) \ge \sum_{i=1}^n f\left(\frac{1-x_i}{n-1}\right). \quad ii)\ \frac{(1-x_1)(1-x_2)\dots(1-x_n)}{x_1x_2\dots x_n} \ge (n-1)^n.$$

Proposed by Neculai Stanciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$i)$ *Since f is a convex function then by Jensen's inequality we have :*

$$\sum_{1\le j\ne i\le n} f(x_j) \ge (n-1)f\left(\frac{1}{n-1}\sum_{1\le j\ne i\le n} x_j\right) = (n-1)f\left(\frac{1-x_i}{n-1}\right),\quad \forall i=\overline{1,n}.$$

Summing up this inequality for $i=1,2,\dots,n$ we get :

$$(n-1)\sum_{i=1}^{n} f(x_i) \ge (n-1)\sum_{i=1}^{n} f\left(\frac{1-x_i}{n-1}\right).$$

$$\textit{Therefore},\qquad \sum_{i=1}^{n} f(x_i) \ge \sum_{i=1}^{n} f\left(\frac{1-x_i}{n-1}\right).$$

$ii)$ *Let $g(x)=-\log(x), x>0$. The function g is convex on $(0,\infty)$.*

$$\textit{By the question } i) \textit{ we have :}\quad \sum_{i=1}^{n} g(x_i) \ge \sum_{i=1}^{n} g\left(\frac{1-x_i}{n-1}\right)$$

$$\Leftrightarrow\ -\sum_{i=1}^{n}\log(x_i) \ge -\sum_{i=1}^{n}\log\left(\frac{1-x_i}{n-1}\right) \Leftrightarrow \log\left(\frac{(1-x_1)(1-x_2)\dots(1-x_n)}{(n-1)^n x_1x_2\dots x_n}\right) \ge 0$$

$$\textit{Therefore},\quad \frac{(1-x_1)(1-x_2)\dots(1-x_n)}{x_1x_2\dots x_n} \ge (n-1)^n.$$

1040. For $a_1, a_2, \dots, a_n \ge 0$ $(n\in N^*), k>0$. Prove that:

$$(a_1^2+k)(a_1^2+a_2^2+k)\dots(a_1^2+a_2^2+\dots+a_n^2+k) \ge \sqrt{k^n(n+1)^{n+1}}a_1a_2\dots a_n$$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $S_0=k$ and $S_i=a_1^2+a_2^2+\dots+a_i^2+k,\ \forall i=\overline{1,n}$.

We have : $S_i=S_{i-1}+a_i^2,\ \forall i$

$=\overline{1,n}$, then by $AM-GM$ inequality we have for any $i=\overline{1,n}$:

$$S_i^{\,n+2-i} = \left[(n+1-i).\frac{S_{i-1}}{n+1-i}+a_i^2\right]^{n+2-i} \ge (n+2-i)^{n+2-i}.\left(\frac{S_{i-1}}{n+1-i}\right)^{n+1-i}.a_i^2$$

$$\textit{Then :}\ \frac{S_i^{\,n+2-i}}{S_{i-1}^{\,n+1-i}} \ge \frac{(n+2-i)^{n+2-i}}{(n+1-i)^{n+1-i}}.a_i^2,\quad \forall i=\overline{1,n}.$$

Multiplying this inequality for $i=1,2,\dots,n$ we get :

$$\frac{S_1^{\,n+1}}{S_0^{\,n}}\cdot\frac{S_2^{\,n}}{S_1^{\,n-1}}\cdot\frac{S_3^{\,n-1}}{S_2^{\,n-2}}\cdots\frac{S_n^{\,2}}{S_{n-1}}=\frac{(S_1S_2\dots S_n)^2}{S_0^{\,n}}\geq\frac{(n+2-1)^{n+2-1}}{(n+1-n)^{n+1-n}}.a_1^{\,2}a_2^{\,2}\dots a_n^{\,2}$$

$$=(n+1)^{n+1}.a_1^{\,2}a_2^{\,2}\dots a_n^{\,2}$$

$$Therefore,\ (a_1^{\,2}+k)(a_1^{\,2}+a_2^{\,2}+k)\dots(a_1^{\,2}+a_2^{\,2}+\dots+a_n^{\,2}+k)=S_1S_2\dots S_n\geq$$

$$\geq\sqrt{S_0^{\,n}.(n+1)^{n+1}.a_1^{\,2}a_2^{\,2}\dots a_n^{\,2}}=\sqrt{k^n(n+1)^{n+1}}a_1a_2\dots a_n.$$

$$Equality\ holds\ if\ a_i=\sqrt{\frac{S_{i-1}}{n+1-i}},\quad\forall i=\overline{1,n}.$$

1041. Find Max value of

$$P=\left(\sqrt{ab+bc+ca}-\frac{1}{2}\right)^2+\frac{(b-a)\sqrt{bc}+(c-b)\sqrt{ca}+(a-c)\sqrt{ab}}{2},$$

for all non-negative real numbers a,b,c such that : $a+b+c=1$.

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:$$

$$2P=\frac{1}{2}-2\sqrt{ab+bc+ca}+2(ab+bc+ca)+\left(\sqrt{a^3b}+\sqrt{b^3c}+\sqrt{c^3a}\right)$$
$$-\sqrt{abc}\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right).$$

$$Assume\ that\ a=max\{a,b,c\}.\ By\ CBS\ inequality\ we\ have:$$

$$\sqrt{ab+bc+ca}=(a+b+c)\sqrt{ab+bc+ca}$$
$$=\sqrt{[(a+b-c)^2+4c(a+b)][ab+c(a+b)]}\geq$$
$$\geq\sqrt{ab}(a+b-c)+2c(a+b).Equality\ holds\ when\ c=0\ or\ a+b-c=2\sqrt{ab}.$$

$$Similarly\ we\ have:\ \sqrt{ab+bc+ca}\geq\sqrt{ca}(c+a-b)+2b(c+a).Equality\ for\ b$$
$$=0\ or\ c+a-b=2\sqrt{ca}.$$

$$Summing\ up\ these\ inequalities\ we\ get:$$

$$2\sqrt{ab+bc+ca}\geq2(ab+bc+ca)+\left(\sqrt{a^3b}+\sqrt{c^3a}\right)-\sqrt{abc}\left(\sqrt{b}+\sqrt{c}\right)+\sqrt{ab^3}+\sqrt{ca^3}+2bc$$

$$Then:\ \ 2P\leq\frac{1}{2}+\sqrt{b^3c}-a\sqrt{bc}-\sqrt{ab^3}-\sqrt{ca^3}-2bc$$

$$=\frac{1}{2}-\sqrt{b^3}\left(\sqrt{a}-\sqrt{c}\right)-a\sqrt{bc}-\sqrt{ca^3}-2bc\leq\frac{1}{2}.$$

$$Then:\ \ P\leq\frac{1}{4}\ with\ equality\ for\ a=1,b=c=0\ and\ their\ permutation.$$

$$Therefore,\quad Max\{P\}=\frac{1}{4}.$$

1042. **Let $a,b,c>0$, $abc=1$. Prove that :**

$$\sqrt{\frac{a}{a+6b+2bc}}+\sqrt{\frac{b}{b+6c+2ca}}+\sqrt{\frac{c}{c+6a+2ab}}\ge 1.$$

Proposed by Phan Ngoc Chau-Vietnam

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $a=\frac{y}{x},\ b=\frac{z}{y},\ c=\frac{x}{z},\ x,y,z>0$. The inequality in the statement becomes

$$\frac{y}{\sqrt{y^2+6zx+2x^2}}+\frac{z}{\sqrt{z^2+6xy+2y^2}}+\frac{x}{\sqrt{x^2+6yz+2z^2}}\ge 1.$$

By Hölder's inequality we have :

$$\left(\sum_{cyc}\frac{x}{\sqrt{x^2+6yz+2z^2}}\right)^2\left(\sum_{cyc}x(x^2+6yz+2z^2)\right)\ge (x+y+z)^3$$

So it suffices to proves :

$$(x+y+z)^3\ge\sum_{cyc}x(x^2+6yz+2z^2)\quad or\quad \sum_{cyc}x^2y+3\sum_{cyc}xy^2\ge 12xyz.$$

$$Which\ is\ true\ by\ AM-GM\ inequality\ \therefore \sum_{cyc}x^2y,\sum_{cyc}xy^2\ge 3xyz.$$

So the proof is completed. Equality holds $a=b=c=1$.

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\text{Let } a=\frac{x}{y},b=\frac{y}{z},c=\frac{z}{x}\Rightarrow\sum_{cyc}\sqrt{\frac{a}{a+6b+2bc}}=\sum_{cyc}\sqrt{\frac{x^2z}{x^2z+6y^2z+2y^2z}}=$$

$$=\sum_{cyc}\sqrt{\frac{x^2z^3}{x^3z^3+6x^2y^2z^2+2xy^2z^3}}\ge 1$$

$$\text{Iff } \frac{(xy+yz+zx)^{\frac{3}{2}}}{\left(\sum(xy)^3+18x^2y^2z^2+2\sum xy^2z^3\right)^{\frac{1}{2}}}\ge\frac{1}{2}\Leftrightarrow$$

$$\frac{(xy+yz+zx)^3}{\sum(xy)^3+18x^2y^2z^2+2\sum xy^2z^3}\ge 1\Leftrightarrow$$

$$\frac{\sum(xy)^3+3\sum xy^2z^3+3\sum x^2y^3z+6x^2y^2z^2}{\sum(xy)^3+18x^2y^2z^2+2\sum xy^2z^3}\ge 1\Leftrightarrow$$

$$\sum xy^2z^3+3\sum x^3y^2z\ge 12(xyz)^2 \text{ true.}$$

1043. **Let $a,b,c\ge 0: a+b+c=ab+bc+ca=k>0$. Prove that :**

$$a\sqrt{4a^2+5b^2}+b\sqrt{4b^2+5c^2}+c\sqrt{4c^2+5a^2}\le 2k^2-k-6.$$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By AM − GM inequality we have :

$$2\sqrt{4a^2+5b^2}\le\frac{4a^2+5b^2}{2a+b}+(2a+b)=\left(\frac{4a^2+5b^2}{2a+b}-(2a+5b)\right)+4a+6b$$

$$=4a+6b-\frac{12ab}{2a+b}$$

$$Then: \sqrt{4a^2+5b^2}\le 2a+3b-\frac{6ab}{2a+b}\quad (and\ analogs)$$

Summing up this inequality with similar ones we get :

$$\sum_{cyc}a\sqrt{4a^2+5b^2}\le\sum_{cyc}a\left(2a+3b-\frac{6ab}{2a+b}\right)$$

$$=2(a+b+c)^2-(ab+bc+ca)-6\sum_{cyc}\frac{(ab)^2}{b(2a+b)}\le$$

$$\overset{CBS}{\le} 2k^2-k-6.\frac{(ab+bc+ca)^2}{(a+b+c)^2}=2k^2-k-6,\ \ as\ desired.$$

Equality holds iff $a=b=c=1$.

1044.

If $a,b,c>0,\ \ a+b+c=3$ then :

$$\sum_{cyc}\frac{a^2}{a+4b^5}\ge\frac{3}{5}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:\sum_{cyc}\frac{a^2}{a+4b^5}=\sum_{cyc}\left(a-\frac{4ab^5}{a+4b^5}\right)\overset{AM-GM}{\geq}3-\sum_{cyc}\frac{4ab^5}{5\sqrt[5]{a(b^5)^4}}$$
$$=3-\frac{4}{5}\sum_{cyc}\sqrt[5]{a^4b^5}\geq$$

$$\overset{AM-GM}{\geq}3-\frac{4}{5}\sum_{cyc}\frac{b+4ab}{5}=3-\frac{4.3}{25}-\frac{16}{25}\sum_{cyc}ab\geq\frac{63}{25}-\frac{16}{25}\cdot\frac{\left(\sum_{cyc}a\right)^2}{3}=\frac{3}{5}.$$

$$Therefore,\ \sum_{cyc}\frac{a^2}{a+4b^5}\geq\frac{3}{5}.\ Equality\ holds\ iff\ a=b=c=1.$$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\sum_{cyc}\frac{a^2}{a+4b^5}=\sum_{cyc}\frac{a^4}{a^3+4a^2b^5}\geq\frac{\left(\sum\frac{a^2}{b^2}\right)^2}{\sum\frac{a^3}{b^4}+4a^2b}\geq\frac{3}{5}$$

$$5\left(\sum_{cyc}\frac{a^4}{b^4}+2\sum_{cyc}\frac{a^2}{c^2}\right)\geq3\sum_{cyc}\frac{a^3}{b^4}+12\sum_{cyc}a^2b$$

$$3\sum_{cyc}\frac{a^4}{b^4}\geq\sum_{cyc}\frac{a^3}{b^4}\cdot\sum_{cyc}a=2$$

$$2\sum_{cyc}\frac{a^4}{b^4}\geq\sum_{cyc}\frac{ab^3}{c^4}+\sum_{cyc}\frac{a^3}{b^3}$$

$$2\sum_{cyc}a^8c^4\geq\sum_{cyc}a^5b^7+\sum_{cyc}a^7bc^4$$

$$\sum_{cyc}(ab)^4(a-b)^2(a^2+ab+b^2)\geq0\ \text{true and}$$

$$\sum_{cyc}\frac{a^2}{c^2}\geq\sum_{cyc}a^2b\Leftrightarrow\frac{1}{3}(a^2b+b^2c+c^2a)^2\geq abc(a^2b+b^2c+c^2a)$$

$$a^2b+b^2c+c^2a\geq 3abc.$$

1045. Let $a,b,c>0: abc=1$. Prove that :

$$\frac{b+c}{a^3+1}+\frac{c+a}{b^3+1}+\frac{a+b}{c^3+1}\geq a+b+c.$$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Since:\ \frac{b+c}{a^3+1}=\frac{bc(b+c)}{a^2+bc}=b+c-\frac{a^2(b+c)}{a^2+bc}$$

$$then\ the\ given\ inequality\ can\ be\ written\ as\ follows:$$

$$\frac{a^2(b+c)}{a^2+bc}+\frac{b^2(c+a)}{b^2+ca}+\frac{c^2(a+b)}{c^2+ca}\leq a+b+c.$$

$$We\ have:\ \sum_{cyc}\frac{a^2(b+c)}{a^2+bc}=\sum_{cyc}\frac{a^2(b+c)^2}{(a^2+bc)(b+c)}=\sum_{cyc}a^2.\frac{(b+c)^2}{b(c^2+a^2)+c(a^2+b^2)}\leq$$

$$\overset{CBS}{\leq}\sum_{cyc}a^2\left(\frac{b}{c^2+a^2}+\frac{c}{a^2+b^2}\right)=\sum_{cyc}\left(\frac{a^2b}{c^2+a^2}+\frac{a^2c}{a^2+b^2}\right)=$$

$$=\sum_{cyc}\left(\frac{c^2a}{b^2+c^2}+\frac{b^2a}{b^2+c^2}\right)=\sum_{cyc}a.$$

$$So\ the\ proof\ is\ completed.\ Equality\ holds\ iff\ a=b=c=1.$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$The\ given\ inequality\ can\ be\ written\ as\ follows:$$

$$\sum_{cyc}\frac{bc(b+c)}{a^2+bc}\geq a+b+c.$$

$$By\ CBS\ inequality\ we\ have:$$

$$\sum_{cyc}\frac{bc(b+c)}{a^2+bc}\geq\frac{\left[\sum_{cyc}bc(b+c)\right]^2}{\sum_{cyc}bc(b+c)\,(a^2+bc)}$$

$$So\ it\ suffices\ to\ prove:$$

$$\left(\sum\nolimits_{cyc}bc(b+c)\right)^2\geq(a+b+c)\left(\sum\nolimits_{cyc}bc(b+c)\,(a^2+bc)\right)$$

Which, after expanding and simplifying equivalent to :

$$2abc\sum_{cyc} a^3 \ge abc\sum_{ayc} ab(a+b) \;\; or \;\; abc\sum_{cyc}(a+b)(a-b)^2 \ge 0 \;\; which\ is\ true.$$

Equality holds iff $a=b=c=1$.

Solution 3 by Mohamed Amine Ben Ajiba-Tanger-Morocco

The given inequality is successively equivalent to :

$$\sum_{cyc}\left(\frac{bc(b+c)}{a^2+bc}-b-c+a\right)\ge 0 \Leftrightarrow \sum_{cyc}\frac{a(a^2+bc-ab-ca)}{a^2+bc}\ge 0 \Leftrightarrow$$

$$\sum_{cyc}\frac{a^2(b-a)(c-a)}{a^3+1}\ge 0$$

$$\overset{abc=1}{\Leftrightarrow} \frac{a^3(bc-ca)(bc-ab)}{a^3+1}+\frac{b^3(ca-ab)(ca-bc)}{b^3+1}+\frac{c^3(ab-bc)(ab-ca)}{c^3+1}\ge 0$$

WLOG, we assume that $a\ge b\ge c$ then we have $ab\ge ca\ge bc$ also,

$$(bc-ab)(bc-ca)\ge(ab-ca)(ca-bc) \;\; and \;\; (ab-bc)(ab-ca)\ge 0$$

So it suffices to prove that :

$$\frac{a^3(ab-ca)(ca-bc)}{a^3+1}+\frac{b^3(ca-ab)(ca-bc)}{b^3+1}\ge 0$$

$$\Leftrightarrow (ab-ca)(ca-bc)\left(\frac{a^3}{a^3+1}-\frac{b^3}{b^3+1}\right)\ge 0$$

Which is true because $ab\ge ca\ge bc$ and $\dfrac{a^3}{a^3+1}\ge\dfrac{b^3}{b^3+1}$.

Equality holds iff $a=b=c=1$.

1046. Let $a,b,c>0$. Prove that :

$$\frac{b+c}{\sqrt{a^2+3bc}}+\frac{c+a}{\sqrt{b^2+3ca}}+\frac{a+b}{\sqrt{c^2+3ab}}\ge 3.$$

Proposed by Tran Quoc Thinh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By Hölder's inequality we have :

$$\left(\sum_{cyc}\frac{b+c}{\sqrt{a^2+3bc}}\right)^2\left(\sum_{cyc}(b+c)(a^2+3bc)(3a+2b+2c)^3\right)$$

$$\ge\left(\sum_{cyc}(b+c)(3a+2b+2c)\right)^3$$

So it suffices to prove that :

$$\left(\sum_{cyc}(b+c)(3a+2b+2c)\right)^3\geq 9\sum_{cyc}(b+c)(a^2+3bc)(3a+2b+2c)^3$$

Which, after expanding and simplifying, becomes :

$$64\sum_{cyc}a^6+21\sum_{sym}a^5b+16\sum_{cyc}a^3b^3$$

$$\geq 30\sum_{sym}a^4b^2+36abc\sum_{cyc}a^3+3abc\sum_{sym}a^2b+60a^2b^2c^2$$

By Muirhead's inequality we have :

$$60\sum_{cyc}a^6\geq 30\sum_{sym}a^4b^2,\qquad 18\sum_{sym}a^5b\geq 36abc\sum_{cyc}a^3,$$

$$3\sum_{sym}a^5b\geq 3abc\sum_{sym}a^2b,\qquad 4\sum_{cyc}a^6+16\sum_{cyc}a^3b^3\geq 60a^2b^2c^2.$$

Summing up these inequalities yields the desired inequality.

Equality holds iff $a=b=c$.

1047. Let $a,b,c>0: a+b+c=7.$ Prove that :

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq\sqrt[3]{(a^2+b^2+c^2)^2+6abc}.$$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By CBS inequality we have : $\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq\frac{(a^2+b^2+c^2)^2}{a^2b+b^2c+c^2a}.$

By AM − GM inequality we have :

$$\sum_{cyc}a^2b\leq\sum_{cyc}\frac{a^3+ab^2+a^2b}{3}=\frac{(a+b+c)(a^2+b^2+c^2)}{3}=\frac{7}{3}(a^2+b^2+c^2)$$

Then :

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq\frac{3}{7}(a^2+b^2+c^2)=\sqrt[3]{\frac{3^3}{7^3}(a^2+b^2+c^2)^3}\overset{CBS}{\geq}\sqrt[3]{\frac{3^3}{7^3}\cdot\frac{(a+b+c)^2}{3}\cdot(a^2+b^2+c^2)^2}=$$

$$=\sqrt[3]{\frac{9}{7}(a^2+b^2+c^2)^2}\overset{CBS}{\geq}\sqrt[3]{(a^2+b^2+c^2)^2+\frac{2}{7}\cdot\frac{(a+b+c)^2(a^2+b^2+c^2)}{3}}\geq$$

$$\overset{AM-GM}{\geq} \sqrt[3]{(a^2+b^2+c^2)^2+\frac{2}{7}\cdot\frac{7.3\sqrt[3]{abc}.3\sqrt[3]{(abc)^2}}{3}} = \sqrt[3]{(a^2+b^2+c^2)^2+6abc}$$

$$Equality\ holds\ iff\ a=b=c=\frac{7}{3}.$$

1048. Let $a,b,c \geq 0 : a+b+c=3.$ Prove that :

$$\frac{a-1}{\sqrt{b+3}}+\frac{b-1}{\sqrt{c+3}}+\frac{c-1}{\sqrt{a+3}} \geq 0.$$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$The\ given\ inequality\ is\ successively\ equivalent\ to:$$

$$\sum_{cyc}\frac{3a-(a+b+c)}{\sqrt{b+3}} \geq 0 \Leftrightarrow \sum_{cyc}\left(\frac{a-b}{\sqrt{b+3}}-\frac{c-a}{\sqrt{b+3}}\right) \geq 0 \Leftrightarrow \sum_{cyc}\left(\frac{a-b}{\sqrt{b+3}}-\frac{a-b}{\sqrt{c+3}}\right) \geq 0$$

$$\Leftrightarrow \sum_{cyc}\frac{(a-b)\left(\sqrt{c+3}-\sqrt{b+3}\right)}{\sqrt{(b+3)(c+3)}} \geq 0 \Leftrightarrow \sum_{cyc}\frac{\sqrt{a+3}.(a-b)(c-b)}{\sqrt{b+3}+\sqrt{c+3}} \geq 0$$

$$Let\ x=\frac{\sqrt{c+3}}{\sqrt{a+3}+\sqrt{b+3}},\ \ y=\frac{\sqrt{a+3}}{\sqrt{b+3}+\sqrt{c+3}},\ \ z=\frac{\sqrt{b+3}}{\sqrt{c+3}+\sqrt{a+3}}.\ The\ inequality\ becomes,$$

$$\sum_{cyc}x(a-b)(a-c) \geq 0 \Leftrightarrow \sum_{cyc}[(x+y-z)+(z+x-y)](a-b)(a-c) \geq 0$$

$$\Leftrightarrow \sum_{cyc}[(x+y-z)(a-b)(a-c)+(z+x-y)(a-b)(a-c)] \geq 0$$

$$\Leftrightarrow \sum_{cyc}[(x+y-z)(a-b)(a-c)+(x+y-z)(b-c)(b-a)] \geq 0$$

$$\Leftrightarrow \sum_{cyc}(x+y-z)(a-b)^2 \geq 0.$$

$$Since\ a+b+c=3\ then\ a,b,c \in [0,3]\ and$$

$$x+y\geq 2.\frac{\sqrt{3}}{2\sqrt{3}+3}=\frac{\sqrt{6}}{2\sqrt{3}}\geq z\ (and\ analogs)$$

So the proof is completed. Equality holds iff $a=b=c=1$.

1049. Prove that:

$$0<(\sin x+\cos x)^4-3(\sin x+\cos x)^3+4(\sin x+\cos x)+2\leq 6+2\sqrt{2},\quad \forall x\in R.$$

Proposed by Hikmat Mammadov-Azerbaijan

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:\ \sin x+\cos x=\sqrt{2}\cos\left(x-\frac{\pi}{4}\right)\in\left[-\sqrt{2},\sqrt{2}\right].$$

$$Let\ f(t)=t^4-3t^3+4t+2,\ \ t\in\left[-\sqrt{2},\sqrt{2}\right].$$

$$We\ have:\ f'(t)=4t^3-9t^2+4=(t-2)(4t^2-t-2)$$
$$=4(t-2)\left(t-\frac{1-\sqrt{33}}{8}\right)\left(t-\frac{1+\sqrt{33}}{8}\right).$$

$$Then\ f\ is\ decreasing\ on\ \left[-\sqrt{2},\frac{1-\sqrt{33}}{8}\right]\ and\ on\ \left[\frac{1+\sqrt{33}}{8},\sqrt{2}\right],$$

$$increasing\ on\ \left[\frac{1-\sqrt{33}}{8},\frac{1+\sqrt{33}}{8}\right].$$

$$Then\ \min_{t\in[-\sqrt{2},\sqrt{2}]}\{f(t)\}=min\left\{f\left(\frac{1-\sqrt{33}}{8}\right),f(\sqrt{2})\right\}$$
$$=min\left\{\frac{1141-165\sqrt{33}}{512},2(3-\sqrt{2})\right\}>0$$

$$Because\ \ 3>\sqrt{2}\ \ and\ \ 165\sqrt{33}<165.6=990<1141.$$

$$And: \max_{t\in[-\sqrt{2},\sqrt{2}]}\{f(t)\} = max\left\{f(-\sqrt{2}), f\left(\frac{1+\sqrt{33}}{8}\right)\right\}$$
$$= max\left\{6+2\sqrt{2}, \frac{1141+165\sqrt{33}}{512}\right\} = 6+2\sqrt{2}$$

$$\frac{1141+165\sqrt{33}}{512} < \frac{1141+165.6}{512} = \frac{2131}{512} < 5 < 6+2\sqrt{2}.$$

$$Therefore,\ 0 < (\sin x+\cos x)^4 - 3(\sin x+\cos x)^3 + 4(\sin x+\cos x)+2 \le 6+2\sqrt{2},\ \forall x\in R.$$

1050. If $a, b > 0$ then:

$$\frac{2a}{2a+3b}\log\left(1+\frac{3b}{2a}\right) + \frac{3b}{2a+3b}\log\left(1+\frac{2a}{3b}\right) \le \log 2$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Nikos Ntorvas-Greece

Let be the function $f(t) = t\log t, t > 0, f$ –is a strictly convex function on $(0,\infty)$, as a continuous function on $(0,\infty)$ where $f''(t) = \frac{1}{t} > 0, \forall t > 0$

For $t_1 + t_1 = 1$ we have from Jensen's inequality that:

$$f(t_1)+f(t_2) \ge 2f\left(\frac{t_1+t_2}{2}\right) \Leftrightarrow t_1\log t_1 + t_2\log t_2 \ge 2f\left(\frac{1}{2}\right) \Leftrightarrow$$
$$t_1\log t_1 + t_2\log t_2 \ge -\log 2 \Leftrightarrow -t_1\log t_1 - t_2\log t_2 \le \log 2 \Leftrightarrow$$
$$t_1\log\left(\frac{1}{t_1}\right) + t_2\log\left(\frac{1}{t_2}\right) \le \log 2\,;(1)$$

For $t_1 = \frac{2a}{2a+3b}; (2)$ and $t_2 = \frac{3b}{2a+3b}; (3), a, b > 0$ we have:

$$t_1+t_2 = 1, 0 < t_1, t_2 < 1$$

Combining (1), (2) and (3) we have:

$$\frac{2a}{2a+3b}\log\left(1+\frac{3b}{2a}\right) + \frac{3b}{2a+3b}\log\left(1+\frac{2a}{3b}\right) \le \log 2$$

Equality holds for $2a = 3b$.

Solution 2 by Tapas Das-India

Let $f(x) = \log x, x > 0$, then $f'(x) = \frac{1}{x}, f''(x) = -\frac{1}{x^2} < 0 \Rightarrow f$ –concave on $(0,\infty)$

Let $p = \frac{2a+3b}{2a}, q = \frac{2a+3b}{3b}$. Using Jensen's inequality:

$$\frac{1}{p}f(p) + \frac{1}{q}f(q) \le \left(\frac{1}{p} + \frac{1}{q}\right) f\left(\frac{p \cdot \frac{1}{p} + q \cdot \frac{1}{q}}{\frac{1}{p} + \frac{1}{q}}\right)$$

$$\frac{1}{p}f(p) + \frac{1}{q}f(q) \le \left(\frac{1}{p} + \frac{1}{q}\right) f\left(\frac{2}{\frac{1}{p} + \frac{1}{q}}\right)$$

$$\frac{2a}{2a+3b}\log\left(\frac{2a+3b}{2a}\right) + \frac{3b}{2a+3b}\log\left(\frac{2a+3b}{3b}\right) \Leftrightarrow$$

$$\frac{2a}{2a+3b}\log\left(1 + \frac{3b}{2a}\right) + \frac{3b}{2a+3b}\log\left(1 + \frac{2a}{3b}\right) \le \log 2$$

Equality holds for $2a = 3b$.

1051. If $a, b, c > 0, \frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} = 2$ then

$$a + b + c - \frac{abc}{ab + bc + ca} \ge 2$$

Proposed by Marin Chirciu-Romania

Solution 1 by Vivek Kumar-India

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} = 2 \Rightarrow$$

$$(a+b)(b+c) + (b+c)(c+a) + (c+a)(a+b) = 2(a+b)(b+c)(c+a)$$

$$a^2 + b^2 + c^2 + 3(ab + bc + ca) = 2(a+b+c)(ab+bc+ca) - 2abc$$

$$(a+b+c)(ab+bc+ca) = \frac{2abc + (a+b+c)^2 + ab + bc + ca}{2}$$

Given inequality is:

$$a + b + c - \frac{abc}{ab+bc+ca} \ge 2 \Leftrightarrow$$

$$(a+b+c)(ab+bc+ca) - abc \ge 2(ab+bc+ca)$$

$$\frac{2abc + (a+b+c)^2 + ab + bc + ca}{2} - abc \ge 2(ab+bc+ca)$$

$$2abc + (a+b+c)^2 + ab + bc + ca - 2abc \ge 4(ab+bc+ca) \Leftrightarrow$$

$(a+b+c)^2 \ge 3(ab+bc+ca)$, which is true from AM-GM inequality.

Solution 2 by Tapas Das-India

$$\text{Let } p = a + b + c, q = ab + bc + ca, r = abc$$

$$(a + b)(b + c) + (b + c)(c + a) + (c + a)(a + b) = p^2 + q \text{ and}$$

$$(a + b)(b + c)(c + a) = pq - r$$

$$\frac{1}{a + b} + \frac{1}{b + c} + \frac{1}{c + a} = 2 \Rightarrow$$

$$(a + b)(b + c) + (b + c)(c + a) + (c + a)(a + b) = 2(a + b)(b + c)(c + a)$$

$$p^2 + q = 2(pq - r) \Rightarrow r = \frac{2pq - p^2 - q}{2}$$

Now,

$$a + b + c - \frac{abc}{ab + bc + ca} - 2 = p - \frac{r}{q} - 2 = \frac{pq - r - 2q}{q} =$$

$$= \frac{1}{q}\left[pq - 2q - \frac{1}{2}(2pq - p^2 - q)\right] = \frac{1}{2q}(2pq - 4q - 2pq + p^2 + q) =$$

$$= \frac{1}{2q}(p^2 + q - 4q) \geq 0 \Rightarrow a + b + c - \frac{abc}{ab + bc + ca} \geq 2$$

$$\text{Since } p^2 + q - 4q =$$

$$= (a + b + c)^2 + (ab + bc + ca) - 4(ab + bc + ca) \geq 0$$

$$\Rightarrow (a + b + c)^2 \geq 3(ab + bc + ca), \text{ which is true from AM-GM}$$

Solution 3 by Alex Szoros-Romania

$$\frac{1}{a + b} + \frac{1}{b + c} + \frac{1}{c + a} = 2 \Rightarrow \sum_{cyc}(a + b)(a + c) = 2(a + b)(b + c)(c + a); (1)$$

$$a + b + c - \frac{abc}{ab + bc + ca} \geq 2 \Leftrightarrow$$

$$(a + b + c)(ab + bc + ca) - abc \geq 2(ab + bc + ca)$$

$$(a + b)(b + c)(c + a) \geq 2(ab + bc + ca)$$

$$2(a + b)(b + c)(c + a) \geq 4(ab + bc + ca)$$

$$\sum_{cyc}(a + b)(a + c) \geq 4\sum_{cyc} ab$$

$$\sum_{cyc}(a^2 + ac + ab + bc) \geq 4\sum_{cyc} ab$$

$$\sum_{cyc} a^2 \geq \sum_{cyc} ab$$

Solution 4 by Namig Mammadov-Azerbaijan

Since $\sum \frac{1}{a+1} = 2$ then by Cauchy's inequality:

$$2\sum a = \sum (a+b) \geq \frac{9}{\sum \frac{1}{a+b}} = \frac{9}{2}$$

$$\Rightarrow a+b+c \geq \frac{9}{4} \text{ and } \sum ab \sum a \geq 9abc \Rightarrow$$

$$\frac{abc}{\sum ab} \leq \frac{1}{9}\sum a$$

Hence,

$$LHS \geq \sum a - \frac{1}{9}\sum a = \frac{8}{9}\sum a \geq \frac{8}{9}\cdot\frac{9}{4} = 2$$

1052. If $a, b, c > 0$ and $\lambda \geq 2$ then:

$$\sum_{cyc} \frac{3a+b+c}{2a+\lambda b+\lambda c} \geq \frac{15}{2(\lambda+1)}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Mohammed Diai-Morocco

$$(*): \sum_{cyc} \frac{3a+b+c}{2a+\lambda b+\lambda c} \geq \frac{15}{2(\lambda+1)} \Leftrightarrow \sum_{cyc} \frac{a+\frac{1}{3}(b+c)}{a+\frac{\lambda}{2}(b+c)} \geq \frac{5}{\lambda+1} \Leftrightarrow$$

$$3 - \left(\frac{\lambda}{2} - \frac{1}{3}\right)\sum_{cyc} \frac{b+c}{a+\frac{\lambda}{2}(b+c)} \geq \frac{5}{\lambda+1} \Leftrightarrow \sum_{cyc} \frac{b+c}{a+\frac{\lambda}{2}(b+c)} \leq \frac{6}{\lambda+1}; (*)$$

$$\sum_{cyc} \frac{b+c}{a+\frac{\lambda}{2}(b+c)} = \sum_{cyc} \frac{b+c}{p+\frac{\lambda-2}{2}(b+c)}, \text{where } p = a+b+c$$

Since the function $x \to \dfrac{x}{p+\frac{\lambda-2}{2}x}$ is concave, we have:

$$\sum_{cyc} \frac{b+c}{p+\frac{\lambda-2}{2}(b+c)} \leq 3\,\frac{\sum \frac{b+c}{3}}{p+\frac{\lambda-2}{2}\sum \frac{b+c}{3}} = \frac{6}{\lambda+1} \Rightarrow (**) \Rightarrow (*) \text{ true.}$$

Solution 2 by George Titakis-Greece

$$\sum_{cyc}\frac{3a+b+c}{2a+\lambda b+\lambda c}=\sum_{cyc}\frac{2a+(a+b+c)}{(2-\lambda)a+\lambda(a+b+c)}=\sum_{cyc}\frac{2\frac{a}{a+b+c}+1}{(2-\lambda)\frac{a}{a+b+c}+\lambda}\overset{(*)}{=}$$

$$(*)\ x=\frac{a}{a+b+c},y=\frac{b}{a+b+c},z=\frac{c}{a+b+c}$$

$$\overset{(*)}{=}\sum_{cyc}\frac{2x+1}{(2-\lambda)x+\lambda}=\sum_{cyc}\frac{2x+1}{(2-\lambda)x+\lambda(x+y+z)}=\sum_{cyc}\frac{2x+1}{2x+\lambda(y+z)}=$$

$$=2\sum_{cyc}\frac{x}{2x+\lambda(y+z)}+\sum_{cyc}\frac{1}{2x+\lambda(y+z)}=$$

$$=2\sum_{cyc}\frac{x^2}{2x^2+\lambda(xy+xz)}+\sum_{cyc}\frac{1}{2x+\lambda(y+z)}=$$

$$=2\sum_{cyc}\frac{x^2}{2x^2+(\lambda-2)(xy+xz)+2(xy+xz)}+\sum_{cyc}\frac{1}{2x+\lambda(y+z)}\ge$$

$$\ge\frac{2(x+y+z)^2}{2[x^2+y^2+z^2+2(xy+yz+zx)]+2(\lambda-2)(xy+yz+zx)}+\frac{9}{2(x+y+z)(\lambda+1)}$$

$$=\frac{2(x+y+z)^2}{2(x+y+z)^2+2(\lambda-2)(xy+yz+zx)}+\frac{9}{2(\lambda+1)}\ge$$

$$\ge\frac{2(x+y+z)^2}{2(x+y+z)^2+2(\lambda-2)\cdot\frac{(x+y+z)^2}{3}}+\frac{9}{2(\lambda+1)}=$$

$$=\frac{3}{\lambda+1}+\frac{9}{2(\lambda+1)}=\frac{15}{2(\lambda+1)}$$

Equality holds for $x=y=z\Leftrightarrow a=b=c$.

Solution 3 by Alex Szoros-Romania

We show that for $\lambda\ge 2$ holds:

$$(1):\sum_{cyc}\frac{a}{2a+\lambda(b+c)}\ge\frac{3}{2(\lambda+1)},\text{ where } a,b,c>0$$

$$\sum_{cyc}\frac{a}{2a+\lambda(b+c)}=\sum_{cyc}\frac{a^2}{2a^2+\lambda(ab+ac)}\ge\frac{(\sum a)^2}{2\sum a^2+2\lambda\sum ab}=$$

$$=\frac{\sum a^2+2\sum ab}{2(\sum a^2+\lambda\sum ab)}\ge\frac{3}{2(\lambda+1)}\Leftrightarrow$$

$$(\lambda+1)\sum_{cyc}a^2+(2\lambda+2)\sum_{cyc}ab\ge 3\sum_{cyc}a^2+3\lambda\sum_{cyc}ab\Leftrightarrow$$

$$(\lambda-2)\sum_{cyc}a^2\ge(\lambda-2)\sum_{cyc}ab\Leftrightarrow$$

$$(\lambda-2)\sum_{cyc}(a-b)^2\ge 0;\forall\,\lambda\ge 2;\ \ (1)$$

On the other hand:

$$\frac{a}{2a+\lambda(b+c)}=\frac{3a}{2a+\lambda b+\lambda c}+\frac{1}{\lambda}\cdot\frac{\lambda b+\lambda c}{2a+\lambda(b+c)}=$$

$$=\frac{3a}{2a+\lambda(b+c)}+\frac{1}{\lambda}\left(1-\frac{2a}{2a+\lambda(b+c)}\right)=\left(3-\frac{2}{\lambda}\right)\cdot\frac{a}{2a+\lambda(b+c)}+\frac{1}{\lambda}$$

$$\sum_{cyc}\frac{a}{2a+\lambda(b+c)}=\left(3-\frac{2}{\lambda}\right)\sum_{cyc}\frac{a}{2a+\lambda(b+c)}+\frac{3}{\lambda}\overset{(1)}{\ge}$$

$$\ge\left(3-\frac{2}{\lambda}\right)\cdot\frac{3}{2(\lambda+1)}+\frac{3}{\lambda}=\frac{3}{\lambda}\left(\frac{3\lambda-2}{2\lambda+2}+1\right)=\frac{3}{\lambda}\cdot\frac{5\lambda}{2\lambda+2}=\frac{15}{2(\lambda+1)}$$

1053. Prove that:

$$\sum_{k=1}^{n}\frac{\tan^2\frac{x}{2^k}}{2^{2k}F_k^2}\ge\frac{\left(\cot\frac{x}{2^n}-2^{n+1}\cot 2x\right)^2}{2^{2n}F_nF_{n+1}},x\in\left(0,\frac{\pi}{4}\right)$$

where F_k is $k-th$ Fibonacci number.

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania

Solution by Florică Anastase-Romania

Using identity: $2\cot 2a-\cot a=-\tan a,\ a\ne\frac{k\pi}{2}$ and taking

$$a=\left\{x;\frac{x}{2},\dots,\frac{x}{2^n}\right\}\Rightarrow\begin{cases}2\cot x-\cot x=-\tan x\\ 2\cot x-\cot\frac{2}{x}=-\tan\frac{x}{2}\\ \vdots\\ 2\cot\frac{x}{2^{n-1}}-\cot\frac{x}{2^n}=-\tan\frac{x}{2^n}\end{cases}$$

By multiplying the above relationships with $1,\frac{1}{2},\frac{1}{2^2},\dots,\frac{1}{2^n}$ and by adding, we have:

$$\sum_{k=1}^{n}\frac{1}{2^k}\tan\frac{x}{2^k}=\frac{1}{2^n}\cot\frac{x}{2^n}-2\cot 2x;(1)$$

$$F_kF_{k+1}=F_k(F_k+F_{k-1})=F_k^2+F_kF_{k-1}$$

$$\sum_{k=1}^{n}F_kF_{k+1}=\sum_{k=1}^{n}F_k^2+\sum_{k=1}^{n}F_{k-1}F_k$$

$$\sum_{k=1}^{n}F_k^2=\sum_{k=1}^{n}F_kF_{k+1}-\sum_{k=1}^{n}F_{k-1}F_k=\sum_{k=1}^{n}F_kF_{k+1}-\sum_{k=0}^{n-1}F_kF_{k+1}=$$

$$=F_nF_{n+1}-F_0F_1=F_nF_{n+1};(2)$$

From (1) and (2) it follows:

$$\sum_{k=1}^{n}\frac{\tan^2\frac{x}{2^k}}{2^{2k}F_k^2}=\sum_{k=1}^{n}\frac{1}{F_k^2}\left(\frac{\tan\frac{x}{2^k}}{2^k}\right)^2\overset{Bergstrom}{\geq}\frac{\left(\sum_{k=1}^{n}\frac{1}{2^k}\tan\frac{x}{2^k}\right)^2}{\sum_{k=1}^{n}F_k^2}=\frac{\left(\frac{1}{2^n}\cot\frac{x}{2^n}-2\cot 2x\right)^2}{F_nF_{n+1}}$$

$$=\frac{\left(\cot\frac{x}{2^n}-2^{n+1}\cot 2x\right)^2}{2^{2n}F_nF_{n+1}}$$

1054. If $a,b,c\geq 0, a+b+c=\lambda, 0<\lambda\leq 1$ then find minimum and maximum value of expression

$$P=\sqrt{a^2+\lambda a}+\sqrt{b^2+\lambda b}+\sqrt{c^2+\lambda c}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:$$

$$P=\sum_{cyc}\sqrt{a^2+\lambda a}=\frac{1}{4}\sum_{cyc}2\sqrt{4a(a+\lambda)}\overset{AM-GM}{\leq}\frac{1}{4}\sum_{cyc}[4a+(a+\lambda)]$$

$$=\frac{5(a+b+c)+3\lambda}{4}=2\lambda.$$

$$Then\ the\ maximum\ value\ of\ P\ is\ 2\lambda\ when\ a=b=c=\frac{\lambda}{3}.$$

$$Now\ we\ have:\ P=\sum_{cyc}\sqrt{a^2+\lambda a}=\sum_{cyc}\sqrt{2a^2+(\lambda-a)a}\overset{0\le a\le\lambda}{\ge}\sum_{cyc}\sqrt{2a^2}$$
$$=\sqrt{2}(a+b+c)=\sqrt{2}\lambda.$$

Then the minimum value of P *is* $\sqrt{2}\lambda$ *when* $a=\lambda$ *and* $b=c=0$.

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

$$P=\sqrt{a^2+\lambda a}+\sqrt{b^2+\lambda b}+\sqrt{c^2+\lambda c}\le\sqrt{a(a+\lambda)}+\sqrt{b(b+\lambda)}+\sqrt{c(c+\lambda)}\le$$
$$\le\sqrt{(a+b+c)(a+b+c+3\lambda)}=\sqrt{4\lambda^2}=2\lambda;\ (1)$$
$$P=\sqrt{a^2+\lambda a}+\sqrt{b^2+\lambda b}+\sqrt{c^2+\lambda c}=$$
$$=\sqrt{a^2+\left(\sqrt{\lambda a}\right)^2}+\sqrt{b^2+\left(\sqrt{\lambda b}\right)^2}+\sqrt{c^2+\left(\sqrt{\lambda c}\right)^2}\ge$$
$$\ge\sqrt{(a+b+c)^2+\left(\sqrt{\lambda a}+\sqrt{\lambda b}+\sqrt{\lambda c}\right)^2}=\sqrt{\lambda^2+\lambda\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2}=$$
$$=\sqrt{\lambda^2+\lambda\left[(a+b+c)+2\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)\right]}=$$
$$=\sqrt{\lambda^2+\lambda^2+2\lambda\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)}\ge\sqrt{2\lambda^2}=\lambda\sqrt{2}$$

Hence, the minimum is $\sqrt{2}\lambda$ and maximum is 2λ.

1055. Let $a,b,c\ge 0: ab+bc+ca+2abc\ge 1$. Find the minimum value of

$$P=\sqrt{a+1}+\sqrt{b+1}+\sqrt{c+1}.$$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

The given condition can be written as follows: $\dfrac{1}{a+1}+\dfrac{1}{b+1}+\dfrac{1}{c+1}\le 2.$

$$\text{Then}:\ 2P^2 \ge \left(\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}\right)\left(\sqrt{a+1}+\sqrt{b+1}+\sqrt{c+1}\right)^2 \overset{\text{Hölder}}{\ge} 3^3 = 27.$$

$$\text{Therefore, the minimum value of } P \text{ is } \sqrt{\frac{27}{2}} = \frac{3\sqrt{6}}{2} \text{ when } a=b=c=\frac{1}{2}.$$

Solution 2 by Hikmat Mammadov-Azerbaijan

$$P = \sqrt{a+1}+\sqrt{b+1}+\sqrt{c+1} \overset{AGM}{\ge} 3[(1+a)(1+b)(1+c)]^{\frac{1}{6}}$$

Equality holds for $a=b=c$.

Now we need to find $\min(1+a)(1+b)(1+c)$, for $a,b,c\in\mathbb{R}_+$

Since $a,b,c>0$ the minimum is achievable via reducing a,b,c towards zero while satisfying the constant the optimal is when $ab+bc+ca+2abc=1$.

From the application of AM-GM inequality, $a=b=c\Rightarrow 2a^3+3a^2=1$

$$2a^3+3a^2-1=0\Rightarrow (a+1)(2a^2+a-1)=0$$

$$a_{1,2}=\frac{-1\pm\sqrt{9}}{4}=\frac{1}{2}, a_3=-1$$

$$2a^3+3a^2-1=(a+1)^2(2a-1)=0\Rightarrow a=b=c=\frac{1}{2}.$$

The minimum value of P is $3\sqrt{\frac{3}{2}}$ achievable with $a=b=c=\frac{1}{2}$.

1056. **Let $a,b,c\ge 0: ab+bc+ca=1$. Find the minimum value of P:**

$$P=\frac{ab+2c\sqrt{ab}+1}{c+\sqrt{ab}}+\frac{bc+2a\sqrt{bc}+1}{a+\sqrt{bc}}+\frac{ca+2b\sqrt{ca}+1}{b+\sqrt{ca}}$$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$WLOG, \text{we may assume that } a\ge b\ge c.$$

$$We\ have:\ P=\frac{c\sqrt{ab}+1}{c+\sqrt{ab}}+\frac{a\sqrt{bc}+1}{a+\sqrt{bc}}+\frac{b\sqrt{ca}+1}{b+\sqrt{ca}}+\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\ge$$

$$\overset{a,b,c\ge 0}{\ge} \frac{1}{a+\sqrt{bc}}+\left(\frac{1}{b+\sqrt{ca}}+\left(b+\sqrt{ca}\right)\right)+\left(\frac{1}{c+\sqrt{ab}}+\left(c+\sqrt{ab}\right)\right)+\sqrt{bc}-b-c\ge$$

$$\overset{AM-GM}{\geq}\frac{ab+bc+ca}{a+\sqrt{bc}}+2+2+\sqrt{bc}-b-c$$

$$=4+\frac{\sqrt{bc}(a+2\sqrt{bc}-b-c)}{a+\sqrt{bc}}\overset{a\geq b\,\&\,\sqrt{bc}\geq c}{\geq}4.$$

So the minimum value of P is 4 for $(a,b,c)=(1,1,0)$ and their permutation.

1057. If $a,b,c,d,e,f>0$ then:

$$3abcdef(abc+def-5)+(ab+bc+ca)(de+ef+fd)\geq 0$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Vivek Kumar-India

If $a,b,c,d,e,f>0$ then

$$3abcdef(abc+def-5)+(ab+bc+ca)(de+ef+fd)\overset{AGM}{\geq}$$

$$\geq 3abcdef\cdot 2\sqrt{abcdef}+3\sqrt[3]{(abc)^2}\cdot 3\sqrt[3]{(def)^2}-15abcdef=$$

$$=6(abcdef)^{\frac{3}{2}}+9(abcdef)^{\frac{2}{3}}-15abcdef\overset{AGM}{\geq}$$

$$\geq 15\Bigg[\underbrace{(abcdef)^{\frac{3}{2}}\cdot\ldots\cdot(abcdef)^{\frac{3}{2}}}_{6-times}\underbrace{(abcdef)^{\frac{2}{3}}\cdot\ldots\cdot(abcdef)^{\frac{2}{3}}}_{9-times}\Bigg]^{\frac{1}{15}}-15abcdef=$$

$$=15\left((abcdef)^{\frac{3}{2}\cdot 6}\cdot(abcdef)^{\frac{2}{3}\cdot 9}\right)^{\frac{1}{15}}-15abcdef=$$

$$=15((abcdef)^9\cdot(abcdef)^6)^{\frac{1}{15}}-15abcdef=15abcdef-15abcdef=0$$

Equality holds for $a=b=c=d=e=f=1$.

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$(ab+bc+ca)(de+ef+fd)=(abc)(def)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(\frac{1}{d}+\frac{1}{e}+\frac{1}{f}\right)\overset{AGM}{\geq}$$

$$\geq(abc)(def)\cdot\frac{9}{\sqrt[3]{(abc)(def)}}\overset{(*)}{\geq}$$

Let $abc=x, def=y$; $(*)$

$$\overset{(*)}{\geq}3xy(x+y-5)+\frac{9xy}{\sqrt[3]{xy}}=3xy\left(x+y-5+\frac{3}{\sqrt[3]{xy}}\right)=$$

$$= 3xy\left(x + y + \frac{1}{\sqrt[3]{xy}} + \frac{1}{\sqrt[3]{xy}} + \frac{1}{\sqrt[3]{xy}} - 5\right) \overset{AGM}{\geq} 3xy\left(5\sqrt[5]{xy\left(\frac{1}{\sqrt[3]{xy}}\right)^3} - 5\right) =$$

$$= 3xy(5 - 5) = 0$$

Equality holds for $a = b = c = d = e = f = 1$.

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

$$3abcdef(abc + def - 5) + (ab + bc + ca)(de + ef + fd) - 15abcdef \geq 0 \Leftrightarrow$$

$$3abcdef(abc + def - 5) + (ab + bc + ca)(de + ef + fd) \geq 15abcdef$$

$$3(abc + def) + \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)\left(\frac{1}{d} + \frac{1}{e} + \frac{1}{f}\right) \geq 15$$

$$3(abc + def) + \frac{1}{ad} + \frac{1}{ac} + \frac{1}{af} + \frac{1}{bd} + \frac{1}{bc} + \frac{1}{bf} + \frac{1}{cd} + \frac{1}{ce} + \frac{1}{cf} \geq 15$$

$$15\sqrt[15]{\frac{(abcdef)^3}{(abcdef)^3}} \geq 15, \qquad 15 \geq 15$$

Equality holds for $a = b = c = d = e = f = 1$.

1058. If $a, x > 0, b, c, y, z \in \mathbb{R}$ then:

$$\frac{(a + x)^2 - (b + y)^2 - (c + z)^2}{a + x} \geq \frac{a^2 - b^2 - c^2}{a} + \frac{x^2 - y^2 - z^2}{x}$$

Proposed by Daniel Sitaru-Romania

Solution by Vivek Kumar-India

$$\frac{(a + x)^2 - (b + y)^2 - (c + z)^2}{a + x} \geq \frac{a^2 - b^2 - c^2}{a} + \frac{x^2 - y^2 - z^2}{x}$$

$$a + x - \frac{(b + y)^2 + (c + z)^2}{a + x} \geq a - \frac{b^2 + c^2}{a} + x - \frac{y^2 + z^2}{x}$$

$$\frac{b^2 + c^2}{a} + \frac{y^2 + z^2}{x} \geq \frac{(b + y)^2 + (c + z)^2}{a + x}$$

$$\frac{b^2 + c^2}{a} + \frac{y^2 + z^2}{x} = \left(\frac{b^2}{a} + \frac{y^2}{x}\right) + \left(\frac{c^2}{a} + \frac{z^2}{x}\right) \geq$$

$$\geq \frac{(b + y)^2}{a + x} + \frac{(c + z)^2}{a + x} = \frac{(b + y)^2 + (c + z)^2}{a + x}$$

1059. If $a, b, c \geq 0 : a + b + c = 2$ then :

$$\sqrt{\frac{a+b}{c^2+1}} + \sqrt{\frac{b+c}{a^2+1}} + \sqrt{\frac{c+a}{b^2+1}} \geq 2\sqrt{2}.$$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have: \sqrt{\frac{a+b}{c^2+1}} = \sqrt{\frac{2-c}{c^2+1}} \overset{?}{\geq} \sqrt{2}\left(1-\frac{c}{2}\right) \Leftrightarrow$$

$$2 \geq (c^2+1)(2-c) \Leftrightarrow 0 \geq -c(c-1)^2$$

$$Which\ is\ true.\ Equality\ holds\ iff\ c = 0, c = 1\ or\ c = 2.$$

$$Similarly\ we\ get:$$

$$\sqrt{\frac{b+c}{a^2+1}} \geq \sqrt{2}\left(1-\frac{a}{2}\right)\ and\ \sqrt{\frac{c+a}{b^2+1}} \geq \sqrt{2}\left(1-\frac{b}{2}\right).$$

$$Summing\ up\ these\ inequalities\ we\ get:$$

$$\sqrt{\frac{a+b}{c^2+1}} + \sqrt{\frac{b+c}{a^2+1}} + \sqrt{\frac{c+a}{b^2+1}} \geq \sqrt{2}\left(3-\frac{a+b+c}{2}\right) = 2\sqrt{2}.$$

$$Equality\ holds\ iff\ (a,b,c) = (1,1,0)\ or\ (2,0,0)\ and\ their\ permutation.$$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

If $a = 0, b + c = 2$, we have:

$$\sqrt{\frac{b}{c^2+1}} + \sqrt{\frac{b+c}{1}} + \sqrt{\frac{c}{b^2+1}} \geq 2\sqrt{2}$$

$$\sqrt{\frac{b}{c^2+1}}+\sqrt{\frac{c}{b^2+1}}\geq\sqrt{2}$$

$$\frac{(b+c)^{\frac{3}{2}}}{\left(b^2(c^2+1)+c^2(b^2+1)\right)^{\frac{1}{2}}}\geq\sqrt{2}$$

$$(b+c)^3\geq 2(2b^2c^2+b^2+c^2)$$

$$b^2+b^2+2bc\geq 2b^2c^2+b^2+c^2 \text{ true}, b+c=2, bc\leq 1.$$

If $a=b=0\Rightarrow\sqrt{\frac{c}{1}}+\sqrt{\frac{c}{1}}=\sqrt{2}+\sqrt{2}=2\sqrt{2}$

If $a,b,c>0$, then: $\sum_{cyc}\sqrt{\frac{(a+b)^3}{(a+b)^2(c^2+1)}}\geq 2\sqrt{2}\Leftrightarrow$

$$\frac{(a+b+b+c+c+a)^{\frac{3}{2}}}{\left((a+b)^2(c^2+1)+(b+c)^2(a^2+1)+(c+a)^2(b^2+1)\right)^{\frac{1}{2}}}\geq 2\sqrt{2}$$

$$4^3\geq 8[(a+b)^2(c^2+1)+(b+c)^2(a^2+1)+(c+a)^2(b^2+1)]$$

$$8\geq 2(a^2c^2+b^2c^2+a^2b^2)+2(abc^2+bca^2+cab^2)$$
$$+2(a^2+b^2+c^2+ab+bc+ca)$$

$$[a^2+b^2+c^2+2(ab+bc+ca)](ab+bc+ca)\geq$$
$$\geq 4(a^2b^2+b^2c^2+c^2a^2+abc^2+bca^2+cab^2)$$

1060. Let $a,b,c>0$. Prove that:

$$\sum_{cyc}\frac{1}{a+\sqrt{bc}}+\frac{3}{2\left(a+b+c-\sqrt[3]{abc}\right)}\geq\frac{27}{4(a+b+c)}$$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{a+\sqrt{bc}}+\frac{1}{b+\sqrt{ca}}+\frac{1}{c+\sqrt{ab}}+\frac{3}{2(a+b+c-\sqrt[3]{abc})}\overset{A-G}{\geq}\frac{1}{\sqrt{a}.\sqrt{a}+\sqrt{bc}}+\frac{1}{\sqrt{b}.\sqrt{b}+\sqrt{ca}}$$
$$+\frac{1}{\sqrt{c}.\sqrt{c}+\sqrt{ab}}+\frac{3}{2\left(a+b+c-\frac{a+b+c}{3}\right)}$$

$$\overset{CBS}{\geq}\sum_{cyc}\frac{1}{\sqrt{a+b}.\sqrt{a+c}}+\frac{9}{4\sum_{cyc}a}\overset{?}{\geq}\frac{27}{4(a+b+c)}\Leftrightarrow\frac{\left(\sum_{cyc}\sqrt{b+c}\right)}{\sqrt{\prod_{cyc}(b+c)}}\overset{?}{\geq}\frac{9}{2\sum_{cyc}a}$$

$$\Leftrightarrow\left(\sum_{cyc}\sqrt{b+c}\right)\left(\sum_{cyc}(b+c)\right)\underset{(*)}{\overset{?}{\geq}}9.\sqrt{\prod_{cyc}(b+c)}$$

$$\text{Now,}\left(\sum_{cyc}\sqrt{b+c}\right)\left(\sum_{cyc}(b+c)\right)$$

$$=\frac{1}{3}\left(\sum_{cyc}\sqrt{b+c}\right)\left(\sum_{cyc}(b+c)\right)\left(\sum_{cyc}1\right)\overset{Holder}{\geq}\frac{1}{3}\left(\sum_{cyc}\sqrt{b+c}\right)^3\overset{A-G}{\geq}\frac{1}{3}.27\prod_{cyc}\sqrt{b+c}$$

$$=9.\sqrt{\prod_{cyc}(b+c)}\Rightarrow(*)\text{ is true}$$

$$\therefore\forall a,b,c>0,\frac{1}{a+\sqrt{bc}}+\frac{1}{b+\sqrt{ca}}+\frac{1}{c+\sqrt{ab}}+\frac{3}{2\left(a+b+c-\sqrt[3]{abc}\right)}$$

$$\geq\frac{27}{4(a+b+c)},\text{equality iff } a=b=c\ (QED)$$

1061. If $a,b,c>0, a+b+c=\lambda>0$ then:

$$\frac{a}{(b+c)^3}+\frac{b}{(c+a)^3}+\frac{c}{(a+b)^3}\geq\frac{27}{8\lambda^2}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{WLOG assuming } a\geq b\geq c\Rightarrow\frac{1}{(b+c)^3}\geq\frac{1}{(c+a)^3}\geq\frac{1}{(a+b)^3}$$

$$\therefore\sum_{cyc}\frac{a}{(b+c)^3}\overset{Chebyshev}{\geq}\frac{1}{3}\left(\sum_{cyc}a\right)\sum_{cyc}\frac{1^4}{(b+c)^3}\overset{Radon}{\geq}$$

$$\frac{1}{3}\left(\sum_{cyc}a\right)\frac{3^4}{\left(\sum_{cyc}(b+c)\right)^3}\overset{\sum_{cyc}a=\lambda}{=}\frac{27\lambda}{8\lambda^3}=\frac{27}{8\lambda^2}\ (QED)$$

1062. Let $a,b,c>0: a+b+c=1.$ Prove that :

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\geq 6\left(\frac{1}{4-(a-b)^2}+\frac{1}{4-(b-c)^2}+\frac{1}{4-(c-a)^2}\right).$$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc}\frac{1}{b+c}\overset{a+b+c=1}{\triangleq}\sum_{cyc}\left(\frac{a-b}{2(b+c)}-\frac{c-a}{2(b+c)}+\frac{3}{2}\right)=\sum_{cyc}\left(\frac{a-b}{2(b+c)}-\frac{a-b}{2(c+a)}\right)+\frac{9}{2}=$$

$$=\frac{1}{2}\sum_{cyc}\frac{(a-b)^2}{(b+c)(c+a)}+\frac{9}{2}\geq\frac{1}{2}\sum_{cyc}\frac{(a-b)^2}{(a+b+c)^2}+\frac{9}{2}=\frac{1}{2}\sum_{cyc}(a-b)^2+\frac{9}{2}.$$

$$Then:\ \frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\geq\frac{1}{2}\sum_{cyc}(a-b)^2+\frac{9}{2}\quad(1)$$

$$Also\ we\ have:$$

$$6\sum_{cyc}\frac{1}{4-(a-b)^2}=\frac{3}{2}\sum_{cyc}\left(\frac{(a-b)^2}{4-(a-b)^2}+1\right)=\frac{3}{2}\sum_{cyc}\frac{(a-b)^2}{4-(a-b)^2}+\frac{9}{2}\leq$$

$$\leq\frac{3}{2}\sum_{cyc}\frac{(a-b)^2}{4-(a+b+c)^2}+\frac{9}{2}=\frac{1}{2}\sum_{cyc}(a-b)^2+\frac{9}{2}.$$

$$Then:\ \frac{1}{2}\sum_{cyc}(a-b)^2+\frac{9}{2}\geq 6\left(\frac{1}{4-(a-b)^2}+\frac{1}{4-(b-c)^2}+\frac{1}{4-(c-a)^2}\right)\quad(2)$$

$$From\ (1)\ and\ (2)\ yields\ the\ desired\ inequality.$$

$$Equality\ holds\ iff\ a=b=c=\frac{1}{3}.$$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

For $a,b,c>0$ and $a+b+c=1$, we will obtain that:

$$(a+b)+(b+c)+(c+a)=2\Leftrightarrow\frac{2}{(a+b)+(b+c)+(c+a)}=1$$

$$\frac{2}{9}\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right)\geq 1\Rightarrow\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\geq\frac{9}{2}\ \text{and since}$$

$$\frac{6}{4+2(a-b)}\le\frac{6}{4}\Rightarrow 4\le 4+(a-b)\Rightarrow(a-b)^2\ge 0,\text{similarly}$$

$$\frac{6}{4+(b-c)^2}\le\frac{6}{4}\text{ and }\frac{6}{4+(c-a)^2}\le\frac{6}{4}$$

$$\text{Therefore, }\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\ge\frac{9}{2}=\frac{6}{4}+\frac{6}{4}+\frac{6}{4}\ge$$

$$\ge\frac{6}{4+(a-b)^2}+\frac{6}{4+(b-c)^2}+\frac{6}{3+(c-a)^2}$$

Solution 3 by Vivek Kumar-India

$$\sum_{cyc}\frac{1}{a+b}\ge 6\sum_{cyc}\frac{1}{4-(a-b)^2}=6\sum_{cyc}\frac{1}{(2a+2b+2c)^2-(a-b)^2}=$$

$$=6\sum_{cyc}\frac{1}{(3a+b+2c)(a+3b+2c)};(a+b+c=1)$$

$$=6\sum_{cyc}\frac{1}{\big(a+b+2(c+a)\big)\big(a+b+2(b+c)\big)}$$

$$\text{Let: }b+c=x,c+a=y,a+b=z\Rightarrow x+y+z=2$$

$$\sum_{cyc}\frac{1}{x}\ge 6\sum_{cyc}\frac{1}{(z+2y)(z+2x)}\Leftrightarrow\sum_{cyc}\frac{1}{x}\ge\frac{3}{2}\sum_{cyc}\frac{4}{(z+2y)(z+2x)}$$

$$\sum_{cyc}\frac{1}{x}\ge\frac{3}{2}\sum_{cyc}\frac{2x+2y+2z}{(z+2y)(z+2x)}\Leftrightarrow 2\sum_{cyc}\frac{1}{x}\ge\frac{3}{2}\sum_{cyc}\frac{z+2y+z+2x}{(z+2y)(z+2x)}$$

$$2\sum_{cyc}\frac{1}{x}\ge 3\sum_{cyc}\left(\frac{1}{z+2x}+\frac{1}{z+2y}\right)$$

$$2\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\ge 3\left(\frac{1}{z+2x}+\frac{1}{z+2y}+\frac{1}{x+2y}+\frac{1}{x+2z}+\frac{1}{y+2x}+\frac{1}{y+2z}\right)$$

$$\sum_{cyc}\left(\frac{1}{x}+\frac{1}{y}\right)\ge 3\sum_{cyc}\left(\frac{1}{x+2y}+\frac{1}{2x+y}\right)$$

$$\sum_{cyc}\frac{x+y}{xy}\ge 9\sum_{cyc}\frac{x+y}{(x+2y)(2x+y)}$$

$$\sum_{cyc}(x+y)\big((x+2y)(2x+y)-9xy\big)\ge 0$$

$$\sum_{cyc}(x+y)(2x^2+2y^2+5xy-9xy)\geq 0, \quad 2\sum_{cyc}(x+y)(x-y)^2\geq 0$$

1063. If $a,b,c>0$, $a^2+b^2+c^2=3abc$ and $\lambda>0$ then :

$$\sum_{cyc}\sqrt{\frac{a}{(\lambda+1)a^2+1}}\leq\frac{3}{2}\sqrt{\frac{1}{2}+\frac{1}{\lambda}}.$$

Proposed by Marin Chirciu-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\frac{a}{(\lambda+1)a^2+1}=\frac{a}{\lambda a^2+(a^2+1)}\overset{AM-GM}{\leq}\frac{a}{\lambda a^2+2a}=\frac{1}{\lambda a+2}\overset{CBS}{\leq}\frac{\frac{\lambda}{a}+2}{(\lambda+2)^2}.$$

$$Then:\sqrt{\frac{a}{(\lambda+1)a^2+1}}\leq\frac{1}{\lambda+2}\sqrt{\frac{\lambda}{a}+2}\ (and\ analogs)$$

$$Then:\sum_{cyc}\sqrt{\frac{a}{(\lambda+1)a^2+1}}\leq\frac{1}{\lambda+2}\cdot\sum_{cyc}\sqrt{\frac{\lambda}{a}+2}\overset{CBS}{\leq}\frac{1}{\lambda+2}\cdot\sqrt{3\sum_{cyc}\left(\frac{\lambda}{a}+2\right)}$$

$$=\frac{1}{\lambda+2}\cdot\sqrt{3\left(\frac{\lambda(ab+bc+ca)}{abc}+6\right)}\leq$$

$$\leq\frac{1}{\lambda+2}\cdot\sqrt{3\left(\frac{\lambda(a^2+b^2+c^2)}{abc}+6\right)}=\frac{\sqrt{3(3\lambda+6)}}{\lambda+2}\overset{AM-GM}{\leq}\frac{3\sqrt{\lambda+2}}{2\sqrt{2\lambda}}$$

$$=\frac{3}{2}\sqrt{\frac{1}{2}+\frac{1}{\lambda}},\ as\ desired.$$

$$Equality\ holds\ iff\ a=b=c=1\ and\ \lambda=2.$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\sqrt{\frac{a}{(\lambda+1)a^2+1}}=\sum_{cyc}\frac{\sqrt{4a}.\sqrt{(\lambda+1)a^2+1}}{2\left((\lambda+1)a^2+1\right)}\overset{A-G}{\leq}\sum_{cyc}\frac{4a+(\lambda+1)a^2+1}{4\left((\lambda+1)a^2+1\right)}$$

$$=\frac{3}{4}+\sum_{cyc}\frac{a}{(\lambda+1)a^2+1}\overset{A-G}{\leq}\frac{3}{4}+\sum_{cyc}\frac{a}{\lambda a^2+2a}=\frac{3}{4}+\frac{1}{2}\sum_{cyc}\frac{2+\lambda a-\lambda a}{2+\lambda a}$$

$$=\frac{3}{4}+\frac{3}{2}-\frac{\lambda}{2}\cdot\sum_{cyc}\frac{a}{2+\lambda a}=\frac{9}{4}-\frac{\lambda}{2}\cdot\sum_{cyc}\frac{1}{\frac{2}{a}+\lambda}\overset{\text{Bergstrom}}{\leq}\frac{9}{4}-\frac{9\lambda}{2}\cdot\frac{1}{\frac{2\sum_{cyc}ab}{abc}+3\lambda}$$

$$\leq\frac{9}{4}-\frac{9\lambda}{2}\cdot\frac{1}{\frac{2\sum_{cyc}a^2}{abc}+3\lambda}\overset{a^2+b^2+c^2=3abc}{=}\frac{9}{4}-\frac{9\lambda}{2}\cdot\frac{1}{\frac{6abc}{abc}+3\lambda}$$

$$=\frac{9}{4}-\frac{3\lambda}{2}\cdot\frac{1}{\lambda+2}\boxed{\overset{?}{\leq}}\frac{3}{2}\cdot\sqrt{\frac{1}{2}+\frac{1}{\lambda}}$$

$$\Leftrightarrow\boxed{\frac{\lambda}{\lambda+2}+\sqrt{\frac{\lambda+2}{2\lambda}}\underset{(*)}{\overset{?}{\geq}}\frac{3}{2}}$$

Now, LHS of $(*)=\frac{1}{2}\cdot\frac{2\lambda}{\lambda+2}+\frac{1}{2}\cdot\sqrt{\frac{\lambda+2}{2\lambda}}+\frac{1}{2}\cdot\sqrt{\frac{\lambda+2}{2\lambda}}\overset{A-G}{\geq}3\sqrt[3]{\frac{1}{8}\cdot\sqrt{\frac{\lambda+2}{2\lambda}}\cdot\sqrt{\frac{\lambda+2}{2\lambda}}\cdot\frac{2\lambda}{\lambda+2}}=\frac{3}{2}$

$\Rightarrow(*)$ is true

$\therefore\forall a,b,c>0\,|\,a^2+b^2+c^2=3abc$ and $\lambda>0,\ \sum_{cyc}\sqrt{\frac{a}{(\lambda+1)a^2+1}}$

$\leq\frac{3}{2}\cdot\sqrt{\frac{1}{2}+\frac{1}{\lambda}}$, equality iff $a=b=c=1$ and $\lambda=2$ (QED)

1064. If $x,y,z>0$ and $\lambda\geq 0$ then:

$$\sum_{cyc}\frac{1}{y(x+\lambda y)}\geq\frac{27}{(\lambda+1)(x+y+z)^2}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Vivek Kumar-India

$$\sum_{cyc}\frac{1}{y(x+\lambda y)}\sum_{cyc}y\sum_{cyc}(x+\lambda y)\overset{Holder}{\geq}(1+1+1)^3$$

$$\sum_{cyc}\frac{1}{y(x+\lambda y)}\geq\frac{27}{\sum y\sum(x+\lambda y)}=\frac{27}{(\sum x)(\sum x+\lambda\sum x)}\geq\frac{27}{(\lambda+1)(x+y+z)^2}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\frac{1}{y(x+\lambda y)}+\frac{1}{z(y+\lambda z)}+\frac{1}{x(z+\lambda x)}$$

$$=\frac{1^3}{\left(\sqrt{y(x+\lambda y)}\right)^2}+\frac{1^3}{\left(\sqrt{z(y+\lambda z)}\right)^2}$$

$$+\frac{1^3}{\left(\sqrt{x(z+\lambda x)}\right)^2}\overset{\text{Radon}}{\geq}\frac{(1+1+1)^3}{\left(\sqrt{y(x+\lambda y)}+\sqrt{z(y+\lambda z)}+\sqrt{x(z+\lambda x)}\right)^2}$$

$$\overset{CBS}{\geq}\frac{27}{(y+z+x)(x+\lambda y+y+\lambda z+z+\lambda x)}=\frac{27}{(x+y+z)\big((y+z+x)+\lambda(x+y+z)\big)}$$

$$=\frac{27}{(\lambda+1)(x+y+z)^2}\text{ (QED)}$$

1065. If $a,b,c>0, a+b+c=3$ then:

$$\frac{ac^3}{(c^4+1)(c^2+c+1)}+\frac{ba^3}{(a^4+1)(a^2+a+1)}+\frac{cb^3}{(b^4+1)(b^2+b+1)}\leq\frac{1}{2}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Vivek Kumar-India

$$\frac{ac^3}{(c^4+1)(c^2+c+1)}\overset{AGM}{\leq}\frac{ac^3}{2\sqrt{c^4\cdot 1}\cdot 3\sqrt[3]{c^2\cdot c\cdot 1}}=\frac{ac^3}{bc^2\cdot c}=\frac{a}{6}$$

Analogously,

$$\frac{ba^3}{(a^4+1)(a^2+a+1)}\leq\frac{b}{6}\text{ and }\frac{cb^3}{(b^4+1)(b^2+b+1)}\leq\frac{c}{6}$$

Therefore,

$$\frac{ac^3}{(c^4+1)(c^2+c+1)}+\frac{ba^3}{(a^4+1)(a^2+a+1)}+\frac{cb^3}{(b^4+1)(b^2+b+1)}\leq$$

$$\leq\frac{a+b+c}{6}=\frac{1}{2}$$

Solution 2 by Tapas Das-India

$$\frac{ac^3}{(c^4+1)(c^2+c+1)}+\frac{ba^3}{(a^4+1)(a^2+a+1)}+\frac{cb^3}{(b^4+1)(b^2+b+1)}\leq\frac{1}{2}\Leftrightarrow$$

$$\frac{ac^3}{(c^4+1)(c^2+c+1)}+\frac{ba^3}{(a^4+1)(a^2+a+1)}+\frac{cb^3}{(b^4+1)(b^2+b+1)}\leq\frac{3}{6}\Leftrightarrow$$

$$\frac{ac^3}{(c^4+1)(c^2+c+1)}+\frac{ba^3}{(a^4+1)(a^2+a+1)}+\frac{cb^3}{(b^4+1)(b^2+b+1)}\leq\frac{a+b+c}{6}$$

$$\frac{ac^3}{(c^4+1)(c^2+c+1)}-\frac{a}{6}=a\cdot\frac{6c^3-(c^4+1)(c^2+c+1)}{(c^4+1)(c^2+c+1)}\leq 0\text{ since}$$

$$(c^4+1)(c^2+c+1)\geq 2\sqrt{c^4\cdot 1}\cdot 3\sqrt[3]{c^2\cdot c\cdot 1}=6c^3$$

$$\frac{ac^3}{(c^4+1)(c^2+c+1)}\leq\frac{a}{6}$$

Analogously,

$$\frac{ba^3}{(a^4+1)(a^2+a+1)} \le \frac{b}{6} \text{ and } \frac{cb^3}{(b^4+1)(b^2+b+1)} \le \frac{c}{6}$$

Therefore,

$$\frac{ac^3}{(c^4+1)(c^2+c+1)} + \frac{ba^3}{(a^4+1)(a^2+a+1)} + \frac{cb^3}{(b^4+1)(b^2+b+1)} \le$$

$$\le \frac{a+b+c}{6} = \frac{1}{2}$$

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\frac{ac^3}{(c^4+1)(c^2+c+1)} + \frac{ba^3}{(a^4+1)(a^2+a+1)} + \frac{cb^3}{(b^4+1)(b^2+b+1)} =$$

$$= \frac{a}{\left(b^2+\frac{1}{b^2}\right)\left(b+1+\frac{1}{b}\right)} + \frac{b}{\left(c^2+\frac{1}{c}\right)\left(c+1+\frac{1}{c}\right)} + \frac{c}{\left(a^2+\frac{1}{a^2}\right)\left(a+1+\frac{1}{a}\right)} \le \frac{1}{2} \Leftrightarrow$$

$$\frac{a}{2\cdot 3} + \frac{b}{2\cdot 3} + \frac{c}{2\cdot 3} \le \frac{1}{2} \Leftrightarrow \frac{a+b+c}{6} \le \frac{1}{2} \Leftrightarrow a+b+c \le 3 \text{ ok.}$$

Solution 4 by Sakthi Vel-India

$$\frac{ac^3}{(c^4+1)(c^2+c+1)} + \frac{ba^3}{(a^4+1)(a^2+a+1)} + \frac{cb^3}{(b^4+1)(b^2+b+1)} \le \frac{1}{2}$$

$$\frac{ac^3}{(c^4+1)(c^2+c+1)} = \frac{ac^3}{c^2\left(c^2+\frac{1}{c^2}\right)c\left(c+\frac{1}{c}+1\right)} = \frac{a}{\left(c^2+\frac{1}{c^2}\right)\left(c+\frac{1}{c}+1\right)} \overset{AGM}{\le}$$

$$\le \frac{a}{2\sqrt{c^2\cdot\frac{1}{c^2}}\cdot 3\sqrt[3]{c\cdot\frac{1}{c}\cdot 1}} \le \frac{a}{6}$$

$$\frac{ba^3}{(a^4+1)(a^2+a+1)} \le \frac{b}{6} \text{ and } \frac{cb^3}{(b^4+1)(b^2+b+1)} \le \frac{c}{6}$$

Therefore,

$$\frac{ac^3}{(c^4+1)(c^2+c+1)} + \frac{ba^3}{(a^4+1)(a^2+a+1)} + \frac{cb^3}{(b^4+1)(b^2+b+1)} \le \frac{a+b+c}{6} = \frac{1}{2}$$

1066. If $a,b,c>0, ab+bc+ca=3$ then:

$$\frac{a}{3b^4+1} + \frac{b}{3c^4+1} + \frac{c}{3a^4+1} \ge \frac{3}{4}$$

Proposed by Marin Chirciu-Romania

Solution by Alex Szoros-Romania

For all $a,b,c>0$ holds:

$$\left(\sum a\right)^2\ge 3\sum ab\Rightarrow\left(\sum a\right)^2\ge 9\Rightarrow\sum a\ge 3;\ (1)$$

We show that for all $x>0$ holds:

$$\frac{1}{3x^4+1}\ge\frac{-3x}{4}+1;\ (2)$$

$$(2)\Leftrightarrow\frac{1}{3x^4+1}\ge\frac{-3x+4}{4}\Leftrightarrow 4\ge(3x^4+1)(-3x+4)\Leftrightarrow$$

$$4\ge-9x^5+12x^4-3x+4\Leftrightarrow 9x^5-12x^4+3x\ge 0\Leftrightarrow$$

$$3x(3x^4-4x^3+1)\ge 0\Leftrightarrow 3x(x-1)^2(3x^2+2x+1)\ge 0$$

Equality holds for $x=1$.

$$(2)\Rightarrow\frac{1}{3b^4+1}\ge\frac{-3b}{4}+1\,\Big|\cdot a\Rightarrow\frac{a}{3a^4+1}\ge\frac{-3ab}{4}+a$$

$$\sum_{cyc}\frac{a}{3b^4+1}\ge-\frac{3}{4}\sum_{cyc}ab+\sum_{cyc}a$$

$$\sum_{cyc}\frac{a}{3b^4+1}\ge-\frac{3}{4}\cdot 3+\sum_{cyc}a\overset{(1)}{\ge}-\frac{9}{4}+3=\frac{3}{4}$$

1067. **Let $a,b,c>0$, $ab+bc+ca=3$. Prove that:**

$$\frac{\sqrt{4-ab}+\sqrt{ab}}{2c+\sqrt{ab}}+\frac{\sqrt{4-bc}+\sqrt{bc}}{2a+\sqrt{bc}}+\frac{\sqrt{4-ca}+\sqrt{ca}}{2b+\sqrt{ca}}\ge\sqrt{3}+1.$$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$(a+b+c)\sqrt{3(4-ab)}=(a+b+c)\sqrt{ab+4bc+4ca}$$
$$=\sqrt{[(a+b-c)^2+4c(a+b)][ab+4c(a+b)]}\ge$$

$$\overset{CBS}{\ge}(a+b-c)\sqrt{ab}+4c(a+b)=2\left(2c+\sqrt{ab}\right)(a+b)-(a+b+c)\sqrt{ab}$$

$$Then:\ \sqrt{4-ab}+\sqrt{ab}\ge\frac{2(2c+\sqrt{ab})(a+b)}{\sqrt{3}(a+b+c)}+\left(1-\frac{1}{\sqrt{3}}\right)\sqrt{ab}\ \ (and\ analogs)$$

$$Therefore,\ \sum_{cyc}\frac{\sqrt{4-ab}+\sqrt{ab}}{2c+\sqrt{ab}}\ge\sum_{cyc}\left(\frac{2(a+b)}{\sqrt{3}(a+b+c)}+\left(1-\frac{1}{\sqrt{3}}\right).\frac{\sqrt{ab}}{2c+\sqrt{ab}}\right)=$$

$$=\frac{4}{\sqrt{3}}+\left(1-\frac{1}{\sqrt{3}}\right).\sum_{cyc}\frac{ab}{ab+2c\sqrt{ab}}\overset{CBS}{\ge}\frac{4}{\sqrt{3}}+\left(1-\frac{1}{\sqrt{3}}\right).\frac{\left(\sum_{cyc}\sqrt{ab}\right)^2}{\sum_{cyc}(ab+2c\sqrt{ab})}$$

$$=\frac{4}{\sqrt{3}}+\left(1-\frac{1}{\sqrt{3}}\right).1=\sqrt{3}+1.\ Equality\ holds\ iff\ a=b=c=1.$$

1068. If $a_1,a_2,\dots,a_n>0$ then prove that :

$$3\sum_{k=1}^{n}{a_k}^3+9\sum_{cyc}\frac{1}{a_1+a_2+a_3}\ge 4n\cdot\sqrt[4]{3}$$

Proposed by Neculai Stanciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$3\sum_{k=1}^{n}{a_k}^3+9\sum_{cyc}\frac{1}{a_1+a_2+a_3}\overset{Hölder\ \&\ CBS}{\ge}3.\frac{\left(\sum_{cyc}a_k\right)^3}{n^2}+9.\frac{n^2}{\sum_{cyc}(a_1+a_2+a_3)}=$$

$$=\frac{3\left(\sum_{cyc}a_k\right)^3}{n^2}+3.\frac{n^2}{\sum_{cyc}a_k}\overset{AM-GM}{\ge}4\sqrt[4]{\frac{3\left(\sum_{cyc}a_k\right)^3}{n^2}.\left(\frac{n^2}{\sum_{cyc}a_k}\right)^3}=4n.\sqrt[4]{3}.$$

$$Equality\ holds\ iff\ a_1=a_2=\dots=a_k=\frac{1}{\sqrt[4]{3}}.$$

1069. Let $a,b,c>0:ab+bc+ca=3.$ Prove that :

$$\sqrt{\frac{a(b+c)}{a^2+bc}}+\sqrt{\frac{b(c+a)}{b^2+ca}}+\sqrt{\frac{c(a+b)}{c^2+ab}}\ge(abc)^2+2$$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:$$

$$\sum_{cyc}\sqrt{\frac{a(b+c)}{a^2+bc}}=\sum_{cyc}\frac{2a(b+c)}{2\sqrt{(a^2+bc).a(b+c)}}\overset{AM-GM}{\geq}\sum_{cyc}\frac{2a(b+c)}{(a^2+bc)+a(b+c)}=$$

$$=\sum_{cyc}\frac{2a(b+c)}{(a+b)(a+c)}=\frac{2\sum_{cyc}a(b+c)^2}{(a+b)(b+c)(c+a)}=2+\frac{8abc}{(a+b)(b+c)(c+a)}=$$

$$=2+\frac{8(abc)^2}{c(a+b).a(b+c).b(c+a)}\overset{AM-GM}{\geq}2+\frac{8(abc)^2}{\left(\frac{c(a+b)+a(b+c)+b(c+a)}{3}\right)^3}$$

$$=2+(abc)^2.\text{Equality holds iff } a=b=c=1.$$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\sum_{cyc}\sqrt{\frac{a(b+c)}{a^2+bc}}\geq\frac{2}{1+\frac{a^2+bc}{a(b+c)}}+\frac{2}{1+\frac{b^2+ca}{b(c+a)}}+\frac{2}{1+\frac{c^2+ba}{c(a+b)}}=$$

$$=2\left[\frac{a(b+c)}{(a+b)(a+c)}+\frac{b(c+a)}{(b+a)(b+c)}+\frac{c(a+b)}{(c+a)(c+b)}\right]\geq(abc)^2+2$$

$$2[a(b+c)^2+b(c+a)^2+c(a+b)^2]\geq((abc)^2+2)(a+b)(b+c)(c+a)$$

$$2(a^2b+ab^2+b^2c+bc^2+c^2a+ca^2+6abc)\geq$$

$$\geq((abc)^2+2)(a^2b+ab^2+b^2c+bc^2+c^2a+ca^2+2abc)$$

$$8abc\geq(abc)^2(a^2b+ab^2+b^2c+bc^2+c^2a+ca^2+2bac)$$

$$8\geq abc(a^2b+ab^2+b^2c+bc^2+c^2a+ca^2+2abc)$$

$8(ab+bc+ca)^3\geq 27abc(a^2b+ab^2+b^2c+bc^2+c^2a+ca^2+2abc)$ true because

$$ab+bc+ca=3.$$

1070. Let $a,b,c\geq 0,\ a+b+c=5.$ Prove that :

$$\frac{1}{(a^2+bc)(b+c)}+\frac{1}{(b^2+ca)(c+a)}+\frac{1}{(c^2+ab)(a+b)}\geq\frac{1}{ab+bc+ca}$$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\text{We have}:\ (a+b+c)(ab+bc+ca)\geq(a+b)(b+c)(c+a),\ \forall a,b,c\geq 0$$

$$\text{Then}:\ \frac{1}{ab+bc+ca}\leq\frac{5}{(a+b)(b+c)(c+a)},\ \text{so it suffices to prove}:$$

$$\sum_{cyc}\frac{1}{(a^2+bc)(b+c)}\ge\frac{5}{(a+b)(b+c)(c+a)}\Leftrightarrow$$

$$\sum_{cyc}\frac{(a+b)(a+c)}{a^2+bc}\ge 5\Leftrightarrow\sum_{cyc}\left(1+\frac{a(b+c)}{a^2+bc}\right)\ge 5\Leftrightarrow\sum_{cyc}\frac{a(b+c)}{a^2+bc}\ge 2\Leftrightarrow$$

$$\sum_{cyc}a(b+c)\left(b^2+ca\right)(c^2+ab)\ge 2(a^2+bc)\left(b^2+ca\right)(c^2+ab)$$

$$\Leftrightarrow\sum_{cyc}(a^4b^2+a^2b^4)+3abc\sum_{cyc}(a^2b+ab^2)\ge 2\sum_{cyc}(ab)^3+2abc\sum_{cyc}a^3+4(abc)^2$$

$$\Leftrightarrow (a-b)^2(b-c)^2(c-a)^2+abc\sum_{cyc}(a^2b+ab^2)+2(abc)^2\ge 0,\ \ \textit{which is true.}$$

So the proof is completed.

Equality holds iff $(a,b,c)=\left(\frac{5}{2},\frac{5}{2},0\right)$ *and their permutation.*

1071. Let $a,b,c\ge 0: ab+bc+ca=1.$ Prove that :

$$\frac{1}{a+\sqrt{bc}}+\frac{1}{b+\sqrt{ca}}+\frac{1}{c+\sqrt{ab}}\ge 2\left(\frac{1}{1+\sqrt{ab}}+\frac{1}{1+\sqrt{bc}}+\frac{1}{1+\sqrt{ca}}-1\right)$$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

WLOG, assume that $a\ge b\ge c.$ *We have :*

$$\frac{1}{1+\sqrt{ab}}=1-\frac{\sqrt{ab}}{1+\sqrt{ab}}\overset{ab\le 1}{\le}1-\frac{\sqrt{ab}}{2}\ (\textit{and analogs})$$

$$\textit{Then}:\ \ 2\left(\frac{1}{1+\sqrt{ab}}+\frac{1}{1+\sqrt{bc}}+\frac{1}{1+\sqrt{ca}}-1\right)\le 4-\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)\ \ (1)$$

Also by AM − GM inequality we have :

$$\frac{1}{b+\sqrt{ca}}+\left(b+\sqrt{ca}\right)\ge 2\ \&\ \frac{1}{c+\sqrt{ab}}+\left(c+\sqrt{ab}\right)\ge 2$$

$$\text{Then}: \frac{1}{a+\sqrt{bc}}+\frac{1}{b+\sqrt{ca}}+\frac{1}{c+\sqrt{ab}} \ge 4+\frac{1}{a+\sqrt{bc}}-\left(b+c+\sqrt{ab}+\sqrt{ca}\right)\ (2)$$

From (1) *and* (2) *it suffices to prove* :

$$\frac{1}{a+\sqrt{bc}}-(b+c) \ge -\sqrt{bc}$$

$$\Leftrightarrow \frac{ab+bc+ca}{a+\sqrt{bc}}-(b+c)+\sqrt{bc} \ge 0 \Leftrightarrow \frac{\sqrt{bc}\left(a+2\sqrt{bc}-b-c\right)}{a+\sqrt{bc}} \ge 0$$

Which is true because $a \ge b$ *and* $\sqrt{bc} \ge c$.

So the proof is completed. Equality holds iff $(a,b,c) = (1,1,0)$ *and their permutation.*

1072. Let $a,b,c>0 : a+b+c=ab+bc+ca$. **Prove that :**

$$\frac{1}{1+\sqrt{ab}}+\frac{1}{1+\sqrt{bc}}+\frac{1}{1+\sqrt{ca}} \ge \frac{\sqrt{a}+\sqrt{b}+\sqrt{c}+3}{\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{abc}}$$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

The given inequality is equivalent to :

$$\sum_{cyc}\frac{\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{abc}}{1+\sqrt{ab}} \ge \sqrt{a}+\sqrt{b}+\sqrt{c}+3 \quad or \quad \sum_{cyc}\frac{\sqrt{a}+\sqrt{b}}{1+\sqrt{ab}} \ge 3.$$

Using the well known inequality : $x+y+z \ge \sqrt{3(xy+yz+zx)}$, *we get* :

$$\sum_{cyc}\frac{\sqrt{a}+\sqrt{b}}{1+\sqrt{ab}} \ge \sqrt{3\sum_{cyc}\frac{(\sqrt{b}+\sqrt{c})(\sqrt{c}+\sqrt{a})}{(1+\sqrt{bc})(1+\sqrt{ca})}} \overset{?}{\ge} 3$$

$$\Leftrightarrow \sum_{cyc}(\sqrt{b}+\sqrt{c})(\sqrt{c}+\sqrt{a})(1+\sqrt{ab}) \ge 3(1+\sqrt{ab})(1+\sqrt{bc})(1+\sqrt{ca})$$

$$\Leftrightarrow (a+b+c)+(ab+bc+ca) \ge 3+3abc, \quad \text{which is true because}:$$

$$(a+b+c)^2 \ge 3(ab+bc+ca) = 3(a+b+c) \Rightarrow a+b+c \ge 3,$$

$$(ab+bc+ca)^2 \geq 3abc(a+b+c) = 3abc(ab+bc+ca) \Rightarrow ab+bc+ca \geq 3abc.$$

So the proof is completed. Equality holds iff $a=b=c=1$.

1073. **If x,y,z are different positives real numbers such that**

$$\sqrt{x}+\sqrt{y}+\sqrt{z}=1, \text{then prove that}:$$

$$\frac{xy}{x-y}\log\left(\frac{x}{y}\right)+\frac{yz}{y-z}\log\left(\frac{y}{z}\right)+\frac{zx}{z-x}\log\left(\frac{z}{x}\right) \leq \frac{1}{3}$$

Proposed by Hikmat Mammadov-Azerbaijan

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Applying the inequality between the geometric mean and the logarithmic mean,

$$We\ have: \sqrt{xy} \leq \frac{x-y}{\log x - \log y}\quad then: \frac{\log\left(\frac{x}{y}\right)}{x-y} \leq \frac{1}{\sqrt{xy}}\quad (and\ analogs)$$

$$Therefore:$$

$$\frac{xy}{x-y}\log\left(\frac{x}{y}\right)+\frac{yz}{y-z}\log\left(\frac{y}{z}\right)+\frac{zx}{z-x}\log\left(\frac{z}{x}\right) \leq \sqrt{xy}+\sqrt{yz}+\sqrt{zx} \leq$$

$$\leq \frac{\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)^2}{3} = \frac{1}{3}\quad and\ the\ proof\ is\ competed.$$

1074. **If $a,b,c>0, a+b+c=1$ then:**

$$\sum_{cyc}(a+b)\sqrt[a+b]{a^ab^b} \leq a^2+b^2+c^2+a^ab^bc^c$$

Proposed by Daniel Sitaru-Romania

Solution by Hikmat Mammadov-Azerbaijan

Let $f: I \to \mathbb{R}$ a convex function, $x,y,z \in I$ and $a,b,c>0$, then:

$$af(x)+bf(y)+cf(z)+(a+b+c)f\left(\frac{ax+by+cz}{a+b+c}\right) \geq$$

$$\geq (a+b)f\left(\frac{ax+by}{a+b}\right)+(b+c)f\left(\frac{bx+cy}{b+c}\right)+(c+a)f\left(\frac{cz+ax}{c+a}\right); (Popoviciu)$$

If $a+b+c=1$, we obtain:

$$af(x)+bf(y)+cf(z)+f(ax+by+cz) \geq$$

$$\geq (a+b)f\left(\frac{ax+by}{a+b}\right)+(b+c)f\left(\frac{bx+cy}{b+c}\right)+(c+a)f\left(\frac{cz+ax}{c+a}\right)$$

Let $I=\mathbb{R}, f(x)=e^x$ and $x=\log a, y=\log b, z=\log c$, then:

$$ae^{\log a}+be^{\log b}+ce^{\log c}+e^{a\log a+b\log b+c\log c}\geq$$

$$\geq (a+b)e^{\frac{a\log a+b\log b}{a+b}}+(b+c)e^{\frac{b\log b+c\log c}{b+c}}+(c+a)e^{\frac{c\log c+a\log a}{c+a}}\Leftrightarrow$$

$$a^2+b^2+c^2+a^ab^bc^c\geq (a+b)\left(a^ab^b\right)^{\frac{1}{a+b}}+(b+c)\left(b^bc^c\right)^{\frac{1}{c+b}}+(c+a)(c^ca^a)^{\frac{1}{c+a}}$$

$$\sum_{cyc}(a+b)\sqrt[a+b]{a^ab^b}\leq a^2+b^2+c^2+a^ab^bc^c$$

1075. If $a,b,c>0$ then prove that:

$$\sqrt{\frac{a+b}{a^2+b^2}}+\sqrt{\frac{b^2+c^2}{b^3+c^3}}+\sqrt{\frac{c^3+a^3}{c^4+a^4}}\leq\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}}$$

Proposed by Tran Quoc Thinh-Vietnam

Solution 1 by Soumitra Mandal-Chandar Nagore-India

$$\sqrt{\frac{a+b}{a^2+b^2}}+\sqrt{\frac{b^2+c^2}{b^3+c^3}}+\sqrt{\frac{c^3+a^3}{c^4+a^4}}\leq$$

$$\leq\sqrt{\frac{2(a+b)}{(a+b)^2}}+\sqrt{\frac{2(b^2+c^2)}{(b+c)(b^2+c^2)}}+\sqrt{\frac{2(c^3+a^3)}{(c+a)(c^3+a^3)}}=$$

$$\because 2(x^{n+1}+y^{n+1})\geq (x+y)(x^n+y^n), n\geq 1$$

$$=\sqrt{\frac{2}{a+b}}+\sqrt{\frac{2}{b+c}}+\sqrt{\frac{2}{c+a}}\leq\frac{1}{2}\sum_{cyc}\left(\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}\right)=\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}}$$

$$\because x\to\frac{1}{\sqrt{x}}\text{ is convex function.}$$

Equality holds for $a=b=c$.

Solution 2 by Tapas Das-India

$$\frac{a+b}{a^2+b^2}-\frac{2}{a+b}=\frac{(a+b)^2-2(a^2+b^2)}{(a^2+b^2)(a+b)}=\frac{2ab-(a^2+b^2)}{(a^2+b^2)(a+b)}$$

$$\because a^2+b^2\geq 2ab$$

$$\sqrt{\frac{a+b}{a^2+b^2}} \le \sqrt{\frac{2}{a+b}}$$

$$\frac{b^2+c^2}{b^3+c^3} - \frac{2}{b+c} = \frac{(b^2+c^2)(b+c)-2(b^3+c^3)}{(b+c)(b^3+c^3)} =$$

$$= \frac{bc(b+c)-(b^3+c^3)}{(b+c)(b^3+c^3)} \le 0$$

$$\because b^3+c^3 = (b+c)(b^2-bc+c^2) \ge bc(b+c)$$

$$\sqrt{\frac{b^2+c^2}{b^3+c^3}} \le \sqrt{\frac{2}{b+c}}$$

$$\frac{c^3+a^3}{c^4+a^4} - \frac{2}{c+a} = \frac{(c^3+a^3)(c+a)-2(c^4+a^4)}{(c^4+a^4)(c+a)} =$$

$$= \frac{ca(c^2+a^2)-(c^4+a^4)}{(c^4+a^4)(c+a)} \le 0$$

$$\because ca(c^2+a^2) \le \frac{c^2+a^2}{2}(c^2+a^2) = \frac{(c^2+a^2)^2}{2}$$

$$(c^4+a^4)(1^2+1^2) \ge (c^2+a^2)^2 \Rightarrow c^4+a^4 \ge \frac{1}{2}(c^2+a^2)^2$$

$$\sqrt{\frac{c^3+a^3}{c^4+a^4}} \le \sqrt{\frac{2}{c+a}}$$

$$\sqrt{\frac{a+b}{a^2+b^2}} + \sqrt{\frac{b^2+c^2}{b^3+c^3}} + \sqrt{\frac{c^3+a^3}{c^4+a^4}} \le \sqrt{\frac{2}{a+b}} + \sqrt{\frac{2}{b+c}} + \sqrt{\frac{2}{c+a}} \le$$

$$\le \frac{1}{4}\sum_{cyc}\frac{1}{\sqrt{a}} \le \sum_{cyc}\frac{1}{\frac{2}{\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}}} = \frac{1}{2}\sum_{cyc}\left(\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}\right) = \frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}}$$

Equality holds for $a = b = c$.

1076. If $a, b, c > 0, abc = 1$ then:

$$\frac{1+a}{2+b} + \frac{1+b}{2+c} + \frac{1+c}{2+a} \le \frac{2(a+b+c)}{3}$$

Proposed by Tran Quoc Thinh-Vietnam

Solution 1 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\frac{1+a}{2+b}+\frac{1+b}{2+c}+\frac{1+c}{2+a}\le\frac{2(a+b+c)}{3}$$

$$\frac{3-a-2ab}{b+2}+\frac{3-b-2bc}{c+2}+\frac{3-c-2ac}{a+2}\le 0$$

$$(3-a-2ab)(a+2)(c+2)+(3-b-2bc)(a+2)(b+2)$$
$$+(3-c-2ac)(b+2)(c+2)\le 0$$

$$b(a+b+c)+24\le$$
$$\le 2(a^2+b^2+c^2)+7(ab+bc+ca)+a^2c+c^2b+b^2a+4(a^2b+b^2c+c^2a)$$

$$b(a+b+c)\le 2(a^2+b^2+c^2)+a^2c+c^2b+b^2a+3(a^2b+b^2c+c^2a)$$

$$6\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)^2\le 2\left(\left(\frac{x}{y}\right)^2+\left(\frac{y}{z}\right)^2+\left(\frac{z}{x}\right)^2\right)+\frac{xz}{y^2}+\frac{zy}{x^2}+\frac{xy}{z^2}+3\left(\frac{x^2}{yz}+\frac{y^2}{zx}+\frac{z^2}{xy}\right)$$

$$6(x^2z+z^2y+y^2x)\le 2(x^3z+y^3x+z^3y)+\frac{(xz)^2}{y}+\frac{(yz)^2}{x}+\frac{(xy)^2}{z}+$$
$$+3(x^3+y^3+z^3)\text{ which is true, because}$$

$$\frac{(xz)^2}{y}+\frac{(yz)^2}{x}+\frac{(xy)^2}{z}+3(x^3+y^3+z^3)\ge 2(x^2z+z^2y+y^2x)\text{ and}$$

$$2\left(\frac{x^3z}{y}+\frac{y^3x}{z}+\frac{z^3y}{x}\right)+2(x^3+y^3+z^3)\ge 4(x^2z+z^2y+y^2x)$$

$$\left(\frac{x^3z}{y}+\frac{y^3x}{z}+\frac{z^3y}{x}\right)+(x^3+y^3+z^3)\ge 2(x^2z+z^2y+y^2x)$$

$$x^4z^2+y^4x^2+z^4y^2+x^4yz+y^4zx+z^4xy\ge 2(x^3y^2z+z^3y^2x+y^3x^2z)$$

Solution 2 by proposer

$$\frac{1+a}{2+b}+\frac{1+b}{2+c}+\frac{1+c}{2+a}\le\frac{2(a+b+c)}{3};(*)$$

$$(*)\Leftrightarrow\frac{2}{3}(1+a)-\frac{1+a}{2+b}+\frac{2}{3}(1+b)-\frac{1+b}{2+c}+\frac{2}{3}(1+c)-\frac{1+c}{2+a}\ge 2$$

$$\frac{(1+a)(1+2b)}{2(2+b)}+\frac{(1+b)(1+2c)}{3(2+c)}+\frac{(1+c)(1+2a)}{3(2+a)}\ge 2;(1)$$

$$\text{Let }\frac{1+2x}{2+x}\ge\frac{1+5x}{1+x};(2)\Leftrightarrow(x-1)^2\ge 0,\forall x>0$$

Use (2) and AM-GM, we have:

$$\frac{(1+a)(1+2b)}{2(2+b)}+\frac{(1+b)(1+2c)}{3(2+c)}+\frac{(1+c)(1+2a)}{3(2+a)}\geq$$

$$\geq\frac{(1+a)(1+5b)}{9(1+b)}+\frac{(1+b)(1+5c)}{9(1+c)}+\frac{(1+c)(1+5a)}{9(1+a)}\geq$$

$$\geq 3\sqrt[3]{\frac{(1+a)(1+5b)}{9(1+b)}\cdot\frac{(1+b)(1+5c)}{9(1+c)}\cdot\frac{(1+c)(1+5a)}{9(1+a)}}=$$

$$=\frac{1}{3}\sqrt[3]{(1+5a)(1+5b)(1+5c)}=$$

$$=\frac{1}{3}\sqrt[3]{1+5(a+b+c)+25(ab+bc+ca)+125abc}\geq$$

$$\geq\frac{1}{3}\sqrt[3]{1+15\sqrt[3]{abc}+75\sqrt[3]{a^2b^2c^2}+125abc}=\frac{1}{3}\sqrt[3]{1+15+75+125}=2$$

Solution 3 by Tran Quoc Anh-Vietnam

Let $S=\frac{a+1}{b+2}+\frac{b+1}{c+2}+\frac{c+1}{a+2}$ and let $a\geq b\geq c$ then $a+1\geq b+1\geq c+1$

$$\text{or}\ \frac{1}{a+2}\leq\frac{1}{b+2}\leq\frac{1}{c+2}$$

$$S=(a+1)\cdot\frac{1}{b+2}+(b+1)\cdot\frac{1}{c+2}+(c+1)\cdot\frac{1}{a+2}\overset{Chebyshev}{\leq}$$

$$\leq\frac{a+b+c+3}{3}\left(\frac{1}{a+2}+\frac{1}{b+2}+\frac{1}{c+2}\right)$$

We must show: $\frac{a+b+c+3}{3}\left(\frac{1}{a+2}+\frac{1}{b+2}+\frac{1}{c+2}\right)\leq\frac{2}{3}(a+b+c)$

$$\frac{1}{a+2}+\frac{1}{b+2}+\frac{1}{c+2}\leq 2\left(\frac{a+b+c}{a+b+c+3}\right)=2-\frac{6}{a+b+c+3}$$

$$\frac{1}{a+2}+\frac{1}{b+2}+\frac{1}{c+2}+\frac{6}{a+b+c+3}\leq 2$$

But from AM − GM: $a+b+c+3\geq 3\sqrt[3]{abc}+3=6$, hence

$$\frac{1}{a+2}+\frac{1}{b+2}+\frac{1}{c+2}+\frac{6}{a+b+c+3}\leq\frac{1}{a+2}+\frac{1}{b+2}+\frac{1}{c+2}+1$$

$$\frac{1}{a+2}+\frac{1}{b+2}+\frac{1}{c+2}+1\leq 2\Leftrightarrow\frac{1}{a+2}+\frac{1}{b+2}+\frac{1}{c+2}\leq 1$$

$$(a+2)(b+2)+(a+2)(c+2)+(b+2)(c+2)-(a+2)(b+2)(c+2)\leq 0$$

$$4-abc-ba-bc-ca\leq 0$$

$$3\leq ab+bc+ca;(1)$$

$$ab+bc+ca\geq 3\sqrt[3]{(abc)^2}=3$$

1077. In ΔABC the following relationship holds:

$$2\leq\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{4abc}{(a+b)(b+c)(c+a)}\leq\frac{R^2}{2r^2}$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution 1 by Alex Szoros-Romania

$$\sum_{cyc}\frac{a}{b+c}+\frac{4abc}{(a+b)(b+c)(c+a)}\geq 2;(1)$$

$$\Leftrightarrow\sum_{cyc}a(a+b)(a+c)+4abc\geq 2(a+b)(b+c)(c+a)$$

$$\sum_{cyc}a[a^2+(b+c)a+bc]+4abc\geq 2\left[2abc+\sum_{cyc}ab(a+b)\right]$$

$$\sum_{cyc}a^3+\sum_{cyc}(b+c)a^2+3abc+4abc\geq 4abc+2\sum_{cyc}ab(a+b)$$

$$\sum_{cyc}a^3+\sum_{cyc}(b+c)a^2+3abc\geq 2\sum_{cyc}(b+c)a^2$$

$$\sum_{cyc}a^3-\sum_{cyc}(b+c)a^2+3abc\geq 0,\qquad \sum_{cyc}a(a-b)(a-c)\geq 0\ (Schur)$$

$$\because \frac{R}{r}\geq\frac{b}{c}+\frac{c}{b}\text{ in any }\Delta ABC\ (Bandila)$$

$$\Rightarrow\frac{3R}{r}\geq\sum_{cyc}\left(\frac{b}{c}+\frac{c}{b}\right)=\sum_{cyc}\left(\frac{a}{b}+\frac{a}{c}\right)\geq\sum_{cyc}\frac{4a}{b+c}$$

$$\frac{3R}{4r}\geq\sum_{cyc}\frac{a}{b+c};(2)$$

$$\because R\geq 2r\ (Euler)\Rightarrow\frac{R}{4r}\geq\frac{1}{4}$$

$$(a+b)(b+c)(c+a)\ge 8abc\ (Cesaro)\Rightarrow \frac{1}{2}\ge\frac{4abc}{(a+b)(b+c)(c+a)}$$

$$\frac{R}{4r}\ge\frac{4abc}{(a+b)(b+c)(c+a)};(3)$$

From (2), (3):

$$\frac{3R}{4r}+\frac{R}{4r}\ge\sum_{cyc}\frac{a}{b+c}+\frac{4abc}{(a+b)(b+c)(c+a)}$$

$$\frac{R}{r}\ge\sum_{cyc}\frac{a}{b+c}+\frac{4abc}{(a+b)(b+c)(c+a)};\quad (4)$$

$$\frac{R^2}{2r^2}=\frac{R}{r}\cdot\frac{R}{2r}\ge\frac{R}{r}\cdot 1=\frac{R}{r};(5)$$

From (4), (5):

$$\frac{R^2}{2r^2}\ge\sum_{cyc}\frac{a}{b+c}+\frac{4abc}{(a+b)(b+c)(c+a)};(6)$$

From (1), (6):

$$2\le\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{4abc}{(a+b)(b+c)(c+a)}\le\frac{R^2}{2r^2}$$

Solution 2 by Tapas Das-India

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}=\frac{(a+b+c)(a^2+b^2+c^2)+3abc}{(a+b+c)(ab+bc+ca)-abc}=$$

$$=\frac{2s\cdot 2(s^2+r^2-4Rr)+3\cdot 4Rrs}{2s(s^2+r^2+4Rr)-4Rrs}=2-\frac{r(6R+4r)}{s^2+r^2+2Rr}$$

$$\frac{4abc}{(a+b)(b+c)(c+a)}=\frac{4abc}{(a+b+c)(ab+bc+ca)-abc}=$$

$$=\frac{4\cdot 4Rrs}{2s(s^2+r^2+4Rr)-4Rrs}=\frac{8Rr}{s^2+r^2+2Rr}$$

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{4abc}{(a+b)(b+c)(c+a)}=$$

$$=2-\frac{r(6R+4r)}{s^2+r^2+2Rr}+\frac{8Rr}{s^2+r^2+2Rr}=2+\frac{2Rr-4r^2}{s^2+r^2+2Rr}\ge 0;(\because R\ge 2r)$$

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{4abc}{(a+b)(b+c)(c+a)}\ge 2$$

Now, we have:

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{4abc}{(a+b)(b+c)(c+a)}=$$

$$=2+\frac{2Rr-4r^2}{s^2+r^2+2Rr}\overset{Gerretsen}{\leq}2+\frac{2Rr-4r^2}{16Rr-5r^2+r^2+2Rr}=$$

$$=2+\frac{2Rr-4r^2}{18Rr-4r^2}=2+\frac{R-2r}{9R-2r}\overset{Euler}{\leq}2+\frac{R-2r}{16r}$$

We need to show:

$$2+\frac{R-2r}{16r}\leq\frac{R^2}{2r^2}\Leftrightarrow(R-2r)(8R+15r)\geq 0 \text{ true.}$$

Therefore,

$$2\leq\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{4abc}{(a+b)(b+c)(c+a)}\leq\frac{R^2}{2r^2}$$

1078. Let $a,b,c\geq 0, ab+bc+ca>0$. Prove that:

$$\frac{1}{2a^2+5a\sqrt{bc}+2bc}+\frac{1}{2b^2+5b\sqrt{ca}+2ca}+\frac{1}{2c^2+5c\sqrt{ab}+2ab}\geq\frac{1}{ab+bc+ca}$$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution by Duc Nguyen-Vietnam

Set $(a,b,c)\to(a^2,b^2,c^2)$, we need to prove:

$$\sum_{cyc}\frac{1}{2a^4+5a^2bc+2(bc)^2}\geq\frac{1}{a^2b^2+b^2c^2+c^2a^2}$$

Case 1: $abc=0$ it is obviously true.

Case 2: $abc>0$ we can assume that $abc=1$.

Set $t=b+c, s=bc$ with $a=\min\{a,b,c\}\Rightarrow as=1$

Hence, the inequality becomes proving:

$$f(s)=\frac{2t^4+4s^2-8st^2+5t+2a^2t^2-4a}{(4s^2+a+2a^3t-6t)(s^2+4a+2a^3t-6t)}+\frac{1}{2a^4+5a+2s^2}-\frac{1}{(at)^2-2a+s^2}\geq 0$$

$$f'(s)<0, s\in\left[1;\frac{t^2}{4}\right]\Rightarrow f(s)\geq f\left(\frac{t^2}{4}\right)\geq 0$$

Equality holds iff $a=b=c>0$.

1079. If $a,b,c,\lambda,n>0,\ \ a+b+c=3$ then:

$$\left(n+\lambda\frac{b}{a}\right)^{\frac{1}{b^2}}\left(n+\lambda\frac{c}{b}\right)^{\frac{1}{c^2}}\left(n+\lambda\frac{a}{c}\right)^{\frac{1}{a^2}}\geq(\lambda+n)^3$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:\ \ ln\left(\left(n+\lambda\frac{b}{a}\right)^{\frac{1}{b^2}}\left(n+\lambda\frac{c}{b}\right)^{\frac{1}{c^2}}\left(n+\lambda\frac{a}{c}\right)^{\frac{1}{a^2}}\right)=\sum_{cyc}\frac{ln\left(n+\lambda\frac{b}{a}\right)}{b^2}$$

$$=ln(n)\sum_{cyc}\frac{1}{b^2}+\sum_{cyc}\frac{ln\left(1+\frac{\lambda}{n}.\frac{b}{a}\right)}{b^2}=$$

$$=ln(n)\sum_{cyc}\frac{1}{b^2}+\int_0^1\left(\sum_{cyc}\frac{\frac{\lambda}{n}}{b\left(\frac{\lambda b}{n}x+a\right)}\right)dx\overset{H\ddot{o}lder}{\geq}\frac{3^3.ln(n)}{\left(\sum_{cyc}b\right)^2}$$

$$+\frac{\lambda}{n}.\int_0^1\frac{3^3}{\left(\sum_{cyc}b\right)\left[\sum_{cyc}\left(\frac{\lambda b}{n}x+a\right)\right]}dx=$$

$$=3ln(n)+\frac{\lambda}{n}.\int_0^1\frac{3dx}{\frac{\lambda}{n}x+1}=3ln(n)+\left[3ln\left(\frac{\lambda}{n}x+1\right)\right]_0^1=3ln(\lambda+n)=ln[(\lambda+n)^3].$$

$$Therefore,\ \ \left(n+\lambda\frac{b}{a}\right)^{\frac{1}{b^2}}\left(n+\lambda\frac{c}{b}\right)^{\frac{1}{c^2}}\left(n+\lambda\frac{a}{c}\right)^{\frac{1}{a^2}}\geq(\lambda+n)^3.$$

1080. Let $a,b,c>0:a+b+c=\frac{2}{3}$. Prove that:

$$\frac{1}{a^3+abc}+\frac{1}{b^3+abc}+\frac{1}{c^3+abc}\geq\frac{1}{a^2b^2+b^2c^2+c^2a^2}$$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution 1 by Vivek Kumar-India

$$\text{Let } a=\frac{1}{3x},b=\frac{1}{3y},c=\frac{1}{3z}\Rightarrow\frac{1}{3x}+\frac{1}{3y}+\frac{1}{3z}=\frac{2}{3}$$

$$xy+yz+zx=2xyz$$

$$\frac{1}{a^3+abc}+\frac{1}{b^3+abc}+\frac{1}{c^3+abc}\ge\frac{1}{a^2b^2+b^2c^2+c^2a^2}\Leftrightarrow$$

$$\sum_{cyc}\frac{1}{\left(\frac{1}{3x}\right)^3+\frac{1}{3x}\cdot\frac{1}{3y}\cdot\frac{1}{3z}}\ge\frac{1}{\left(\frac{1}{3x}\right)^2\left(\frac{1}{3y}\right)^2+\left(\frac{1}{3y}\right)^2\left(\frac{1}{3z}\right)^2+\left(\frac{1}{3z}\right)^2\left(\frac{1}{3x}\right)^2}\Leftrightarrow$$

$$\sum_{cyc}\frac{27x^3yz}{x^2+yz}\ge\frac{81x^2y^2z^2}{x^2+y^2+z^2},\qquad\sum_{cyc}\frac{x^2}{x^2+yz}\ge\frac{3xyz}{x^2+y^2+z^2}$$

Now, we have:

$$\sum_{cyc}\frac{x^2}{x^2+yz}\overset{CBS}{\ge}\frac{(x+y+z)^2}{x^2+y^2+z^2+xy+yz+zx}$$

$$\frac{(x+y+z)^2}{x^2+y^2+z^2+xy+yz+zx}\ge\frac{3xyz}{x^2+y^2+z^2};(*)$$

$$\frac{x^2+y^2+z^2+2(xy+yz+zx)}{x^2+y^2+z^2+xy+yz+zx}\ge\frac{\frac{3}{2}(xy+yz+zx)}{x^2+y^2+z^2}$$

$$2(x^2+y^2+z^2)+(x^2+y^2+z^2)(xy+yz+zx)-3(xy+yz+zx)^2\ge 0$$

$$\Big((x^2+y^2+z^2)-(xy+yz+zx)\Big)\Big(2(x^2+y^2+z^2)+3(xy+yz+zx)\Big)\ge 0$$

Which is true because $x^2+y^2+x^2\ge xy+yz+zx$.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc}\frac{1}{a^3+abc}=\sum_{cyc}\frac{(bc)^2}{abc(a^2bc+b^2c^2)}\overset{AM-GM}{\ge}\frac{1}{abc}\sum_{cyc}\frac{(bc)^2}{\frac{c^2a^2+a^2b^2}{2}+b^2c^2}\ge$$

$$\overset{CBS}{\ge}\frac{\left(\sum_{cyc}bc\right)^2}{abc.2\sum_{cyc}b^2c^2}\overset{(\sum x)^2\ge 3\sum xy}{\ge}\frac{3abc\sum_{cyc}a}{2abc\sum_{cyc}b^2c^2}=\frac{1}{a^2b^2+b^2c^2+c^2a^2},\ \textit{as desired.}$$

$$\textit{Equality holds iff } a=b=c=\frac{2}{9}.$$

1081. **Let $a,b,c\ge 0:a+b+c=5.$ Prove that:**

$$\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}-\frac{5}{2}\right)^2+5\ge 5\left(\sqrt{ab+bc+ca}-\frac{3}{2}\right)^2$$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

The given inequality is successively equivalent to :

$$\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)^2-5\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)\geq 5(ab+bc+ca)-15\sqrt{ab+bc+ca}$$

Or

$$15\sqrt{ab+bc+ca}\geq 5(ab+bc+ca)+5\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)-\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)^2.$$

We have :

$$5\sqrt{ab+bc+ca}=(a+b+c)\sqrt{ab+bc+ca}=$$

$$=\sqrt{[(-a+b+c)^2+4a(b+c)][bc+a(b+c)]}\overset{CBS}{\geq}\sqrt{bc}(-a+b+c)+2a(b+c)$$

Similarly we have :

$$5\sqrt{ab+bc+ca}\geq\sqrt{ca}(a-b+c)+2b(c+a)$$

$$And,\ \ 5\sqrt{ab+bc+ca}\geq\sqrt{ab}(a+b-c)+2c(a+b)$$

Summing up these inequalities we get :

$$15\sqrt{ab+bc+ca}\geq(a+b+c)\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)-\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)^2+5(ab+bc+ca)=$$

$$=5(ab+bc+ca)+5\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)-\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)^2,\ \ as\ desired.$$

$$Equality\ hold\ iff\ (a,b,c)=\left(\frac{5}{2},\frac{5}{2},0\right)\ or\ (5,0,0)\ and\ their\ permutations.$$

1082. If $0\leq a,b,c\leq 1$ and $\lambda\geq 1$ then :

$$\frac{1}{\lambda+a}+\frac{1}{\lambda+b}+\frac{1}{\lambda+c}\leq\frac{3}{\lambda+\sqrt[3]{abc}}$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$If\ c=0\ the\ problem\ becomes\ \frac{1}{\lambda+a}+\frac{1}{\lambda+b}\leq\frac{2}{\lambda}\ which\ is\ true\ because$$

$$\frac{1}{\lambda+a}+\frac{1}{\lambda+b}\leq\frac{1}{\lambda}+\frac{1}{\lambda}=\frac{2}{\lambda}.$$

$$Assume\ now\ that\ a,b,c>0\ and\ let\ f(x)=\frac{1}{\lambda+e^x},\ \ x\leq 0.$$

$$We\ have:\ f'(x)=-\frac{e^x}{(\lambda+e^x)^2}\ and\ f''(x)=-\frac{e^x(\lambda-e^x)}{(\lambda+e^x)^2}\overset{\lambda\geq 1\geq e^x}{\leq}0,\ \ \forall x\leq 0.$$

$$Then\ f\ is\ concave\ and\ by\ Jensen's\ inequality\ we\ have:$$

$$\sum_{cyc}\frac{1}{\lambda+a}=\sum_{cyc}f(ln(a))\leq 3f\left(\frac{ln(a)+ln(b)+ln(c)}{3}\right)=3f\left(ln(\sqrt[3]{abc})\right)=\frac{3}{\lambda+\sqrt[3]{abc}}$$

$$Equality\ holds\ iff\ a=b=c.$$

1083. **Let $a,b,c>0: ab+bc+ca=3.$ Prove that:**

$$\frac{a+\sqrt{bc}}{\sqrt{a^2+3}+\sqrt{bc}}+\frac{b+\sqrt{ca}}{\sqrt{b^2+3}+\sqrt{ca}}+\frac{c+\sqrt{ab}}{\sqrt{c^2+3}+\sqrt{ab}}\leq 2$$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:$$

$$\frac{a+\sqrt{bc}}{\sqrt{a^2+3}+\sqrt{bc}}=\frac{a+\sqrt{bc}}{\sqrt{a^2+ab+bc+ca}+\sqrt{bc}}=$$

$$\frac{\left(a+\sqrt{bc}\right)\left(\sqrt{(a+b)(a+c)}-\sqrt{bc}\right)}{a(a+b+c)}=$$

$$=\frac{2\left(a+\sqrt{bc}\right)\sqrt{(a+b)(a+c)}-2\left(a+\sqrt{bc}\right)\sqrt{bc}}{2a(a+b+c)}\overset{AM-GM}{\leq}$$

$$\leq\frac{\left(a+\sqrt{bc}\right)^2+(a+b)(a+c)-2\left(a+\sqrt{bc}\right)\sqrt{bc}}{2a(a+b+c)}$$

$$Then:\ \frac{a+\sqrt{bc}}{\sqrt{a^2+3}+\sqrt{bc}}\leq\frac{2a+b+c}{2(a+b+c)}.$$

$$\frac{b+\sqrt{ca}}{\sqrt{b^2+3}+\sqrt{ca}}\leq\frac{a+2b+c}{2(a+b+c)}\ and\ \frac{c+\sqrt{ab}}{\sqrt{c^2+3}+\sqrt{ab}}\leq\frac{a+b+2c}{2(a+b+c)}.$$

$$Summing\ up\ these\ inequalities\ yields\ the\ desired\ results.$$

$$Equality\ holds\ iff\ a=b=c=1.$$

1084. If $a_1, a_2, \dots, a_n > 0, a_1 + a_2 + \dots + a_n = n, n \geq 2$, then:

$$\frac{1}{n-a_1}+\frac{1}{n-a_2}+\dots+\frac{1}{n-a_n}\geq\frac{n}{n-1}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Tapas Das-India

$$\text{Let } f(x)=\frac{1}{n-x}, x>0 \text{ then } f'(x)=\frac{1}{(n-x)^2}, f''(x)=\frac{2}{(n-x)^3}>0$$

$$f-\text{is convex function in } (0,\infty).$$

$$\text{By Jensen's: } f(a_1)+f(a_2)+\dots+f(a_n)\geq n\,f\left(\frac{a_1+a_2+\dots+a_n}{n}\right)$$

$$\frac{1}{n-a_1}+\frac{1}{n-a_2}+\dots+\frac{1}{n-a_n}\geq nf(1)=\frac{n}{n-1}$$

Solution 2 by Ertan Yildirim-Turkiye

$$\frac{n-a_1+n-a_2+\dots+n-a_n}{n}\overset{HM-AM}{\geq}\frac{n}{\frac{1}{n-a_1}+\frac{1}{n-a_2}+\dots+\frac{1}{n-a_n}}$$

$$\frac{n\cdot n-(a_1+a_2+\dots+a_n)}{n}\geq\frac{n}{\frac{1}{n-a_1}+\frac{1}{n-a_2}+\dots+\frac{1}{n-a_n}}$$

$$\frac{n^2-n}{n}\geq\frac{n}{\frac{1}{n-a_1}+\frac{1}{n-a_2}+\dots+\frac{1}{n-a_n}}$$

$$\frac{1}{n-a_1}+\frac{1}{n-a_2}+\dots+\frac{1}{n-a_n}\geq\frac{n^2}{n^2-n}=\frac{n}{n-1}$$

Solution 3 by Vivek Kumar-India

$$\frac{1}{n-a_1}+\frac{1}{n-a_2}+\dots+\frac{1}{n-a_n}\overset{CBS}{\geq}\frac{(1+1+\dots+1)^2}{(n-a_1)+(n-a_2)+\dots+(n-a_n)}=$$

$$=\frac{n^2}{(n+n+\dots+n)-(a_1+a_2+\dots+a_n)}=\frac{n^2}{n^2-n}=\frac{n}{n-1}$$

Solution 4 by Fayssal Abdelli-Algerie

$$\frac{1}{n-a_1}+\frac{1}{n-a_2}+\cdots+\frac{1}{n-a_n}\overset{Bergstrom}{\geq}\frac{(1+1+\cdots+1)^2}{(n-a_1)+(n-a_2)+\cdots+(n-a_n)}=$$

$$=\frac{n^2}{(n+n+\cdots+n)-(a_1+a_2+\cdots+a_n)}=\frac{n^2}{n^2-n}=\frac{n}{n-1}$$

1085. If $m,n,p\in\mathbb{N}-\{0\}$ then:

$$\sum_{k=1}^{n}\frac{1}{k}+\sum_{k=1}^{m}\frac{1}{k}+\sum_{k=1}^{p}\frac{1}{k}\leq 1+\sum_{k=1}^{mnp}\frac{1}{k}$$

Proposed by Daniel Sitaru-Romania

Solution by Ravi Prakash-New Delhi-India

Let us first show that if $m,n\geq 1$, then:

$$\sum_{k=1}^{n}\frac{1}{k}+\sum_{k=1}^{m}\frac{1}{k}\leq 1+\sum_{k=1}^{mn}\frac{1}{k}\Leftrightarrow\sum_{k=2}^{n}\frac{1}{k}\leq\sum_{k=m+1}^{mn}\frac{1}{k};\quad(1)$$

$$\sum_{k=m+1}^{mn}\frac{1}{k}=\left(\frac{1}{m+1}+\frac{1}{m+2}+\cdots+\frac{1}{2m}\right)+\left(\frac{1}{2m+1}+\frac{1}{2m+2}+\cdots+\frac{1}{3m}\right)+$$

$$+\left(\frac{1}{3m+1}+\frac{1}{3m+2}+\cdots+\frac{1}{4m}\right)+\cdots+\left(\frac{1}{(n-1)m+1}+\frac{1}{(n-1)m+2}+\cdots+\frac{1}{nm}\right)\geq$$

$$\geq\frac{m}{2m}+\frac{m}{3m}+\cdots+\frac{m}{nm}=\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}=\sum_{k=1}^{n}\frac{1}{k}$$

Now,

$$\sum_{k=1}^{n}\frac{1}{k}+\sum_{k=1}^{m}\frac{1}{k}+\sum_{k=1}^{p}\frac{1}{k}=\left(\sum_{k=1}^{n}\frac{1}{k}+\sum_{k=1}^{m}\frac{1}{k}\right)+\sum_{k=1}^{p}\frac{1}{k}\leq$$

$$\leq 1+\sum_{k=1}^{mn}\frac{1}{k}+\sum_{k=1}^{p}\frac{1}{k}\leq 1+1+\sum_{k=1}^{mnp}\frac{1}{k}=2+\sum_{k=1}^{mnp}\frac{1}{k}$$

1086. For $a_1,a_2,\ldots,a_n\geq 0:\ \sum_{1\leq i<j\leq n}a_ia_j=1$. Set $t=\sum_{1\leq i\leq n}a_i$ then:

$$\sum_{i=1}^{n}\frac{1-a_i}{a_i+\sqrt{a_i^2-a_it+1}}\ge n-2.$$ **When does equality hold?**

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$For\ k\in\overline{1,n},\ \ we\ have:\ \ {a_k}^2-a_kt+1=1-a_k(t-a_k)$$
$$=\sum_{1\le i<j\le n}a_ia_j-\sum_{1\le i\le n,i\ne k}a_ka_i\ge 0.$$

$$Now\ we\ have:$$

$$t=\sqrt{t^2.1}=\sqrt{[(t-2a_i)^2+4a_i(t-a_i)][({a_i}^2-a_it+1)+a_i(t-a_i)]}\ge$$

$$\overset{CBS}{\ge}(t-2a_i)\sqrt{{a_i}^2-a_it+1}+2a_i(t-a_i)=(t-2a_i)\left(\sqrt{{a_i}^2-a_it+1}+a_i\right)+a_it$$

$$Then:\ \ (1-a_i)t\ge(t-2a_i)\left(\sqrt{{a_i}^2-a_it+1}+a_i\right)\ \ or$$

$$\frac{1-a_i}{a_i+\sqrt{{a_i}^2-a_it+1}}\ge 1-\frac{2a_i}{t},\ \ \forall i\in\overline{1,n}.$$

$$Therefore,\ \sum_{i=1}^{n}\frac{1-a_i}{a_i+\sqrt{{a_i}^2-a_it+1}}\ge\sum_{i=1}^{n}\left(1-\frac{2a_i}{t}\right)=n-2.$$

$$Equality\ holds\ iff\ (a_1,a_2,a_3,\dots,a_n)=(1,1,0,\dots,0)\ and\ their\ permutation.$$

1087. If $a,b,c,d>0$ then:

$$\frac{(a^3+b^3)(a^4+b^4+c^4)(a^5+b^5+c^5+d^5)}{a^3b^3c^2d(a+b)(a+b+c)(a+b+c+d)}\ge 1$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Vivek Kumar-India

$$a^3+b^3=\frac{a^4}{a}+\frac{b^4}{b}\overset{CBS}{\ge}\frac{(a^2+b^2)^2}{a+b}\overset{CBS}{\ge}\frac{\left(\frac{1}{2}(a+b)^2\right)^2}{a+b}$$

$$\Rightarrow a^3+b^3\ge\frac{1}{4}(a+b)^3;(1)$$

$$a^4+b^4+c^4\overset{CBS}{\geq}\frac{1}{3}(a^2+b^2+c^2)^2\overset{CBS}{\geq}\frac{1}{3}\left(\frac{1}{3}(a+b+c)^2\right)^2$$

$$\Rightarrow a^4+b^4+c^4\geq\frac{1}{27}(a+b+c)^4;(2)$$

$$a^5+b^5+c^5+d^5=\frac{a^6}{a}+\frac{b^6}{b}+\frac{c^6}{c}+\frac{d^6}{d}\overset{CBS}{\geq}\frac{(a^3+b^3+c^3+d^3)^2}{a+b+c+d}=$$

$$=\frac{\left(\frac{a^4}{a}+\frac{b^4}{b}+\frac{c^4}{c}+\frac{d^4}{d}\right)^2}{a+b+c+d}\overset{CBS}{\geq}\frac{(a^2+b^2+c^2+d^2)^2}{a+b+c+d}\overset{CBS}{\geq}$$

$$\geq\frac{\left(\frac{1}{4}(a+b+c+d)^2\right)^4}{(a+b+c+d)^3}$$

$$\Rightarrow a^5+b^5+c^5+d^5\geq\frac{1}{256}(a+b+c+d)^5;(3)$$

Now, from (1), (2) and (3) we get:

$$\frac{(a^3+b^3)(a^4+b^4+c^4)(a^5+b^5+c^5+d^5)}{a^3b^3c^2d(a+b)(a+b+c)(a+b+c+d)}\geq$$

$$\geq\frac{\frac{1}{4}(a+b)^3\cdot\frac{1}{27}(a+b+c)^4\cdot\frac{1}{256}(a+b+c+d)^5}{a^3b^3c^2d(a+b)(a+b+c)(a+b+c+d)}=$$

$$=\frac{1}{4\cdot 27\cdot 256}\cdot\frac{(a+b)^2(a+b+c)^3(a+b+c+d)^4}{a^3b^3c^2d}\overset{AGM}{\geq}$$

$$\geq\frac{1}{4\cdot 27\cdot 256}\cdot\frac{(2\sqrt{ab})^2(3\sqrt[3]{abc})^3\cdot(4\sqrt[4]{abcd})^4}{a^3b^3c^2d}=$$

$$=\frac{1}{4\cdot 27\cdot 256}\cdot\frac{4ab\cdot 27abc\cdot 256abcd}{a^3b^3c^2d}=1$$

Solution 2 by Tapas Das-India

$$\frac{a^3+b^3}{2}\geq\frac{a^2+b^2}{2}\cdot\frac{a+b}{2}\Rightarrow a^3+b^3\geq\frac{(a^2+b^2)(a+b)}{2}$$

$$\frac{a^4+b^4+c^4}{3}\geq\frac{a^3+b^3+c^3}{3}\cdot\frac{a+b+c}{3}\Rightarrow a^4+b^4+c^4\geq\frac{(a+b+c)(a^3+b^3+c^3)}{3}$$

$$\frac{a^5+b^5+c^5+d^5}{4}\geq\frac{a^4+b^4+c^4}{4}\cdot\frac{a+b+c+d}{4}$$

$$\Rightarrow a^5+b^5+c^5+d^5 \ge \frac{(a^4+b^4+c^4+d^4)(a+b+c+d)}{4}$$

$$\frac{(a^3+b^3)(a^4+b^4+c^4)\left(a^5+b^5+c^5+d^5\right)}{a^3b^3c^2d(a+b)(a+b+c)(a+b+c+d)} \ge$$

$$\ge \frac{(a^2+b^2)(a+b)(a+b+c)(a^3+b^3+c^3)(a+b+c+d)(a^4+b^4+c^4+d^4)}{24a^3b^3c^2d(a+b)(a+b+c)(a+b+c+d)} =$$

$$= \frac{(a^2+b^2)(a^3+b^3+c^3)(a^4+b^4+c^4+d^4)}{24a^3b^3c^2d} \ge \frac{2ab\cdot 3abc\cdot 4abcd}{24a^3b^3c^2d} = 1$$

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

$$a^3+b^3=(a+b)(a^2-ab+b^2)\ge (a+b)(2ab-ab)=ab(a+b)$$

$$a^4+b^4+c^4 \ge \frac{(a+b+c)^4}{3^3} \ge \frac{a+b+c}{27}\cdot 27abc = abc(a+b+c)$$

$$a^5+b^5+c^5+d^5 \ge \frac{(a+b+c+d)^4(a+b+c+d)}{4} \ge \frac{a+b+c+d}{4}\cdot 4\sqrt[4]{(abcd)^4}$$

$$= abcd(a+b+c+d)$$

$$\Rightarrow a^5+b^5+c^5+d^5 \ge \frac{(a^4+b^4+c^4+d^4)(a+b+c+d)}{4}$$

Hence,

$$(a^3+b^3)(a^4+b^4+c^4)\left(a^5+b^5+c^5+d^5\right) \ge$$

$$\ge a^3b^3c^2d(a+b)(a+b+c)(a+b+c+d)$$

Therefore,

$$\frac{(a^3+b^3)(a^4+b^4+c^4)\left(a^5+b^5+c^5+d^5\right)}{a^3b^3c^2d(a+b)(a+b+c)(a+b+c+d)} \ge 1$$

Solution 4 by Hikmat Mammadov-Azerbaijan

$$a^3+b^3=\frac{1}{2}(a+b)(a^2+b^2)\ge ab(a+b)$$

$$a^4+b^4+c^4 \ge \frac{(a+b+c)(a^3+b^3+c^3)}{3} \ge abc(a+b+c)$$

$$a^5+b^5+c^5+d^5 \ge \frac{(a^4+b^4+c^4+d^4)(a+b+c+d)}{4} \ge abcd(a+b+c+d)$$

Hence,

$$(a^3+b^3)(a^4+b^4+c^4)(a^5+b^5+c^5+d^5)\geq$$

$$\geq a^3b^3c^2d(a+b)(a+b+c)(a+b+c+d)$$

Therefore,

$$\frac{(a^3+b^3)(a^4+b^4+c^4)(a^5+b^5+c^5+d^5)}{a^3b^3c^2d(a+b)(a+b+c)(a+b+c+d)}\geq 1$$

1088. If $a,b,c,d,e,f>0, a+b+c=d+e+f=1$ then:

$$\sqrt[5]{ad^4}+\sqrt[5]{be^4}+\sqrt[5]{cf^4}\leq 1$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Tapas Das-India

$$\sqrt[5]{ad^4}+\sqrt[5]{be^4}+\sqrt[5]{cf^4}\overset{AGM}{\leq}\frac{a+4d}{5}+\frac{b+4e}{5}+\frac{c+4f}{5}=$$

$$=\frac{1}{4}[(a+b+c)+4(a+b+c)]=\frac{1}{5}(1+4)=1$$

Solution 2 by Hikmat Mammadov-Azerbaijan

$$\sqrt[5]{ad^4}+\sqrt[5]{be^4}+\sqrt[5]{cf^4}\overset{AGM}{\leq}\frac{a+4d}{5}+\frac{b+4e}{5}+\frac{c+4f}{5}=$$

$$=\frac{1}{4}[(a+b+c)+4(a+b+c)]=\frac{1}{5}(1+4)=1$$

1089. Let $a,b,c>0, ab+bc+ca=1$. Find Min. value of the following expression:

$$P=\frac{\sqrt{bc}(2a-1)}{2a+\sqrt{bc}}+\frac{\sqrt{ca}(2b-1)}{2b+\sqrt{ca}}+\frac{\sqrt{ab}(2c-1)}{2c+\sqrt{ab}}+2(a+b+c-1)^2$$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution by proposer

$$P=\frac{\sqrt{bc}(2a-1)}{2a+\sqrt{bc}}+\frac{\sqrt{ca}(2b-1)}{2b+\sqrt{ca}}+\frac{\sqrt{ab}(2c-1)}{2c+\sqrt{ab}}+2(a+b+c-1)^2$$

$$=\frac{2a(\sqrt{bc}+1)}{2a+\sqrt{bc}}+\frac{2b(\sqrt{ca}+1)}{2b+\sqrt{ca}}+\frac{2c(\sqrt{ab}+1)}{2c+\sqrt{ab}}+2(a+b+c-1)^2-3$$

Using AM-GM inequality:

$$2=2\sqrt{\frac{ab+bc+ca}{a+\sqrt{bc}}(a+\sqrt{bc})}\le a+\sqrt{bc}+\frac{ab+bc+ca}{a+\sqrt{bc}}=2\sqrt{bc}+\frac{a(a+b+c)}{a+\sqrt{bc}}$$

$$\Rightarrow a(1+\sqrt{bc})+b+c\ge(b+c)(2a+\sqrt{bc})$$

$$\frac{2a(\sqrt{bc}+1)}{2a+\sqrt{bc}}+\frac{2(b+c)}{2a+\sqrt{bc}}\ge 2(b+c)$$

Also by HM-GM:

$$\sqrt{bc}\ge\frac{2bc}{b+c}\Rightarrow 2a+\sqrt{bc}\ge\frac{2}{b+c}$$

Hence,

$$\frac{2a(\sqrt{bc}+1)}{2a+\sqrt{bc}}\ge 2(b+c)-(b+c)^2$$

Summing up similar inequalities, we get:

$$P\ge 2(a+b+c-1)^2-3+4(a+b+c)-2(a^2+b^2+c^2+1)=1$$

$$P_{Min}=1\Leftrightarrow(a,b,c)\in\{(1,1,0);(1,0,1);(0,1,1)\}$$

1090. Let $a,b,c>0: a+b+c=1$. Prove that:

$$\frac{a+\sqrt{bc}}{a+1}+\frac{b+\sqrt{ca}}{b+1}+\frac{c+\sqrt{ab}}{c+1}\le 1+\frac{\sqrt{ab}+\sqrt{bc}+\sqrt{ca}}{2}$$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$1=a+b+c\Rightarrow\frac{a+\sqrt{bc}}{a+1}+\frac{b+\sqrt{ca}}{b+1}+\frac{c+\sqrt{ab}}{c+1}\le 1+\frac{\sqrt{ab}+\sqrt{bc}+\sqrt{ca}}{2}$$

$$\Leftrightarrow\frac{a+\sqrt{bc}}{a+1}+\frac{b+\sqrt{ca}}{b+1}+\frac{c+\sqrt{ab}}{c+1}\le a+b+c+\frac{\sqrt{ab}+\sqrt{bc}+\sqrt{ca}}{2}$$

$$\Leftrightarrow\sum_{cyc}\left(a-\frac{a}{a+1}\right)\ge\sum_{cyc}\left(\sqrt{bc}\left(\frac{1}{a+1}-\frac{1}{2}\right)\right)\Leftrightarrow\sum_{cyc}\frac{a^2}{a+1}\ge\frac{1}{2}\sum_{cyc}\left(\sqrt{bc}\left(\frac{2-a-1}{a+1}\right)\right)$$

$$\Leftrightarrow\sum_{cyc}\frac{a^2}{a+1}\overset{(*)}{\ge}\frac{1}{2}\sum_{cyc}\left(\sqrt{bc}\left(\frac{1-a}{a+1}\right)\right)$$

Now, $1-a=b+c>0$ *and analogs* $\Rightarrow \frac{1}{2}\sum_{cyc}\left(\sqrt{bc}\left(\frac{1-a}{a+1}\right)\right) \overset{A-G}{\le} \frac{1}{2}\sum_{cyc}\left(\left(\frac{b+c}{2}\right)\left(\frac{1-a}{a+1}\right)\right)$

$$= \sum_{cyc}\frac{\frac{(1-a)^2}{4}}{a+1} \overset{?}{\le} \sum_{cyc}\frac{a^2}{a+1} \Leftrightarrow \sum_{cyc}\frac{a^2-\left(\frac{1-a}{2}\right)^2}{a+1} \overset{?}{\ge} 0$$

$$\Leftrightarrow \sum_{cyc}\frac{\left(\frac{2a+1-a}{2}\right)\left(\frac{2a-1+a}{2}\right)}{a+1} \overset{?}{\ge} 0 \Leftrightarrow \frac{1}{4}\sum_{cyc}(3a-1) \overset{?}{\ge} 0$$

$$\Leftrightarrow 3(a+b+c)-3 \overset{?}{\ge} 0 \overset{a+b+c=1}{\Leftrightarrow} 3-3 \overset{?}{\ge} 0 \to \text{true} \therefore \frac{1}{2}\sum_{cyc}\left(\sqrt{bc}\left(\frac{1-a}{a+1}\right)\right)$$

$$\le \sum_{cyc}\frac{a^2}{a+1} \Rightarrow (*) \text{ is true}$$

$$\therefore \forall a,b,c>0 \mid a+b+c=1, \frac{a+\sqrt{bc}}{a+1}+\frac{b+\sqrt{ca}}{b+1}+\frac{c+\sqrt{ab}}{c+1}$$

$$\le 1+\frac{\sqrt{ab}+\sqrt{bc}+\sqrt{ca}}{2}, \text{with equality iff } a=b=c=\frac{1}{3} \text{ (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have : $\sum_{cyc}\left(\frac{a+\sqrt{bc}}{a+1}-\frac{\sqrt{bc}}{2}\right)$

$$= \sum_{cyc}\frac{2a+(1-a)\sqrt{bc}}{2(a+1)} \overset{AM-GM,\ a<1}{\le} \sum_{cyc}\frac{2a+(1-a).\frac{b+c}{2}}{2(a+1)} =$$

$$= \sum_{cyc}\frac{4a+(1-a)^2}{4(a+1)} = \sum_{cyc}\frac{a+1}{4} = \frac{1+3}{4} = 1.$$

Therefore, $$\frac{a+\sqrt{bc}}{a+1}+\frac{b+\sqrt{ca}}{b+1}+\frac{c+\sqrt{ab}}{c+1} \le 1+\frac{\sqrt{ab}+\sqrt{bc}+\sqrt{ca}}{2}.$$

$$\textit{Equality holds iff } a=b=c=\frac{1}{3}.$$

1091. If $x,y,z>0$ such that $xyz=1$, then prove that :

$$(x^7+y^7+z^7)^2 \ge 3(x^9+y^9+z^9)$$

Proposed by Marius Drăgan, Neculai Stanciu-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\textit{The given inequality is successively equivalent to}:$$

$$x^{14}+y^{14}+z^{14}+2(xy)^7+2(yz)^7+2(zx)^7\geq 3(x^9+y^9+z^9)$$

$$\Leftrightarrow x^{14}+y^{14}+z^{14}+\frac{2}{z^7}+\frac{2}{x^7}+\frac{2}{y^7}\geq 3(x^9+y^9+z^9)$$

$$\Leftrightarrow \left(x^{14}-3x^9+\frac{2}{x^7}\right)+\left(y^{14}-3y^9+\frac{2}{y^7}\right)+\left(z^{14}-3z^9+\frac{2}{z^7}\right)\geq 0.$$

$$\textit{Let } f(x)=x^{14}-3x^9+\frac{2}{x^7}+27\ln(x),\ \ x>0.$$

$$\textit{We have}:\ f'(x)=14x^{13}-27x^8-\frac{14}{x^8}+\frac{27}{x}\ \ \textit{and}\ f''(x)$$
$$=182x^{12}-216x^8+\frac{112}{x^8}-\frac{27}{x^2},\forall x>0.$$

$$\textit{By AM}-\textit{GM inequality we have}:$$

$$4x^{12}+\frac{1}{x^8}\geq 5\sqrt[5]{(x^{12})^4.\frac{1}{x^8}}=5x^8\ \ \textit{and}\ \ 3x^{12}+\frac{7}{x^8}\geq 10\sqrt[5]{(x^{12})^3.\left(\frac{1}{x^8}\right)^7}=\frac{10}{x^2}.$$

$$216x^8+\frac{27}{x^2}\leq\frac{216}{5}\left(4x^{12}+\frac{1}{x^8}\right)+\frac{27}{10}\left(3x^{12}+\frac{7}{x^8}\right)=\frac{1809}{10}x^{12}+\frac{621}{10x^8}$$
$$<182x^{12}+\frac{112}{x^8}.$$

$$\textit{So } f''(x)>0,\ \ \forall x>0 \textit{ and } f' \textit{ is increasing on } (0,\infty),\ \ \textit{since } f'(1)=0 \textit{ then}:$$

$$f'(x)\leq 0,\forall x\in(0,1]\ \textit{and}\ f'(x)\geq 0,\ \ \forall x\in[1,\infty),\textit{so } f(x)\geq f(1)=0,\ \ \forall x\in(0,\infty).$$

$$\textit{Then}:\ x^{14}-3x^9+\frac{2}{x^7}\geq -27\ln(x)\ \ (\textit{and analogs})$$

$$\textit{Therefore,}$$

$$\sum_{cyc}\left(x^{14}-3x^9+\frac{2}{x^7}\right)\geq\sum_{cyc}\left(-27\ln(x)\right)=-27\ln(xyz)=0,$$

$$\textit{and the proof is complete.}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$(x+1)^3(y+1)^3(z+1)^3 \le (x+1)^{\frac{1}{x}}(y+1)^{\frac{1}{y}}(z+1)^{\frac{1}{z}} \Leftrightarrow \prod_{cyc}(x+1)^{\frac{1}{x}-3} \ge 1$$

$$\Leftrightarrow \sum_{cyc}\left(\left(\frac{1}{x}-3\right).\ln(x+1)\right) \overset{(*)}{\ge} 0$$

$$\text{Let } f(x) = \left(\frac{1}{x}-3\right).\ln(x+1)\ \forall x>0 \Rightarrow f''(x) = \frac{2\ln(x+1)}{x^3} - \frac{2}{x^2(x+1)} - \frac{1-3x}{x(x+1)^2} > 0$$

$$\Leftrightarrow \frac{2\ln(x+1)}{x^3} > \frac{2(x+1)+x(1-3x)}{x^2(x+1)^2} \Leftrightarrow \ln(x+1) > \frac{-3x^3+3x^2+2x}{2(x+1)^2}$$

$$\Leftrightarrow \ln(x+1) - \frac{-3x^3+3x^2+2x}{2(x+1)^2} \overset{(**)}{>} 0$$

$$\text{Let } P(x) = \ln(x+1) - \frac{-3x^3+3x^2+2x}{2(x+1)^2}\ \forall x \ge 0 \therefore P'(x) = \frac{x^2(3x+11)}{2(x+1)^3} \ge 0 \Rightarrow P(x) \text{ is}$$

$$\uparrow \text{ on } [0,\infty) \Rightarrow \forall x \ge 0, P(x) \ge P(0) = 0, ''='' \text{ iff } x=0 \Rightarrow \forall x>0, P(x)>0$$

$$\Rightarrow (**) \text{ is true } \forall x>0 \Rightarrow f''(x) > 0 \Rightarrow f(x) \text{ is convex} \therefore \sum_{cyc}\left(\left(\frac{1}{x}-3\right).\ln(x+1)\right) \overset{\text{Jensen}}{\ge}$$

$$3\left(\left(\frac{1}{\frac{x+y+z}{3}}-3\right).\ln\left(\frac{x+y+z}{3}+1\right)\right) \overset{x+y+z=1}{=} 0 \Rightarrow (*) \text{ is true}$$

$$\therefore (x+1)^3(y+1)^3(z+1)^3 \le (x+1)^{\frac{1}{x}}(y+1)^{\frac{1}{y}}(z+1)^{\frac{1}{z}}\ \forall x,y,z>0 \mid x+y+z=1, ''='' \text{ iff } x = y = z = \frac{1}{3} \text{ (QED)}$$

1092. If $a,b,c>0, abc=1$ then :

$$\left(5\sqrt{a+3}-8\right)\left(5\sqrt{b+3}-8\right)\left(5\sqrt{c+3}-8\right) \ge 8$$

Proposed by Tran Quoc Thinh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\textit{We have}: \prod_{cyc}\left(5\sqrt{a+3}-8\right) = \prod_{cyc}\left(\frac{25a+11}{5\sqrt{a+3}+8}\right) \overset{AM-GM}{\ge} \prod_{cyc}\left(\frac{25a+11}{5.\frac{(a+3)+4}{4}+8}\right)$$

$$=$$

$$=64\prod_{cyc}\left(\frac{25a+11}{5a+67}\right)\overset{?}{\geq}8\Leftrightarrow 8\prod_{cyc}(25a+11)\geq\prod_{cyc}(5a+67)$$

$$\Leftrightarrow 13(a+b+c)+395(ab+bc+ca)\geq 1224,\ \textit{which is true because}:$$

$$a+b+c\geq 3\sqrt[3]{abc}=3\ \text{and}\ ab+bc+ca\geq 3\sqrt[3]{(abc)^2}=3.$$

So the proof is completed. Equality holds iff $a=b=c=1$.

1093. Let $a,b,c\geq 0: ab+bc+ca>0$. Prove that:

$$\frac{\sqrt{a^4+4abc(b+c)}}{a^2+2bc}+\frac{\sqrt{b^4+4abc(c+a)}}{b^2+2ca}+\frac{\sqrt{c^4+4abc(a+b)}}{c^2+2ab}\geq\frac{(a+b+c)^2}{a^2+b^2+c^2}$$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\textit{We have}:\ (ab+bc+ca)\sqrt{a^4+4abc(b+c)}$$
$$=\sqrt{[(ab-bc+ca)^2+4abc(b+c)][a^4+4abc(b+c)]}\geq$$

$$\overset{CBS}{\geq}(ab-bc+ca)a^2+4abc(b+c)$$
$$=(ab-bc+ca)(a^2+2bc)+2bc(ab+bc+ca)$$

Then:

$$\frac{\sqrt{a^4+4abc(b+c)}}{a^2+2bc}\geq\frac{ab-bc+ca}{ab+bc+ca}+\frac{2bc}{a^2+2bc}\overset{AM-GM}{\geq}\frac{ab-bc+ca}{ab+bc+ca}+\frac{2bc}{a^2+b^2+c^2}.$$

Similarly we have:

$$\frac{\sqrt{b^4+4abc(c+a)}}{b^2+2ca}\geq\frac{ab+bc-ca}{ab+bc+ca}+\frac{2ca}{a^2+b^2+c^2}\ \&\ \frac{\sqrt{c^4+4abc(a+b)}}{c^2+2ab}$$
$$\geq\frac{-ab+bc+ca}{ab+bc+ca}+\frac{2ab}{a^2+b^2+c^2}.$$

Summing up these inequalities we get:

$$\frac{\sqrt{a^4+4abc(b+c)}}{a^2+2bc}+\frac{\sqrt{b^4+4abc(c+a)}}{b^2+2ca}+\frac{\sqrt{c^4+4abc(a+b)}}{c^2+2ab}$$
$$\ge 1+\frac{2(ab+bc+ca)}{a^2+b^2+c^2}=\frac{(a+b+c)^2}{a^2+b^2+c^2}.$$

Equality holds iff $(a=b=c)$ *or* $(a=b,\ c=0)$ *and their permutation.*

1094. If $x,y,z>0$ and $xy+yz+zx=1$ then prove that:

$$(x+y)(y+z)(z+x)\left(x+\sqrt{x^2+1}\right)\left(y+\sqrt{y^2+1}\right)\left(z+\sqrt{z^2+1}\right)\ge 8$$

Proposed by Hikmat Mammadov-Azerbaijan

Solution 1 by Sanong Huayrerai-Nakon Pathom-Thailand

$$(x+y)(y+z)(z+x)\left(x+\sqrt{x^2+1}\right)\left(y+\sqrt{y^2+1}\right)\left(z+\sqrt{z^2+1}\right)=$$

$$=(x+y)(y+z)(z+x)\left(x+(x+z)\sqrt{x+y}\right)\left(y+(y+z)\sqrt{y+z}\right)\left(z+(z+y)\sqrt{z+x}\right)$$

$$\ge(x+y)(y+z)(z+x)\left(x+x+\sqrt{yz}\right)\left(y+y+\sqrt{zx}\right)\left(z+z+\sqrt{xy}\right)\ge$$

$$\ge(x+y)(y+z)(z+x)\left(2x+\frac{2yz}{y+z}\right)\left(2y+\frac{2zx}{z+x}\right)\left(2z+\frac{2xy}{x+y}\right)\ge 8$$

$$(x+y)(y+z)(z+x)\frac{x(y+z)+yz}{y+z}\frac{y(z+x)+zx}{z+x}\frac{z(x+y)+xy}{x+y}\ge 1$$

$$(xy+yz+zx)^3\ge 1$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$x+\sqrt{x^2+1}=x+\sqrt{x^2+xy+yz+zx}=x+\sqrt{(x+y)(x+z)}\overset{CBS}{\ge}x+\left(x+\sqrt{yz}\right)\ge$$

$$\overset{GM-HM}{\ge}2x+\frac{2yz}{y+z}=\frac{2(xy+yz+zx)}{y+z}=\frac{2}{y+z}.\ Then:\ (y+z)\left(x+\sqrt{x^2+1}\right)\ge 2.$$

Similarly we have: $(z+x)\left(y+\sqrt{y^2+1}\right)\ge 2$ & $(x+y)\left(z+\sqrt{z^2+1}\right)\ge 2.$

Therefore, $(x+y)(y+z)(z+x)\left(x+\sqrt{x^2+1}\right)\left(y+\sqrt{y^2+1}\right)\left(z+\sqrt{z^2+1}\right)\ge 8.$

1095. If $x, y, z > 0$ such that $x + y + z = 1$, then prove that:

$$(x+1)^3 \cdot (y+1)^3 \cdot (z+1)^3 \le (x+1)^{\frac{1}{x}} \cdot (y+1)^{\frac{1}{y}} \cdot (z+1)^{\frac{1}{z}}$$

Proposed by Marius Drăgan, Neculai Stanciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

The given inequality is successively equivalent to :

$$\sum_{cyc} 3ln(x+1) \le \sum_{cyc} \frac{ln(x+1)}{x} \Leftrightarrow \sum_{cyc} (3x-1).\frac{ln(x+1)}{x} \le 0$$

WLOG, we assume that $x \ge y \ge z$. *Let* $g(x) = \frac{ln(x+1)}{x}, x \in (0,1)$.

$$We\ have:\ g'(x) = \frac{\frac{x}{x+1} - ln(x+1)}{x^2} \le 0, because$$

$$ln(x+1) \ge 1 - \frac{1}{x+1} = \frac{x}{x+1}, \forall x \in (0,1).$$

Then g is decreasing on $(0,1)$ *and since :* $x \ge y \ge z$ *then :*

$$3x - 1 \ge 3y - 1 \ge 3z - 1\ \ and\ \ \frac{ln(x+1)}{x} \le \frac{ln(y+1)}{y} \le \frac{ln(z+1)}{z}$$

By Chebyshev's inequality we get :

$$\sum_{cyc} (3x-1).\frac{ln(x+1)}{x} \le \frac{1}{3}\left(\sum_{cyc}(3x-1)\right)\left(\sum_{cyc}\frac{ln(x+1)}{x}\right)$$
$$= (x+y+z-1).\sum_{cyc}\frac{ln(x+1)}{x} = 0.$$

1096. Let $a, b, c \ge 0 :\ (ab)^2 + (bc)^2 + (ca)^2 = 1.$ Prove that :

$$\frac{1}{a^2+bc} + \frac{1}{b^2+ca} + \frac{1}{c^2+ab} + \frac{(a+b+c)^3 + 9abc}{(a+b)(b+c)(c+a)} \ge 7$$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\textit{Lemma}: \ \textit{If } a,b,c \geq 0 \textit{ then}: \sum_{cyc} \frac{a^2}{a^2+bc} + \frac{(a+b+c)^3+9abc}{(a+b)(b+c)(c+a)} \geq 6 \ \ (*)$$

$$\textit{Proof}: \ \textit{We have}: (*) \Leftrightarrow$$

$$\prod_{cyc}(b+c).\sum_{cyc}\frac{a^2}{a^2+bc} + (a+b+c)^3 + 9abc \geq 6\prod_{cyc}(b+c)$$

$$\Leftrightarrow \sum_{cyc}\left(1+\frac{a(b+c)}{a^2+bc}\right)a^2(b+c) + a^3+b^3+c^3+3abc \geq 3\sum_{cyc}a^2(b+c)$$

$$\Leftrightarrow \sum_{cyc}\left(\frac{a^3(b+c)^2}{a^2+bc} + a(a^2+bc)\right) \geq 2\sum_{cyc}a^2(b+c),$$

$$\textit{which is true by } AM-GM \textit{ inequality}$$

$$\therefore \frac{a^3(b+c)^2}{a^2+bc} + a(a^2+bc) \geq 2a^2(b+c) \ \ (\textit{and analogs})$$

$$\textit{From the lemma, it suffices to prove}: \ \frac{1-a^2}{a^2+bc} + \frac{1-b^2}{b^2+ca} + \frac{1-c^2}{c^2+ab} \geq 1.$$

$$\textit{We have}:$$

$$a^2+b^2+c^2 = \sqrt{[(-a^2+b^2+c^2)^2+4a^2(b^2+c^2)][(bc)^2+a^2(b^2+c^2)]} \geq$$

$$\overset{CBS}{\geq} bc(-a^2+b^2+c^2)+2a^2(b^2+c^2)$$
$$= (a^2+bc)(-a^2+b^2+c^2)+a^2(a^2+b^2+c^2)$$

$$\textit{Then}: \ \frac{1-a^2}{a^2+bc} \geq \frac{-a^2+b^2+c^2}{a^2+b^2+c^2} \ \ (\textit{and analogs})$$

$$\textit{Therefore}: \sum_{cyc}\frac{1-a^2}{a^2+bc} \geq \sum_{cyc}\frac{-a^2+b^2+c^2}{a^2+b^2+c^2} = 1 \ \textit{ and the proof is completed.}$$

$$\textit{Equality holds iff } (a,b,c) = (1,1,0) \textit{ and their permutation.}$$

1097. If $f:\ (0,\infty)\to(0,e)$ is increasing function with $\sum_{i=1}^{n} f(x_i)=n$,

then prove that:

$$f(x_1).f(x_2)\dots f(x_n)\geq f(x_1)^{\frac{1}{f(x_1)}}.f(x_2)^{\frac{1}{f(x_2)}}\dots f(x_n)^{\frac{1}{f(x_n)}}$$

Proposed by Marius Drăgan, Neculai Stanciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

The given inequality is successively equivalent to:

$$\sum_{i=1}^{n} ln(f(x_i))\geq\sum_{i=1}^{n}\frac{ln(f(x_i))}{f(x_i)}\ \Leftrightarrow\ \sum_{i=1}^{n}(f(x_i)-1).\frac{ln(f(x_i))}{f(x_i)}\geq 0$$

WLOG, we assume that $x_1\geq x_2\geq\cdots\geq x_n$ *then we have:*

$$f(x_1)\geq f(x_2)\geq\cdots\geq f(x_n).$$

Let $g(x)=\frac{ln(x)}{x}, x\in(0,e)$. *We have:* $g'(x)=\frac{1-ln(x)}{x^2}>0,\forall x\in(0,e)$.

Then g is increasing on $(0,e)$ *and since:*

$$e>f(x_1)\geq f(x_2)\geq\cdots\geq f(x_n)>0 \text{ then:}$$

$\frac{ln(f(x_1))}{f(x_1)}\geq\frac{ln(f(x_2))}{f(x_2)}\geq\cdots\geq\frac{ln(f(x_n))}{f(x_n)}$ *and by Chebyshev's inequality we get:*

$$\sum_{i=1}^{n}(f(x_i)-1).\frac{ln(f(x_i))}{f(x_i)}\geq\frac{1}{n}\left(\sum_{i=1}^{n}(f(x_i)-1)\right)\left(\sum_{i=1}^{n}\frac{ln(f(x_i))}{f(x_i)}\right)$$
$$=\frac{1}{n}\left(\sum_{i=1}^{n} f(x_i)-n\right)\left(\sum_{i=1}^{n}\frac{ln(f(x_i))}{f(x_i)}\right)=0.$$

1098. Let $x,y,z,a,b,c>0$ such that $abc=xyz$,

$$max\{a,b,c\}\geq max\{x,y,z\},\ \ min\{a,b,c\}\leq min\{x,y,z\}.$$

Prove that: $\ a+b+c\geq x+y+z.$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution 1 Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\textit{Assume that}: \ a \geq b \geq c, \ \ x \geq y \geq z. \ \textit{We have}: \ \ a \geq x \ \textit{and} \ z \geq c \ \ (1)$$

$$(a+b+c)-(x+y+z)=x\left(\frac{a}{x}-1\right)+y\left(\frac{b}{y}-1\right)+z\left(\frac{c}{z}-1\right)\geq$$

$$\overset{t-1\geq ln(t),\ \forall t>0.}{\geq} x.ln\left(\frac{a}{x}\right)+y.ln\left(\frac{b}{y}\right)+z.ln\left(\frac{c}{z}\right)=(x-y)ln\left(\frac{a}{x}\right)+y.ln\left(\frac{ab}{xy}\right)-z.ln\left(\frac{z}{c}\right)$$

$$=$$

$$\overset{abc=xyz}{=} (x-y)ln\left(\frac{a}{x}\right)+(y-z).ln\left(\frac{z}{c}\right)\overset{(1)}{\geq} 0 \ \textit{and the proof is complete.}$$

Solution 2 by Marian Dincă-Romania

$$\text{Let } a \geq b \geq c, x \geq y \geq z$$

$$a \geq x, c \leq z$$

$$a+b+c-x-y-z=Ax+By+Cz-x-y-z$$

$$\text{Let } \frac{x}{a}=A \geq 1, \frac{b}{y}=B, \frac{c}{z}=C \leq 1=(A-1)x+(B-1)y+(C-1)z=$$

$$=(A-1)(x-y)+(A-1+B-1)(y-z)+(A-1+B-1+C-1)z \geq 0$$

$$ABC=1, AB=\frac{1}{C} \geq 1$$

$$A-1 \geq 0, A-1+B-1=A+B-2 \geq 2\sqrt{AB}-2 \geq 0$$

$$A-1+B-1+C-1=A+B+C \geq 3\sqrt[3]{ABC}-3=0$$

Solution 3 by Hikmat Mammadov-Azerbaijan

We may assume $a \geq b \geq c$ and $x \geq y \geq z$ then

$$\log a \geq \log x$$

$$\log a+\log b \geq \log x+\log y$$

$$\log a+\log b+\log c \geq \log x+\log y+\log z$$

Now with majorization principle (Hardy-Litllewood-Polya inequality)

$$e^{\log a}+e^{\log b}+e^{\log c}\geq e^{\log x}+e^{\log y}+e^{\log z}$$

$$a+b+c\geq x+y+z$$

1099. Let $\lambda\geq 0$ fixed. $If\ a,b,c\geq 0, a^2+b^2+c^2=1$

then find min and max of:

$$P=\sqrt{a+\lambda b^2}+\sqrt{b+\lambda c^2}+\sqrt{c+\lambda a^2}$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$We\ have:$

$$P\overset{CBS}{\leq}\sqrt{3[a+b+c+\lambda(a^2+b^2+c^2)]}\overset{CBS}{\leq}\sqrt{3\left[\sqrt{3(a^2+b^2+c^2)}+\lambda\right]}$$

$$=\sqrt{3(\sqrt{3}+\lambda)}.$$

$$So\ the\ maximum\ value\ of\ P\ is\ \sqrt{3(\sqrt{3}+\lambda)}\ for\ a=b=c=\frac{\sqrt{3}}{3}.$$

$$Also\ we\ have:\ \ P^2=\sum_{cyc}(a+\lambda b^2)+2\sum_{cyc}\sqrt{(a+\lambda b^2)(b+\lambda c^2)}\geq$$

$$\overset{a,b,c\geq 0}{\geq}\sum_{cyc}a+\lambda+2\sum_{cyc}\sqrt{\lambda b^2.b}\overset{0\leq a,b,c\leq 1}{\geq}\sum_{cyc}a^2+\lambda+2\sqrt{\lambda}\sum_{cyc}b^2=\left(1+\sqrt{\lambda}\right)^2$$

$$So\ the\ minimum\ value\ of\ P\ is\ 1+\sqrt{\lambda}\ for\ a=1\ and$$
$$b=c=0\ and\ their\ permutation.$$

1100. If $x,y,z\in\left[0,\frac{\pi}{2}\right)$ then:

$$exp\left(\frac{2}{3}\sum_{cyc}sinx(4-cos^2x)\right)\leq\prod_{cyc}\left(\frac{1+sinx}{1-sinx}\right)$$

Proposed by Daniel Sitaru-Romania

Solution by Ravi Prakash-New Delhi-India

$0\leq x<\frac{\pi}{2}\rightarrow 0\leq sina<1$. Denote $a=sinx$.

$$\log\left(\frac{1+a}{1-a}\right)=2a+\frac{2}{3}a^3+\cdots\geq 2a+\frac{2}{3}a^3=\frac{2}{3}(3a+a^3)$$

$$\log\left(\frac{1+\sin x}{1-\sin x}\right)\geq\frac{2}{3}(3\sin x+\sin^3 x)=\frac{2}{3}\sin x(4-\cos^2 x)$$

$$\frac{1+\sin x}{1-\sin x}\geq\exp\left(\frac{2}{3}\sin x(4-\cos^2 x)\right)$$

$$\prod_{cyc}\frac{1+\sin x}{1-\sin x}\geq\prod_{cyc}\exp\left(\frac{2}{3}\sin x(4-\cos^2 x)\right)$$

$$\prod_{cyc}\frac{1+\sin x}{1-\sin x}\geq\exp\left(\sum_{cyc}\frac{2}{3}\sin x(4-\cos^2 x)\right)$$

Equality holds for $x=y=z=0$.

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru