A SIMPLE PROOF FOR LEHMER'S INEQUALITY

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Abstract. In this paper it's presented a simple proof for Lehmer's inequality, corollaries and applications.

LEHMER'S INEQUALITY:
If $a, b, x > 0$ then:

$$0 < \log\left(1 + \frac{x}{a}\right) \log\left(1 + \frac{b}{x}\right) < \frac{b}{a} \tag{1}$$

Proof.
Lemma.
If $x > 0$ then:

$$\log(1 + x) < x$$

Proof of lemma.

Let be the function $f : (0, \infty) \to \mathbb{R}; f(x) = \log(1 + x) - x$, then

$$f'(x) = \frac{1}{1+x} - 1 = \frac{-x}{1+x} < 0 \Rightarrow f \text{ - decreasing}$$

For $x > 0$:

$$\sup f(x) = \lim_{x \to 0_+} f(x) = \lim_{x \to 0_+} [\log(1 + x) - x] = 0 \Rightarrow f(x) < 0, > 0.$$

By Lemma:

$$0 = \log 1 < \log\left(1 + \frac{x}{a}\right) < \frac{x}{a}; \tag{2}$$

$$0 = \log 1 < \log\left(1 + \frac{b}{x}\right) < \frac{b}{x} \tag{3}$$

By multiplying (2) and (3):

$$0 < \log\left(1 + \frac{x}{a}\right) \log\left(1 + \frac{b}{x}\right) < \frac{x}{a} \cdot \frac{b}{x} = \frac{b}{a}$$

□

Corollary 1.
If $a, b > 0$ then:

$$0 < \log\left(1 + \frac{1}{a}\right) \log(1 + b) < \frac{b}{a}$$

Proof.
We take $x = 1$ in (1). □

Corollary 2.
If $a, b > 0$ then:

$$0 < \log^2\left(1 + \sqrt{\frac{b}{a}}\right) < \frac{b}{a}$$

Proof.
We take $x = \sqrt{ab}$ in (1):

$$0 < \log\left(1 + \frac{\sqrt{ab}}{a}\right)\log\left(1 + \frac{b}{\sqrt{ab}}\right) < \frac{b}{a}$$

$$0 < \log\left(1 + \sqrt{\frac{b}{a}}\right)\log\left(1 + \sqrt{\frac{a}{b}}\right) < \frac{b}{a}$$

Hence

$$0 < \log^2\left(1 + \sqrt{\frac{b}{a}}\right) < \frac{b}{a}$$

□

□

Application 1.
If $a, b > 0$ then:

$$e^{2\sqrt{\frac{b}{a}}} > 1 + 2\sqrt{\frac{b}{a}} + \frac{b}{a}$$

Proof.
By Lemma:

$$\log(1 + x) < x \Rightarrow e^x > 1 + x, \forall x > 0$$

$$e^{\frac{x}{a}} > 1 + \frac{x}{a}; \tag{4}$$

$$e^{\frac{b}{x}} > 1 + \frac{b}{x} \tag{5}$$

By multiplying (4) and (5):

$$e^{\frac{x}{a}} \cdot e^{\frac{b}{x}} > \left(1 + \frac{x}{a}\right)\left(1 + \frac{b}{x}\right)$$

$$e^{\frac{x}{a} + \frac{b}{x}} > 1 + \frac{x}{a} + \frac{b}{x} + \frac{b}{a}; \tag{6}$$

For $x = \sqrt{ab}$ in (6):

$$e^{\frac{\sqrt{ab}}{a} + \frac{b}{\sqrt{ab}}} > 1 + \frac{\sqrt{ab}}{a} + \frac{b}{\sqrt{ab}} + \frac{b}{a}$$

□

Application 2.

$$\Omega = \lim_{x \to \infty} \log\left(1 + \frac{1}{x^4}\right)\log(1 + x^2) = 0$$

Proof.
By (1):

$$0 < \log\Big(1+\frac{1}{x^4}\Big)\log\Big(1+\frac{x^2}{1}\Big) < \frac{1}{x^4}\cdot x^2 = \frac{1}{x^2}$$

$$0 < \log\Big(1+\frac{1}{x^4}\Big)\log\Big(1+\frac{x^2}{1}\Big) < \frac{1}{x^2}$$

Therefore,

$$0 \le \lim_{x\to\infty}\log\Big(1+\frac{1}{x^4}\Big)\log(1+x^2) \le \lim_{x\to 0}\frac{1}{x^2} = 0$$

$$\Omega = \lim_{x\to\infty}\log\Big(1+\frac{1}{x^4}\Big)\log(1+x^2) = 0$$

□

Application 3.

$$\Omega = \sum_{n=1}^{\infty}\log\Big(1+\frac{1}{n^4}\Big)\log(1+n^2) < \frac{\pi^2}{6}$$

Proof.
By (1):

$$0 < \log\Big(1+\frac{1}{n^4}\Big)\log(1+n^2) < \frac{1}{n^4}\cdot n^2 = \frac{1}{n^2}$$

$$\Omega = \sum_{n=1}^{\infty}\log(1+\frac{1}{n^4}\Big)\log(1+n^2) < \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

□

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