A SIMPLE PROOF FOR WANG'S INEQUALITY AND APPLICATIONS

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Abstract. In this paper it's given a simple proof for the famous Wang's inequality and a few applications.

WANG'S INEQUALITY:

If $x > 0, x \neq 1$ then:

$$\log x < n(\sqrt[n]{x} - 1) < \sqrt[n]{x}\log x; n \in \mathbb{N}, n \geq 1 \tag{1}$$

Proof.

Let be the function $f : (0, \infty) \to \mathbb{R}, f(x) = \log x - n\sqrt[n]{x} - n$, then

$$f'(x) = \frac{1}{x} - n \cdot \frac{1}{n} \cdot x^{\frac{1}{n}-1} = \frac{1 - \sqrt[n]{x}}{x}$$

$$f'(x) = 0 \Rightarrow 1 - \sqrt[n]{x} = 0 \Rightarrow 1 = \sqrt[n]{x} \Rightarrow x = 1$$

$$\lim_{x \to 0_+} f(x) = \lim_{x \to 0_+} (\log x - n\sqrt[n]{x} - n) = -\infty$$

$$\lim_{x \to \infty} \frac{\log x - n}{\sqrt[n]{x}} = \lim_{x \to \infty} \frac{\frac{1}{x}}{\frac{1}{n}x^{\frac{1}{n}-1}} = \lim_{x \to \infty} \frac{n}{x^{2-\frac{1}{n}}} = 0$$

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} (\log x - n\sqrt[n]{x} - n) = \lim_{x \to \infty} \sqrt[n]{x}\left(\frac{\log x - n}{\sqrt[n]{x}} - n\right) = \infty(0 - n) = -\infty$$

$$M = \sup_{x > 0} f(x) = f(1) = -n - 1 < 0$$

$$f(x) < 0, \forall x > 0, x \neq 1$$

Let be $g : (0, \infty) \to \mathbb{R}, g(x) = \log x - n + x^{-\frac{1}{n}}$, then

$$g'(x) = \frac{1}{x} - \frac{1}{n} \cdot \frac{x^{\frac{1}{n}}}{x} = \frac{n - \sqrt[n]{x}}{nx}$$

$$g'(x) = 0 \Rightarrow n - \sqrt[n]{x} = 0 \Rightarrow x = n^n$$

$$g(n^n) = \log(n^n) - n + (n^n)^{-\frac{1}{n}} = \frac{1}{n} > 0$$

$$\lim_{x \to \infty} g(x) = \lim_{x \to \infty} \left(\log x - n + \frac{1}{\sqrt[n]{x}}\right) = \infty$$

$$\lim_{x \to 0_+} \frac{\log x}{x^{-\frac{1}{n}}} = \lim_{x \to 0_+} \frac{\frac{1}{x}}{-\frac{1}{n}x^{-\frac{1}{n}-1}} = -n\sqrt[n]{0} = 0$$

$$\lim_{x \to 0_+} g(x) = \lim_{x \to 0_+} \left(\log x - n + \frac{1}{\sqrt[n]{x}}\right) = -n + \lim_{x \to 0_+} \frac{\sqrt[n]{x}\log x + 1}{\sqrt[n]{x}} = -n + \frac{1}{0_+} = \infty$$

$$m = \inf_{x > 0} g(x) = g(n^n) = \frac{1}{n} > 0 \Rightarrow g(x) > 0$$

$$\begin{cases} f(x) < 0 \\ g(x) > 0 \end{cases} \Rightarrow \begin{cases} \log x - n\sqrt[n]{x} - n < 0 \\ \log x - n + \frac{1}{\sqrt[n]{x}} > 0 \end{cases} \Rightarrow \begin{cases} n(\sqrt[n]{x} - n) > \log x \\ \sqrt[n]{x}\log x > n(\sqrt[n]{x} - 1) \end{cases}$$

□

Application 1.
If $x, y, z > 0$ then:

$$xyz\exp(3n) \leq \exp(\sqrt[n]{x} + \sqrt[n]{y} + \sqrt[n]{z}); n \in \mathbb{N}^*$$

Proof.
By (1):

$$\log x \leq n(\sqrt[n]{x} - n); \tag{2}$$

$$\log y \leq n(\sqrt[n]{y} - n); \tag{3}$$

$$\log z \leq n(\sqrt[n]{z} - n); \tag{4}$$

By adding (2), (3) and (4):

$$\log x + \log y + \log z \leq n(\sqrt[n]{x} + \sqrt[n]{y} + \sqrt[n]{z}) - 3n$$
$$\log(xyz) \leq n(\sqrt[n]{x} + \sqrt[n]{y} + \sqrt[n]{z}) - 3n$$
$$xyz \leq \exp(n(\sqrt[n]{x} + \sqrt[n]{y} + \sqrt[n]{z}) - 3n)$$
$$xyz\exp(3n) \leq \exp(\sqrt[n]{x} + \sqrt[n]{y} + \sqrt[n]{z}); n \in \mathbb{N}^*$$

Equality holds for $x = y = z = 1$. □

Application 2.

If $x, y, z > 0; m, n, p \in \mathbb{N}^*$ then:

$$x^3\exp(m + n + p) \leq \exp(n\sqrt[n]{x} + m\sqrt[m]{x} + p\sqrt[p]{x})$$

Proof.
By (1):

$$\log x \leq n\sqrt[n]{x} - n; \tag{5}$$

$$\log x \leq m\sqrt[m]{x} - m; \tag{6}$$

$$\log x \leq p\sqrt[p]{x} - p; \tag{7}$$

By adding (5), (6) and (7):

$$3\log x \leq n\sqrt[n]{x} + m\sqrt[m]{x} + p\sqrt[p]{x} - (m + n + p)$$
$$\log(x^3) \leq n\sqrt[n]{x} + m\sqrt[m]{x} + p\sqrt[p]{x} - (m + n + p)$$
$$x^3 \leq \exp(n\sqrt[n]{x} + m\sqrt[m]{x} + p\sqrt[p]{x} - (m + n + p))$$
$$x^3\exp(m + n + p) \leq \exp(n\sqrt[n]{x} + m\sqrt[m]{x} + p\sqrt[p]{x})$$

Equality holds for $x = 1$. □

Application 3.
If $x, y, z > 0, n \in \mathbb{N}^*$ then:

$$xyz \exp\left(n\left(\frac{1}{\sqrt[n]{x}} + \frac{1}{\sqrt[n]{y}} + \frac{1}{\sqrt[n]{z}}\right)\right) \geq \exp(3n)$$

Proof.
By (1):

$$n(\sqrt[n]{x} - 1) \leq \sqrt[n]{x} \log x$$

$$\log x \geq n\left(1 - \frac{1}{\sqrt[n]{x}}\right); \tag{8}$$

$$\log y \geq \left(1 - \frac{1}{\sqrt[n]{y}}\right); \tag{9}$$

$$\log z \geq n\left(1 - \frac{1}{\sqrt[n]{z}}\right); \tag{10}$$

By adding (8), (9) and (10):

$$\log(xyz) \geq n\left(3 - \frac{1}{\sqrt[n]{x}} + \frac{1}{\sqrt[n]{y}} + \frac{1}{\sqrt[n]{z}}\right)$$

$$xyz \geq \exp\left(3n - n\left(\frac{1}{\sqrt[n]{x}} + \frac{1}{\sqrt[n]{y}} + \frac{1}{\sqrt[n]{z}}\right)\right)$$

$$xyz \exp\left(n\left(\frac{1}{\sqrt[n]{x}} + \frac{1}{\sqrt[n]{y}} + \frac{1}{\sqrt[n]{z}}\right)\right) \geq \exp(3n)$$

Equality holds for $x = y = z = 1$. $\square$

Application 4.

If $x > 0; m, n, p \in \mathbb{N}^*$ then:

$$m\sqrt[m]{x} + n\sqrt[n]{x} + p\sqrt[p]{x} \leq m + n + p + (m\sqrt[m]{x} + n\sqrt[n]{x} + p\sqrt[p]{x}) \log x$$

Proof.
By (1):

$$n(\sqrt[n]{x} - 1) < \sqrt[n]{x} \log x; \tag{11}$$

$$m(\sqrt[m]{x} - 1) < \sqrt[m]{x} \log x; \tag{12}$$

$$p(\sqrt[p]{x} - 1) < \sqrt[p]{x} \log x; \tag{13}$$

By adding (11), (12) and (13):

$$m\sqrt[m]{x} + n\sqrt[n]{x} + p\sqrt[p]{x} - (m + n + p) \leq (m\sqrt[m]{x} + n\sqrt[n]{x} + p\sqrt[p]{x}) \log x$$

$$m\sqrt[m]{x} + n\sqrt[n]{x} + p\sqrt[p]{x} \leq m + n + p + (m\sqrt[m]{x} + n\sqrt[n]{x} + p\sqrt[p]{x}) \log x$$

Equality holds for $x = 1$. $\square$

Application 5.
Prove without any software:

$$\sqrt[3]{3}(3-\log 3) < 3$$

Proof.
Replace $x = n = 3$ in (1):

$$3(\sqrt[3]{3}-1) < \sqrt[3]{3}\log 3$$
$$3\sqrt[3]{3} - \sqrt[3]{3}\log 3 < 3$$
$$\sqrt[3]{3}(3-\log 3) < 3$$

□

Application 6.
Prove without any software:

$$2\sqrt{2} - \log 2 > 2$$

Proof.
Replace $x = n = 2$ in (1):

$$\log 2 < 2(\sqrt{2}-1)$$
$$\log 2 < 2\sqrt{2} - 2$$
$$2\sqrt{2} - \log 2 > 2$$

□

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