ABOUT A PROPOSED PROBLEM FROM CRUX MATHEMATICORUM

D.M.BĂTINEŢU-GIURGIU, DANIEL SITARU, FLORICĂ ANASTASE

In Crux Mathematicorum Magazine, Vol. 34, No.3, Mihály Bencze proposed the problem $3326(a)$:
Let $a, b, c > 0$ then:

$$\prod_{cyc}(a^2+2)+4\prod_{cyc}(a^2+1) \geq 6(a+b+c)^2; \qquad (B-M)$$

We will developed this problem.
Proposition 1.
If $u, v, x > 0$ then:

$$(u^2+x)(v^2+x) \geq \frac{3}{4}x((u+v)^2+x); \qquad (1)$$

Proof. We have:

$$(u^2+x)(v^2+x) \geq \frac{3}{4}x((u+v)^2+x) \Leftrightarrow$$
$$4u^2v^2+4x(u^2+v^2)+4^2 \geq 3x(u^2+v^2)+6xuv+3x^2 \Leftrightarrow$$
$$4u^2v^2-4xuv+x^2+x(u^2+v^2-2uv) \geq 0 \Leftrightarrow$$
$$(2uv-x)^2+x(u-v)^2 \geq 0$$

Equality holds for $2uv = x$ and $u = v = \sqrt{\frac{x}{2}}$.
Proposition 2.
If $v, w, x > 0$ then:

$$(v^2+x)(w^2+x) \geq x(v+w)^2; \qquad (2)$$

Proof. We have:

$$(v^2+x)(w^2+x) \geq x(v+w)^2 \Leftrightarrow$$
$$v^2w^2+x(v^2+w^2)+x^2 \geq x(v^2+w^2)+2xvw \Leftrightarrow$$
$$v^2w^2-2xvw+x^2 \geq 0 \Leftrightarrow (vw-x)^2 \geq 0$$

Equality holds for $vw = x$.
Proposition 3.
If $u, v, w, x > 0$ then:

$$(u^2+x)(v^2+x)(w^2+x) \geq \frac{3}{4}x^2(u+v+w)^2; \qquad (3)$$

Proof. We have:

$$(u^2+x)(v^2+x)(w^2+x) \overset{(1)}{\geq} \frac{3}{4}x((u+v)^2+x)(w^2+x) \overset{(2)}{\geq}$$

$$\geq \frac{3}{4}x^2((u+v)+w)^2 = \frac{3}{4}x^2(u+v+w)^2 \geq \frac{9}{4}x^2(uv+vw+wu)$$

Theorem.

If $a, b, c, m, n, t, s > 0$ and $p, q > 0, p+q > 0$ then:

$$\boldsymbol{p(m^2a^2+t)(m^2b^2+t)(m^2c^2+t)+q(n^2a^2+s)(n^2b^2+s)(n^2c^2+s) \geq}$$

$$\boldsymbol{\geq \frac{3}{4}(m^2t^2p+b^2s^2q)(a+b+c)^2 \geq \frac{9}{4}(m^2t^2p+n^2s^2q)(ab+bc+ca);} \qquad \textbf{(4)}$$

Proof.

In relation (3), taking $u = ma, v = mb, w = mc, x = t$, it follows:

$$\boldsymbol{(m^2a^2+t)(m^2b^2+t)(m^2c^2+t) \geq \frac{3}{4}t^2(ma+mb+mc)^2 =}$$

$$\boldsymbol{\frac{3}{4}m^2t^2(a+b+c)^2 \geq \frac{9}{4}m^2t^2(ab+bc+ca);} \qquad \textbf{(5)}$$

Analogous, in relation (3) taking $u = na, v = nb, w = nc$ and $x = s$, we get:

$$\boldsymbol{(n^2a^2+s)(n^2b^2+s)(n^2c^2+s) \geq \frac{3}{4}s^2(na+nb+nc)^2 =}$$

$$\boldsymbol{= \frac{3}{4}n^2s^2(a+b+c)^2 \geq \frac{9}{4}n^2s^2(ab+bc+ca);} \qquad \textbf{(6)}$$

From (5) and (6), it follows:

$$p(m^2a^2+t)(m^2b^2+t)(m^2c^2+t)+q(n^2a^2+s)(n^2b^2+s)(n^2c^2+s) \geq$$

$$\geq \frac{3}{4}m^2t^2p(a+b+c)^2 + \frac{3}{4}n^2s^2q(a+b+c)^2 =$$

$$= \frac{3}{4}(m^2t^2p+n^2s^2q)(a+b+c)^2 \geq \frac{9}{4}(m^2t^2p+n^2s^2q)(ab+bc+ca)$$

If in (4), we take $t = 2, s = 1, m = n = 1, p = 1, q = 4$, we get:

$$(a^2+2)(b^2+2)(c^2+2)+4(a^2+1)(b^2+1)(c^2+1) \geq$$

$$\frac{3}{4}(1\cdot 4\cdot 1+1\cdot 1\cdot 4)(a+b+c)^2 = 6(a+b+c)^2 \geq$$

$$\geq 18(ab+bc+ca) \text{ i.e. (M-B)}$$

If in (4), we take $q = 0, m = 1, p = 1$ we get:

$$\boldsymbol{(a^2+t)(b^2+t)(c^2+t) \geq \frac{3}{4}t^2(a+b+c)^2 \geq \frac{9}{4}t^2(ab+bc+ca);} \qquad \textbf{(7)}$$

from which taking $t \to t^2$, results:

$$\boldsymbol{(a^2+t^2)(b^2+t^2)(c^2+t^2) \geq \frac{3}{4}t^4(a+b+c)^4 \geq \frac{9}{4}t^4(ab+bc+ca);} \qquad \boldsymbol{(A-A)}$$

i.e. Arkady M. Alt inequality published in [1].

If in (7) we take $t = 2$, we obtain:

$$\boldsymbol{(a^2+2)(b^2+2)(c^2+2) \geq 3(a+b+c)^2 \geq 9(ab+bc+ca);} \qquad \boldsymbol{(H-L)}$$

i.e. Hojoo Lee's inequality proposed to APMO, 2004.

References

[1] lt M. Arkady-ABOUT ONE INEQUALITY FROM APMO, 2004-NEW SOLUTION AND GENERALIZATIONS, Octogon Mathematical Magazine, Vol.2,No.1, April 2019, pages 228-232