About upper and lower bounds of reciprocal Fibonacci and Lucas series

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Abstract in this paper, introduces upper and lower bounds for $\sum_{n=1}^{\infty} \frac{1}{(F_1.F_2.\cdots F_n)^{\frac{1}{n}}}$ and $\sum_{n=1}^{\infty} \frac{1}{(L_1.L_2.\cdots L_n)^{\frac{1}{n}}}$ series

Keywords Fibonacci Numbers, Lucas Numbers, Reciprocal Fibonacci series, Reciprocal Lucas series

1 Introduction

The Fibonacci numbers were described in work by Italian mathematician Leonardo Fibonacci, which has a lot of applications in cryptology along with mathematics. Many studies have been done by mathematicians about Fibonacci numbers.Fibonacci numbers are strongly related to Lucas numbers which $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}, n \geq 2$,$L_0 = 2$, $L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}, n \geq 2$ are Fibonacci and Lucas numbers, respectively. These n^{th} numbers can be found by the Binet's formula given as[1]

$$F_n = \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}}, L_n = \varphi^n + (-\varphi)^{-n}, \varphi = \frac{\sqrt{5}+1}{2}$$

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2 preliminaires

Lemma[1][2]If F_n and L_n are Fibonacci and Lucas numbers, respectively. Then the following inequalities are satisfied

- $\varphi^{n-1} \leq F_n \leq \varphi^n$
- $\varphi^{n-1} \leq L_n \leq 2\varphi^n$

3 mean results

Theorem 2.1 If F_n are Fibonacci numbers then the following inequality is

$$\frac{1}{\sqrt{\varphi}(\sqrt{\varphi}-1)} < \sum_{n=1}^{\infty} \frac{1}{(F_1.F_2.\cdots F_n)^{\frac{1}{n}}} < \frac{\sqrt{\varphi}}{\sqrt{\varphi}-1} \tag{1}$$

proof : we know for all $n \geq 1, F_n \geq \varphi^{n-1}$ then we obtain $F_1.F_2.\cdots F_n \geq \varphi^0.\varphi^1.\cdots\varphi^{n-1} = \varphi^{\frac{(n-1)n}{2}}$ That means

$$\sum_{n=1}^{\infty} \frac{1}{(F_1.F_2.\cdots F_n)^{\frac{1}{n}}} < \frac{\sqrt{\varphi}}{\sqrt{\varphi}-1} \tag{2}$$

and for all $n \geq 1, F_n \leq \varphi^n$ then we obtain $F_1.F_2.\cdots F_n \leq \varphi^1.\varphi^2.\cdots\varphi^n = \varphi^{\frac{n(n+1)}{2}}$ That means

$$\sum_{n=1}^{\infty} \frac{1}{(F_1.F_2.\cdots F_n)^{\frac{1}{n}}} > \frac{1}{\sqrt{\varphi}(\sqrt{\varphi}-1)} \tag{3}$$

from (3) and (2) we obtain : $\frac{1}{\sqrt{\varphi}(\sqrt{\varphi}-1)} < \sum_{n=1}^{\infty} \frac{1}{(F_1.F_2.\cdots F_n)^{\frac{1}{n}}} < \frac{\sqrt{\varphi}}{\sqrt{\varphi}-1}$

Theorem 2.2 If L_n are Lucas numbers then the following inequality is satisfied

$$\frac{1}{\sqrt{2\varphi}(\sqrt{2\varphi}-1)} < \sum_{n=1}^{\infty} \frac{1}{(L_1.L_2.\cdots L_n)^{\frac{1}{n}}} < \frac{\sqrt{\varphi}}{\sqrt{\varphi}-1} \tag{4}$$

proof : we know for all $n \geq 1, L_n \geq \varphi^{n-1}$ then we obtain $L_1.L_2.\cdots L_n \geq \varphi^0.\varphi^1.\cdots\varphi^{n-1} = \varphi^{\frac{(n-1)n}{2}}$ That means

$$\sum_{n=1}^{\infty} \frac{1}{(L_1.L_2.\cdots L_n)^{\frac{1}{n}}} < \frac{\sqrt{\varphi}}{\sqrt{\varphi}-1} \tag{5}$$

and for all $n \geq 1, L_n \leq 2\varphi^n$ then we obtain $L_1.L_2.\cdots L_n \leq 2\varphi^1.2\varphi^2.\cdots 2\varphi^n = (2\varphi)^{\frac{n(n+1)}{2}}$ That means

$$\sum_{n=1}^{\infty} \frac{1}{(L_1.L_2.\cdots L_n)^{\frac{1}{n}}} > \frac{1}{\sqrt{2\varphi}(\sqrt{2\varphi}-1)} \tag{6}$$

from (5) and (6) we obtain : $\frac{1}{\sqrt{2\varphi}(\sqrt{2\varphi}-1)} < \sum_{n=1}^{\infty} \frac{1}{(L_1.L_2.\cdots L_n)^{\frac{1}{n}}} < \frac{\sqrt{\varphi}}{\sqrt{\varphi}-1}$

References

[1] Thomas Koshy. *Fibonacci and Lucas Numbers with Applications, Volume 2*. John Wiley & Sons, 2019.

[2] Runnan Liu and Andrew YZ Wang. Sums of products of two reciprocal fibonacci numbers. *Advances in Difference Equations*, 2016(1):1–26, 2016.