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Problem. Prove that

$$\int_0^1 \frac{\log(1+x)}{3x^2+4x+1}\,dx \;=\; \frac{1}{2}\left(G+\frac{1}{2}\zeta(2)-\mathrm{Li}_2\left(\frac{3}{4}\right)-\log^2(2)\right)$$

where G is the Catalan's constant defined as $G=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}$ and $\mathrm{Li}_k(x)=\sum_{n=1}^{\infty}\frac{x^n}{n^k}$ is a polylogarithm function.

Solution. Before showing the result, we needs to state the following proposition on which the Catalan's constant appears.

Proposition. *We have*

$$\int_0^1 \frac{\log(t)}{1+t^2}\,dx \;=\; -G \tag{1}$$

Proof. Applying the geometric series $\sum_{n=0}^{\infty} z^n=\frac{1}{1-z}$, we get by integrating by parts

$$\begin{aligned}\int_0^1 \frac{\log(t)}{1+t^2}\,dx &= \sum_{n=0}^{\infty}(-1)^n\int_0^1 t^{2n}\log(t)\,dt \\ &= \sum_{n=0}^{\infty}(-1)^n\left(\left[\frac{t^{2n+1}}{2n+1}\log(t)\right]_0^1-\frac{1}{2n+1}\int_0^1 t^{2n}\,dt\right) \\ &= -\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2} \\ &= -G\end{aligned}$$

Now, let $J=\int_0^1 \frac{\log(1+x)}{3x^2+4x+1}\,dx$. So we have by substitute $t=\frac{1-x}{1+x}$, $dt=-\frac{2}{(1+x)^2}\,dx$ in result (1).

$$\begin{aligned}
\int_0^1 \frac{\log(t)}{1+t^2}\,dx &= \int_0^1 \frac{\log(1-t)}{1+(1-t)^2} \\
&= \int_0^1 \frac{\log(1-t)}{t^2-2t+2}\,dx \\
&= \int_0^1 \frac{\log\left(1-\frac{1-x}{1+x}\right)}{\left(\frac{1-x}{1+x}\right)^2-2\left(\frac{1-x}{1+x}\right)+2}\left(\frac{2}{(1+x)^2}\,dt\right) \\
&= 2\int_0^1 \frac{\log(2x)-\log(1+x)}{(1-x)^2-2(1-x)+2(1+x)^2}\,dx \\
&= 2\int_0^1 \frac{\log(2x)-\log(1+x)}{(1-2x+x^2)-2(1-x)+2(1+2x+x^2)}\,dx \\
&= 2\int_0^1 \frac{\log(2x)-\log(1+x)}{3x^2+4x+1}\,dx \\
&= 2\log(2)\int_0^1 \frac{1}{3x^2+4x+1}\,dx + 2\int_0^1 \frac{\log(x)}{3x^2+4x+1}\,dx - 2\underbrace{\int_0^1 \frac{\log(1+x)}{x^2+4x+1}\,dx}_{J}.
\end{aligned}$$

It easy to verify that $\int \frac{1}{3x^2+4x+1}\,dx = \frac{1}{2}\left[\log(1+3x)-\log(1+x)\right]$.

Then

$$\int_0^1 \frac{1}{3x^2+4x+1}\,dx = \frac{\log(2)}{2}.$$

Thereby

$$J = \frac{1}{2}(G+\log^2(2)) + \int_0^1 \frac{\log(x)}{3x^2+4x+1}\,dx.$$

Now, we need to evaluate the integral $\int_0^1 \frac{\log(x)}{3x^2+4x+1}\,dx$.

Indeed, we have

$$\begin{aligned}
\int_0^1 \frac{\log(x)}{3x^2+4x+1}\,dx &= \int_0^1 \frac{\log(x)}{(1+x)(1+3x)}\,dx \\
&= \frac{3}{2}\int_0^1 \frac{\log(x)}{(1+3x)}\,d - \frac{1}{2}\int_0^1 \frac{\log(x)}{(1+x)}\,dx.
\end{aligned}$$

Let $J_a(x) = \displaystyle\int \frac{\log(x)}{(1+ax)}\,dx$, $a \in \mathbb{R}^*$. Integrating by parts, we get

$$\begin{aligned}
J_a(x) &= \frac{1}{a}\log(1+ax)\log(x) - \frac{1}{a}\int \frac{\log(1+ax)}{x}\,dx \\
&= \frac{1}{a}\log(1+ax)\log(x) - \frac{1}{a}\int \frac{\log(1-u)}{u}\,du \ \text{ Substitute } u=-ax,\ dx = -\frac{1}{a}\,du \\
&= \frac{1}{a}\log(1+ax)\log(x) + \frac{1}{a}\mathrm{Li}_2(u) \\
&= \frac{1}{a}\log(1+ax)\log(x) + \frac{1}{a}\mathrm{Li}_2(-ax).
\end{aligned}$$

Because $J_a(0) = 0$, $\forall a$, we thus have

$$\begin{aligned}
\int_0^1 \frac{\log(x)}{3x^2+4x+1}\,dx &= \frac{3}{2}\int_0^1 \frac{\log(x)}{(1+3x)}\,dx - \frac{1}{2}\int_0^1 \frac{\log(x)}{(1+x)}\,dx \\
&= \frac{1}{2}\mathrm{Li}_2(-3) - \frac{1}{2}\mathrm{Li}_2(-1)
\end{aligned}$$

It is known that $\mathrm{Li}_2(-1) = -\frac{1}{2}\zeta(2)$, to provide the value of $\mathrm{Li}_2(-3)$ we use the property of the dilogarithm function $\mathrm{Li}_2(1-u) + \mathrm{Li}_2\left(1-\frac{1}{u}\right) = -\frac{1}{2}\log^2(u)$ for $u = \frac{1}{4}$. So we have $\mathrm{Li}_2(-3) = -\mathrm{Li}_2\left(\frac{3}{4}\right) - 2\log^2(2)$. Hence

$$\int_0^1 \frac{\log(x)}{3x^2+4x+1}\,dx = -\frac{1}{2}\left(\mathrm{Li}_2\left(\frac{3}{4}\right) - \frac{1}{2}\zeta(2) + 2\log^2(2)\right).$$

Finally

$$\begin{aligned}
\int_0^1 \frac{\log(1+x)}{3x^2+4x+1}\,dx &= \frac{1}{2}\left(G + \log^2(2)\right) - \frac{1}{2}\left(\mathrm{Li}_2\left(\frac{3}{4}\right) - \frac{1}{2}\zeta(2) + 2\log^2(2)\right) \\
&= \frac{1}{2}\left(G + \frac{1}{2}\zeta(2) - \mathrm{Li}_2\left(\frac{3}{4}\right) - \log^2(2)\right).
\end{aligned}$$