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Problem. Prove that

$$\int_0^{\pi/2} \sin(2x)\cot^2(x)\log^2(\cos(x))\,dx = \frac{1}{2}\Big(\zeta(3)-1\Big).$$

where $\zeta(3)$ is the Apery's constant defined as $\zeta(3)=\sum_{n=1}^{\infty}\frac{1}{n^3}=1.202056903159594...$

Solution. We have

$$\begin{aligned}
\int_0^{\pi/2} \sin(2x)\cot^2(x)\log^2(\cos(x))\,dx &= \int_0^{\pi/2} \sin(2x)\frac{\cos^2(x)}{\sin^2(x)}\log^2(\cos(x))\,dx \\
&= \int_0^{\pi/2} \sin(\pi-2x)\frac{\sin^2\left(\frac{\pi}{2}-x\right)}{\cos^2\left(\frac{\pi}{2}-x\right)}\log^2\left(\sin\left(\frac{\pi}{2}-x\right)\right)dx \\
&= \int_0^{\pi/2} \sin(2x)\frac{\sin^2(x)}{\cos^2(x)}\log^2(\sin(x))\,dx \\
&\quad \left\{\text{Replace }\left(\frac{\pi}{2}-x\right)\text{ by } x\right\} \\
&= 2\int_0^{\pi/2} \frac{\sin^3(x)}{\cos(x)}\log^2(\sin(x))\,dx.
\end{aligned}$$

Substitute $y=\sin(x)$, $dy=\cos(x)\,dx$, we get

$$\begin{aligned}
\int_0^{\pi/2} \sin(2x)\cot^2(x)\log^2(\cos(x))\,dx &= 2\int_0^{\pi/2} \frac{y^3}{1-y^2}\log^2(y)\,dy \\
&= 2\sum_{n=0}^{\infty}\left(\int_0^1 y^{2n+3}\log^2(y)\,dt\right)
\end{aligned}$$

By substitute $y = e^{-t}$, $dy = -e^{-t}\,dt$ and the fact that $\displaystyle\int_0^\infty t^a e^{-bt}\,dt = \frac{\Gamma(a+1)}{b^{a+1}}$, we get

$$\begin{aligned}
\int_0^{\pi/2} \sin(2x)\cot^2(x)\log^2(\cos(x))\,dx &= 2\sum_{n=0}^{\infty}\left(\int_0^\infty t^2 e^{-2(n+2)t}\,dt\right) \\
&= \frac{1}{4}\sum_{n=0}^{\infty}\frac{\Gamma(3)}{(n+2)^3} \\
&= \frac{1}{4}\sum_{n=2}^{\infty}\frac{2!}{n^3} \\
&= \frac{1}{2}\left(\sum_{n=1}^{\infty}\frac{1}{n^3} - 1\right) \\
&= \frac{1}{2}\Big(\zeta(3) - 1\Big).
\end{aligned}$$