CRUX MATHEMATICORUM CHALLENGES-(VII)

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4679. Let $(x_n)_{n\geq 1}$ be a sequence of real numbers such that $x_1 = \frac{1}{7}, x_2 = \frac{1}{5}$ and for $n \geq 2$,

$$2nx_{n+1} \cdot x_{n-1} = (n+1)x_n \cdot x_{n-1} + (n-1)x_n \cdot x_{n+1}.$$

Find

$$\lim_{n\to\infty}\left(\frac{2}{3} + x_n\right)^{nx_n}$$

Proof.
The answer is $e^{-\frac{4}{27}}$, which is about 0.86230.
Let $y_n = \frac{n}{x_n}$. Then $y_1 = 7, y_2 = 10$ and $2y_n = y_{n+1} + y_{n-1}$, so that $y_n = 3n + 4$, an arithmetic progression with common difference 3. Hence

$$x_n = \frac{n}{3n+4} = \frac{1}{3} - u_n$$

and

$$nx_n = \frac{n^2}{3n+4} = \left(\frac{1}{u_n}\right)\left(\frac{4n^2}{3(3n+4)^2}\right),$$

where $u_n = \frac{4}{3(3n+4)}$. Therefore

$$\lim_{n\to\infty}\left(\frac{2}{3} + x_n\right)^{nx_n} = \lim_{n\to\infty}\left[(1-u_n)^{\frac{1}{u_n}}\right]^{\frac{4n^2}{3(3n+4)^2}} = (e^{-1})^{\frac{4}{27}} = e^{-\frac{4}{27}}.$$

□

B104. Find all real roots of the equation:

$$\mathbf{5x^3 - 9x^2 - 15x + 3 = 0}$$

Proof.

$$5x^3 - 9x^2 - 15x + 3 = 0$$
$$5x^3 - 9x^2 - 15x + 3 + 5(3x - x^3) = 5(3x - x^3)$$
$$3 - 9x^2 = 5(3x - x^3)$$
$$3(1 - 3x^2) = 5(3x - x^3)$$

Let's observe that $x = \pm\frac{\sqrt{3}}{3}$ are not solutions, so we can divide by $1 - 3x^2$ the equation:

(1) $$\frac{3x - x^3}{1 - 3x^2} = \frac{5}{3}$$

$$x \in \mathbb{R}\setminus\left\{\pm\frac{\sqrt{3}}{3}\right\} \Rightarrow (\exists)\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\setminus\left\{\pm\frac{\pi}{6}\right\}$$
$$x = \tan\alpha$$

$$(1) \Leftrightarrow \frac{3\tan\alpha - \tan^3\alpha}{1-3\tan^2\alpha} = \frac{5}{3}$$

$$\tan 3\alpha = \frac{5}{3} \Rightarrow 3\alpha \in \left\{ \arctan\frac{5}{3} + k\pi \middle| k \in \mathbb{Z} \right\}$$

$$\alpha \in \left\{ \frac{1}{3}\arctan\frac{5}{3} + \frac{k\pi}{3} \middle| k \in \mathbb{Z} \right\} \cap \left(-\frac{\pi}{2}, \frac{\pi}{2} \right) \setminus \left\{ \pm\frac{\pi}{6} \right\}$$

$$k = 0 \Rightarrow \alpha_1 = \frac{1}{3}\arctan\frac{5}{3} \Rightarrow x_1 = \tan\left(\frac{1}{3}\arctan\frac{5}{3}\right)$$

$$k = 1 \Rightarrow \alpha_2 = \frac{1}{3}\arctan\frac{5}{3} + \frac{\pi}{3} \Rightarrow x_2 = \tan\left(\frac{1}{3}\arctan\frac{5}{3} + \frac{\pi}{3}\right)$$

$$k = -1 \Rightarrow \alpha_3 = \frac{1}{3}\arctan\frac{5}{3} - \frac{\pi}{3} \Rightarrow x_3 = \tan\left(\frac{1}{3}\arctan\frac{5}{3} - \frac{\pi}{3}\right)$$

□

B107. In ΔABC (a, b, c - sides; s - semiperimeter; r - inradii) the following relationship holds:

$$\frac{a}{\sqrt{s+b}} + \frac{b}{\sqrt{s+c}} + \frac{c}{\sqrt{s+a}} \geq \frac{108r^2}{a\sqrt{s+b} + b\sqrt{s+c} + c\sqrt{s+a}}$$

Solution 1 by proposer.

$$\left(\sum_{cyc} a\right)\left(\sum_{cyc} \frac{a}{\sqrt{s+b}}\right)\left(\sum_{cyc} a\sqrt{s+b}\right) =$$

$$= \left(\sum_{cyc} (\sqrt[3]{a})^3\right)\left(\sum_{cyc} \left(\sqrt[3]{\frac{a}{\sqrt{s+b}}}\right)^3\right)\left(\sum_{cyc} \left(\sqrt[3]{a\sqrt{s+b}}\right)^3\right) \geq$$

$$\overset{\text{HÖLDER}}{\geq} \left(\sum_{cyc} \sqrt[3]{a} \cdot \sqrt[3]{\frac{a}{\sqrt{s+b}}} \cdot \sqrt[3]{a\sqrt{s+b}}\right)^3 =$$

$$= \left(\sum_{cyc} (\sqrt[3]{a})^3\right)^3 = (a+b+c)^3$$

$$\left(\sum_{cyc} a\right)\left(\sum_{cyc} \frac{a}{\sqrt{s+b}}\right)\left(\sum_{cyc} a\sqrt{s+b}\right) \geq \left(\sum_{cyc} a\right)^3$$

$$\left(\sum_{cyc} \frac{a}{\sqrt{s+b}}\right)\left(\sum_{cyc} a\sqrt{s+b}\right) \geq \left(\sum_{cyc} a\right)^2 = (2s)^2 =$$

$$= 4s^2 \overset{\text{MITRINOVICI}}{\geq} 4(3\sqrt{3}r)^2 = 108r^2$$

$$\left(\sum_{cyc} \frac{a}{\sqrt{s+b}}\right)\left(\sum_{cyc} a\sqrt{s+b}\right) \geq 108r^2$$

$$\sum_{cyc} \frac{a}{\sqrt{s+b}} \geq \frac{108r^2}{\sum_{cyc} a\sqrt{s+b}}$$

$$\frac{a}{\sqrt{s+b}} + \frac{b}{\sqrt{s+c}} + \frac{c}{\sqrt{s+a}} \geq \frac{108r^2}{a\sqrt{s+b} + b\sqrt{s+c} + c\sqrt{s+a}}$$

□

Solution 2 by Marin Chirciu - Romania.
Inequality can be written:

$$\left(\frac{a}{\sqrt{s+b}}+\frac{b}{\sqrt{s+c}}+\frac{c}{\sqrt{s+a}}\right)(a\sqrt{s+b}+b\sqrt{s+c}+c\sqrt{s+a})\geq 108r^2$$

Which results by CBS:

$$\left(\frac{a}{\sqrt{s+b}}+\frac{b}{\sqrt{s+c}}+\frac{c}{\sqrt{s+a}}\right)(a\sqrt{s+b}+b\sqrt{s+c}+c\sqrt{s+a})\overset{CBS}{\geq}$$

$\geq (a+b+c)^2 = 4p^2 \overset{(1)}{\geq} 108r^2$, where $(1) \Leftrightarrow 4p^2 \geq 108r^2 \Leftrightarrow p^2 \geq 27r^2$, (Mitrinovici).
Equality holds for an equilateral triangle.

□

B120. If $x, y, z \in (0, 1); xy + yz + zx = 1$ then:

$$\frac{(1+x^2)(1+y^2)}{x^2y^2}+\frac{(1+y^2)(1+z^2)}{y^2z^2}+\frac{(1+z^2)(1+x^2)}{z^2x^2}\geq\frac{48}{x^2+y^2+z^2}$$

Solution 1 by proposer.
Lemma 1:
In ΔABC the following relationship holds:

$$\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}\geq 1$$

Proof. Let be $f:(0,\pi)\to\mathbb{R}; f(x)=\tan^2\frac{x}{2}$

$$f'(x)=\frac{\sin\frac{x}{2}}{\cos^3\frac{x}{2}}; f''(x)=\frac{1}{2\cos^4\frac{x}{2}}>0; f \text{ convexe}$$

By Jensen:

$$\sum_{cyc}\tan^2\frac{A}{2}\geq 3\tan^2\left(\frac{\frac{A+B+C}{3}}{2}\right)=$$

$$=3\tan^2\left(\frac{\pi}{6}\right)=3\cdot\left(\frac{1}{\sqrt{3}}\right)^2=1$$

□

Lemma 2:
In ΔABC the following relationship holds:

$$\frac{1}{\sin^2\frac{B}{2}\sin^2\frac{C}{2}}+\frac{1}{\sin^2\frac{C}{2}\sin^2\frac{A}{2}}+\frac{1}{\sin^2\frac{A}{2}\sin^2\frac{B}{2}}\geq 48$$

Proof.

$$\sum_{cyc}\frac{1}{\sin^2\frac{B}{2}\sin^2\frac{C}{2}}=\sum_{cyc}\frac{ac\cdot ab}{(s-a)(s-c)\cdot(s-a)(s-b)}=$$

$$=\frac{abc}{(s-a)(s-b)(s-c)}\cdot\sum_{cyc}\frac{a}{s-a}=$$

$$=\frac{abc}{(s-a)^2(s-b)^2(s-c)^2}\cdot\sum_{cyc}a(s-b)(s-c)=$$

$$= \frac{abcs^2}{s^2(s-a)^2(s-b)^2(s-c)^2} \cdot \sum_{cyc} a(s^2 - s(b+c) + bc) =$$

$$= \frac{4RFs^2}{F^4} \sum_{cyc} (as^2 - as(2s-a) + abc) =$$

$$= \frac{4Rs^2}{F^3} \sum_{cyc} (a^2 s - as^2 + abc) =$$

$$= \frac{4Rs^2}{F \cdot r^2 s^2} \cdot \left(s \sum_{cyc} a^2 - s^2 \sum_{cyc} a + 3abc \right) =$$

$$= \frac{4R}{Fr^2} (s \cdot 2(s^2 - r^2 - 4Rr) - 2s^3 + 12RF) =$$

$$= \frac{4R}{Fr^2} (2s^3 - 2sr^2 - 8Rrs - 2s^3 + 12RF) =$$

$$= \frac{4R}{Fr^2} (-2rF - 8RF + 12RF) =$$

$$= \frac{4R}{r^2} (-2r - 8R + 12R) = \frac{4R(4R - 2r)}{r^2} =$$

$$= \frac{8R(2R - r)}{r^2} \overset{\text{EULER}}{\geq} \frac{8 \cdot 2r \cdot (2 \cdot 2r - r)}{r^2} =$$

$$= \frac{16r \cdot 3r}{r^2} = \frac{48r^2}{r^2} = 48$$

□

Back to the main problem:

$$x, y, z \in (0,1) \Rightarrow (\exists)\alpha, \beta, \gamma \in \left(0, \frac{\pi}{4}\right);$$

$$x = \tan\alpha; y = \tan\beta; z = \tan\gamma$$

$$xy + yz + zx = 1 \Rightarrow x(y+z) = 1 - yz \Rightarrow x = \frac{1 - yz}{y + z}$$

$$\tan\alpha = \frac{1 - \tan\beta\tan\gamma}{\tan\beta + \tan\gamma} \Rightarrow \tan\alpha = \cot(\beta + \gamma)$$

$$\tan\alpha = \tan\left(\frac{\pi}{2} - (\beta + \gamma)\right) \Rightarrow \alpha = \frac{\pi}{2} - (\beta + \gamma)$$

$$2\alpha + 2\beta + 2\gamma = \pi$$

Denote: $A = 2\alpha; B = 2\beta; C = 2\gamma$

$$\alpha = \frac{A}{2}; \beta = \frac{B}{2}; \gamma = \frac{C}{2}; A + B + C = \pi$$

$$x = \tan\frac{A}{2}; y = \tan\frac{B}{2}; z = \tan\frac{C}{2}$$

$$\sin^2\frac{A}{2} = \frac{x^2}{1 + x^2}; \sin^2\frac{B}{2} = \frac{y^2}{1 + y^2}; \sin^2\frac{C}{2} = \frac{z^2}{1 + z^2}$$

$$\frac{(1 + x^2)(1 + y^2)}{x^2 y^2} + \frac{(1 + y^2)(1 + z^2)}{y^2 z^2} + \frac{(1 + z^2)(1 + x^2)}{z^2 x^2} \geq \frac{48}{x^2 + y^2 + z^2} \Leftrightarrow$$

$$\Leftrightarrow \left(\sum_{cyc} x^2\right) \cdot \left(\sum_{cyc} \frac{(1 + x^2)(1 + y^2)}{x^2 y^2}\right) \geq 48$$

$$\Leftrightarrow\left(\sum_{cyc}\tan^2\frac{A}{2}\right)\cdot\left(\sum_{cyc}\frac{1}{\frac{x^2}{1+x^2}\cdot\frac{y^2}{1+y^2}}\right)\geq 48$$

$$\Leftrightarrow\left(\sum_{cyc}\tan^2\frac{A}{2}\right)\cdot\left(\sum_{cyc}\frac{1}{\sin^2\frac{A}{2}\sin^2\frac{B}{2}}\right)\geq 48$$

By lemma 1 and lemma 2:

$$\sum_{cyc}\tan^2\frac{A}{2}\geq 1 \tag{1}$$

$$\sum_{cyc}\frac{1}{\sin^2\frac{A}{2}\sin^2\frac{B}{2}}\geq 48 \tag{2}$$

By multiplying $(1);(2)$:

$$\left(\sum_{cyc}\tan^2\frac{A}{2}\right)\cdot\left(\sum_{cyc}\frac{1}{\sin^2\frac{A}{2}\sin^2\frac{B}{2}}\right)\geq 48$$

Equality holds if $A=B=C=\frac{\pi}{3}\Rightarrow x=y=z=\frac{\sqrt{3}}{3}$. □

Solution 2 by Marin Chirciu - Romania.

$$\text{In the identity }\sum\tan\frac{B}{2}\tan\frac{C}{2}=1\text{ replace:}$$

$$(x,y,z)=\left(\tan\frac{A}{2},\tan\frac{B}{2},\tan\frac{C}{2}\right)$$

$$\sum\frac{(1+y^2)(1+z^2)}{y^2z^2}=\sum\frac{(1+\tan^2\frac{B}{2})(1+\tan^2\frac{C}{2})}{\tan^2\frac{B}{2}\tan^2\frac{C}{2}}=$$

$$=\frac{\sum\sin^2\frac{A}{2}}{\prod\sin^2\frac{A}{2}}=\frac{1-\frac{r}{2R}}{\frac{r^2}{4R^2}}=\frac{8R(2R-r)}{r^2}$$

$$x^2+y^2+z^2=\sum\tan^2\frac{A}{2}=\frac{(4R+r)^2}{p^2}-2$$

Inequality can be written:

$$\frac{8R(2R-r)}{r^2}\geq\frac{48}{\frac{(4R+r)^2}{p^2}-2}\Leftrightarrow\frac{8R(2R-r)}{r^2}\cdot\left[\frac{(4R+r)^2}{p^2}-2\right]\geq 48$$

Which results by Blundon-Gerretsen:

$$p^2\leq\frac{R(4R+r)^2}{2(2R-r)}$$

Remains to prove:

$$\frac{8R(2R-r)}{r^2}\left[\frac{(4R+r)^2}{\frac{R(4R+r)^2}{2(2R-r)}}-2\right]\geq 48\Leftrightarrow\frac{R(2R-r)}{r^2}\cdot\left[\frac{2(2R-r)}{R}-2\right]\geq 6\Leftrightarrow$$

$$\Leftrightarrow(2R-r)(R-r)\geq r^2\Leftrightarrow 2R^2-3Rr-2r^2\geq 0\Leftrightarrow(R-2r)(2R+r)\geq 0,$$

obviously by Euler $R \geq 2r$.
Equality holds by an equilateral triangle. In the initial inequality the equality holds for:

$$x = y = z = \frac{1}{\sqrt{3}}.$$

□

B124. In ΔABC the following relationship holds:

$$\frac{a^4}{w_a} + \frac{b^4}{w_b} + \frac{c^4}{w_c} \geq 144r^3$$

(w_a, w_b, w_c - internal bisectors; r - inradii)

Solution 1 by proposer.

$$w_a = \frac{2bc}{b+c}\cos\frac{A}{2} = \frac{2\sqrt{bc}}{b+c}\sqrt{s(s-a)} \leq \sqrt{s(s-a)}$$

$$\text{because } \frac{2\sqrt{bc}}{b+c} \leq 1 \Leftrightarrow 0 \leq (\sqrt{b} - \sqrt{c})^2$$

$$w_a \leq \sqrt{s(s-a)} \text{ and analogous:}$$

$$w_b \leq \sqrt{s(s-b)}; w_c \leq \sqrt{s(s-c)}$$

$$(w_a + w_b + w_c)^2 \leq (\sqrt{s(s-a)} + \sqrt{s(s-b)} + \sqrt{s(s-c)})^2 \leq$$

$$\overset{\text{CAUCHY-SCHWARZ}}{\leq} (1^2 + 1^2 + 1^2)(s(s-a) + s(s-b) + s(s-c)) =$$

$$= 3(3s^2 - s(a+b+c)) = 3(3s^2 - s \cdot 2s) = 3s^2$$

$$(w_a + w_b + w_c)^2 \leq 3s^2 = \frac{3}{4}(a+b+c)^2 \leq$$

$$\leq \frac{3}{4}(3a^2 + 3b^2 + 3c^2) = \frac{9}{4}(a^2 + b^2 + c^2)$$

(1) $$w_a + w_b + w_c \leq \frac{3}{2}\sqrt{a^2 + b^2 + c^2}$$

$$\frac{a^4}{w_a} + \frac{b^4}{w_b} + \frac{c^4}{w_c} = \frac{(a^2)^2}{w_a} + \frac{(b^2)^2}{w_b} + \frac{(c^2)^2}{w_c} \geq$$

$$\overset{\text{BERGSTÖM}}{\geq} \frac{(a^2 + b^2 + c^2)^2}{w_a + w_b + w_c} \overset{(1)}{\geq} \frac{(a^2 + b^2 + c^2)}{\frac{3}{2}\sqrt{a^2 + b^2 + c^2}} =$$

$$= \frac{2}{3}(a^2 + b^2 + c^2)^{\frac{3}{2}} \overset{\text{WEITZENBOCK}}{\geq} \frac{2}{3}(4F\sqrt{3})^{\frac{3}{2}} =$$

$$= \frac{2}{3}(4\sqrt{3}rs)^{\frac{3}{2}} \overset{\text{MITRINOVICI}}{\geq} \frac{2}{3}(4\sqrt{3}r \cdot 3\sqrt{3}r)^{\frac{3}{2}} =$$

$$= \frac{2}{3}(36r^2)^{\frac{3}{2}} = \frac{2}{3} \cdot (6r)^3 = \frac{2}{3} \cdot 216r^3 = 144r^3$$

Equality holds for $a = b = c$. □

Solution 2 by Marin Chirciu - Romania.
By Bergström's inequality:

$$LHS = \sum \frac{a^4}{w_a} \overset{\text{CS}}{\geq} \frac{(\sum a^2)^2}{\sum w_a} \overset{m_a \geq w_a}{\geq} \frac{(\sum a^2)^2}{\sum m_a} \overset{\text{Leuenberger}}{\geq}$$

$$\geq \frac{(\sum a^2)^2}{4R+r} = \frac{(2(p^2-r^2-4Rr))^2}{4R+r} \overset{\text{Gerretsen}}{\geq}$$

$$\geq \frac{4(16Rr-5r^2-r^2-4Rr)^2}{4R+r} = \frac{4(12Rr-6r^2)^2}{4R+r} = \frac{4\cdot 36r^2(2R-r)^2}{4R+r} \overset{(1)}{\geq} 144r^3,$$

$$\text{where (1)} \Leftrightarrow \frac{4\cdot 36r^2(2R-r)^2}{4R+r} \geq 144r^3 \Leftrightarrow (2R-r)^2 \geq r(4R+r) \Leftrightarrow$$

$$4R^2-4Rr+r^2 \geq 4Rr+r^2 \Leftrightarrow R \geq 2r, \text{ (Euler)}.$$

Equality holds for an equilateral triangle.
Remark.
The problem can be generalized:
In ΔABC:

$$\frac{a^{2n}}{w_a}+\frac{b^{2n}}{w_b}+\frac{c^{2n}}{w_c} \geq 12^n r^{2n-1}, n \in \mathbb{N}^*.$$

Proposed by Marin Chirciu - Romania

Proof.
By Hölder:

$$LHS = \sum \frac{a^{2n}}{w_a} \overset{CS}{\geq} \frac{(\sum a^2)^n}{3^{n-2}\sum w_a} \overset{m_a \geq w_a}{\geq} \frac{(\sum a^2)^n}{3^{n-2}\sum m_a} \overset{\text{Leuenberger}}{\geq}$$

$$\geq \frac{(\sum a^2)^n}{3^{n-2}(4R+r)} = \frac{(2(p^2-r^2-4Rr))^n}{3^{n-2}(4R+r)} \overset{\text{Gerretsen}}{\geq}$$

$$\geq \frac{2^n(16Rr-5r^2-r^2-4Rr)^n}{3^{n-2}(4R+r)} = \frac{2^n(12Rr-6r^2)^n}{3^{n-2}(4R+r)} =$$

$$= \frac{2^n\cdot(6r)^n(2R-r)^n}{3^{n-2}(4R+r)} \overset{(1)}{\geq} 12^n r^{2n-1},$$

$$\text{where (1)} \Leftrightarrow \frac{2^n\cdot(6r)^n(2R-r)^n}{3^{n-2}(4R+r)} \geq 12^n r^{2n-1} \Leftrightarrow \frac{(6r)^n(2R-r)^n}{3^{n-2}(4R+r)} \geq 6^n r^{2n-1} \Leftrightarrow$$

$$\Leftrightarrow \frac{r^n(2R-r)^n}{3^{n-2}(4R+r)} \geq r^{2n-1} \Leftrightarrow \frac{(2R-r)^n}{3^{n-2}(4R+r)} \geq r^{n-1} \Leftrightarrow \left(\frac{2R-r}{3r}\right)^n \geq \frac{4R+r}{9r},$$

Which will be proved by mathematical induction:

$$P(n) : \left(\frac{2R-r}{3r}\right)^n \geq \frac{4R+r}{9R}, n \in \mathbb{N}^*.$$

$$P(1) : \frac{2R-r}{3r} \geq \frac{4R+r}{9r} \Leftrightarrow R \geq 2r, \text{ (Euler)}.$$

$P(k) \Rightarrow P(k+1), k \geq 1$ it is equivalent with

$$P(1) : \frac{2R-r}{3r} \geq \frac{4R+r}{9r} \Leftrightarrow R \geq 2r, \text{ (Euler)}.$$

Equality holds for an equilateral triangle.
Note.
For $n = 2$ it's obtained Problem B124 from Crux Mathematicorum, No. 48 (3),

March 2022, proposed by Daniel Sitaru.
In ΔABC:

$$\frac{a^4}{w_a}+\frac{b^4}{w_b}+\frac{c^4}{w_c}\geq 144r^3$$

Proposed by Daniel Sitaru - Romania

Remark.
Problem can be developed:
In ΔABC:

$$\frac{a^{2n}}{m_a}+\frac{b^{2n}}{m_b}+\frac{c^{2n}}{m_c}\geq 12^n r^{2n-1}, n\in\mathbb{N}^*.$$

Proposed by Marin Chirciu - Romania

Proof.
By Hölder:

$$LHS=\sum\frac{a^{2n}}{m_a}\overset{CS}{\geq}\frac{(\sum a^2)^n}{3^{n-2}\sum m_a}\overset{\text{Leuenberger}}{\geq}\frac{(\sum a^2)^n}{3^{n-2}(4R+r)}=$$

$$=\frac{(2(p^2-r^2-4Rr))^n}{3^{n-2}(4R+r)}\overset{\text{Gerretsen}}{\geq}\frac{2^n(16Rr-5r^2-r^2-4Rr)^n}{3^{n-2}(4R+r)}=$$

$$=\frac{2^n(12Rr-6r^2)^n}{3^{n-2}(4R+r)}=\frac{2^n\cdot(6r)^n(2R-r)^n}{3^{n-2}(4R+r)}\overset{(1)}{\geq}12^n r^{2n-1},$$

$$\text{where } (1)\Leftrightarrow\frac{2^n\cdot(6r)^n(2R-r)^n}{3^{n-2}(4R+r)}\geq 12^n r^{2n-1}\Leftrightarrow\frac{(6r)^n(2R-r)^n}{3^{n-2}(4R+r)}\geq 6^n r^{2n-1}\Leftrightarrow$$

$$\frac{r^n(2R-r)^n}{3^{n-2}(4R+r)}\geq r^{2n-1}\Leftrightarrow\frac{(2R-r)^n}{3^{n-2}(4R+r)}\geq r^{n-1}\Leftrightarrow\left(\frac{2R-r}{3r}\right)^n\geq\frac{4R+r}{9r},$$

Which can be proved by mathematical induction.

$$P(n):\left(\frac{2R-r}{3r}\right)^n\geq\frac{4R+r}{9R}, n\in\mathbb{N}^*.$$

$$P(1):\frac{2R-r}{3r}\geq\frac{4R+r}{9r}\Leftrightarrow R\geq 2r, \text{ (Euler)}.$$

$P(k)\Rightarrow P(k+1), k\geq 1$ it's equivalent with

$$P(1):\frac{2R-r}{3r}\geq\frac{4R+r}{9r}\Leftrightarrow R\geq 2r, \text{ (Euler)}.$$

Equality holds for an equilateral triangle.

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