R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

ELEGANT, CLASSIC AND NEW IN INTEGRAL CALCULATION

By D.M. Bătineţu-Giurgiu and Neculai Stanciu-Romania

Proposition 1. If $a\in R_+^*$ $f,g,h:R\to R$ are continue functions with f and g odd and h even, then $\int\limits_{-a}^{a} f(x)\cdot\ln\left(1+e^{g(x)}\right)\cdot arctg(h(x))dx=\int\limits_{0}^{a} f(x)g(x)arctg(h(x))dx$.

Proof. $I=\int\limits_{-a}^{a} f(x)\ln\left(1+e^{g(x)}\right)arctg(h(x))dx$, and putting $x=u(t)=-t\,,u'(t)=-1$, $u(a)=-a,\ u(-a)=a$ we obtain

$$I=\int\limits_{-a}^{a} f(-x)\ln\left(1+e^{g(-x)}\right)arctg(h(-x))dx=-\int\limits_{-a}^{a} f(x)\ln\left(1+e^{-g(x)}\right)arctg(h(x))dx=$$

$$=-\int\limits_{-a}^{a} f(x)\ln\left(\frac{1+e^{g(x)}}{e^{g(x)}}\right)arctg(h(x))dx=-\int\limits_{-a}^{a} f(x)\ln\left(1+e^{g(x)}\right)arctg(h(x))dx+$$

$$+\int\limits_{-a}^{a} f(x)g(x)arctg(h(x))dx=-I+2\cdot\int\limits_{0}^{a} f(x)g(x)arctg(h(x))dx\text{, so}$$

$$I=\int\limits_{0}^{a} f(x)g(x)arctg(h(x))dx\text{, q.e.d.}$$

Proposition 2. If $a,b\in R,\ a<b\,,c\in R_+^*$ iar $f:R\to R_+^*$ is continue, then Să se calculeze:

$$\int\limits_{a}^{b}\frac{e^{f(x-a)}\left(f(x-a)\right)^{\frac{1}{c}}}{e^{f(x-a)}\left(f(x-a)\right)^{\frac{1}{c}}+e^{f(b-x)}\left(f(b-x)\right)^{\frac{1}{c}}}dx=\frac{b-a}{2}.$$

Proof. $I=\int\limits_{a}^{b}\frac{e^{f(x-a)}\left(f(x-a)\right)^{\frac{1}{c}}}{e^{f(x-a)}\left(f(x-a)\right)^{\frac{1}{c}}+e^{f(b-x)}\left(f(b-x)\right)^{\frac{1}{c}}}dx$, where we putting

$x=u(t)=a+b-t,\ u'(t)=-1,u(a)=b,u(b)=a$ and we obtain

$$I=\int\limits_{a}^{b}\frac{e^{f(a+b-x-a)}\left(f(a+b-x-a)\right)^{\frac{1}{c}}}{e^{f(a+b-x-a)}\left(f(a+b-x-a)\right)^{\frac{1}{c}}+e^{f(b-a-b+x)}\left(f(b-a-b+x)\right)^{\frac{1}{c}}}dx=$$

$$= \int_a^b \frac{e^{f(b-x)}\left(f(b-x)\right)^{\frac{1}{c}}}{e^{f(b-x)}\left(f(b-x)\right)^{\frac{1}{c}} + e^{f(x-a)}\left(f(x-a)\right)^{\frac{1}{c}}} dx \text{ , so}$$

$$2I = I + I = \int_a^b \frac{e^{f(x-a)}\left(f(x-a)\right)^{\frac{1}{c}} + e^{f(b-x)}\left(f(b-x)\right)^{\frac{1}{c}}}{e^{f(x-a)}\left(f(x-a)\right)^{\frac{1}{c}} + e^{f(b-x)}\left(f(b-x)\right)^{\frac{1}{c}}} dx = \int_a^b dx = x\Big|_a^b = b - a \text{ , hence}$$

$$I = \frac{b-a}{2}.$$

Proposition 3. If $a \in R_+$, $b, c \in (1, \infty)$ and $f, g : R \to R$ are continue and odd, then

$$\int_{-a}^{a} f(x) \ln\left(b^{g(x)} + c^{g(x)}\right) dx = \ln(bc) \int_0^a f(x) g(x) dx.$$

Proof. $I = \int_{-a}^{a} f(x) \ln\left(b^{g(x)} + c^{g(x)}\right) dx$, where we putting $x = u(t) = -t$,

$u'(t) = -1, u(a) = -a, u(-a) = a$, so

$$I = \int_a^{-a} - f(t) \ln\left(b^{-g(t)} + c^{-g(t)}\right)(-1) dt = -\int_{-a}^{a} f(x) \ln \frac{b^{g(x)} + c^{g(x)}}{(bc)^{g(x)}} dx =$$

$$= -I + \int_{-a}^{a} f(x) \ln(bc)^{g(x)} dx = -I + \ln(bc) \int_{-a}^{a} f(x) g(x) dx \qquad (1)$$

Also we have $(fg)(-x) = f(-x)g(-x) = -f(x)(-g(x)) = (fg)(x)$, i.e. $fg : R \to R$, is even – so by (1) we obtain $2I = \ln(bc) \int_{-a}^{a} (fg)(x) dx = 2\ln(bc) \int_0^a f(x) g(x) dx$, hence

$$I = \ln(bc) \int_0^a f(x) g(x) dx.$$

Proposition 4. If $a, b \in R$, $a < b$ and $f : R \to R$ is derivanble with the derivative continue and $g : R \to R_+$ is such that $f(a + b - x) = f(x), g(a + b - x) g(x) = 1, \forall x \in R$, then

$$\int_a^b \left(\frac{f(x)}{1 + g(x)} + f'(x) \ln(1 + g(x)) \right) dx = \frac{1}{2} \int_a^b \left(f(x) + f'(x) \ln g(x) \right) dx.$$

Proof. $f(a + b - x) = f(x), \forall x \in R$, so: $f'(a + b - x) = -f'(x), \forall x \in R$.

$I=\int_a^b\left(\frac{f(x)}{1+g(x)}+f'(x)\ln(1+g(x))\right)dx$, where we putting

$x=u(t)=a+b-t,\ u'(t)=-1,u(a)=b,u(b)=a$, therefore

$$I=-\int_b^a\left(\frac{f(a+b-t)}{1+g(a+b-t)}+f'(a+b-t)\ln(1+g(a+b-t))\right)dt=$$

$$=\int_a^b\left(\frac{f(x)}{1+\frac{1}{g(x)}}-f'(x)\ln\left(1+\frac{1}{g(x)}\right)\right)dt=$$

$$=\int_a^b\left(\frac{f(x)g(x)}{1+g(x)}-f'(x)\ln(1+g(x))+f'(x)\ln g(x)\right)dt.$$

Hence,

$2I=\int_a^b\left(\frac{(1+g(x))f(x)}{1+g(x)}+f'(x)\ln g(x)\right)dx=\int_a^b(f(x)+f'(x)\ln g(x))dx$, q.e.d.

Proposition 5. If $a,b\in R,\ a<b$ and $f,g,h:R\to R$ are continue, such that

$f(a+b-x)=-f(x),g(a+b-x)=g(x),h(a+b-x)=-h(x),\forall x\in R$, then

$\int_a^b f(x)(arctgg(x))\ln\left(1+e^{h(x)}\right)dx=\frac{1}{2}\int_a^b f(x)h(x)arctgg(x)dx$.

Proof. $I=\int_a^b f(x)(arctgg(x))\ln\left(1+e^{h(x)}\right)dx$, where we make the changes

$x=u(t)=a+b-t,\ u'(t)=-1,u(a)=b,u(b)=a$, then

$$I=-\int_b^a f(a+b-t)(arctgg(a+b-t))\ln\left(1+e^{h(a+b-t)}\right)dt=$$

$$=-\int_a^b f(x)(arctgg(x))\ln\left(1+e^{-h(x)}\right)dx=-\int_a^b f(x)(arctgg(x))\ln\frac{1+e^{h(x)}}{e^{h(x)}}dx=$$

$$=-\int_a^b f(x)(arctgg(x))\ln\left(1+e^{h(x)}\right)dx+\int_a^b f(x)(arctgg(x))h(x)dx.$$

Hence, $2I = \int_a^b f(x)h(x)arctgg(x)dx$, q.e.d

Proposition 6. If $a, b \in R_+^*$ and $f : R \to R$ is continue and even, then

$$\int_{-a}^{a} \frac{f(x)}{b^2 + arctgx + \sqrt{b^4 + arctg^2 x}} dx = \frac{1}{b^2} \int_0^a f(x)dx .$$

Proof. Putting $x = u(t) = -t$, $u'(t) = -1, u(a) = -a, u(-a) = a$, then

$$I = \int_{-a}^{a} \frac{f(x)}{b^2 + arctgx + \sqrt{b^4 + arctg^2 x}} dx = \int_{a}^{-a} \frac{f(-t)}{b^2 - arctgt + \sqrt{b^4 + arctg^2 t}} (-1)dt =$$

$$= \int_{-a}^{a} \frac{f(t)}{b^2 - arctgt + \sqrt{b^4 + arctg^2 t}} dt .$$

So,

$$2I = I + I = \int_{-a}^{a} f(x) \left(\frac{1}{b^2 + arctgx + \sqrt{b^4 + arctg^2 x}} + \frac{1}{b^2 - arctgx + \sqrt{b^4 + arctg^2 x}} \right) dx =$$

$$= \int_{-a}^{a} f(x) \cdot \frac{2\left(b^2 + \sqrt{b^4 + arctg^2 x}\right)}{\left(b^4 + \sqrt{b^4 + arctg^2 x}\right)^2 - arctg^2 x} dx =$$

$$= \int_{-a}^{a} f(x) \cdot \frac{2\left(b^2 + \sqrt{b^4 + arctg^2 x}\right)}{2b^2\left(b^2 + \sqrt{b^4 + arctg^2 x}\right)} dx = \frac{1}{b^2} \int_{-a}^{a} f(x)dx = \frac{2}{b^2} \int_0^a f(x)dx ,$$

hence:

$$I = \frac{1}{b^2} \int_0^a f(x)dx .$$

Proposition 7. If $a \in R_+^*$ and $f, g, h : R \to R$ are continue and odd and $k : R \to (1, \infty)$ is continue and even, then

$$\int_{-a}^{a} f(x) \ln\left((k(x)^{g(x)} + (k(x))^{h(x)}\right) dx = \int_0^a f(x)\left(g(x) + h(x)\right) \ln k(x) dx .$$

Proof. Putting $x = u(t) = -t$, $u'(t) = -1, u(a) = -a, u(-a) = a$, then

$$I=\int_{-a}^{a} f(x)\ln\left((k(x)^{g(x)}+(k(x))^{h(x)}\right)dx=\int_{a}^{-a} f(-t)\ln\left((k(-t)^{g(-t)}+(k(-t))^{h(-t)}\right)(-1)dt=$$

$$=-\int_{-a}^{a} f(x)\ln\left((k(x)^{-g(x)}+(k(x))^{-h(x)}\right)dx=-\int_{-a}^{a} f(x)\ln\frac{(h(x))^{g(x)}+(k(x))^{h(x)}}{(k(x))^{g(x)+h(x)}}dx=$$

$$=-I+\int_{-a}^{a} f(x)\ln(k(x))^{g(x)+h(x)}dx=-I+\int_{-a}^{a} f(x)\big(g(x)+h(x)\big)\ln(k(x))dx,$$

so, $2I=\int_{-a}^{a} f(x)\big(g(x)+h(x)\big)\ln(k(x))dx$, and

$f(-x)\big(g(-x)+h(-x)\big)\ln(k(-x))=-f(x)\big(-g(x)-h(x)\big)\ln(k(x))=f(x)\big(g(x)+h(x)\big)\ln(k(x))$, i.e. $f(x)\big(g(x)+h(x)\big)\ln(k(x))$ is even. Therefore,

$$2I=2\int_{0}^{a} f(x)\big(g(x)+h(x)\big)\ln(k(x))dx, \text{ so } I=\int_{0}^{a} f(x)\big(g(x)+h(x)\big)\ln(k(x))dx,$$

Proposition 8. If $f:R\to R$ is continue such that $f(x)=f(1-x)$, $\forall x\in R$, then

$$\int_0^1 \frac{\sqrt{1-x}+\sqrt{x}}{1+\sqrt{2x}}f(x)dx=\sqrt{2}\cdot\int_0^1 f(x)dx.$$

Proof. Let $x=u(t)=1-t$, $u'(t)=-1, u(0)=1, u(1)=0$. Therefore,

$$I=\int_0^1 \frac{\sqrt{1-x}+\sqrt{x}}{1+\sqrt{2x}}f(x)dx=-\int_1^0 \frac{\sqrt{t}+\sqrt{1-t}}{1+\sqrt{2(1-t)}}f(1-t)dt=\int_0^1 \frac{\sqrt{t}+\sqrt{1-t}}{1+\sqrt{2(1-t)}}f(t)dt.$$

Hence,

$$2I=\int_0^1\left(\frac{\sqrt{x}+\sqrt{1-x}}{1+\sqrt{2x}}+\frac{\sqrt{x}+\sqrt{1-x}}{1+\sqrt{2(1-x)}}\right)f(x)dx=$$

$$=\int_0^1(\sqrt{x}+\sqrt{1-x})f(x)\left(\frac{1}{1+\sqrt{2x}}+\frac{1}{1+\sqrt{2(1-x)}}\right)dx=$$

$$=\int_0^1(\sqrt{x}+\sqrt{1-x})f(x)\cdot\frac{2+\sqrt{2x}+\sqrt{2(1-x)}}{1+\sqrt{2x}+\sqrt{2(1-x)}+2\sqrt{x(1-x)}}dx=$$

$$=\int_0^1 \frac{(\sqrt{x}+\sqrt{1-x})f(x)\sqrt{2}(\sqrt{2}+\sqrt{x}+\sqrt{1-x})}{\sqrt{2}(\sqrt{x}+\sqrt{1-x})+1+2\sqrt{x(1-x)}}dx=$$

$$=\int_0^1 \frac{(\sqrt{x}+\sqrt{1-x})f(x)\sqrt{2}(\sqrt{2}+\sqrt{x}+\sqrt{1-x})}{\sqrt{2}(\sqrt{x}+\sqrt{1-x})+(\sqrt{x}+\sqrt{1-x})^2}dx=$$

$$=\int_0^1 \frac{\sqrt{2}\left(\sqrt{2}+\sqrt{x}+\sqrt{1-x}\right)f(x)}{\sqrt{2}+\sqrt{x}+\sqrt{1-x}}dx=\sqrt{2}\cdot\int_0^1 f(x)dx.$$

Proposition 9. If $a,b\in R$, $c\in R-\{1\}$ and $f,g:R\to R$ are continue such that $f(a+b-x)=cf(x)$, $g(a+b-x)=-g(x)$, $\forall x\in R$, then

$$\int_a^b f(x)\ln\left(1+e^{g(x)}\right)dx=\frac{c}{c-1}\int_a^b f(x)g(x)dx.$$

Proof. $I=\int_a^b f(x)\ln\left(1+e^{g(x)}\right)dx$, $x=u(t)=a+b-t$, $u(a)=b, u(b)=a, u'(t)=-1$, then

$$I=\int_b^a f(a+b-t)\ln\left(1+e^{g(a+b-t)}\right)(-1)dt=\int_a^b cf(t)\ln\left(1+e^{-g(t)}\right)dt=$$

$$=c\cdot\int_a^b f(t)\ln\frac{1+e^{g(t)}}{e^{g(t)}}dt=c\cdot\int_a^b f(t)\ln\left(1+e^{g(t)}\right)dt-c\cdot\int_a^b f(t)\ln e^{g(t)}dt=$$

$$=cI-c\cdot\int_a^b f(x)g(x)dx\Leftrightarrow(1-c)I=-c\cdot\int_a^b f(x)g(x)dx\Leftrightarrow$$

$$\Leftrightarrow(c-1)I=c\cdot\int_a^b f(x)g(x)dx\Leftrightarrow I=\frac{c}{c-1}\int_a^b f(x)g(x)dx.$$

Proposition 10. If $a,\ m\in R_+^*$ şi $f:R\to R$ is continue and odd, then

$$\int_{\frac{1}{a}}^{a}\frac{f\left(\ln^{2n+1}x\right)}{\left(1+x^{2m}\right)^{\frac{1}{m}}}dx=0.$$

Proof. Putting $x=u(t)=\frac{1}{t}$, $u'(t)=-\frac{1}{t^2}$, $u(a)=\frac{1}{a}$, $u\left(\frac{1}{a}\right)=a$ we obtain

$$I=\int_{\frac{1}{a}}^{a}\frac{f\left(\ln^{2n+1}x\right)}{\left(1+x^{2m}\right)^{\frac{1}{m}}}dx=\int_{a}^{\frac{1}{a}}\frac{f\left(-\ln^{2n+1}t\right)}{\left(1+\frac{1}{t^{2m}}\right)^{\frac{1}{m}}}\left(-\frac{1}{t^2}\right)dt=$$

$$=\int_{\frac{1}{a}}^{a}\frac{f\left(-\ln^{2n+1}t\right)}{\frac{\left(1+t^{2m}\right)^{\frac{1}{m}}}{t^2}}\cdot\frac{1}{t^2}dt=-\int_{\frac{1}{a}}^{a}\frac{f\left(\ln^{2n+1}t\right)}{\left(1+t^{2m}\right)^{\frac{1}{m}}}dt=-I\ ,$$

Hence, $I=0$.

References:

[*] Romanian Mathematical Magazine-www.ssmrmh.ro