Elliptic Integral Inequality

Introduction

In this article, we will find a simple inequality for the complete elliptic integral of the second kind.

Prerequisite

Definition 1

If $|k| < 1$, then,

$$E(k) = \int_0^1 \sqrt{\frac{1-k^2t^2}{1-t^2}}dt$$

where, $E(k)$ is the complete elliptic integral of the second kind and k is the modulus.

Lemma 1

The perimeter P of an ellipse is given by,

$$P = 4aE(e)$$

where a is the length of its semi-major axis and e is its eccentricity.
Proof: It is known from calculus that, the length L of a curve $y = f(x)$ between $x = x_1$ and $x = x_2$ is given by,

$$L = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx$$

substituting $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the above equation and choosing the limits as $x_1 = 0$ and $x_2 = a$, we obtain,

$$L = \int_0^a \sqrt{1 + \left(\frac{-bx}{a\sqrt{a^2-x^2}}\right)^2}dx = \int_0^a \sqrt{\frac{a^2-(1-b^2/a^2)x^2}{a^2-x^2}}dx$$

let $e^2 = 1 - b^2/a^2$, then we have the following equation,

$$L = \int_0^a \sqrt{\frac{a^2 - e^2x^2}{a^2 - x^2}} dx \overset{x=at}{=} a \int_0^1 \sqrt{\frac{1 - e^2t^2}{1 - t^2}} dt$$

using definition 1, we finally have,

$$L = aE(e)$$

Observe that, L is the length of the ellipse which lies in the first quadrant, therefore the perimeter P of the entire ellipse is given by,

$$P = 4aE(e)$$

which completes the proof of lemma 1.

The Inequality

Theorem: If $n \in \mathbb{N}$, $|k| < 1$ and $k^2 + k'^2 = 1$, then we have the following result,

$$E(k) > \frac{1}{\sqrt{n}} \sum_{m=0}^{n-1} \sqrt{(2m + 1 - 2\sqrt{m^2 + m}) + k'^2(2(n-m) - 1 - 2\sqrt{(n-m)^2 - (n-m)})}$$

Proof: Consider an ellipse of semi-major axis a and semi-minor axis b centered at the origin. From lemma 1, we have,

$$P = 4aE(e)$$

Let us find a lower bound for the length of the part of the ellipse lying in the first quadrant, that is, $aE(e)$.

Observe that, $aE(e)$ is strictly greater than the length of the line joining the points $A(0, b)$ and $B(a, 0)$, thus,

$$aE(e) > \sqrt{a^2 + b^2} \implies E(e) > \sqrt{2 - e^2}$$

Now consider the point $A_1(a/\sqrt{2}, b/\sqrt{2})$ which lies on this ellipse. It is clear from geometry that,

$$aE(e) > \overline{AA_1} + \overline{A_1B} = \sqrt{a^2/2 + (b - b/\sqrt{2})^2} + \sqrt{(a - a/\sqrt{2})^2 + b^2/2}$$

after some simplification, we obtain,

$$E(e) > \sqrt{1/2 + (1-e^2)(1-1/\sqrt{2})^2} + \sqrt{(1-1/\sqrt{2})^2 + (1-e^2)/2}$$

Continuing the above procedure, we obtain $(n-1)$ points (between the points A and B) on the ellipse dividing it into n parts. Thus the inequality becomes,

$$aE(e) > \overline{AA_1} + \overline{A_1A_2} + \overline{A_2A_3} + \ldots + \overline{A_{n-1}B}$$

where,

$$A_m := \left(\frac{a\sqrt{m}}{\sqrt{n}}, \frac{b\sqrt{n-m}}{\sqrt{n}}\right)$$

and

$$\overline{A_mA_{m+1}} = \sqrt{\frac{a^2}{n}(2m+1-2\sqrt{m^2+m}) + \frac{b^2}{n}(2(n-m)-1-2\sqrt{(n-m)^2-(n-m)})}$$

where, $0 \leq m < n$, $A_0 := A$ and $A_n := B$.
Therefore,

$$aE(e) > \sum_{m=0}^{n-1} \sqrt{\frac{a^2}{n}(2m+1-2\sqrt{m^2+m}) + \frac{b^2}{n}(2(n-m)-1-2\sqrt{(n-m)^2-(n-m)})}$$

hence

$$E(e) > \frac{1}{\sqrt{n}} \sum_{m=0}^{n-1} \sqrt{(2m+1-2\sqrt{m^2+m}) + (1-e^2)(2(n-m)-1-2\sqrt{(n-m)^2-(n-m)})}$$

which completes the proof.

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