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ROMANIAN MATHEMATICAL MAGAZINE
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GAKOPOULOS' LEMMAS and THEOREMS

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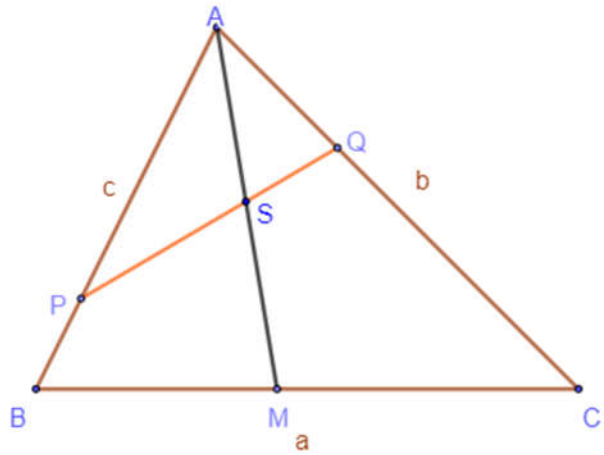
Introduction. *I have compiled into an article some basic exercises then I have created, proved and published in the past.*

These exercises have been used by me and other team members as auxiliary exercises-lemmas-theorems to solve geometric problems. I think it is useful.

<u>1. GAKOPOULOS' LEMMA or THEOREM:</u>



$$\frac{PS}{SQ}=\frac{BM}{MC}\cdot\frac{AP}{AB}\cdot\frac{AC}{AQ}$$

Or

$$\frac{PS}{SQ}\cdot\frac{QA}{AP}=\frac{BM}{MC}\cdot\frac{CA}{AB}$$

Proof.

$*\ \Delta PQR,\overline{AST}-(\text{Menelaus th.}):$

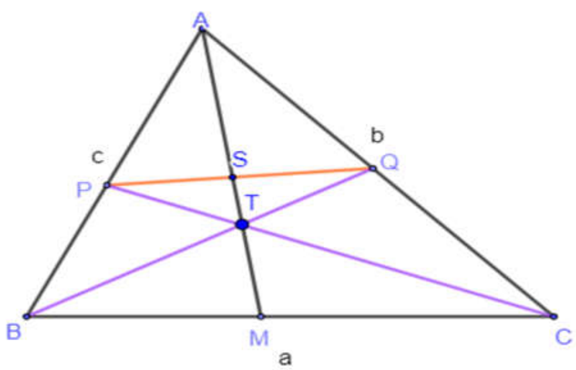
$$\frac{SP}{SQ}\cdot\frac{TR}{TP}\cdot\frac{AQ}{AR}=1;\quad (1)$$

$$*\ PR\parallel BC\Rightarrow\frac{AP}{AR}=\frac{AB}{AC}\Rightarrow AR=AP\cdot AB\cdot AC;\quad (2)$$

$$\text{From (1)and (2)}\Rightarrow\frac{SP}{SQ}\cdot\frac{MC}{MB}\cdot\frac{AQ}{\frac{AP}{AB}\cdot AC}=1\Rightarrow\frac{PS}{PQ}=\frac{BM}{MC}\cdot\frac{AP}{AB}\cdot\frac{AC}{AQ}\text{ or}$$

$$\frac{PS}{SQ}\cdot\frac{QA}{AC}=\frac{BM}{MC}\cdot\frac{CA}{AB}$$

Application 1.



$$\frac{PS}{SQ}=\frac{PB}{QC}\cdot\frac{AC}{AB}$$

Proof.

$$\begin{cases}\dfrac{PS}{SQ}=\dfrac{BM}{MC}\cdot\dfrac{AP}{AB}\cdot\dfrac{AC}{AQ}\ (Gakopoulos\ th.)\\ \dfrac{BM}{MC}\cdot\dfrac{CQ}{QA}\cdot\dfrac{AP}{PB}=1\ (Ceva\ th.)\end{cases}$$

$$\frac{PS}{SQ}=\frac{AQ}{CQ}\cdot\frac{PB}{AP}\cdot\frac{AP}{AB}\cdot\frac{AC}{AQ}\text{ or}$$

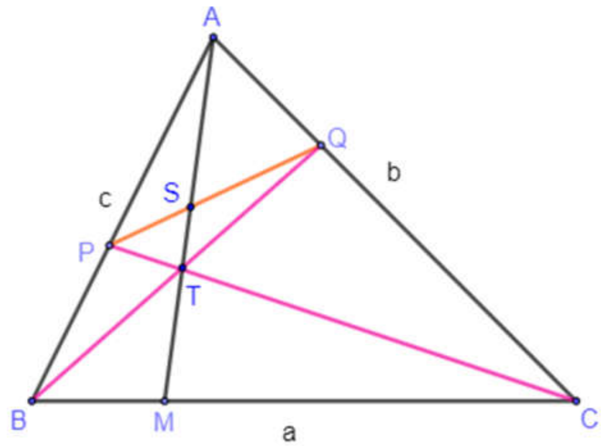
$$\frac{PS}{SQ}=\frac{PB}{QC}\cdot\frac{AC}{AB}$$

Application 2.

$$\boldsymbol{AB=AC;\frac{PS}{SQ}=\frac{PB}{QC}}$$

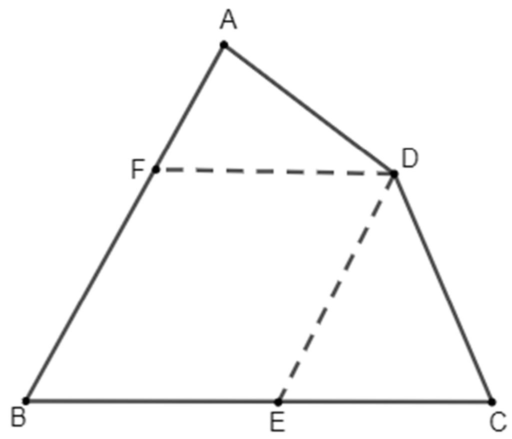
Proof.

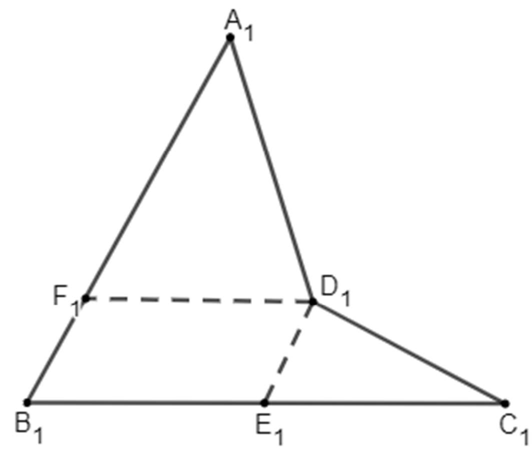
$$\frac{PS}{SQ}=\frac{PB}{QC}\cdot\frac{AC}{AB}\overset{AB=AC}{\Longrightarrow}\frac{PS}{SQ}=\frac{PB}{QC}$$



2. GAKOPOULOS-BLATSIS formulae:

$\boldsymbol{DE\parallel AB, DF\parallel BC}$ $\boldsymbol{D_1E_1\parallel A_1B_1, D_1F_1\parallel B_1CC_1}$





$$[ABCD]=\frac{\sin B}{2}(BC\cdot BF+BA+BE);\ [A_1B_1C_1D_1]=\frac{\sin B}{2}(B_1C_1\cdot B_1F_1+B_1A_1+B_1E_1)$$

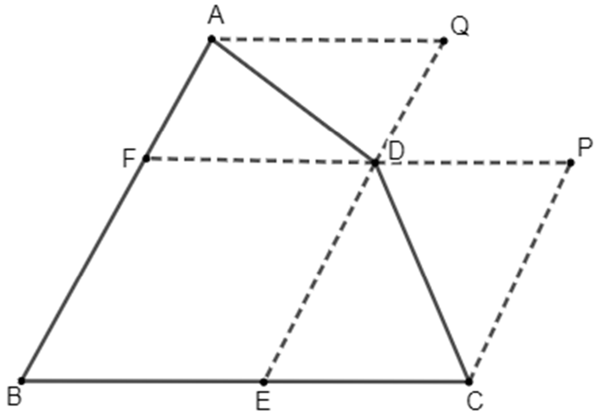
Proof.

$2[ABCD] = 2[BFDE] + 2[ECD] + 2[ADF]$

$= [BFDE] + [DPCE] + [BFDE] + [AQDF]$

$= [BCPF] + [ABEQ] =$

$$= \frac{\sin B}{2}(BC \cdot BF) + \frac{\sin B}{2}(BA \cdot BE)$$



Hence,

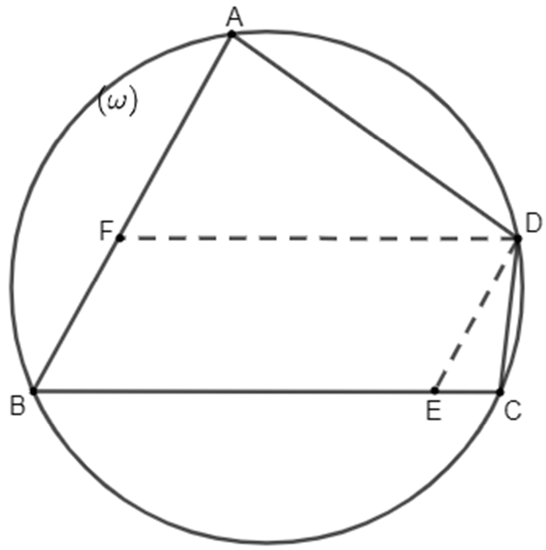
$$[ABCD] = \frac{\sin B}{2}(BC \cdot BF + BA + BE);$$

3. $NCCQ$1-NEW CRITERION FOR CYCLIC QUADRILATERAL-(1)

$$DE \parallel AB; \ \ FD \parallel BC$$

***ABCD* –is cyclic⇔**

$$BD^2 = BC \cdot BE + BA \cdot BF$$



Proof.

Let $BC = a, BA = c, BE = d_1$,

$BF = d_2$

(ω) –circumcircle of ΔABC.

Plagiognal system: $BC \equiv BX, BA \equiv By$

$B(0,0), C(a,0), A(0,c), D(d_1, d_2)$

$$(\omega): x^2 + y^2 + 2xy \cdot \cos B - ax - cy = 0; \ \ (1)$$

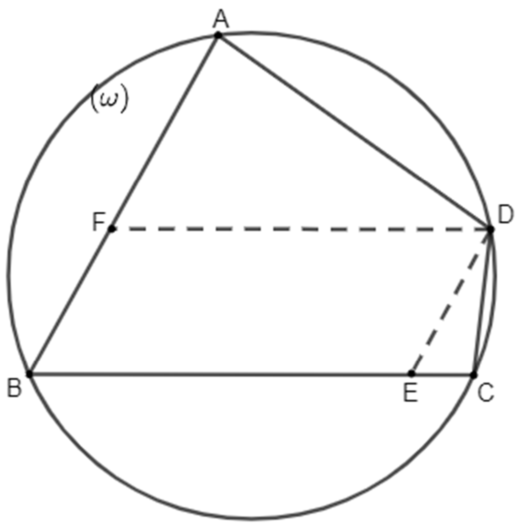
$$BD^2 = d_1^2 + d_2^2 + 2d_1 d_2 \cdot \cos B; \ \ (2)$$

$ABCD$ –is cyclic⇔ $D \in (\omega) \Leftrightarrow d_1^2 + d_2^2 + 2d_1 d_2 \cdot \cos B - ad_1 - cd_2 = 0 \Leftrightarrow$

$$BD^2 = BC \cdot BE + BA \cdot BF$$

4. $NCCQ$2-NEW CRITERION FOR CYCLIC QUADRILATERAL-(2)

$$DE \parallel AB; \ \ FD \parallel BC; \ ABCD - \text{is cyclic} \Leftrightarrow \cos B = \frac{1}{2}\left(\frac{EC}{BF} + \frac{FA}{BE}\right)$$



Proof. $BD^2 = BC \cdot BE + BA \cdot BF \ (NCCQ1); \quad (1)$

$$BD^2 = BE^2 + BF^2 + 2BE \cdot BF \cdot \cos B \,; (2)$$

$$ABCD - \text{cyclic} \overset{(1)}{\Rightarrow} BD^2 = BC \cdot BE + BA \cdot BF \overset{(2)}{\Rightarrow}$$

$$BE^2 + BF^2 + 2BE \cdot BF \cos B = BC \cdot BE + BA \cdot BF$$

$$2BE \cdot BF \cos B = BC \cdot BE - BE^2 + BA \cdot BF - BF^2$$

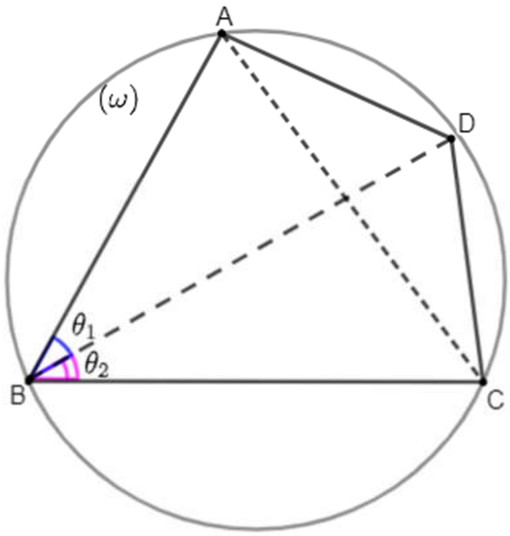
$$2BE \cdot BF \cos B = BE(BC - BE) + BF(BA - BF)$$

$$2BE \cdot BF \cos B = BE \cdot EC + BF \cdot FA$$

5. $NCCQ3$-NEW CRITERION FOR CYCLIC QUADRILATERAL-(3)

$$\boldsymbol{ABCD - \text{cyclic} \Leftrightarrow}$$

$$\boldsymbol{BD = \frac{BA \cdot \sin\theta_1 + BC \sin\theta_2}{\sin(\theta_1 + \theta_2)}}$$



Proof.(by Mansur Mansurov)

$$AD = 2R\sin\theta_2 \,; (1) \quad CD = 2R\sin\theta_1 \,; (2)$$

$$AC = 2R\sin(\theta_1 + \theta_2) \,; \quad (3)$$

$$ABCD - \text{cyclic} \Leftrightarrow$$

$$BC \cdot 2R\sin\theta_2 + BA \cdot 2R\sin\theta_1 = BD \cdot 2R\sin(\theta_1 + \theta_2)$$

$$\Leftrightarrow BD = \frac{BA \cdot \sin\theta_1 + BC\sin\theta_2}{\sin(\theta_1 + \theta_2)}$$

6. $NCCQ4$-NEW CRITERION FOR CYCLIC QUADRILATERAL-(4)

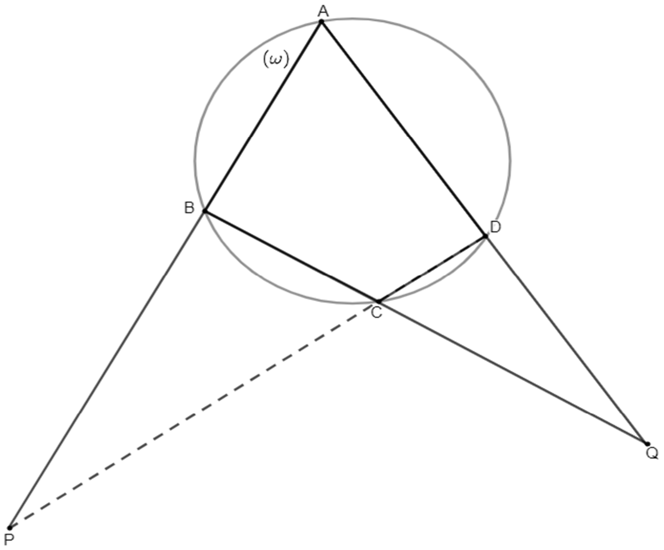
$$\boldsymbol{ABCD - \text{cyclic} \Leftrightarrow \cos A = \frac{1}{2} \cdot \frac{AB \cdot AP + AD \cdot AQ}{AP \cdot AQ}}$$

Proof. Let $AB = b, AP = p, AD = d, AQ = q$

Plagiogonal system: $AB \equiv Ax, AD \equiv Ay$

$$A(0,0), B(b,0), P(p,0), D(0,d), Q(0,q), C(c_1, c_2)$$

$$\begin{cases} BQ: \dfrac{x}{b}+\dfrac{y}{q}=1 \\ PD: \dfrac{x}{p}+\dfrac{y}{d}=1 \end{cases} \Rightarrow \begin{cases} x=c_1=\dfrac{bp(q-d)}{pq-bd} \\ y=c_2=\dfrac{qd(p-d)}{pq-bd} \end{cases}; (1)$$



From $NCCQ1$, we have $ABCD$ is cyclic$\Leftrightarrow AC^2=AB\cdot c_1+AD\cdot c_2 \Leftrightarrow$

$$c_1^2+c_2^2+2c_1c_2\cos A=bc_1+dc_2 \Leftrightarrow \cos A=\frac{1}{2}\cdot\frac{bp+dq}{pq}$$

$$\Leftrightarrow \cos A=\frac{1}{2}\cdot\frac{AB\cdot AP+AD\cdot AQ}{AP\cdot AQ}$$

7. AREA OF CYCLIC QUADRILATERAL

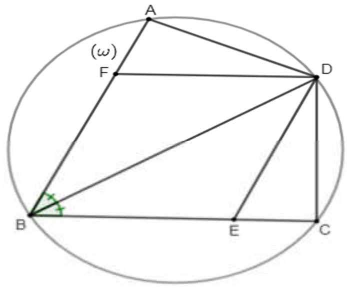
$$[ABCD]=BD^2\cdot\frac{\sin B}{2}$$

Proof. $BD^2=BE\cdot BC+BF\cdot BA$(by $NCCQ1$)

$$[ABCD]=\frac{\sin B}{2}(BC\cdot BF+BA\cdot BE); (Gakopoulos-Blatsis\ formulae)$$

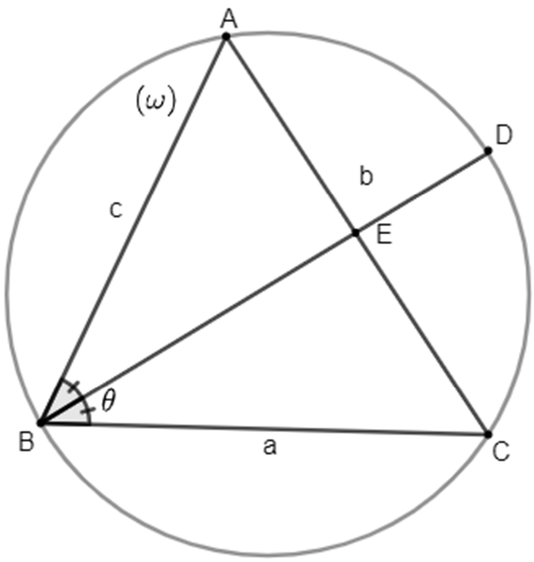
$$BE=BF\Rightarrow\frac{[ABCD]}{BD^2}=\frac{\sin B}{2}$$

$$[ABCD]=BD^2\cdot\frac{\sin B}{2}$$



8. LENGTH OF ANGLE BISECTOR OF TRIANGLE

$$BE = \frac{BA \cdot BC}{BD}$$



Proof. From $NCCQ3$, we have:

$$BD = \frac{BA \cdot \sin\theta + BC \sin\theta}{\sin(2\theta)} =$$

$$= \frac{BA + BC}{2\cos\theta}; (1)$$

$$BE = \frac{2BA \cdot BC}{BA + BC}\cos\theta$$

$$\frac{BA + BC}{2\cos\theta} = \frac{BA \cdot BC}{BE}; (2)$$

From (1) and (2): $BE = \frac{BA \cdot BC}{BD}$

9. CIRCUMRADIUS OF TRIANGLE

$$R^2 = \frac{OP^2 + OQ^2 - PQ^2}{2}$$

Proof. From $NCCQ4$, we have:

$$\cos A = \frac{1}{2} \cdot \frac{AB \cdot AP + AD \cdot AQ}{AP \cdot AQ}; (1)$$

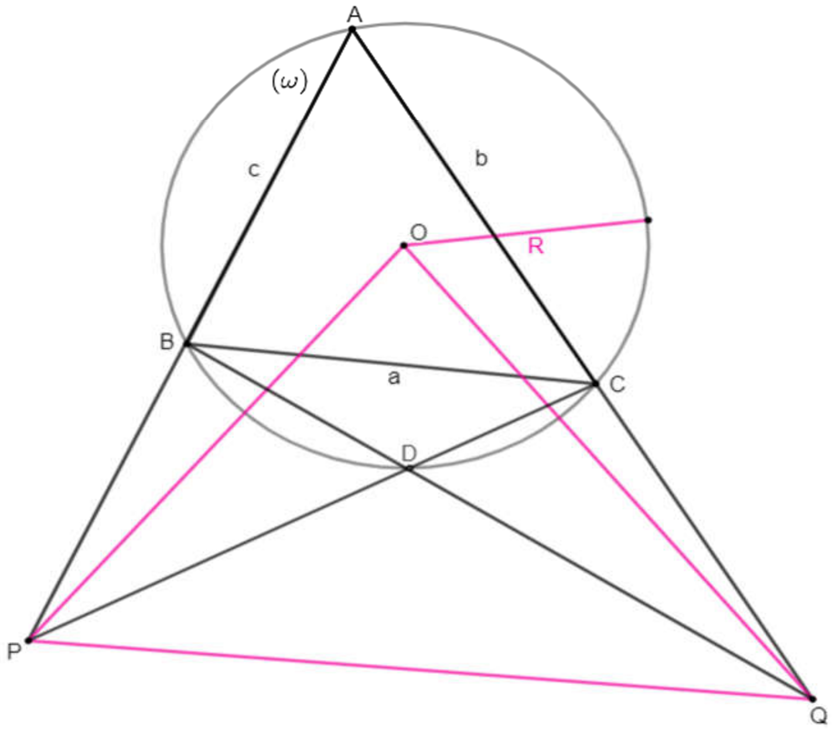
$$\cos A = \frac{AP^2 + AQ^2 - PQ^2}{2AP \cdot PQ}; (2)$$

From (1) and (2), we get: $AP^2 + AQ^2 - PQ^2 = AB \cdot AP + AC \cdot AQ \Leftrightarrow$

$$AP(AP - AB) + AQ(AQ - AC) - PQ^2 = 0 \Leftrightarrow$$

$$AP \cdot PB + AQ \cdot QC - PQ^2 = 0 \Leftrightarrow$$

$$OP^2 - R^2 + OQ^2 - R^2 - PQ^2 = 0 \Leftrightarrow R^2 = \frac{OP^2 + OQ^2 - PQ^2}{2}$$



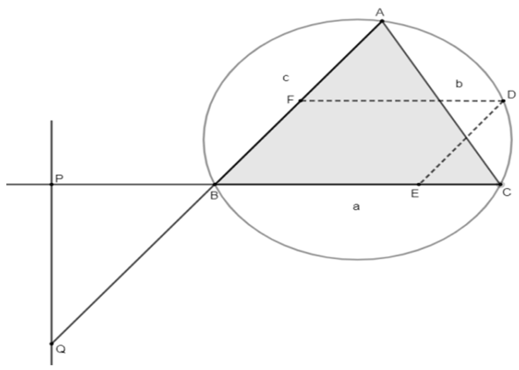
10. PERPENDICULAR CRITERION

$$D \in (\widehat{AC}), DE \parallel AB, \quad DF \parallel BC$$

$$PQ \perp BC \Leftrightarrow \frac{EF}{BF} + \frac{FA}{BE} = 2 \cdot \frac{BP}{PQ}$$

Proof. From $NCCQ2$, we have: $\frac{1}{2} \cdot \frac{EC}{BF} + \frac{FA}{BE} = \cos B$; (1)

$$PQ \perp BC \Leftrightarrow \cos B = \frac{BP}{BQ} \Rightarrow \frac{EC}{BF} + \frac{FA}{BE} = 2 \cdot \frac{BP}{BQ}$$



11. LINE PASSES THROUGH THE INCENTER OF TRIANGLE

$$\boldsymbol{D, I, F - \text{collinear} \Leftrightarrow \left(\frac{1}{BD}-\frac{1}{BC}\right)+\left(\frac{1}{BF}-\frac{1}{BA}\right)=\frac{AC}{BC\cdot BA}}$$

Proof.

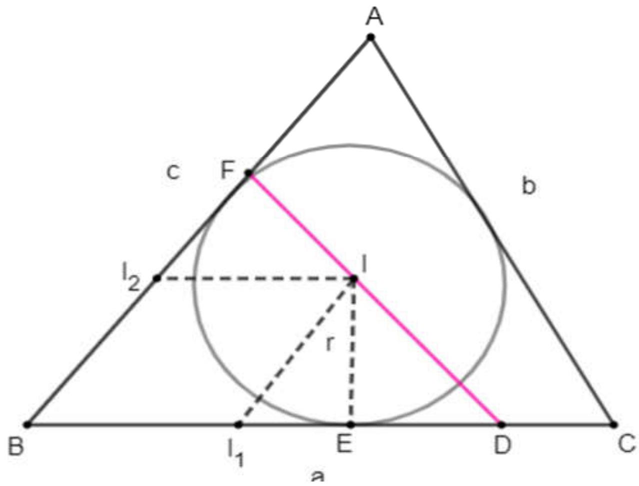
$$BI_1 = BI_2 = II_1 = II_2 = i$$

$$\sphericalangle II_1C = B,$$

$$\Delta IEI_1: \sin(II_1E) = \frac{r}{i}$$

$$i = \frac{F}{s}\cdot\frac{1}{\sin B} = \frac{ac\sin B\cdot 1/2}{\frac{a+b+c}{2}\sin B}$$

$$i = \frac{ac}{a+b+c}; (1)$$



$$\begin{cases}\Delta DII_1 \sim \Delta DBF\\ \Delta FII_2 \sim \Delta FDB\end{cases} \Rightarrow \begin{cases}\dfrac{DI}{DF} = \dfrac{i}{BF}\\ \dfrac{FI}{FD} = \dfrac{i}{BD}\end{cases}$$

$$\Rightarrow \frac{i}{BF} + \frac{i}{BD}$$

$$= 1; (2)$$

$$D, I, F - \text{collinear} \Leftrightarrow \frac{i}{BF} + \frac{i}{BD} = 1 \Leftrightarrow \frac{1}{BF} + \frac{1}{BD} = \frac{1}{i}$$

$$\frac{1}{BF} + \frac{1}{BD} = \frac{AB+BC+CA}{BC\cdot BA} \Leftrightarrow \frac{1}{BD} + \frac{1}{BF} = \frac{1}{BC} + \frac{1}{BA} + \frac{AC}{BC\cdot BA} \Leftrightarrow$$

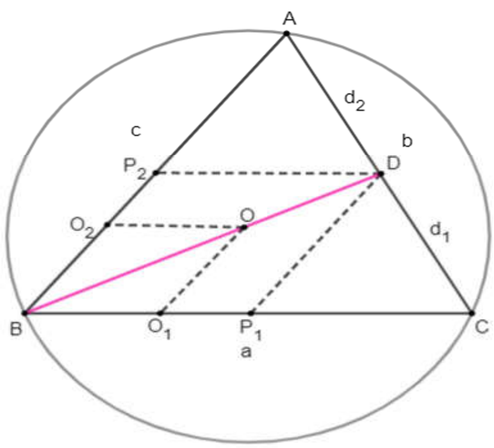
$$\left(\frac{1}{BD}-\frac{1}{BC}\right)+\left(\frac{1}{BF}-\frac{1}{BA}\right)=\frac{AC}{BC\cdot BA}$$

12. LINE PASSES THROW CIRCUMCENTER OF TRIANGLE

$\boldsymbol{B, O, D}$ **–collinear**$\Leftrightarrow$

$$\boldsymbol{\frac{AD}{DC} = \frac{c(a - c\cos B)}{a(c - a\cos B)}} \Leftrightarrow$$

$$\boldsymbol{\cos B = \frac{ac(AD - DC)}{a^2AD - c^2DC}}$$



Proof.

Let $AD = d_2, DC = d_1$. From Plagiogonal system theory, we have:

$$BD_1 = \frac{ad_2}{d_1 + d_2}$$

$$BD_2 = \frac{cd_1}{d_1 + d_2}$$

$$BO_1 = \frac{a - c\cos B}{2\sin^2 B}$$

$$BO_2 = \frac{c - a\cos B}{2\sin^2 B}$$

$$B, O, D - \text{collinear} \Leftrightarrow \frac{BD_1}{BD_2} = \frac{BO_1}{BO_2} \Leftrightarrow \frac{ad_2}{cd_1} = \frac{a - c\cos B}{c - a\cos B} \Leftrightarrow$$

$$\frac{d_2}{d_1} = \frac{AD}{DC} = c(a - c\cos B)/a(c - a\cos B) \Leftrightarrow \cos B = \frac{ac(AD - DC)}{a^2AD - c^2DC}$$

REFERENCES: ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro