# HOJOO LEE'S INEQUALITY REVISITED

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**Abstract:** *In this paper are presented an revisited form of Hojoo Lee's inequality and generalization.*

**Proposition 1.**
**If $u, v, x > 0$ then:**

$$\boldsymbol{(u^2+x)(v^2+x) \geq \frac{3}{4}x((u+v)^2+x);} \qquad \mathbf{(1)}$$

*Proof.* We have:

$$(u^2+x)(v^2+x) \geq \frac{3}{4}x((u+v)^2+x) \Leftrightarrow$$
$$4u^2v^2 + 4x(u^2+v^2) + 4^2 \geq 3x(u^2+v^2) + 6xuv + 3x^2 \Leftrightarrow$$
$$4u^2v^2 - 4xuv + x^2 + x(u^2+v^2-2uv) \geq 0 \Leftrightarrow$$
$$(2uv-x)^2 + x(u-v)^2 \geq 0$$

Equality holds for $2uv = x$ and $u = v = \sqrt{\frac{x}{2}}$.

**Proposition 2.**
**If $v, w, x > 0$ then:**

$$\boldsymbol{(v^2+x)(w^2+x) \geq x(v+w)^2;} \qquad \mathbf{(2)}$$

*Proof.* We have:

$$(v^2+x)(w^2+x) \geq x(v+w)^2 \Leftrightarrow$$
$$v^2w^2 + x(v^2+w^2) + x^2 \geq x(v^2+w^2) + 2xvw \Leftrightarrow$$
$$v^2w^2 - 2xvw + x^2 \geq 0 \Leftrightarrow (vw-x)^2 \geq 0$$

Equality holds for $vw = x$.

**Proposition 3.**
**If $u, v, w, x > 0$ then:**

$$\boldsymbol{(u^2+x)(v^2+x)(w^2+x) \geq \frac{3}{4}x^2(u+v+w)^2;} \qquad \mathbf{(3)}$$

*Proof.* We have:

$$(u^2+x)(v^2+x)(w^2+x) \overset{(1)}{\geq} \frac{3}{4}x((u+v)^2+x)(w^2+x) \overset{(2)}{\geq}$$
$$\geq \frac{3}{4}x^2((u+v)+w)^2 = \frac{3}{4}x^2(u+v+w)^2 \geq \frac{9}{4}x^2(uv+vw+wu)$$

**Theorem.**
**If $a, b, c, m, n, t, s > 0$ and $x, y, x+y \geq 0$, then:**

$$\boldsymbol{x(ma^2+t)(mb^2+t)(mc^2+t) + y(na^2+s)(nb^2+s)(nc^2+s) \geq}$$

$$\geq \frac{3}{4}(mxt^2+nys^2)(a+b+c)^2 \geq \frac{9}{4}(mxt^2+nys^2)(ab+bc+ca); \qquad (4)$$

*Proof.* In relation (3), taking $u = a\sqrt{m}; v = b\sqrt{m}; w = c\sqrt{m}$ and $x = t$ then:

$$(ma^2+t)(mb^2+t)(mc^2+t) \geq \frac{3}{4}t^2(a\sqrt{m}+b\sqrt{m}+c\sqrt{m})^2 =$$
$$= \frac{3}{4}mt^2(a+b+c)^2 \geq \frac{9}{4}mt^2(ab+bc+ca); \qquad (5)$$

Analogous, in (3) taking $u = a\sqrt{n}; v = b\sqrt{n}; w = c\sqrt{n}$ and $y = s$, we get:

$$(na^2+s)(nb^2+s)(nc^2+s) \geq \frac{3}{4}s^2(a\sqrt{n}+b\sqrt{n}+c\sqrt{n})^2 =$$
$$= \frac{3}{4}ns^2(a+b+c)^2 \geq \frac{9}{4}ns^2(ab+bc+ca); \qquad (6)$$

From (5) and (6) we obtain:

$$x(ma^2+t)(mb^2+t)(mc^2+t)+y(na^2+s)(nb^2+s)(nc^2+s) \geq$$
$$\geq \frac{3}{4}mxt^2(a+b+c)^2+\frac{3}{4}nys^2(a+b+c)^2 =$$
$$= \frac{3}{4}(mxt^2+nys^2)(a+b+c)^2 \geq \frac{9}{4}(mxt^2+nys^2)(ab+bc+ca)$$

If in (4) we take $t = 2, s = 1, m = n = 1$ and $x = 1, y = 4$ we obtain:

$$(a^2+2)(b^2+2)(c^2+2)+4(a^2+1)(b^2+1)(c^2+1) \geq$$
$$\geq \frac{3}{4}\cdot 4(a^2+2)(b^2+2)(c^2+2)+\frac{3}{4}\cdot 4(a^2+1)(b^2+1)(c^2+1) \geq$$
$$= \frac{3}{4}(1\cdot 1\cdot 4+1\cdot 4\cdot 1)(a+b+c)^2 = 6(a+b+c)^2 \geq 18(ab+bc+ca) \Leftrightarrow$$
$$(a^2+2)(b^2+2)(c^2+2)+4(a^2+1)(b^2+1)(c^2+1) \geq$$
$$\geq 6(a+b+c)^2 \geq 18(ab+bc+ca)$$

i.e. the proposed problem 3326($a$) by Mihály Bencze from Crux Mathematicorum Magazine.
If in (4) we take $y = 0, m = 1, x = 1$ we get:

$$(a^2+t)(b^2+t)(c^2+t) \geq \frac{3}{4}t^2(a+b+c)^2 \geq \frac{3}{4}t^2(ab+bc+ca); \qquad (7)$$

from which taking $t \to t^2$, results:

$$(a^2+t^2)(b^2+t^2)(c^2+t^2) \geq \frac{3}{4}t^4(a+b+c)^4 \geq \frac{9}{4}t^4(ab+bc+ca); \qquad (8)$$

i.e. Arkady M. Alt inequality published in [1].
If in (7) we take $t = 2$, we obtain:

$$(a^2+2)(b^2+2)(c^2+2) \geq 3(a+b+c)^2 \geq 9(ab+bc+ca); \qquad (9)$$

i.e. Hojoo Lee's inequality proposed to APMO, 2004.

## References

[1] lt M. Arkady-ABOUT ONE INEQUALITY FROM APMO, 2004-NEW SOLUTION AND GENERALIZATIONS, Octogon Mathematical Magazine, Vol.2,No.1, April 2019, pages 228-232