NEW BĂTINEŢU - PEDOE - BOTTEMA - TSINTSIFAS TYPE INEQUALITIES IN TRIANGLE

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Abstract. In this paper we present some New Bătineţu - Tsintsifas type inequalities in triangle.

Inequality 1. In any triangle ABC with area F, usual notations and $M \in (BC), N \in (CA), P \in (AB)$ holds the inequality:

$$a^3(BN+CP)+b^3(CP+AM)+c^3(AM+BN) \geq 16\sqrt{3}\cdot F^2$$

Proof.

$$AM \geq h_a, BN \geq h_b, CP \geq h_c \Rightarrow \sum a^3(BN+CP) \geq \sum a^3(h_b+h_c) =$$

$$= \sum a^3 h_a\left(\frac{h_b+h_c}{h_a}\right) = 2F\sum \frac{h_b+h_c}{h_a}a^2 \overset{\text{Bătineţu-Giurgiu}}{\geq} 2F\cdot 8\sqrt{3}F = 16\sqrt{3}\cdot F^2$$

□

Inequality 2. If $x, y, z > 0$, then in any triangle ABC with usual notations i.e. $a = BC, b = CA, c = AB, R$ circumradius, r inradius, r_a, r_b, r_c exradii, s the semiperimeter and F the area of the triangle, the following inequalities are true:

$$\frac{v+w}{u}\cdot bc+\frac{w+u}{v}\cdot ca+\frac{u+v}{w}\cdot ab \geq 8\sqrt{3}\cdot F \tag{1}$$

$$\frac{v+w}{u}\cdot r_a+\frac{w+u}{v}\cdot r_b+\frac{u+v}{w}\cdot r_c \geq 6\cdot\sqrt[3]{sF} \tag{2}$$

$$\frac{u+w}{u}\cdot r_a^2+\frac{w+u}{v}\cdot r_b^2+\frac{u+v}{w}\cdot r_c^2 \geq 6\cdot\sqrt{3}\cdot F \tag{3}$$

Proof.

$$\sum_{cyc}\frac{v+w}{u}bc \overset{\text{AM-GM}}{\geq} 2\sum_{cyc}\frac{\sqrt{vw}}{u}bc \overset{\text{AM-GM}}{\geq} 2\cdot 3\cdot\sqrt[3]{\prod_{cyc}\left(\frac{\sqrt{vw}}{u}bc\right)} = 6\cdot\sqrt[3]{(abc)^2} \overset{\text{Carlitz}}{\geq}$$

$$\overset{\text{Carlitz}}{\geq} 6\cdot\frac{4}{\sqrt{3}}\cdot F = 8\sqrt{3}\cdot F$$

$$\sum_{cyc}\frac{v+w}{u}r_a \overset{\text{AM-GM}}{\geq} 2\sum_{cyc}\frac{\sqrt{vw}}{u}r_a \overset{\text{AM-GM}}{\geq} 2\cdot 3\cdot\sqrt[3]{\prod_{cyc}\left(\frac{\sqrt{vw}}{u}r_a\right)} = 6\cdot\sqrt[3]{r_a r_b r_c} =$$

$$= 6\cdot\sqrt[3]{sF}$$

We denote $U^2 = u, V^2 = v, W^2 = w$, then:

$$\sum_{cyc} \frac{v+w}{u} \cdot r_a^2 = \sum_{cyc} \frac{V^2+W^2}{U^2} \cdot r_a^2 \overset{\text{Bergström}}{\geq} \frac{1}{2} \cdot \sum_{cyc} \Big(\frac{V+W}{U} \cdot r_a \Big)^2 \overset{\text{Bergström}}{\geq}$$

$$\overset{\text{Bergström}}{\geq} \frac{1}{2} \cdot \frac{1}{3} \Big(\sum_{cyc} \frac{V+W}{U} \cdot r_a \Big)^2 \overset{(2)}{\geq} \frac{1}{2 \cdot 3} \cdot (6 \cdot \sqrt[3]{sF})^2 = 6 \cdot \sqrt[3]{s^2F^2} = 6 \cdot \sqrt[3]{s \cdot s \cdot F^2} \overset{\text{Mitrinovic}}{\geq}$$

$$\overset{\text{Mitrinovic}}{\geq} 6 \cdot \sqrt[3]{s \cdot (3\sqrt{3}r)F^2} = 6 \cdot \sqrt{3} \cdot F$$

□

Inequality 3. If $x, y, z > 0$, then in any triangle ABC with usual notations i.e. $a = BC, B = CA, c = AB, s$ the semiperimeter and F the area of triangle, the following inequalities are true:

$$(1) \qquad \frac{y+z}{x} \cdot a + \frac{z+x}{y} \cdot b + \frac{x+y}{z} \cdot c \geq 4 \cdot \sqrt[4]{27} \cdot \sqrt{F}$$

$$(2) \qquad \frac{y+z}{x} \cdot a^2 + \frac{z+x}{y} \cdot b^2 + \frac{x+y}{z} \cdot c^2 \geq 8\sqrt{3} \cdot F$$

$$(3) \qquad \frac{y+z}{x} \cdot a^3 + \frac{z+x}{y} \cdot b^3 + \frac{x+y}{z} \cdot c^3 \geq 16 \cdot \sqrt[4]{3} \cdot F \cdot \sqrt{F}$$

$$(4) \qquad \frac{y+z}{x} \cdot a^4 + \frac{z+x}{y} \cdot b^4 + \frac{x+y}{z} \cdot c^4 \geq 32 \cdot F^2$$

Proof.

$$\sum_{cyc} \frac{v+w}{u} a \overset{\text{AM-GM}}{\geq} 2 \cdot \sum_{cyc} \frac{\sqrt{vw}}{u} a \overset{\text{AM-GM}}{\geq} 2 \cdot 3 \cdot \sqrt[3]{\prod_{cyc} \Big(\frac{\sqrt{vw}}{u} a \Big)} = 6 \cdot \sqrt[3]{abc} = 6 \cdot \sqrt[3]{4RF} =$$

$$= 6 \cdot \sqrt[3]{8 \cdot \frac{R}{2} \cdot F} = 12 \cdot \sqrt[3]{\sqrt{\frac{R}{2}} \cdot \frac{R}{2} \cdot F} \overset{\text{Euler}}{\geq} 12 \cdot \sqrt[3]{\sqrt{r} \cdot \sqrt{\frac{R}{2}} \cdot F} \overset{\text{Mitrinovic}}{\geq} 12 \cdot \sqrt[3]{\sqrt{r} \cdot \sqrt{\frac{s}{3\sqrt{3}}} \cdot F} =$$

$$= 12 \cdot \sqrt[3]{\sqrt{r} \cdot \sqrt{s} \cdot \sqrt{\frac{1}{3\sqrt{3}}} \cdot F} = 12 \cdot \sqrt[3]{\Big(\frac{1}{\sqrt[4]{3}} \Big)^3 \cdot F \cdot \sqrt{F}} = \frac{12 \cdot \sqrt{F}}{\sqrt[4]{3}} = 4 \cdot \sqrt[4]{27} \cdot \sqrt{F}$$

We denote $u = \sqrt{x}, v = \sqrt{y}, w = \sqrt{z}$, then:

$$\sum_{cyc} \frac{y+z}{x} a^2 = \sum_{cyc} \frac{v^2+w^2}{u^2} a^2 \overset{\text{Bergström}}{\geq} \frac{1}{2} \sum_{cyc} \Big(\frac{v+w}{u} a \Big)^2 \overset{\text{Bergström}}{\geq} \frac{1}{2} \cdot \frac{1}{3} \cdot \Big(\sum_{cyc} \frac{v+w}{u} a \Big)^2 =$$

$$= \frac{1}{6} \cdot \left(\sum_{cyc} \frac{v+w}{u} a \right)^2 \overset{(1)}{\geq} \frac{1}{6} \cdot (4 \cdot \sqrt[4]{27} \cdot \sqrt{F})^2 = \frac{16}{6} \cdot \sqrt{27} \cdot F = 8\sqrt{3} \cdot F$$

We denote $u^3 = x, v^3 = y, w^3 = z$, then:

$$\sum_{cyc} \frac{y+z}{x} a^3 = \sum_{cyc} \frac{v^3+w^3}{u^3} a^3 \overset{\text{Radon}}{\geq} \frac{1}{2^2} \sum_{cyc} \Big(\frac{v+w}{u} a \Big)^3 \overset{\text{Radon}}{\geq} \frac{1}{4} \cdot \frac{1}{3^2} \cdot \left(\sum_{cyc} \frac{v+w}{u} a \right)^3 \overset{(1)}{\geq}$$

$$\overset{(1)}{\geq} \frac{1}{36} \cdot (4 \cdot \sqrt[4]{27} \cdot \sqrt{F})^3 = \frac{1}{36} \cdot 4^3 \cdot \sqrt[4]{3^9} \cdot (\sqrt{F})^3 = 16 \cdot \sqrt[4]{3} \cdot F \cdot \sqrt{F}$$

We denote $u^4 = x, v^4 = y, w^4 = z$, then:

$$\sum_{cyc} \frac{y+z}{x} a^4 = \sum_{cyc} \frac{v^4 + w^4}{u^4} a^4 \overset{\text{Radon}}{\geq} \frac{1}{2^3} \cdot \sum_{cyc} \left(\frac{v+w}{u} a\right)^4 \overset{\text{Radon}}{\geq} \frac{1}{8} \cdot \frac{1}{3^3} \cdot \left(\sum_{cyc} \frac{v+w}{u} a\right)^4 \overset{(1)}{\geq}$$

$$\overset{(1)}{\geq} \frac{1}{2^3 \cdot 3^3} \cdot (4 \cdot \sqrt[4]{27} \cdot \sqrt{F})^4 = 32 \cdot F^2$$

□

Inequality 4. In any triangle ABC with area S and usual notations the following inequality is true:

$$\frac{a^2}{\sqrt{\cos \frac{B}{2}}} + \frac{b^2}{\sqrt{\cos \frac{C}{2}}} + \frac{c^2}{\sqrt{\cos \frac{A}{2}}} \geq 4 \cdot \sqrt{2} \cdot \sqrt[4]{3} \cdot S$$

Proof.
We have by Bergström's inequality:

(1) $$\sum_{cyc} \frac{a^2}{\sqrt{\cos \frac{B}{2}}} \overset{\text{Bergström's inequality}}{\geq} \frac{(a+b+c)^2}{\sum_{cyc} \sqrt{\cos \frac{B}{2}}} = \frac{4s^2}{\sum_{cyc} \sqrt{\cos \frac{B}{2}}}$$

Then by Mitrinovic's inequality ($s \geq 3\sqrt{3}r$) and the fact that $S = rs$ we deduce that

(2) $$\frac{4s^2}{\sum_{cyc} \sqrt{\cos \frac{B}{2}}} \geq \frac{4s \cdot 3\sqrt{3}r}{\sum_{cyc} \sqrt{\cos \frac{B}{2}}} = \frac{12\sqrt{3}}{\sum_{cyc} \sqrt{\cos \frac{B}{2}}} S$$

We consider the function $f : (0, \frac{\pi}{2}) \to \mathbb{R}_+^*, f(x) = \sqrt{\cos x}$.
We have $f''(x) = -\frac{1}{4} \cdot \frac{2\cos^2 x + \sin^2 x}{\cos x \cdot \cos \sqrt{x}} < 0$, so f is concave on $(0, \frac{\pi}{2})$.
By Jensen's inequality we get that:
(3)

$$f\left(\frac{A}{2}\right) + f\left(\frac{B}{2}\right) + f\left(\frac{C}{2}\right) \leq 3f\left(\frac{A+B+C}{6}\right) = 6f\left(\frac{\pi}{6}\right) = 3\sqrt{\cos \frac{\pi}{6}} = 3\sqrt{\sin \frac{\pi}{3}} = 3\sqrt{\frac{\sqrt{3}}{2}}$$

From (1), (2) and (3) we obtain that

$$\sum_{cyc} \frac{a^2}{\sqrt{\cos \frac{B}{2}}} \geq \frac{12\sqrt{3}}{3\sqrt{\frac{\sqrt{3}}{2}}} S = 4 \cdot \sqrt{2} \cdot \sqrt[4]{3} \cdot S$$

□

Inequality 5. In any triangle ABC with s the semiperimeter and usual notations the following inequality is true

$$\frac{a^m}{(s-b)^m} + \frac{b^m}{(s-c)^m} + \frac{c^m}{(s-a)^m} \geq 3 \cdot 2^m$$

Proof.
By AM-GM inequality, the fact that $abc = 4 \cdot R \cdot S$ and Heron's formula $S = \sqrt{s(s-a)(s-b)(s-c)}$ we have

$$\sum_{cyc} \frac{a^m}{(s-b)^m} \geq 3 \cdot \left(\sqrt[3]{\frac{abc}{(s-a)(s-b)(s-c)}} \right)^m =$$

$$(1) \qquad = 3 \cdot \left(\sqrt[3]{\frac{4RSs}{s(s-a)(s-b)(s-c)}} \right)^m = 3 \cdot \left(\sqrt[3]{\frac{4RSs}{S^2}} \right)^m = 3 \cdot \left(\sqrt[3]{\frac{4Rs}{S}} \right)^m$$

By the formula $S = rs$ and Euler's inequality $R \geq 2r$ from (1) we get:

$$\sum_{cyc} \frac{a^m}{(s-b)^m} \geq 3 \cdot \left(\sqrt[3]{\frac{4Rs}{rs}} \right)^m = 3 \cdot \left(\sqrt[3]{\frac{4R}{r}} \right)^m \geq 3 \cdot (\sqrt[3]{8})^m = 3 \cdot 2^m$$

□

Inequality 6. In any triangle ABC with the semiperimeter s, the inradius r and usual notations, the following inequality is true

$$\frac{a^{m+1}}{(s-b)^m} + \frac{b^{m+1}}{(s-c)^m} + \frac{c^{m+1}}{(s-a)^m} \geq 3 \cdot 2^{m+1} \cdot \sqrt{3} \cdot r$$

Proof.
By AM-GM inequality, the fact that $abc = 4 \cdot R \cdot S$ and Heron's formula $S = \sqrt{s(s-a)(s-b)(s-c)}$ we have

$$\sum_{cyc} \frac{a^{m+1}}{(s-b)^m} \geq 3 \cdot \left(\sqrt[3]{\frac{(abc)^{m+1}}{((s-a)(s-b)(s-c))^m}} \right) =$$

$$(1) \qquad = 3 \cdot \left(\sqrt[3]{\frac{(abc)^{m+1} s^m}{S^{2m}}} \right) = 3 \cdot \left(\sqrt[3]{\frac{(4RS)^{m+1} s^m}{S^{2m}}} \right) = 3 \cdot \left(\sqrt[3]{\frac{4^{m+1} R^{m+1} s^m}{S^{m-1}}} \right)$$

By the formula $S = rs$, Euler's inequality $(R \geq 2r)$ and Mitrinovic's inequality $(s \geq 3\sqrt{3}r)$ from (1) we get:

$$\sum_{cyc} \frac{a^{m+1}}{(s-b)^m} \geq 3 \cdot \left(\sqrt[3]{\frac{4^{m+1} 2^{m+1} r^{m+1} s^n}{r^{m-1} s^{m-1}}} \right) = 3 \cdot \left(\sqrt[3]{\frac{2^{3(m+1)} r^{m+1} s}{r^{m-1}}} \right) = 3 \cdot (\sqrt[3]{r^2 s}) \cdot 2^{m+1} \geq$$

$$\geq 3 \cdot 2^{m+1} \cdot \sqrt[3]{r^2 \cdot 3\sqrt{3} \cdot r} = 3 \cdot 2^{m+1} \cdot \sqrt[3]{r^3 (\sqrt{3})^3} = 3 \cdot 2^{m+1} \cdot \sqrt{3} \cdot r$$

□

Inequality 7. In any triangle ABC with the semiperimeter s, the inradius r and usual notations the following inequality true:

$$\frac{ab}{s-c} + \frac{bc}{s-a} + \frac{ca}{s-b} \geq 12 \cdot \sqrt{3} \cdot r$$

Proof.
By AM-GM inequality, the fact that $abc = 4 \cdot R \cdot S$ and Heron's formula $S = \sqrt{s(s-a)(s-b)(s-c)}$ we have

$$(1)\qquad \sum_{cyc} \frac{ab}{s-c} \geq 3 \cdot \Big(\sqrt[3]{\frac{(abc)^2}{(s-a)(s-b)(s-c)}} = 3 \cdot \sqrt[3]{\frac{16R^2S^2s}{S^2}} = 6 \cdot \sqrt[3]{2R^2s}$$

By Euler's inequality ($R \geq 2r$) and Mitrinovic's inequality ($s \geq 3\sqrt{3}r$) from (1) we get:

$$\sum_{cyc} \frac{ab}{s-c} \geq 6 \cdot \sqrt[3]{2R^2s} \geq 6 \cdot \sqrt[3]{4r^2 \cdot 3\sqrt{3} \cdot r} = 12\sqrt{3}r$$

□

Inequality 8. In any triangle ABC with the are F, the altitudes h_a, h_b, h_c(h_a is the altitude from the vertex A, h_b is the altitude from the vertex B, h_c is the altitude from the vertex C) and usual notations, the follwing inequality is true

$$a^4(h_b^2 + h_c^2) + b^4(h_c^2 + h_a^2) + c^4(h_a^2 + h_b^2) \geq 32\sqrt{3}F$$

Proof.
The inequality is equivalent is equivalent to

$$(1)\qquad \sum_{cyc} \frac{h_b^2 + h_c^2}{4F^2} \geq 8\sqrt{3}F$$

We have

$$\sum_{cyc} \frac{h_b^2 + h_c^2}{4F^2} a^4 = \sum_{cyc} \frac{h_b^2 + h_c^2}{a^2 h_a^2} a^4 = \sum_{cyc} \frac{h_b^2 + h_c^2}{h_a^2} = \sum_{cyc} \Big(\frac{h_b^2}{h_a^2} a^2 + \frac{h_a^2}{h_b^2} b^2 \Big) \overset{\text{AM-GM}}{\geq}$$

$$(2)\qquad \overset{\text{AM-GM}}{\geq} 2 \sum_{cyc} \sqrt{\frac{h_b^2}{h_a^2} \cdot a^2 \cdot \frac{h_a^2}{h_b^2} \cdot b^2} = 2(ab + bc + ca)$$

By well-known inequality of Gordon, i.e. $ab + bc + ca \geq 4\sqrt{3}F$ and (1) and (2) we get the conclusion. □

Inequality 9. In any triangle ABC with the area F, the medians m_a, m_b, m_c(m_a is the mediane from the vertex A, m_b is the mediane from the vertex B, m_c is the mediane from the vertex C) and usual notations, the following inequality is true:

$$\frac{\sqrt{3}}{2}(a^2 + b^2 + c^2) \geq am_a + bm_b + cm_c \geq 6F$$

Proof.
First, we prove that:

$$(1)\qquad a^2 + b^2 + c^2 \geq 2\sqrt{3} \cdot a \cdot m_a$$

Indeed,

$$a^2+b^2+c^2 \geq 2\sqrt{3}\cdot a\cdot m_a \Leftrightarrow (a^2+b^2+c^2)^2 \geq 12\cdot a^2\cdot m_a^2 = 3a^2\cdot 4m_a^2 = 3a^2(2b^2+2c^2-a^2)$$
$$\Leftrightarrow a^4+b^4+c^4+2a^2b^2+2b^2c^2+2c^2a^2 \geq 6a^2b^2+6c^2a^2-3a^4$$
$$\Leftrightarrow 4a^4+b^2+c^2-4a^2b^2-4c^2a^2+2b^2c^2 \geq 0$$
$$\Leftrightarrow (2a^2-b^2-c^2)^2 = (b^2+c^2-2a^2)^2 \geq 0, \text{ true.}$$

Therefore,

$$a^2+b^2+c^2 \geq 2\sqrt{3}\cdot a\cdot m_a$$
$$a^2+b^2+c^2 \geq 2\sqrt{3}\cdot b\cdot m_a$$
$$a^2+b^2+c^2 \geq 2\sqrt{3}\cdot c\cdot m_c$$

which by adding up yielding that

$$3(a^2+b^2+c^2) \geq 2\sqrt{3}(am_a+bm_b+cm_c) \tag{2}$$

Next we have:

$$2\sqrt{3}(am_a+bm_b+cm_c) \geq 2\sqrt{3}(ah_a+bh_b+ch_c) = 12\sqrt{3}F \tag{3}$$

where we denote h_a the altitude from the vertex A, h_b the altitude from the vertex B, h_c the altitude from the vertex C.
From (2) and (3) we obtain the desired inequality. □

Inequality 10. In any triangle ABC with the area F, the medians $m_a, m_b, m_c, (m_a$ is the mediane from the vertex A, m_b is the mediane from the vertex B, m_c is the mediane from the vertex $C)$ and the usual notations the following inequality is true:

$$a^2+b^2+c^2 \geq \frac{2}{\sqrt{3}}(am_a+bm_b+cm_c) \geq 4\sqrt{3}F$$

Proof.
First we prove that

$$a^2+b^2+c^2 \geq 2\sqrt{3}\cdot a\cdot m_a \tag{1}$$

Indeed,

$$a^2+b^2+c^2 \geq 2\sqrt{3}\cdot a\cdot m_a \Leftrightarrow (a^2+b^2+c^2)^2 \geq 12\cdot a^2\cdot m_a^2 =$$
$$= 3a^2\cdot 4m_a^2 = 3a^2(2b^2+2c^2-a^2)$$
$$\Leftrightarrow a^4+b^4+c^4+2a^2b^2+2b^2c^2+2c^2a^2 \geq 6a^2b^2+6c^2a^2-3a^4$$
$$\Leftrightarrow 4a^4+b^2+c^2-4a^2b^2-4c^2a^2+2b^2c^2 \geq 0$$
$$\Leftrightarrow (2a^2-b^2-c^2) = (b^2+c^2-2a^2)^2 \geq 0, \text{ true.}$$

Therefore,

$$a^2+b^2+c^2 \geq 2\sqrt{3}\cdot a\cdot m_a$$
$$a^2+b^2+c^2 \geq 2\sqrt{3}\cdot b\cdot m_b$$
$$a^2+b^2+c^2 \geq 2\sqrt{3}\cdot c\cdot m_c$$

which by adding up yielding that

$$3(a^2+b^2+c^2) \geq 2\sqrt{3}(am_a+bm_b+cm_c) \tag{2}$$

Next we have:

$$2\sqrt{3}(am_a+bm_b+cm_c) \geq 2\sqrt{3}(ah_a+bh_b+ch_c) = 12\sqrt{3}F \tag{3}$$

where we denote h_a the altitude from the vertex A, h_b the altitude from the vertex B, h_c the altitude from the vertex C.
From (2) and (3) we obtain the desired inequality. □

Inequality 11. In any triangle ABC with the area F, the altitudes h_a, h_b, h_c, the interior angles bisectors w_a, w_b, w_c and the usual notations the following inequality is true:

$$a^2 + b^2 + c^2 \geq \frac{4F}{\sqrt{3}}\left(\frac{w_a}{h_a} + \frac{w_b}{h_b} + \frac{w_c}{h_c}\right) \geq 4\sqrt{3}F$$

Proof.
WLOG we can assume that $a \leq b \leq c$, then $w_a \geq w_b \geq w_c$ and $h_a \geq h_b \geq h_c$. By Chebyshev's inequality we get that:

(1) $$\sum_{cyc} \frac{w_a}{h_a} \leq \frac{1}{3}\left(\sum_{cyc} w_a\right)\left(\sum_{cyc} \frac{1}{h_a}\right)$$

But,

(2) $$\sum_{cyc} \frac{1}{h_a} = \sum_{cyc} \frac{a}{ah_a} = \frac{1}{2F}(a+b+c) = \frac{s}{F}$$

And,
(3)
$$w_a = \frac{2bc}{b+c}\cos\frac{A}{2} = \frac{2bc}{b+c}\sqrt{\frac{s(s-a)}{bc}} = \frac{2\sqrt{bc}}{b+c}\cdot\sqrt{s(s-a)} \leq \frac{2\sqrt{bc}}{2\sqrt{bc}}\sqrt{s(s-a)} = \sqrt{s(s-a)},$$

and other two similar.
Using the inequality $x^2 + y^2 + z^2 \geq \frac{(x+y+z)^2}{3}$, (*) and (3) we deduce that

$$\sum_{cyc} s(s-a) \geq \frac{(\sum_{cyc}\sqrt{s(s-a)})^2}{3} \Leftrightarrow 3s^2 \geq \left(\sum_{cyc}\sqrt{s(s-a)}\right)^2 \Leftrightarrow \sum_{cyc}\sqrt{s(s-a)} \leq s\sqrt{3} \Leftrightarrow$$

(4) $$\Leftrightarrow \sum_{cyc} w_a \leq s\sqrt{3}$$

From (1), (2), (4) and (*) we obtain that
(5)
$$\sum_{cyc}\frac{w_a}{h_a} \leq \frac{1}{3}\left(\sum_{cyc} w_a\right)\left(\sum_{cyc}\frac{1}{h_a}\right) \leq \frac{1}{3}\cdot s\sqrt{3}\cdot\frac{s}{F} = \frac{(a+b+c)^2}{4F\sqrt{3}} \leq \frac{3(a^2+b^2+c^2)}{4F\sqrt{3}}$$

Since $w_a \geq h_a, w_b \geq h_b, w_c \geq h_c$ from (5) we get:

$$3 \leq \sum_{cyc}\frac{w_a}{h_a} \leq \frac{3(a^2+b^2+c^2)}{4F\sqrt{3}} \Leftrightarrow a^2+b^2+c^2 \geq \frac{4F}{\sqrt{3}}\left(\frac{w_a}{h_a} + \frac{w_b}{h_b} + \frac{w_c}{h_c}\right) \geq 4\sqrt{3}F$$

□

Inequality 12. If $x, y, z > 0$ and $A_1B_1C_1, A_2B_2C_2$ are two triangles with the areas F_1, respectively F_2, then the following inequality holds:

$$\frac{x+y}{z}a_1b_2 + \frac{y+z}{x}b_1c_2 + \frac{z+x}{y}c_1a_2 \geq 8\sqrt{3F_1F_2}$$

Proof.

$$\sum \frac{x+y}{z}a_1b_2 \geq 2\sum \frac{\sqrt{xy}}{z}c_1b_2 \geq 2\cdot 3\cdot \sqrt[3]{\prod\left(\frac{\sqrt{xy}}{z}a_1b_2\right)} = 6\cdot \sqrt[3]{a_1b_1c_1a_2b_2c_2} =$$

$$= 6\cdot\sqrt{\sqrt[3]{(a_1b_1c_1)^2}\cdot\sqrt[3]{(a_2b_2c_2)^2}} \overset{\text{Carliz}}{\geq} 6\cdot\sqrt{\frac{4F_1\sqrt{3}}{3}\cdot\frac{4F_2\sqrt{3}}{3}} = 6\cdot\frac{4\sqrt{3}}{3}\cdot\sqrt{F_1F_2} = 8\sqrt{3F_1F_2}$$

Remark.
If $x=y=z$, then by the above inequality we obtain:

$$a_1b_2+b_1c_2+c_1a_2 \geq 4\sqrt{3F_1F_2}, \text{ i.e. a Tsintsifas type inequality.}$$

□

Inequality 13. If $x,y,z>0$ and $A_1B_1C_1, A_2B_2C_2$ are two triangle with the circumradius R_1 respectively R_2 then the following inequality holds:

$$\frac{x+y}{z\sqrt{a_1a_2}}+\frac{y+z}{x\sqrt{b_1b_2}}+\frac{z+x}{y\sqrt{c_1c_2}} \geq \frac{2\sqrt{3}}{\sqrt{R_1R_2}}$$

Proof.

$$\sum\frac{x+y}{z\sqrt{a_1a_2}} = 2\sum\frac{\sqrt{xy}}{z\sqrt{a_1a_2}} \geq 2\cdot 3\cdot\sqrt[3]{\prod\frac{\sqrt{xy}}{z\sqrt{a_1a_2}}} = 6\cdot\sqrt[3]{\frac{1}{\sqrt{a_1b_1c_1a_2b_2c_2}}} =$$

$$= \frac{6}{\sqrt[6]{4R_1F_1\cdot 4R_2F_2}} = \frac{6}{\sqrt[6]{16R_1F_1R_2F_2}} = \frac{6}{\sqrt[6]{16R_1s_1r_1R_2s_2r_2}} \overset{\text{Euler}}{\geq} \frac{6}{\sqrt[6]{16R_1R_2s_1s_2\cdot\frac{R_1}{2}\cdot\frac{R_2}{2}}} =$$

$$= \frac{6}{\sqrt[6]{4R_1^2R_2^2s_1s_2}} \overset{\text{Mitrinovic}}{\geq} \frac{6}{\sqrt[6]{4R_1^2R_2^2\cdot\frac{3\sqrt{3}R_1}{2}\cdot\frac{3\sqrt{3}R_2}{2}}} =$$

$$= \frac{6}{\sqrt[6]{4R_1^3R_2^3\cdot(\sqrt{3})^6}} = \frac{6}{\sqrt{3}\cdot\sqrt{R_1R_2}} = \frac{2\sqrt{3}}{\sqrt{R_1R_2}}$$

Remark.
If $\Delta ABC \equiv \Delta A_1B_1C_1 \equiv \Delta A_2B_2C_2$, then by the above inequality we obtain

$$\frac{x+y}{za}+\frac{y+z}{xb}+\frac{z+x}{yc} \geq \frac{2\sqrt{3}}{R}$$

and if $x=y=z$, then by the last inequality we obtain

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq \frac{\sqrt{3}}{R}, \text{ i.e Ionescu - Ţiu - Leuenberger inequality}$$

□

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