NEW PROOF FOR V.O. GORDON'S INEQUALITY

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Abstract: *In this paper are presented a new proof for V.O. Gordon's inequality.*

Let $x, y, z > 0$, then holds:

(N) $$\boldsymbol{xy + yz + zx \geq \sqrt{3xyz(x+y+z)}}$$

i.e. Newton's inequality.

Proof. We have:

$$xy + yz + zx \geq \sqrt{3xyz(x+y+z)} \Leftrightarrow$$

(1) $$(xy + yz + zx)^2 \geq 3xyz(x+y+z)$$

Using:

(2) $$(u+v+w)^2 \geq 3(uv + vw + wu); (\forall) u, v, w > 0$$

taking in (2): $u = xy; v = yz; w = zx$, we get:

$$(xy + yz + zx)^2 \geq 3(xy \cdot yz + yz \cdot zx + zx \cdot xy) = 3xyz(x+y+z)$$

i.e. inequality (N) was proved.
If in (N) we take: $x = a; y = b; z = c$, where a, b, c are lengths sides of triangle ABC with F area and s semiperimeter, we get:

(*) $$ab + bc + ca \geq \sqrt{3abc(a+b+c)} = \sqrt{3} \cdot \sqrt{4RF \cdot 2s} \overset{Euler}{\geq}$$
$$\sqrt{3} \cdot \sqrt{8rF \cdot 2s} = 4\sqrt{3} \cdot \sqrt{rs \cdot F} = 4\sqrt{3} \cdot \sqrt{F^2} = 4\sqrt{3} \cdot F$$

i.e. Gordon's inequality. Equality holds for $x = y = z \Leftrightarrow a = b = c$. □

In ΔABC with a, b, c length sides, F area and s semiperimeter, holds:

(I-W) $$\boldsymbol{a^2 + b^2 + c^2 \geq 4\sqrt{3} \cdot F}$$

i.e. Ionescu-Weitzenbock's inequality.

Proof. We have:

$$a^2 + b^2 + c^2 \geq ab + bc + ca \overset{(N)}{\geq} \sqrt{3abc(a+b+c)} \overset{(*)}{\geq} 4\sqrt{3} \cdot F$$

Equality holds if and only if triangle is equilateral. □

References

[1] Adrian Boroica, Paul Becsi-LEME IN INEGALITATI, ARGUMENT MAGAZINE, Year 22, Baia Mare, 2020, pages 9-13