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RAVI'S SUBSTITUTIONS OR VOICULESCU'S SUBSTITUTIONS' ?

By Dorin Mărghidanu,Marian Dincă-Romania

Edited by Florică Anastase-Romania

In 1964, Dan Voiculescu - then a young student, used a substitution which would later be called a 'Ravi substitution' - just six years before Ravi was born!.. Murray Klamkin also used this substitution at least ten years before Ravi. In order to restore the historical truth, a new name is required for this transformation.

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The notion of *Ravi substitution* is relatively often encountered and used in the mathematical practice of the last 30 years. This substitution is contained in the following very simple equivalence , but which is very useful in applications:

Theorem : ***The numbers a, b, c represents the lengths of the sides of a triangle if and only if there are positive real numbers x, y, z such that $a = y + z, b = z + x, c = x + y$.***

Proof.

If a, b, c represent the lengths of the sides of a triangle, then: $a + b > c, b + c > a, c + a > b$. The system of equations $x + y = c, y + z = a, z + x = b$ has the (unique) solution:

$$x = \frac{-a + b + c}{2}, y = \frac{a - b + c}{2}, z = \frac{a + b - c}{2}$$

so with the condition of triangularity it follows: $x > 0, y > 0, z > 0$.

Reciprocally, if $x > 0, y > 0, z > 0$, then we notice that:

$$a + b = x + y + 2z \overset{(z>0)}{>} x + y = c \Rightarrow a + b > c$$

$$b + c = y + z + 2x \overset{(x>0)}{>} y + z = a \Rightarrow b + c > a$$

$$c + a = z + x + 2y \overset{(y>0)}{>} z + x = b \Rightarrow c + a > b$$

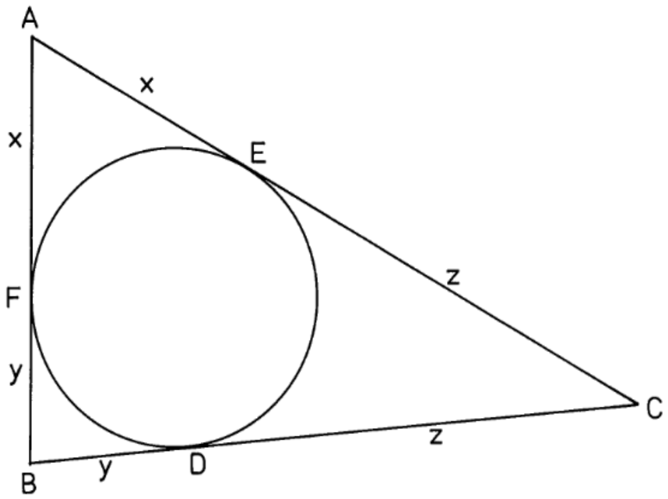
hence a, b, c can be the lengths of the sides of a triangle.

- ***Remark 1.***

With the substitution of the theorem above we can transfer properties (inequalities, identities or other relations) regarding triangles to equivalent properties in positive real numbers - and vice versa. The first implication is obviously interesting , because the properties that take place in the constrained conditions of the triangle are transferred to the properties that take place for positive numbers - for which there are many solving techniques provided by mathematics for a very long time.

- ***Remark 2.***

A very suggestive geometric interpretation of the substitution in the theorem is presented in the figure below. Geometrically x, y, z are the lengths of the segments on the sides of a triangle determined by the tangent points of the circle inscribed in the triangle (using the fact that the tangents from an outer point are equal) (see for example, [5]).



The name of this substitution comes from the name of the canadian mathematician of indian origin *Vakil D. Ravi ,* currently a professor at Stanford University . In adolescence, *Ravi* participated in various school competitions , including three international Olympics, where he won a silver medal (1986) and two gold medals (1987- perfect score and 1988) .

He also had four participations at the Putnam competition and then was the training coordinator in preparation for the Putnam competition at Stanford University. He was one of the founders of the canadian journal *Mathematical Mayhem* (later merged with *Crux Mathematicorum*).

During his youth, *Ravi* often used the substitution that bears his name and successfully converted inequalities in triangle geometry into general inequalities. However, we have no knowledge of any work or reference by *Ravi* dedicated to this substitution. The name *Ravi substitution* was originally used locally - in canadian (olympic) math circles, then proliferated around the world.

But this type of substitution was used at least 10 years before (even several times) by the well-known mathematician *Murray Klamkin* (1921-2004). For example, in each of the papers [1]-[4] is briefly presented the equivalence - as the one presented here, in the Theorem at the beginning of this paper .The equivalence relation is called *duality* (or *principle of duality*) by *Klamkin* and each of the equivalent inequalities is considered dual to the other.

In addition to being one of the greatest problem-solvers in the world (or perhaps because of that) , *Klamkin* was also the coach of the United States IOM team from 1978-1985, and was the editor and coordinator at the *Olympiad corner* of the *Crux Mathematicorum* and then at other magazines. He is also the author of some interesting mathematical articles and notes, as well as some problem-solving books for competitions .

Regarding the *substitution of Ravi*, in [4] *Klamkin* says verbatim that, “this transformation was known and used before he was born”.

And indeed (obviously, without *Klamkin*'s knowledge) , about six years before the birth of *Vakil Ravi* (born in february 22, 1970) , -more precisely in february 1964, in Romania, *Dan V. Voiculescu* -then a student for only 15 years. publishes a short mathematical note [6] , which uses exactly the substitution that is now called the *Ravi substitution* . Being a short work and perhaps harder to find now, we reproduce it below.

2. Asupra unor inegalități într-un triunghi

Condiția necesară și suficientă pentru ca a, b, c să fie laturi ale unui triunghi este să existe x, y, z pozitivi, astfel ca:

$$\left.\begin{aligned} a&=x+y\\ b&=x+z\\ c&=y+z \end{aligned}\right\} \qquad (1)$$

Intr-adevăr, scriind că x, y, z sînt pozitivi avem:

$$x=\frac{a+b-c}{2}>0\,; \quad y=\frac{a-b+c}{2}>0\,; \quad z=\frac{-a+b+c}{2}>0.$$

condiții necesare și suficiente pentru ca a, b, c să fie laturi ale unui triunghi. Rezultă deci, posibilitatea de a obține cu ajutorul formulelor (1), din orice inegalitate în care figurează a, b, c (laturi ale unui triunghi), o inegalitate între numerele pozitive oarecare x, y, z.

Rezultă de asemenea, că pentru ca o inegalitate în a, b, c să fie adevărată este necesar și suficient ca inegalitatea în x, y, z obținută prin formulele (1) să fie adevărată.

Spre a ilustra această metodă vom da în cele ce urmează, cîteva exemple.

a) Cunoscuta inegalitate între raza cercului circumscris şi raza cercului înscris unui triunghi. Din formulele uzuale în care am înlocuit conform lui (1), avem:

$$\frac{R}{r}=\frac{(x+y)(y+z)(z+x)}{4xyz}\geqslant\frac{8xyz}{4xyz}=2$$

unde am aplicat fiecărei paranteze în parte, inegalitatea $u+v\geqslant 2\sqrt{uv}$.

b) Intr-un triunghi avem:

$$a^2+b^2+c^2=2[x(x+y+z)+(y^2+yz+z^2)]\geqslant$$
$$\geqslant 4\sqrt{x(x+y+z)(y^2+yz+z^2)}\geqslant 4\sqrt{3xyz(x+y+z)}=4S\sqrt{3}$$

unde am aplicat mai întîi, inegalitatea $u+v\geqslant 2\sqrt{uv}$, iar apoi, inegalitatea $u+v+w\geqslant 3\sqrt[3]{uvw}$.

c) Vom transforma cunoscuta inegalitate:

$$2(\sin A+\sin B+\sin C)\leqslant 3\sqrt{3}$$

Avem, conform (1):

$$4\sqrt{(x+y+z)xyz}\left[\frac{1}{(x+y)(x+z)}+\frac{1}{(x+y)(y+z)}+\right.$$
$$\left.+\frac{1}{(x+z)(y+z)}\right]\leqslant 3\sqrt{3}$$

adică:

$$64(x+y+z)^3xyz\leqslant 27(x+y)^2(y+z)^2(x+z)^2$$

d) Din inegalitatea dată de Toma Albu în problema nr. 5696:

$$\sqrt{\sin A}+\sqrt{\sin B}+\sqrt{\sin C}\leqslant 3\sqrt[4]{\frac{3}{4}}$$

obţinem:

$$16(x+y+z)(\sqrt{x+y}+\sqrt{x+z}+\sqrt{y+z})^4xyz\leqslant$$
$$\leqslant 243(x+y)^2\cdot(x+z)^2\cdot(y+z)^2.$$

Urmînd această cale se pot stabili lesne şi alte inegalităţi între elemente ale unui triunghi sau între numere pozitive.

Voiculescu Dan, cl. VIII-a, ş.m. 21 Bucureşti

We must add that then, the young *Dan Voiculescu* evolved and performed very well in math competitions. He won (like *Ravi Vakil*) three IMO olympic medals : one silver in 1965 and two gold in 1966 and 1967. *Dan Voiculescu* then has an exceptional mathematical career. After being a brilliant researcher at the Institute of Mathematics of the Romanian Academy, he has been a professor at Berkeley University, California since 1986. He was nominated for the Fields Medal, but he did not obtain it, only because in the previous edition it had been obtained by a mathematician from the same field of research.. He is considered the greatest Romanian mathematician alive.

Returning to the substitution from the beginning of this paper and considering the ones presented , we believe that this type of substitution should be called *Voiculescu substitution*. Perhaps it would be just as correct to call this substitution *Voiculescu - Klamkin substitution* : the first for the chronological precedence of using this type of transformation , the second for the use and intense popularization of this type of relationship (that of *duality* and for the *principle of duality*) . In the last resort, the name of *Voiculescu - Klamkin - Ravi substitution* could be included!

REFERENCES:

[1] **Klamkin M. S. , "*Duality in triangle inequalities*". Notices of the Amer. Math. Soc. p. 782 , 1971.**

[2] **Klamkin M. S. , "*Duality in Spherical Triangles*" , Mathematics Magazine , Volume 46** , - **issue 4** , **1973** .

[3] **Klamkin M. S. , "*Sums of Squares Represesntations for Triangle Inequalities* ", Univ. Beograd. Publ. Electroteh. Fak. Ser. Mat. Fiz., No. 412 –** No. 460 , 1973 .

[4] **Klamkin M. S. , "*On a Problem of the Month*" , Crux Mathematicorum , Vol. 28 , p.86 , 2002 .**

[5] **D.S. Mitrinovic, J.E. Pecaric, V. Volenec**, "***Recent advances in geometric inequalities***", (Chapter II , pp. 26–36) , Kluwer, London , 1989.

[6] **Voiculescu Dan,"Asupra unor inegalitati intr-un triunghi", Gazeta Matematica**, seria B , no.2 , 1964 .

[7] **Wikipedia : a) Ravi Vakil - Wikipedia , b) Murray S. Klamkin - Wikipedia , c) Dan Virgil Voiculescu - Wikipedia .**