RMM - Geometry Marathon 701 - 800

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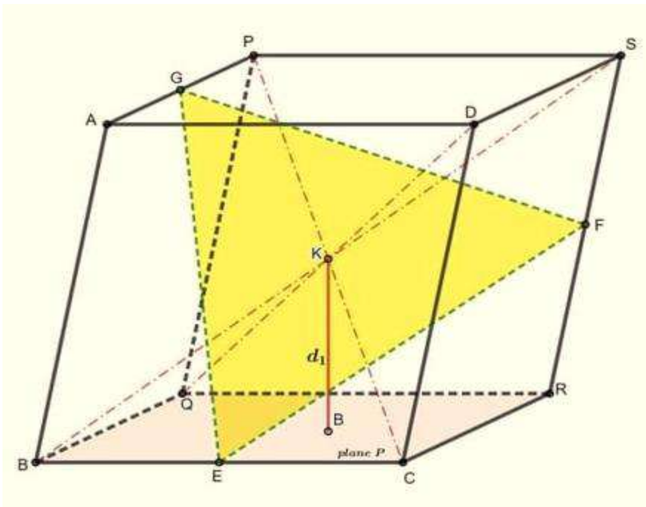
701. $\Omega = (A,B,C,D,P,Q,R,S)$ **−stereo rhomboid (all faces are rhombus).**

Plane $(B,C,R,Q) = P, AB = 2, E,F,G$ **midpoints of** BC, RS, AP**, respectively.**

$$d(K,P) = d_1 = \sqrt{\frac{2}{3}}, d(A,BC) = d_2 = \sqrt{3}, [EFG] = \frac{3}{\sqrt{2}} (area).$$

Find $V_{(\Omega)} = ?\ (volume)$.



Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by proposer

Let $BC = BQ = BA = 2, \widehat{xBy} = \theta_3, \widehat{yBz} = \theta_1, \widehat{zBx} = \theta_2$

Plagiogonal 3D system: $BC \equiv Bx, BQ \equiv By, BA \equiv Bz$

$B(0,0,0), C(2,0,0), A(0,0,2), S(2,2,2), K(1,,1,1)$

Is $P(B,C,R,Q) = P_1 : z = 0$

Let $\vec{u} = (u_1,u_2,u_3), \vec{u}_1 \perp P_1, u_1 = 1 \cdot (-\cos\theta_2 + \cos\theta_3 \cdot \cos\theta_1)$

$u_2 = 1 \cdot (-\cos\theta_1 + \cos\theta_2 \cdot \cos\theta_3), u_3 = 1 \cdot (1 - \cos^2\theta_2)$

$|\vec{u}^2| = u_1^2 + u_2^2 + u_3^2 + 2u_1u_2\cos\theta_3 + 2u_2u_3\cos\theta_1 + 2u_3u_1\cos\theta_2$

$|\vec{u}|^2 = (-b+ca)^2 + (-a+bc)^2 + (1-c^2)^2 + 2(-b+ca)(-a+bc)c +$

$+2(-a+bc)(1-c^2)a + 2(1-c^2)(-b+ca)b$

$|\vec{u}|^2 = c^2a^2 - a^2 - 2abc^3 + 2abc + b^2c^2 - b^2 + c^4 - 2c^2 + 1$

$$d_1^2 = \frac{(1\cdot 1)^2}{1(1-c^2)^2}|u|^2 \Rightarrow d_1^2 = \frac{-a^2 + 2abc - b^2 - c^2 + 1}{1-c^2} = \frac{2}{3}; (1)$$

Let $F\in(BC), F(f,0,0):AF\perp BC, \overrightarrow{AF}=(f,0,-2), \overrightarrow{BC}=(2,0,0)$

$$\overrightarrow{AF}\cdot\overrightarrow{BC}=2f+(-4)\cos\theta_2=0\Rightarrow f-2b=0\Rightarrow f=2b\Rightarrow F(2b,0,0)$$

$$\overrightarrow{AF}=(2b,0,-2), \left|\overrightarrow{AF}\right|^2=4b^2+4+2(2b)(-2)b=-4b^2+4=d_2^2$$

$$\Rightarrow -4b^2+4=3\Rightarrow b=\frac{1}{2} \text{ or } b=-\frac{1}{2};\ (2)$$

$$E(1,0,0), F(2,2,1), G(0,1,2), \overrightarrow{EG}=(-1,1,2), \overrightarrow{EF}=(1,2,1)$$

$$\left|\overrightarrow{EG}\right|^2=1+1+4+2(-1)(1)\cos\theta_3+2(1)(2)\cos\theta_1+2(-1)(2)\cos\theta_2=$$
$$=6-2c+4a-4b$$

$$\left|\overrightarrow{EF}\right|^2=1+4+1+2\cdot1\cdot2\cos\theta_3+2\cdot1\cdot1\cos\theta_1+2\cdot1\cdot1\cos\theta_2=$$
$$=6+4c+4a+2b$$

$$\overrightarrow{EG}\cdot\overrightarrow{EF}=-1+2+2+(-2+1)\cos\theta_3+(1+4)\cos\theta_1+(-1+2)\cos\theta_2=$$
$$=3-c+5a+b$$

$$[EFG]=\frac{1}{2}\sqrt{(6-2c+4a-4b)(6+4c+4a+2b)-(3-c+5a+b)^2}=\frac{3}{\sqrt{2}};(3)$$

$$(1),(3)\overset{b=\frac{1}{2}}{\Longrightarrow}a=0, b=\frac{1}{2}, c=\frac{1}{2}, V=4\sqrt{2}$$

$$a=0.301123, b=\frac{1}{2}, c=0.025, V=6.52992$$

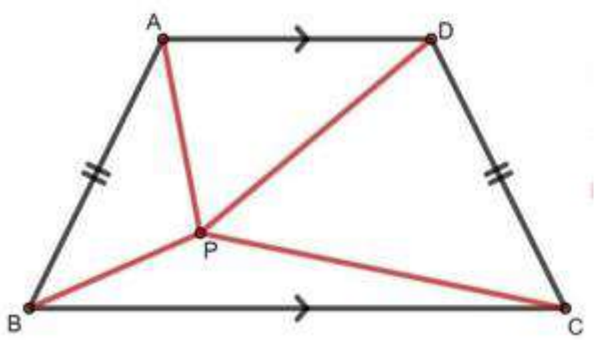
$$(1),(3)\overset{b=-\frac{1}{2}}{\Longrightarrow}a=0, b=-\frac{1}{2}, c=-\frac{1}{2}, V=4\sqrt{2}$$

$a=0.544692, b=-\frac{1}{2}, V=4.95682$. Finally,

$$V=4\sqrt{2}, \theta_1=90°, \theta_2=60°, \theta_3=60° \text{ or } \theta_1=90°, \theta_2=120°, \theta_3=120°$$
$$V=6.52992, \theta_1=72.475°, \theta_2=60°, \theta_3=88.563°$$
$$V=4,95682, \theta_1=56.996°, \theta_2=120°, \theta_3=130.64°$$

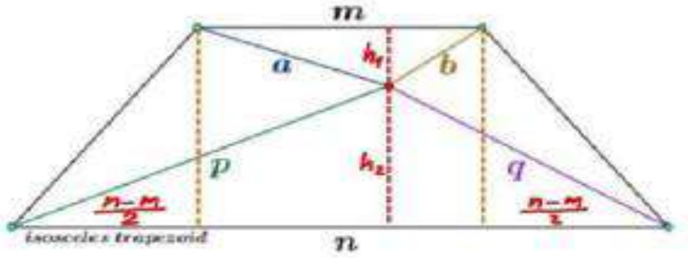
702. P –any point in plane $(ABCD), PB^2-PC^2=PA^2-PD^2$

Prove: $ABCD$ is rectangle.



Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by Hikmat Mammadov-Azerbaijan



$$m=\sqrt{a^2-h_1^2}+\sqrt{b^2-h_1^2}\Rightarrow m-\sqrt{a^2-h_1^2}=\sqrt{b^2-h_1^2}$$

$$m^2+a^2-h_1^2-2m\sqrt{a^2-h_1^2}=b^2-h_1^2,\ \sqrt{a^2-h_1^2}=\frac{m^2+a^2-b^2}{2m}$$

$$\sqrt{p^2-h_2^2}=\frac{n^2+p^2-q^2}{2n},\ \sqrt{p^2-h_2^2}-\sqrt{a^2-h_1^2}=\frac{n-m}{2}$$

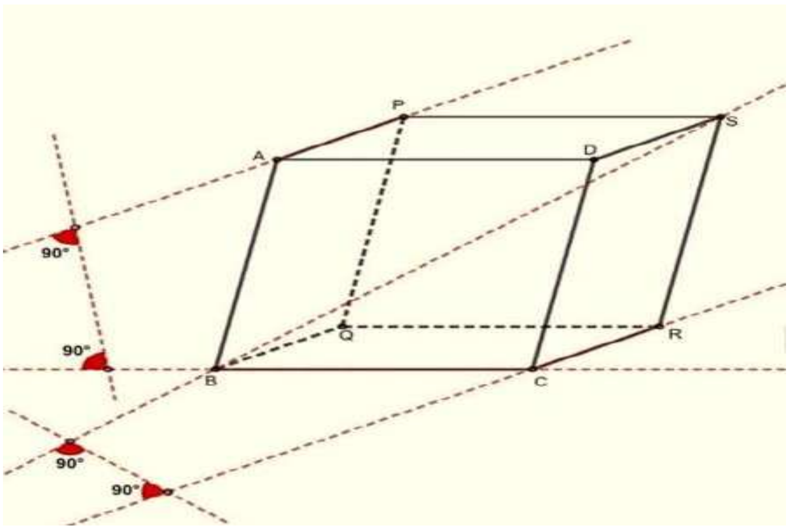
$$\frac{n^2+p^2-q^2}{2n}-\frac{m^2+a^2-b^2}{2m}=\frac{n-m}{2},\qquad \frac{p^2-q^2}{2}-\frac{a^2-b^2}{2}=0\Rightarrow\frac{a^2-b^2}{p^2-q^2}=\frac{m}{n}$$

$$\frac{BC}{AD}=\frac{PB^2-PC^2}{PA^2-PD^2}=1\Rightarrow AD=BC\Rightarrow ABCD\text{ −is rectangle.}$$

703. $(A,B,C,D,P,Q,R,S)=\Omega$ (stereo rhombus) $AB=2$

$$\widehat{ABQ}=\theta_1=90^\circ,\widehat{ABC}=\theta_2=60^\circ,\widehat{CBQ}=\theta_3=60^\circ$$

Find: $d(AP,BC)=?,\ d(BS,CR)=?$



Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by proposer

$$V = 2 \cdot 2 \cdot 2 \cdot \sqrt{1 - \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2} = 4\sqrt{2}$$

$$[BCRQ] = 2 \cdot 2 \cdot \sin 60^\circ = 2\sqrt{3}$$

$$h = \frac{V}{[BCRQ]} = 2\sqrt{\frac{2}{3}}. \text{So}, d(AP, BC) = 2\sqrt{\frac{2}{3}}$$

Plagiogonal 3D system: $BC \equiv Bx, BQ \equiv By, BA \equiv Bz$

$$B(0,0,0), S(2,2,2), C(2,0,0), R(2,2,0)$$

Let $M \in (BC) \Rightarrow M(m,m,m), N \in (CR) \Rightarrow N(2,n,0)$

$$MN^2 = (m-2)^2 + (m-n)^2 + m^2 + (m-2)(m-n) + m(m-2) = f(m,n)$$

$$\begin{cases} \dfrac{df}{dm} = 0 \\ \dfrac{df}{dn} = 0 \end{cases} \Rightarrow \begin{cases} 10m - 3n - 8 = 0 \\ -3m + 2n + 2 = 0 \end{cases} \Rightarrow \left\{m = \frac{10}{11}, n = \frac{4}{11}\right\}$$

$$\min\{MN^2\} = \frac{8}{11} \Rightarrow d(BS, CP) = 2\sqrt{\frac{2}{11}}$$

$$AM = \frac{2}{3}, AN = \frac{4}{3}, MN = 2\sqrt{\frac{2}{3}}, \quad BK = \frac{10}{11}, CL = \frac{4}{11}, KL = 2\sqrt{\frac{2}{11}}$$

704. In acute ΔABC the following relationship holds:

$$\sqrt{2R\left(\frac{1}{m_a} + \frac{1}{m_b} + \frac{1}{m_c}\right)} \geq \sqrt{\frac{r_a + r_b}{r_a + r_c}} + \sqrt{\frac{r_a + r_c}{r_a + r_b}}.$$

Proposed by Bogdan Fuştei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\textit{In any acute } \Delta ABC \textit{ we have}: \quad m_a \leq 2R\cos^2\frac{A}{2} \quad (\textit{and analogs})$$

$$Then:\ 2R\left(\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}\right)\geq\frac{1}{\cos^2\frac{A}{2}}+\frac{1}{\cos^2\frac{B}{2}}+\frac{1}{\cos^2\frac{C}{2}}=\frac{(4R+r)^2}{s^2}+1\ (1)$$

$$Now\ let\ us\ prove\ that\ in\ any\ \Delta ABC:\ \frac{b}{c}+\frac{c}{b}\leq\frac{\sqrt{a^2b^2+b^2c^2+c^2a^2}}{2F}\ (*)$$

$$We\ have:\ (*)\ \Leftrightarrow(b^2+c^2)\sqrt{2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)}\leq 2bc\sqrt{a^2b^2+b^2c^2+c^2a^2}$$

$$\overset{squaring}{\Leftrightarrow}\ \left(2b^2c^2+b^4+c^4\right)\left[2\left(a^2b^2+b^2c^2+c^2a^2\right)-\left(a^4+b^4+c^4\right)\right]\leq 4b^2c^2\left(a^2b^2+b^2c^2+c^2a^2\right)$$

$$\Leftrightarrow\ -a^4(b^2+c^2)^2-2b^2c^2(b^4+c^4)+2(b^4+c^4)(a^2b^2+b^2c^2+c^2a^2)-(b^4+c^4)^2\leq 0$$

$$\Leftrightarrow\ -a^4(b^2+c^2)^2+2(b^4+c^4)(a^2b^2+c^2a^2)-(b^4+c^4)^2=-[a^2(b^2+c^2)-(b^4+c^4)]^2\leq 0$$

Which is true.

Since $\sqrt{r_b+r_c},\sqrt{r_a+r_b},\sqrt{r_a+r_c}$ *can be the sides of a triangle* Δ' *with area* S *where* :

$$16S^2=2\sum_{cyc}\sqrt{r_a+r_b}^2\sqrt{r_a+r_c}^2-\sum_{cyc}\sqrt{r_a+r_b}^4=4\sum_{cyc}r_br_c=4s^2\ \Rightarrow 2S=s.$$

Then $(*)$ *in*

$$\Delta'\Rightarrow\frac{\sqrt{r_a+r_b}}{\sqrt{r_a+r_c}}+\frac{\sqrt{r_a+r_c}}{\sqrt{r_a+r_b}}\leq\frac{\sqrt{\sum_{cyc}\sqrt{r_a+r_b}^2\sqrt{r_a+r_c}^2}}{2S}=\frac{\sqrt{\left(\sum_{cyc}r_a\right)^2+\sum_{cyc}r_ar_b}}{s}.$$

$$\sqrt{\frac{r_a+r_b}{r_a+r_c}}+\sqrt{\frac{r_a+r_c}{r_a+r_b}}\leq\sqrt{\frac{(4R+r)^2}{s^2}+1}\overset{(1)}{\leq}\sqrt{2R\left(\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}\right)},\ as\ desired.$$

705. In any ΔABC holds:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{R^3}{r^3}\geq 8+\frac{a^2}{c^2}+\frac{c^2}{b^2}+\frac{b^2}{a^2}.$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\textit{By Oppenheim's inequality we have}: \ xa^2+yb^2+zc^2 \geq 4F\sqrt{xy+yz+zx}, \ \ \forall x,y,z>0.$$

$$\textit{Let } x=\frac{c^2}{a^2}, \ y=\frac{a^2}{b^2}, \ z=\frac{b^2}{c^2} \textit{ then we have}: \ a^2+b^2+c^2 \geq 4F\sqrt{\frac{a^2}{c^2}+\frac{c^2}{b^2}+\frac{b^2}{a^2}}$$

$$Or \ \frac{a^2}{c^2}+\frac{c^2}{b^2}+\frac{b^2}{a^2} \leq \frac{(a^2+b^2+c^2)^2}{16s^2r^2} = \frac{(s^2-r^2-4Rr)^2}{4s^2r^2} \overset{Gerretsen}{\leq} \frac{(4R^2+2r^2)^2}{4r^2(16Rr-5r^2)} =$$

$$= \frac{(8R^2+4r^2)^2}{8r^3(32R-5.2r)} \overset{Euler}{\leq} \frac{(8R^2+R^2)^2}{8r^3(32R-5R)} = \frac{3R^3}{8r^3}.$$

$$\textit{Therefore,}$$

$$8+\frac{a^2}{c^2}+\frac{c^2}{b^2}+\frac{b^2}{a^2} \leq 3+5+\frac{3R^3}{8r^3} \overset{Euler}{\leq} 3+\frac{5R^3}{8r^3}+\frac{3R^3}{8r^3} \overset{AM-GM}{\leq} \frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{R^3}{r^3}, \ \ \forall \Delta ABC.$$

Solution 2 by Soumava Chakraborty-Kolkata-India

Let $s-a=x, s-b=y$ and $s-c=z \therefore s=x+y+z \Rightarrow a=y+z, b=z+x$ and $c=x+y$

$$\text{Now, } \frac{s^2}{r^2}=\frac{s^4}{\Delta^2}=\frac{s^4}{s(s-a)(s-b)(s-c)}=\frac{\left(\sum_{cyc}x\right)^3}{xyz} \Rightarrow \frac{s^2}{r^2} \overset{(\bullet)}{=} \frac{\left(\sum_{cyc}x\right)^3}{xyz} \text{ and } 1+\frac{4R}{r}$$

$$=1+\frac{4sabc}{4s(s-a)(s-b)(s-c)}=1+\frac{\prod_{cyc}(y+z)}{xyz}$$

$$\Rightarrow 1+\frac{4R}{r} \overset{(\bullet\bullet)}{=} \frac{xyz+\prod_{cyc}(y+z)}{xyz}$$

$$\text{Now, } \sum_{cyc}\frac{b}{a}=\sum_{cyc}\frac{z+x}{y+z} \Rightarrow \sum_{cyc}\frac{b}{a} \overset{(\bullet\bullet\bullet)}{=} \frac{\sum_{cyc}(x+y)^2(y+z)}{\prod_{cyc}(y+z)} \therefore (\bullet),(\bullet\bullet),(\bullet\bullet\bullet) \Rightarrow \frac{s^2}{r^2}$$

$$\geq \left(\sum_{cyc}\frac{b}{a}\right)\left(1+\frac{4R}{r}\right) \Leftrightarrow \frac{\left(\sum_{cyc}x\right)^3}{xyz} \geq \left(\frac{xyz+\prod_{cyc}(y+z)}{xyz}\right)\left(\frac{\sum_{cyc}(x+y)^2(y+z)}{\prod_{cyc}(y+z)}\right)$$

$$\Leftrightarrow \left(\prod_{cyc}(y+z)\right)\left(\sum_{cyc}x\right)^3 \geq \left(xyz+\prod_{cyc}(y+z)\right)\left(\sum_{cyc}(x+y)^2(y+z)\right)$$

$$\Leftrightarrow \sum_{cyc}x^4y^2+\sum_{cyc}x^3y^3 \overset{(i)}{\geq} xyz\left(\sum_{cyc}xy^2\right)+3x^2y^2z^2$$

$$\text{Now}, \forall\, u,v,w>0, u^3+u^3+v^3 \overset{A-G}{\geq} 3u^2v, v^3+v^3+w^3 \overset{A-G}{\geq} 3v^2w \text{ and } w^3+w^3+u^3 \overset{A-G}{\geq} 3w^2u \text{ and summing up}: \sum_{cyc} u^3 \geq \sum_{cyc} u^2v \text{ and choosing } u=xy, v=yz \text{ and } w=zx,$$

$$\sum_{cyc} x^3y^3 \overset{(*)}{\geq} xyz\left(\sum_{cyc} xy^2\right) \text{ and } \sum_{cyc} x^4y^2 \underbrace{\overset{A-G}{\geq}}_{(**)} 3x^2y^2z^2 \therefore (*)+(**) \Rightarrow (i) \text{ is true} \Rightarrow \frac{s^2}{r^2} \geq \left(\sum_{cyc}\frac{b}{a}\right)\left(1+\frac{4R}{r}\right) \Rightarrow \boxed{\sum_{cyc}\frac{b}{a} \overset{(\bullet\bullet\bullet\bullet)}{\leq} \frac{s^2}{r(4R+r)}}$$

$$\text{Now}, \frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{R^3}{r^3} \geq 8+\frac{a^2}{c^2}+\frac{c^2}{b^2}+\frac{b^2}{a^2} \Leftrightarrow \sum_{cyc}\frac{a}{b}+\frac{R^3-8r^3}{r^3} \geq \left(\sum_{cyc}\frac{b}{a}\right)^2 - 2\sum_{cyc}\frac{a}{b}$$

$$\Leftrightarrow \boxed{3\sum_{cyc}\frac{a}{b}+\frac{R^3-8r^3}{r^3} \overset{(***)}{\geq} \left(\sum_{cyc}\frac{b}{a}\right)^2}$$

$$\text{Now, LHS of } (***) \overset{A-G}{\geq} 9+\frac{R^3-8r^3}{r^3} = \frac{R^3+r^3}{r^3} \text{ and RHS of } (***) \overset{via\,(\bullet\bullet\bullet\bullet)}{\leq} \frac{s^4}{r^2(4R+r)^2} \overset{Gerretsen}{\leq} \frac{(4R^2+4Rr+3r^2)^2}{r^2(4R+r)^2}$$

$\therefore$ in order to prove $(***)$, it suffices to prove :

$$\frac{R^3+r^3}{r^3} \geq \frac{(4R^2+4Rr+3r^2)^2}{r^2(4R+r)^2} \Leftrightarrow 16t^5-8t^4-31t^3-24t^2-16t-8 \geq 0 \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(16t^4+24t^3+17t^2+10t+4) \geq 0 \to \text{true} \because t \overset{Euler}{\geq} 2 \Rightarrow (***) \text{ is true}$$

$\therefore$ in any $\Delta\, ABC$

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{R^3}{r^3} \geq 8+\frac{a^2}{c^2}+\frac{c^2}{b^2}+\frac{b^2}{a^2}, \text{ equality iff } \Delta\, ABC \text{ is equilateral (Proved)}$$

706. In ΔABC the following relationship holds:

$$\sqrt{\frac{b}{a}}+\sqrt{\frac{c}{b}}+\sqrt{\frac{a}{c}}+\frac{R}{2r} \geq 1+\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{a}}.$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{a}} \overset{CBS}{\leq} \sqrt{(a+b+c)\left(\frac{1}{b}+\frac{1}{c}+\frac{1}{a}\right)} = \sqrt{2s.\frac{s^2+r^2+4Rr}{4Rrs}} = \sqrt{\frac{s^2+r^2+4Rr}{2Rr}} \leq$$

$$\overset{Gerretsen}{\leq} \sqrt{\frac{(4R^2+4Rr+3r^2)+r^2+4Rr}{2Rr}} = \frac{2(R+r)}{\sqrt{2Rr}} = 2\sqrt{\frac{R}{2r}}+\sqrt{\frac{2r}{R}} \leq$$

$$\overset{AM-GM\ \&\ Euler}{\leq}\left(\frac{R}{2r}+1\right)+1=3+\frac{R}{2r}-1\overset{AM-GM}{\leq}\sqrt{\frac{b}{a}}+\sqrt{\frac{c}{b}}+\sqrt{\frac{a}{c}}+\frac{R}{2r}-1.$$

$$Therefore,\ \sqrt{\frac{b}{a}}+\sqrt{\frac{c}{b}}+\sqrt{\frac{a}{c}}+\frac{R}{2r}\geq 1+\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{a}}.$$

Solution 2 by Soumava Chakraborty-Kolkata-India

Let $s-a=x, s-b=y$ and $s-c=z \therefore s=x+y+z \Rightarrow a=y+z, b=z+x$ and $c=x+y$

$$\text{Now, }\frac{s^2}{r^2}=\frac{s^4}{\Delta^2}=\frac{s^4}{s(s-a)(s-b)(s-c)}=\frac{\left(\sum_{cyc}x\right)^3}{xyz}\Rightarrow\frac{s^2}{r^2}\overset{(\bullet)}{=}\frac{\left(\sum_{cyc}x\right)^3}{xyz}\text{ and }1+\frac{4R}{r}$$

$$=1+\frac{4sabc}{4s(s-a)(s-b)(s-c)}=1+\frac{\prod_{cyc}(y+z)}{xyz}\Rightarrow 1+\frac{4R}{r}\overset{(\bullet\bullet)}{=}\frac{xyz+\prod_{cyc}(y+z)}{xyz}$$

$$\text{Also, }\sum_{cyc}\frac{a}{b}=\sum_{cyc}\frac{y+z}{z+x}\Rightarrow\sum_{cyc}\frac{a}{b}\overset{(\bullet\bullet\bullet)}{=}\frac{\sum_{cyc}(x+y)(y+z)^2}{\prod_{cyc}(y+z)}\ (\bullet),(\bullet\bullet),(\bullet\bullet\bullet)\Rightarrow\frac{s^2}{r^2}\geq\left(\sum_{cyc}\frac{a}{b}\right)\left(1+\frac{4R}{r}\right)$$

$$\Leftrightarrow\frac{\left(\sum_{cyc}x\right)^3}{xyz}\geq\left(\frac{xyz+\prod_{cyc}(y+z)}{xyz}\right)\left(\frac{\sum_{cyc}(x+y)(y+z)^2}{\prod_{cyc}(y+z)}\right)$$

$$\Leftrightarrow\left(\prod_{cyc}(y+z)\right)\left(\sum_{cyc}x\right)^3\geq\left(xyz+\prod_{cyc}(y+z)\right)\left(\sum_{cyc}(x+y)(y+z)^2\right)$$

$$\Leftrightarrow\sum_{cyc}x^2y^4+\sum_{cyc}x^3y^3\overset{(i)}{\geq}xyz\left(\sum_{cyc}x^2y\right)+3x^2y^2z^2$$

Now, $\forall\, u,v,w>0, v^3+v^3+u^3\overset{A-G}{\geq}3v^2u, w^3+w^3+v^3\overset{A-G}{\geq}3w^2v$ and $u^3+u^3+w^3\overset{A-G}{\geq}3u^2w$ and summing up : $\sum_{cyc}u^3\geq\sum_{cyc}uv^2$ and choosing $u=xy, v=yz$ and $w=zx$

$$\sum_{cyc}x^3y^3\overset{(*)}{\geq}xyz\left(\sum_{cyc}x^2y\right)\text{ and }\sum_{cyc}x^2y^4\underbrace{\overset{A-G}{\geq}}_{(**)}3x^2y^2z^2\therefore(*)+(**)\Rightarrow(i)\text{ is true}\Rightarrow\frac{s^2}{r^2}$$

$$\geq\left(\sum_{cyc}\frac{a}{b}\right)\left(1+\frac{4R}{r}\right)\Rightarrow\boxed{\sum_{cyc}\frac{a}{b}\overset{(\bullet\bullet\bullet\bullet)}{\leq}\frac{s^2}{r(4R+r)}}$$

$$\text{Now, }\frac{b}{a}+\frac{c}{b}+\frac{a}{c}+\frac{R}{2r}\overset{?}{\geq}1+\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\Leftrightarrow\sum_{cyc}\frac{a}{b}+\sum_{cyc}\frac{b}{a}+\frac{R-2r}{2r}\overset{?}{\geq}2\sum_{cyc}\frac{a}{b}$$

$$\Leftrightarrow\frac{\sum_{cyc}\left(ab\left(\sum_{cyc}a-c\right)\right)}{4Rrs}+\frac{R-2r}{2r}\overset{?}{\geq}2\sum_{cyc}\frac{a}{b}$$

$$\Leftrightarrow\frac{2s(s^2+4Rr+r^2)-12Rrs}{4Rrs}+\frac{R-2r}{2r}\overset{?}{\geq}2\sum_{cyc}\frac{a}{b}$$

$$\Leftrightarrow\frac{s^2-2Rr+r^2+R(R-2r)}{2Rr}\underbrace{\overset{?}{\geq}}_{(***)}2\sum_{cyc}\frac{a}{b}\text{ and }\because 2\sum_{cyc}\frac{a}{b}\overset{via\,(\bullet\bullet\bullet\bullet)}{\leq}\frac{2s^2}{r(4R+r)}$$

$\therefore$ in order to prove $(***)$, it suffices to prove :

$$\frac{s^2-2Rr+r^2+R(R-2r)}{2Rr}\geq\frac{2s^2}{r(4R+r)}$$

$$\Leftrightarrow rs^2+R(R-2r)(4R+r)\overset{(****)}{\geq}r(2R-r)(4R+r)$$

Now, LHS of $(****)\overset{\text{Gerretsen}}{\geq}r(16Rr-5r^2)+R(R-2r)(4R+r)\overset{?}{\geq}r(2R-r)(4R+r)$

$$\Leftrightarrow 4t^3-15t^2+16t-4\overset{?}{\geq}0\left(t=\frac{R}{r}\right)\Leftrightarrow(t-2)(4t(t-2)+t+2)\overset{?}{\geq}0$$

$\to$ true $\because t\overset{\text{Euler}}{\geq}2\Rightarrow(****)\Rightarrow(***)$ is true $\Rightarrow\frac{b}{a}+\frac{c}{b}+\frac{a}{c}+\frac{R}{2r}\geq1+\frac{a}{b}+\frac{b}{c}+\frac{c}{a}$

$$\Rightarrow\frac{b}{a}+\frac{c}{b}+\frac{a}{c}+\frac{abc(a+b+c)}{16F^2}\geq1+\frac{a}{b}+\frac{b}{c}+\frac{c}{a}$$

$$\Rightarrow\boxed{1+\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)\overset{(l)}{\leq}\frac{abc(a+b+c)}{2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)}}$$

Now, $a+b+2\sqrt{ab}>c\Rightarrow\left(\sqrt{a}+\sqrt{b}\right)^2>c\Rightarrow\sqrt{a}+\sqrt{b}>\sqrt{c}$ and cyclic analogs

$\Rightarrow\sqrt{a},\sqrt{b},\sqrt{c}$ form sides of a triangle XYZ (say) and via (l) on Δ XYZ,

$$1+\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{a}}-\left(\sqrt{\frac{b}{a}}+\sqrt{\frac{c}{b}}+\sqrt{\frac{a}{c}}\right)$$

$$\leq\frac{\sqrt{abc}\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)}{2\sum_{cyc}ab-\sum_{cyc}a^2}\overset{\text{CBS}}{\leq}\frac{\sqrt{12Rrs}.\sqrt{2s}}{2(s^2+4Rr+r^2)-2(s^2-4Rr-r^2)}=\frac{\sqrt{24Rrs^2}}{16Rr+4r^2}\overset{?}{\leq}\frac{R}{2r}$$

$$\Leftrightarrow 24Rrs^2\overset{?}{\leq}4R^2(4R+r)^2$$

$\Leftrightarrow 6rs^2\underset{(*****)}{\overset{?}{\leq}}R(4R+r)^2$ and $\because 6rs^2\overset{\text{Gerretsen}}{\leq}6r(4R^2+4Rr+3r^2)$

$\therefore$ in order to prove $(*****)$, it suffices to prove : $R(4R+r)^2\geq6r(4R^2+4Rr+3r^2)$

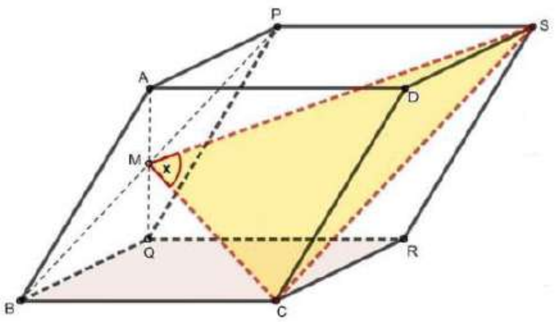
$\Leftrightarrow 16t^3-16t^2-23t-18\geq0\Leftrightarrow(t-2)(16t^2+16t+9)\geq0\to$ true $\because t\overset{\text{Euler}}{\geq}2\Rightarrow(*****)$ is true

$$\therefore 1+\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{a}}-\left(\sqrt{\frac{b}{a}}+\sqrt{\frac{c}{b}}+\sqrt{\frac{a}{c}}\right)\leq\frac{R}{2r}$$

$\Rightarrow$ in any Δ ABC, $\sqrt{\frac{b}{a}}+\sqrt{\frac{c}{b}}+\sqrt{\frac{a}{c}}+\frac{R}{2r}\geq1+\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{a}}$, equality iff Δ ABC is equilateral (QED)

707. All the parallelograms are rhombuses, $\angle CBQ=\angle QBA=\angle ABC=\theta$,

$\cos x=\frac{1}{\sqrt{33}}$. Find: θ.



Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by proposer

$$BC \equiv Bx, BQ \equiv By, BA \equiv Bz$$

Let $BA = 2, B(0,0,0), C(2,0,0), S(2,2,2), M(0,1,1)$

$$\overrightarrow{MS}(2,1,1), \overrightarrow{MC}\,(2,-1,-1)$$

$$\overrightarrow{MS}\cdot\overrightarrow{MC} = 4-1-1+\cos\theta\,(-2+2-1-1-2+2) = 2-2\cos\theta = 2(1-\cos\theta)$$

$$\overrightarrow{MS}^2 = 4+1+1+2\cos\theta\,(2+1+2) \Rightarrow$$

$$\left|\overrightarrow{MS}\right|^2 = \sqrt{6+10\cos\theta} = \sqrt{6}\sqrt{1-\cos\theta}$$

$$\overrightarrow{MC}^2 = 4+1+1+\cos\theta\cdot 2(-2-2+1) \Rightarrow$$

$$\left|\overrightarrow{MC}\right| = \sqrt{6-6\cos\theta} = \sqrt{6}\sqrt{1-\cos\theta}$$

$$\cos x = \frac{\overrightarrow{MS}\cdot\overrightarrow{MC}}{\left|\overrightarrow{MS}\right|\cdot\left|\overrightarrow{MC}\right|} = \frac{2(1-\cos\theta)}{\sqrt{12}\sqrt{(1-\cos\theta)(3+5\cos\theta)}} = \sqrt{\frac{1-\cos\theta}{3(3+5\cos\theta)}}$$

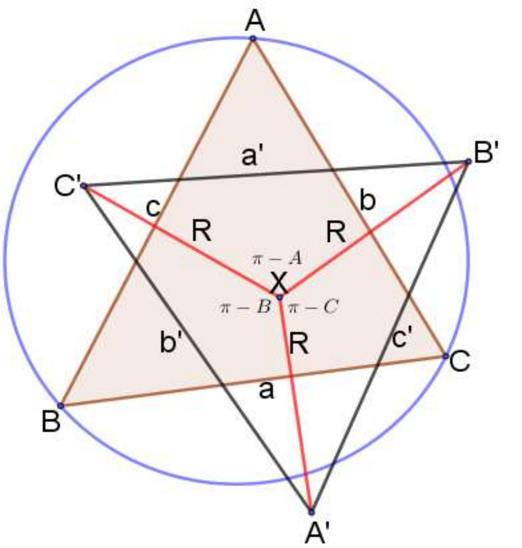
$$\cos x = \frac{1}{\sqrt{33}} \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = 60^\circ$$

708. If $X \in Int(\Delta ABC), XA' = XB' = XC' = R, R$ –circumradii, $XA' \perp BC$, $XB' \perp CA, XC' \perp AB, r'$ –inradii in $\Delta AB'C'$. Prove that:

$$r' = \frac{a+b+c}{4\left(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\right)}; r' \geq \frac{s}{3\sqrt{3}}$$

Proposed by Mehmet Şahin-Ankara-Turkiye

Solution by Ertan Yildirim-Izmir-Turkiye



$$\sum_{cyc}\cos\frac{A}{2}\le\frac{3\sqrt{3}}{2}$$

$$[A'B'C']=\frac{R^2}{2}(\sin(\pi-A)+\sin(\pi-B)+\sin(\pi-C))=$$

$$=\frac{R^2}{2}(\sin A+\sin B+\sin C)=\frac{R^2}{2}\cdot\frac{a+b+c}{2R}=\frac{R}{4}(a+b+c)$$

$$(a')^2=R^2+R^2-2R\cdot R\cdot\cos(\pi-A)$$

$$(a')^2=2R62\ (1+\cos A)=2R^2\cdot 2\cos^2\frac{A}{2}=4R^2\cdot\cos^2\frac{A}{2}$$

$a'=2R\cdot\cos\frac{A}{2}$. Similarly,

$b'=2R\cdot\cos\frac{B}{2}$ and $c'=2R\cdot\cos\frac{C}{2}$ then: $[A'B'C']=\frac{R}{4}(a+b+c)=\frac{(a'+b'+c')r'}{4}$

$$\frac{R}{4}(a+b+c)=\frac{1}{2}\cdot 2R\left(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\right)r'$$

$$r'=\frac{a+b+c}{4\left(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\right)};$$

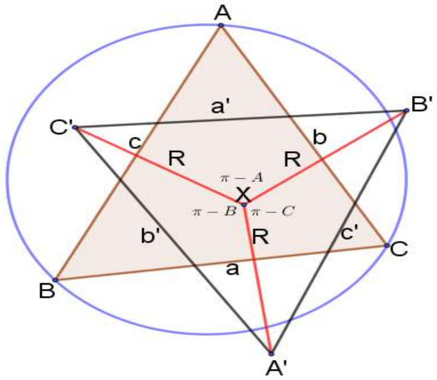
$$r'=\frac{2s}{4\left(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\right)}\ge\frac{s}{2\cdot\frac{3\sqrt{3}}{2}}=\frac{s}{3\sqrt{3}}$$

709. If $X\in Int(\Delta ABC), XA'=XB'=XC'=R, R$ −circumradii, $XA'\perp BC$, $XB'\perp CA, XC'\perp AB, I_a$ −excenter, then:

$$4[A'B'C']=[I_aI_bI_c]$$

Proposed by Mehmet Şahin-Ankara-Turkiye

Solution by Ertan Yildirim-Izmir-Turkiye



$$[A'B'C'] = \frac{R^2}{2}(\sin(\pi - A) + \sin(\pi - B) + \sin(\pi - C)) =$$

$$= \frac{R^2}{2}(\sin A + \sin B + \sin C) = \frac{R^2}{2} \cdot \frac{a+b+c}{2R} = \frac{R}{4}(a+b+c)$$

$$[A'B'C'] = \frac{R}{4}(a+b+c)$$

Since ΔABC is orthic triangle of $\Delta I_a I_b I_c$, then:

$$BC = a = I_b I_c \cos\left(\frac{\pi}{2} - \frac{A}{2}\right) = I_b I_c \sin\frac{A}{2}$$

$$a = 2R\sin A = I_b\, I_c \sin\frac{A}{2}$$

$$2R \cdot 2\sin\frac{A}{2}\cos\frac{A}{2} = I_b I_c \sin\frac{A}{2} \Rightarrow I_b I_c = 4R\cos\frac{A}{2}$$

Hence,

$$I_b I_c = 4R\cos\frac{A}{2}, I_a I_c = 4R\cos\frac{B}{2}, I_a I_b = 4R\cos\frac{C}{2}$$

Lemma: $[I_a I_b I_c] = 8R^2 \cdot \cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = (a+b+c)R$

Proof. $\text{Cos}\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}(\,and\ analogs)$

$$[I_a I_b I_c] = \frac{1}{2} \cdot 4R\cos\frac{B}{2} \cdot 4R\cos\frac{C}{2} \cdot \sin\left(\frac{\pi}{2} - \frac{A}{2}\right) = 8R^2 \cdot \cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} =$$

$$= 8R62 \cdot \sqrt{\frac{s(s-a)}{bc}} \cdot \sqrt{\frac{s(s-b)}{ca}} \cdot \sqrt{\frac{s(s-c)}{ab}} =$$

$$= 8R^2 \cdot \frac{s \cdot sr}{abc} = 8R^2 \cdot \frac{s^2 r}{4Rrs} = 2Rs = (a+b+c)R$$

$$\Rightarrow [A'B'C'] = \frac{R}{4}(a+b+c) = \frac{[I_a I_b I_c]}{4}, \qquad 4[A'B'C'] = [I_a I_b I_c]$$

710. In ΔABC the following relationship holds:

$$2\sum_{cyc} \frac{1}{h_a}\sqrt[4]{m_b m_c (r_a + r_c - h_b)(r_a + r_b - h_c)} \geq \sum_{cyc} \frac{b+c}{a}$$

Proposed by Bogdan Fuştei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:$$

$$\sqrt{m_b m_c} \overset{Tereshin}{\geq} \sqrt{\frac{(c^2+a^2)(a^2+b^2)}{(4R)^2}} \overset{CBS}{\geq} \frac{ca+ab}{4R} = \frac{a(b+c)}{4R} = \frac{F(b+c)}{bc} \quad (1)$$

$$Also,\ r_a + r_c - h_b = \frac{F}{s-a} + \frac{F}{s-c} - \frac{2F}{b} = F\left(\frac{b}{(s-c)(s-a)} - \frac{2}{b}\right) \overset{b=(s-c)+(s-a)}{=} F.\frac{(s-c)^2+(s-a)^2}{b(s-c)(s-a)}$$

$$Then:$$

$$\sqrt{(r_a+r_c-h_b)(r_a+r_b-h_c)} = \sqrt{\frac{F^2}{bc}.\frac{[(s-c)^2+(s-a)^2][(s-a)^2+(s-b)^2]}{(s-a)^2(s-b)(s-c)}} \geq$$

$$\overset{CBS}{\geq} \sqrt{\frac{F^2}{bc}.\frac{[(s-c)(s-a)+(s-a)(s-b)]^2}{(s-a)^2(s-b)(s-c)}} = \sqrt{\frac{F^2}{bc}.\frac{a^2}{(s-b)(s-c)}} = \frac{F.a}{bc.\sin\frac{A}{2}} =$$

$$\overset{Mollweide}{=} \frac{F.(b+c)}{bc.\cos\frac{B-C}{2}} \geq \frac{F(b+c)}{bc},\ then:$$

$$\sqrt{(r_a+r_c-h_b)(r_a+r_b-h_c)} \geq \frac{F(b+c)}{bc} \quad (2)$$

$$From\ (1)\ and\ (2)\ we\ get:$$

$$\frac{2}{h_a}\sqrt[4]{m_b m_c(r_a+r_c-h_b)(r_a+r_b-h_c)} \geq \frac{a}{F}.\frac{F(b+c)}{bc} = \frac{a}{b}+\frac{a}{c} \quad (and\ analogs)$$

$$Summing\ up\ this\ inequality\ with\ similar\ ones\ we\ get:$$

$$2\sum_{cyc}\frac{1}{h_a}\sqrt[4]{m_b m_c(r_a+r_c-h_b)(r_a+r_b-h_c)} \geq \sum_{cyc}\left(\frac{a}{b}+\frac{a}{c}\right) = \sum_{cyc}\frac{b+c}{a}.$$

711. In ΔABC the following relationship holds:

$$\frac{R}{2r}\left(\frac{r_a+r_b}{r_b+r_c}+\frac{r_b+r_c}{r_c+r_a}+\frac{r_c+r_a}{r_a+r_b}\right) \geq \frac{r_b+r_c}{r_a+r_b}+\frac{r_c+r_a}{r_b+r_c}+\frac{r_a+r_b}{r_c+r_a}$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$r_b + r_c = s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}} + \frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right) = \frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = \frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)} = 4R\cos^2\frac{A}{2} \therefore r_b + r_c \overset{(i)}{=} 4R\cos^2\frac{A}{2}$$

$$\frac{R}{2r}\left(\frac{r_a+r_b}{r_b+r_c} + \frac{r_b+r_c}{r_c+r_a} + \frac{r_c+r_a}{r_a+r_b}\right) \overset{A-G}{\geq} \frac{3R}{2r}\cdot\sqrt[3]{\frac{r_a+r_b}{r_b+r_c}\cdot\frac{r_b+r_c}{r_c+r_a}\cdot\frac{r_c+r_a}{r_a+r_b}} \overset{?}{\geq} \frac{r_b+r_c}{r_a+r_b} + \frac{r_c+r_a}{r_b+r_c} + \frac{r_a+r_b}{r_c+r_a}$$

$$= \sum_{cyc}\frac{\left(\sum_{cyc} r_a\right) - r_b}{r_b + r_c}$$

$$\Leftrightarrow \frac{3R}{2r} + \sum_{cyc}\frac{r_b}{r_b+r_c} \overset{?}{\geq} \frac{4R+r}{\prod_{cyc}(r_b+r_c)}\cdot\sum_{cyc}\left((r_c+r_a)(r_a+r_b)\right) \overset{\text{via (i) and analogs}}{\Leftrightarrow} \frac{3R}{2r}$$

$$+ \sum_{cyc}\frac{r_b}{r_b+r_c} \overset{?}{\geq} \frac{4R+r}{\prod_{cyc}\left(4R\cos^2\frac{A}{2}\right)}\cdot\left(3\left(\sum_{cyc} r_a r_b\right) + \sum_{cyc} r_a^2\right)$$

$$\Leftrightarrow \frac{3R}{2r} + \sum_{cyc}\frac{r_b}{r_b+r_c} \overset{?}{\geq} \frac{4R+r}{64R^3\cdot\frac{s^2}{16R^2}}\cdot\left(3s^2 + (4R+r)^2 - 2s^2\right)$$

$$\Leftrightarrow \boxed{\frac{3R}{2r} + \sum_{cyc}\frac{r_b}{r_b+r_c} \underset{(*)}{\overset{?}{\geq}} \frac{(4R+r)\left(s^2 + (4R+r)^2\right)}{4Rs^2}}$$

$$\text{Now, } \frac{3R}{2r} + \sum_{cyc}\frac{r_b}{r_b+r_c}$$

$$= \frac{3R}{2r} + \sum_{cyc}\frac{r_b^2}{r_b^2 + r_b r_c} \overset{\text{Bergstrom}}{\geq} \frac{3R}{2r}$$

$$+ \frac{(4R+r)^2}{(4R+r)^2 - 2s^2 + s^2} \overset{?}{\geq} \frac{(4R+r)\left(s^2+(4R+r)^2\right)}{4Rs^2}$$

$$\Leftrightarrow \frac{3R\left((4R+r)^2 - s^2\right) + 2r(4R+r)^2}{2r\left((4R+r)^2 - s^2\right)} \overset{?}{\geq} \frac{(4R+r)\left(s^2+(4R+r)^2\right)}{4Rs^2}$$

$$\Leftrightarrow 4Rs^2\left(3R\left((4R+r)^2 - s^2\right) + 2r(4R+r)^2\right) \overset{?}{\geq} 2r(4R+r)\left((4R+r)^4 - s^4\right)$$

$$\Leftrightarrow \boxed{\left(6R^2 - 4Rr - r^2\right)s^4 - \left(96R^4 + 112R^3r + 38R^2r^2 + 4Rr^3\right)s^2 + r(4R+r)^5 \underset{(**)}{\overset{?}{\leq}} 0}$$

$$\text{Now, LHS of } (**) \overset{\text{Gerretsen}}{\leq} \left(6R^2 - 4Rr - r^2\right)\left(4R^2 + 4Rr + 3r^2\right)s^2$$

$$- \left(96R^4 + 112R^3r + 38R^2r^2 + 4Rr^3\right)s^2 + r(4R+r)^5 \overset{?}{\leq} 0$$

$$\Leftrightarrow \boxed{\left(72R^4 + 104R^3r + 40R^2r^2 + 20Rr^3 + 3r^4\right)s^2 \underset{(***)}{\overset{?}{\geq}} r(4R+r)^5}$$

$$\text{Again, LHS of } (***) \overset{\text{Gerretsen}}{\geq} \left(72R^4 + 104R^3r + 40R^2r^2 + 20Rr^3 + 3r^4\right)(16Rr$$

$$- 5r^2) \overset{?}{\geq} r(4R+r)^5 \Leftrightarrow 16t^5 + 3t^4 - 65t^3 - 5t^2 - 9t - 2 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(16t^4+35t^3+5t^2+5t+1)\overset{?}{\geq} 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

$$\Rightarrow \text{in any } \Delta\, ABC, \frac{R}{2r}\left(\frac{r_a+r_b}{r_b+r_c}+\frac{r_b+r_c}{r_c+r_a}+\frac{r_c+r_a}{r_a+r_b}\right)$$

$$\geq \frac{r_b+r_c}{r_a+r_b}+\frac{r_c+r_a}{r_b+r_c}+\frac{r_a+r_b}{r_c+r_a}, \text{equality iff } \Delta\, ABC \text{ is equilateral (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\frac{r_b+r_c}{r_a+r_b}+\frac{r_c+r_a}{r_b+r_c}+\frac{r_a+r_b}{r_c+r_a}\overset{CBS}{\leq}$$

$$\sqrt{\left(\sum_{cyc}(r_b+r_c)^2\right)\left(\sum_{cyc}\frac{1}{(r_a+r_b)^2}\right)} \overset{AM-GM}{\leq} \sqrt{2\left(\sum_{cyc}{r_a}^2+\sum_{cyc}r_br_c\right)\left(\sum_{cyc}\frac{1}{4r_br_c}\right)} =$$

$$= \sqrt{[(4R+r)^2-s^2].\frac{4R+r}{2s^2r}} \overset{Gerretsen}{\leq} \sqrt{\frac{[(4R+r)^2-(16Rr-5r^2)](4R+r)}{2(16Rr-5r^2)r}}$$

$$= \sqrt{\frac{32R^3-8R^2r+8Rr^2+3r^3}{(16R-5r)r^2}} \overset{?}{\leq} \frac{3R}{2r}$$

$$\Leftrightarrow 4(32R^3-8R^2r+8Rr^2+3r^3) \leq 9R^2(16R-5r)$$

$$\Leftrightarrow 16R^3-13R^2r-32Rr^2-12r^3 \geq 0$$

$$\Leftrightarrow (R-2r)(16R^2+19Rr+6r^2) \geq 0 \quad \textit{which is true.}$$

$$\frac{r_b+r_c}{r_a+r_b}+\frac{r_c+r_a}{r_b+r_c}+\frac{r_a+r_b}{r_c+r_a} \leq \frac{3R}{2r} \overset{AM-GM}{\leq} \frac{R}{2r}\left(\frac{r_a+r_b}{r_b+r_c}+\frac{r_b+r_c}{r_c+r_a}+\frac{r_c+r_a}{r_a+r_b}\right).$$

712. If in $\Delta ABC, \omega$ −Brocard's angle then :

$$\sin\omega \leq 4\sqrt[3]{\prod_{cyc}\frac{{m_a}^2}{4a^2+b^2+c^2}}.$$

Proposed by Bogdan Fuştei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\textit{Lemma}: \quad \textit{In any } \Delta ABC \textit{ we have}: \frac{b}{c}+\frac{c}{b} \leq \frac{\sqrt{a^2b^2+b^2c^2+c^2a^2}}{2F} \quad (*)$$

$$Proof:\quad Since\ \ 4F=\sqrt{2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)}\ \ then\ we\ have:$$

$$(*)\Leftrightarrow\left(b^2+c^2\right)\sqrt{2\left(a^2b^2+b^2c^2+c^2a^2\right)-\left(a^4+b^4+c^4\right)}\le 2bc\sqrt{a^2b^2+b^2c^2+c^2a^2}$$

$$\overset{squaring}{\Leftrightarrow}\left(2b^2c^2+b^4+c^4\right)\left[2\left(a^2b^2+b^2c^2+c^2a^2\right)-\left(a^4+b^4+c^4\right)\right]\le 4b^2c^2\left(a^2b^2+b^2c^2+c^2a^2\right)$$

$$\Leftrightarrow -a^4\left(b^2+c^2\right)^2-2b^2c^2\left(b^4+c^4\right)+2\left(b^4+c^4\right)\left(a^2b^2+b^2c^2+c^2a^2\right)-\left(b^4+c^4\right)^2\le 0$$

$$\Leftrightarrow -a^4(b^2+c^2)^2+2(b^4+c^4)(a^2b^2+c^2a^2)-(b^4+c^4)^2=-[a^2(b^2+c^2)-(b^4+c^4)]^2\le 0$$

Which is true and the proof of the lemma is completed.

Since m_a, m_b, m_c *can be the sides of a triangle* Δ_m *with area*

$F_m=\dfrac{3F}{4}$ *then using lemma in* Δ_m, *we have:*

$$\frac{m_b}{m_c}+\frac{m_c}{m_b}\le\frac{\sqrt{{m_a}^2{m_b}^2+{m_b}^2{m_c}^2+{m_c}^2{m_a}^2}}{2F_m}=\frac{\sqrt{\frac{9}{16}(a^2b^2+b^2c^2+c^2a^2)}}{\frac{3F}{2}}=\frac{1}{\sin\omega}.$$

$$Then:\quad \sin\omega\le\frac{m_bm_c}{{m_b}^2+{m_c}^2}=\frac{4m_bm_c}{4a^2+b^2+c^2}\quad (and\ analogs)$$

$$Therefore,\ \ 4\sqrt[3]{\prod_{cyc}\frac{{m_a}^2}{4a^2+b^2+c^2}}=\sqrt[3]{\prod_{cyc}\frac{4m_bm_c}{4a^2+b^2+c^2}}\ge\sqrt[3]{\prod_{cyc}\sin\omega}=\sin\omega.$$

713. In ΔABC the following relationship holds:

$$\sum_{cyc}\left(\frac{a}{(b+c)\cos^3\frac{A}{2}}\right)^2\ge\frac{72R^3}{(4R+r)^3-2s^2(2R+r)}$$

Proposed by Neculai Stanciu-Romania

Solution by Tapas Das-India

$$\text{Let } x=\cos^2\frac{A}{2}, y=\cos^2\frac{B}{2}, z=\cos^2\frac{C}{2}, \text{ then:}$$

$$\sum_{cyc}\cos^6\frac{A}{2}=\sum_{cyc}\left(\cos^2\frac{A}{2}\right)^3=x^3+y^3+z^3=$$

$$=(x+y+z)\left((x+y+z)^2-3(xy+yz+zx)\right)+3xyz;\ (1)$$

$$x+y+z=\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}=2+\frac{2r}{R}=\frac{4R+r}{2R}$$

$$xy+yz+zx=\sum_{cyc}\cos^2\frac{A}{2}\cos^2\frac{B}{2}=1+\frac{s^2+r^2+8Rr}{16R^2}=\frac{16R^2+s^2+r^2+8Rr}{16R^2}$$

$$xyz=\prod_{cyc}\cos^2\frac{A}{2}=\left(\frac{s}{4R}\right)^2=\frac{s^2}{16R^2}$$

$$x^3+y^3+z^3\overset{(1)}{=}\frac{4R+r}{2R}\left[\left(\frac{4R+r}{2R}\right)^2-\frac{48R^2+3s^2+3r^2+24Rr}{16R^2}\right]+\frac{3s^2}{16R^2}=$$

$$=\frac{4R+r}{2R}\cdot\frac{4(4R+r)^2-48R^2-3s^2-3r^2-24Rr}{16R^2}+\frac{3s^2}{16R^2}=$$

$$=\frac{4R+r}{2R}\cdot\frac{64R^2+32Rr+4r^2-48R^2-3s^2-3r^2-24Rr}{16R^2}+\frac{3s^2}{16R^2}=$$

$$=\frac{4R+r}{2R}\cdot\frac{16R^2+8Rr+r^2-3s^2}{16R^2}+\frac{3s^2}{16R^2}=$$

$$=\frac{1}{32R^3}(64R^3+48R^2r+12Rr^2+r^3-6Rs^2-3s^3r)=$$

$$=\frac{1}{32R^3}[(4R+r)^3-3s^2(2R+r)]$$

$$\sum_{cyc}\left(\frac{a}{(b+c)\cos^3\frac{A}{2}}\right)^2=\sum_{cyc}\left(\frac{\frac{a}{b+c}}{\cos^3\frac{A}{2}}\right)^2=\sum_{cyc}\frac{\left(\frac{a}{b+c}\right)^2}{\cos^6\frac{A}{2}}\geq$$

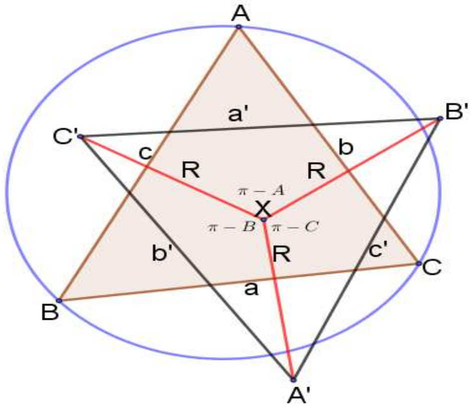
$$\geq\frac{\left(\sum\frac{a}{b+c}\right)^2}{\sum\cos^6\frac{A}{2}}\overset{Nesbitt}{\geq}\frac{9}{4}\cdot\frac{32R^3}{(4R+r)^2-3s^2(2R+r)}=\frac{72R^3}{(4R+r)^3-2s^2(2R+r)}$$

714. If $X\in Int(\Delta ABC), XA'=XB'=XC'=R, R$ –circumradii, $XA'\perp BC$, $XB'\perp CA, XC'\perp AB, a',b',c'$ –sides in $\Delta AB'C'$ then:

$$\frac{a}{a'}+\frac{b}{b'}+\frac{c}{c'}=2\left(\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\right)$$

Proposed by Mehmet Şahin-Ankara-Turkiye

Solution by Ertan Yildirim-Turkiye



$$(a')^2 = R^2 + R^2 - 2R \cdot R \cdot \cos(\pi - A) = 2R^2(1 + \cos A) = 2R^2 \cdot 2\cos^2\frac{A}{2} == 4R^2\cos^2\frac{A}{2}$$

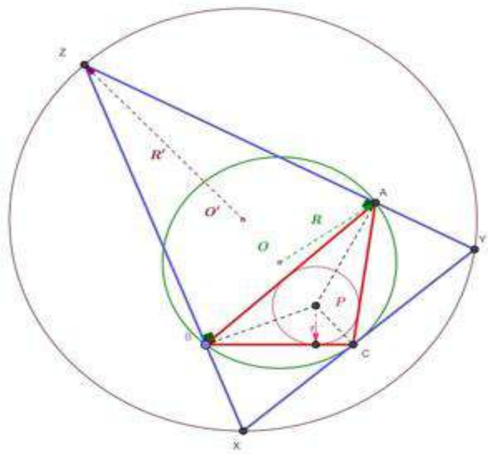
$$a = 2R \cdot \cos\frac{A}{2}, \text{similarly}, b' = 2R \cdot \cos\frac{B}{2} \text{ and } c'2R \cdot \cos\frac{C}{2}$$

$$\frac{a}{a'} + \frac{b}{b'} + \frac{c}{c'} = \frac{a}{2R\cos\frac{A}{2}} + \frac{b}{2R\cos\frac{B}{2}} + \frac{c}{2R\cos\frac{C}{2}} =$$

$$= \sum_{cyc} \frac{2R \cdot \sin A}{2R \cdot \cos\frac{A}{2}} = \sum_{cyc} \frac{2\sin\frac{A}{2}\cos\frac{A}{2}}{\cos\frac{A}{2}} = 2\left(\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}\right)$$

715. ΔABC **scalen,** $P = X(1)$ **of** ΔABC **and** ΔXYZ **antipedal triangle of** P**,** R **and** r **circumradius and inradius of** ΔABC, R' **–circumradius of** ΔXYZ**.**

Prove that: $\frac{[X,Y,Z]}{[A,B,C]} = \frac{2R}{r}$, $R' = 2R$.



Proposed by Juan Jose Isach Mayo-Valencia-Spain

Solution by proposer

ΔABC scalen, $P = X(1)$ of ΔABC and ΔXYZ antipedal triangle of P,

I −incenter of ΔXYZ is the orthocenter of ΔXYZ.

O −circumcenter of ΔXYZ is the nine point center of ΔXYZ.

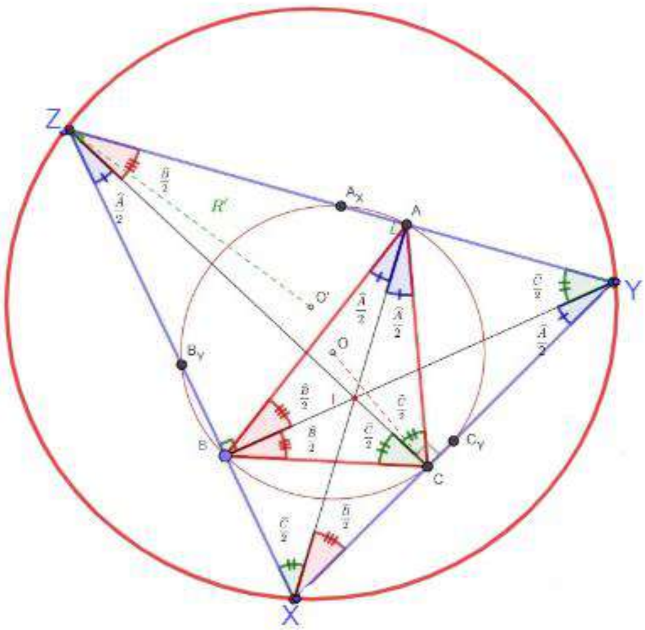
R −circumradius of ΔABC is the radius of Euler circle of ΔXYZ.

R' −is the radius of circumcircle of ΔXYZ.

We know that the circumradius of a triangle is twice the radius of its Euler circle associated, then we can affirm that $R' = 2R$.

$$\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}\ (and\ analogs)$$

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}\ (and\ analogs)$$



$$AI = \sqrt{\frac{(s-a)bc}{s}}\ (and\ analogs)$$

In ΔZAI rectangle in A, $\widehat{IZA} = \frac{B}{2}$

$$AZ = AI \cdot \cot\frac{B}{2} = \sqrt{\frac{(s-b)bc}{s-c}}$$

In ΔYAI rectangle in A, $\widehat{IYA} = \frac{C}{2}$, $\quad AY = AI \cdot \cot\frac{C}{2} = \sqrt{\frac{(s-c)bc}{s-b}}$

$$YZ = AZ + AY = \frac{a\sqrt{bc}}{\sqrt{(s-c)(s-b)}} = \frac{2a\sqrt{bc}}{\sqrt{(a+b-c)(a-b+c)}}$$

Similarly,

$$XZ = \frac{2b\sqrt{ac}}{\sqrt{(a+b-c)(-a+b+c)}} \text{ and } XY = \frac{2c\sqrt{ab}}{\sqrt{(-a+b+c)(a-b+c)}}$$

$$[I,Y,Z] = \frac{YZ \cdot AI}{2} = R(-a+b+c), \qquad [I,X,Z] = \frac{ZX \cdot BI}{2} = R(a-b+c)$$

$$[I,X,Y] = \frac{XY \cdot IC}{2} = R(a+b-c)$$

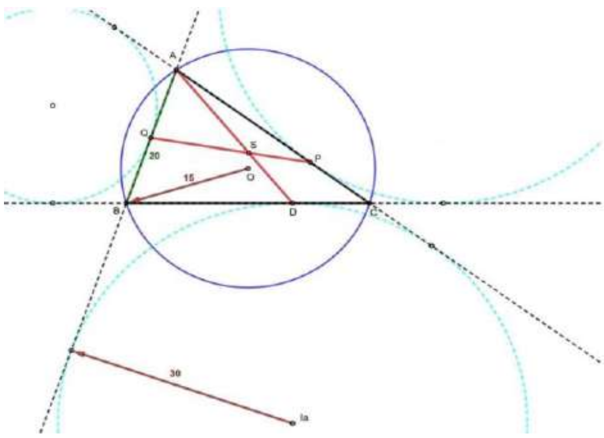
$$[X,Y,Z] = [I,Y,Z] + [I,Z,X] + [I,X,Y] = R(-a+b+c) + R(a-b+c) + R(a+b-c)$$

$$= R(a+b+c) = \frac{2R[A,B,C]}{r}$$

$$\text{How, } XY \cdot XZ \cdot YZ = \frac{8a^2b^2c^2}{(-a+b+c)(a-b+c)(a+b-c)} = 8R^2(a+b+c)$$

$$R' = \frac{XY \cdot XZ \cdot YZ}{4[X,Y,Z]} = 2R$$

716. Find: $\frac{AS}{SD} \cdot \frac{QS}{SP} = ?$



Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by Jose Ferreira Queiroz-Olinda-Brazil

$AQ = s - b, QB = s - a, BD = s - 20, DC = s - b, PC = s - a$ and $AP = s - 20$

$$c = 20, r_a = 30, R = 15, 2s = a + b + c$$

$$[ABC] = \frac{abc}{4R} = r_a(s - a)$$

$$\frac{ab \cdot 20}{4 \cdot 15} = \frac{30(-a + b + 20)}{2}$$

$$ab = 45(b - a + 20)$$

$$\frac{ab}{b - a + 20} = 45$$

Using Gakopoulos' Lemmas

$$\frac{SP}{SQ} = \frac{DC}{DB} \cdot \frac{AP}{AC} \cdot \frac{AB}{AQ}$$

$$\frac{SD}{SA} = \frac{1}{BC}\left(BD \cdot \frac{CP}{PA} + DC \cdot \frac{BQ}{QA}\right)$$

We have:

$$\frac{SP}{SQ} = \frac{s - b}{s - 20} \cdot \frac{s - 20}{b} \cdot \frac{20}{s - b} = \frac{20}{b}$$

$$\frac{SQ}{SP} = \frac{b}{20}, \qquad \frac{SD}{SA} = \frac{1}{a}\left((s - 20) \cdot \frac{s - a}{s - 20} + (s - b) \cdot \frac{s - a}{s - b}\right)$$

$$\frac{SD}{SA} = \frac{1}{a}(2s - 2a) = \frac{1}{a}(b - a + 20), \qquad \frac{SA}{SD} = \frac{a}{b - a + 20}$$

Then,

$$\frac{SA}{SD} \cdot \frac{SQ}{SP} = \frac{b}{20} \cdot \frac{a}{b - a + 20} = \frac{45}{20}$$

$$\frac{SA}{SD} \cdot \frac{SQ}{SP} = \frac{9}{4}$$

717. In ΔABC the following relationship holds:

$$3\sqrt{3}\frac{R}{r} \geq 2\sum_{cyc} \cos A \cot\frac{A}{2} + \sum_{cyc} \cot\frac{C}{2}(\cos A + \cos B) \geq 6\sqrt{3}$$

Proposed by Alex Szoros-Romania

Solution 1 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$\because \sum_{cyc}\cot\frac{A}{2}=\prod_{cyc}\cot\frac{A}{2}=\frac{s}{r}, \quad \because \sum_{cyc}\cos A=1+\frac{r}{R} \ \because \sum_{cyc}\sin A=\frac{s}{R}$

$$2\sum_{cyc}\cos A\cot\frac{A}{2}+\sum_{cyc}\cot\frac{C}{2}(\cos A+\cos B)=$$

$$=2\sum_{cyc}\cos A\cot\frac{A}{2}+\sum_{cyc}\cot\frac{C}{2}(\cos A+\cos B+\cos C)-\sum_{cyc}\cos A\cot\frac{A}{2}=$$

$$=\sum_{cyc}\cos A\cot\frac{A}{2}+\sum_{cyc}\cos A\cdot\sum_{cyc}\cot\frac{A}{2}=\sum_{cyc}\left(1-2\sin^2\frac{A}{2}\right)\cot\frac{A}{2}+\left(1+\frac{r}{R}\right)\cdot\frac{s}{r}=$$

$$=\sum_{cyc}\cot\frac{A}{2}-\sum_{cyc}\sin\frac{A}{2}+\left(1+\frac{r}{R}\right)\cdot\frac{s}{r}=\frac{s}{r}-\frac{s}{R}+\frac{s}{r}-\frac{s}{R}=\frac{2s}{r}$$

$$6\sqrt{3}\le\frac{2s}{r}\le 3\sqrt{3}\cdot\frac{s}{r}, \quad 3\sqrt{3}\cdot r\le s\le\frac{3\sqrt{3}}{2}\cdot R(Mitrinovic)$$

Solution 2 by Tapas Das-India

$$\cot\frac{C}{2}(\cos A+\cos B)=\cot\frac{C}{2}\cdot 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}=$$
$$=\cot\frac{C}{2}\cdot 2\cos\left(\frac{\pi}{2}-\frac{C}{2}\right)\cos\frac{A-B}{2}=\cot\frac{C}{2}\cdot 2\sin\frac{C}{2}\cos\frac{A-B}{2}=$$
$$=\frac{\cos\frac{C}{2}}{\sin\frac{C}{2}}\cdot 2\sin\frac{C}{2}\cos\frac{A-B}{2}=2\cos\frac{C}{2}\cos\frac{A-B}{2}=$$
$$=2\cos\left(\frac{\pi}{2}-\frac{A+B}{2}\right)\cos\frac{A-B}{2}=2\sin\frac{A+B}{2}\cos\frac{A-B}{2}=$$
$$=\sin A+\sin B$$
$$\sum_{cyc}\cot\frac{C}{2}(\cos A+\cos B)=2\sum_{cyc}\sin A;(1)$$

$$\cos A\cot\frac{A}{2}=\left(1-2\sin^2\frac{A}{2}\right)\cot\frac{A}{2}=\cot\frac{A}{2}-2\sin^2\frac{A}{2}\cdot\frac{\cos\frac{A}{2}}{\sin\frac{A}{2}}=$$
$$=\cot\frac{A}{2}-2\sin\frac{A}{2}\cos\frac{A}{2}=\cot\frac{A}{2}-\sin A$$
$$2\sum_{cyc}\cos A\cot\frac{A}{2}=2\sum_{cyc}\left(\cot\frac{A}{2}-\sin A\right);(2)$$

From (1), (2):

$$2\sum_{cyc}\cos A\cot\frac{A}{2}+\sum_{cyc}\cot\frac{C}{2}(\cos A+\cos B)=2\sum_{cyc}\sin A+2\sum_{cyc}\left(\cot\frac{A}{2}-\sin A\right)=$$

$$=2\sum_{cyc}\cot\frac{A}{2}\geq 2\cdot 3\sqrt{3},\text{ because}$$

$$\sum_{cyc}\cot\frac{A}{2}\geq 3\sqrt{3}\text{ and }\sum_{cyc}\cot\frac{A}{2}=\prod_{cyc}\cot\frac{A}{2}$$

Now, we have:

$$2\sum_{cyc}\cos A\cot\frac{A}{2}+\sum_{cyc}\cot\frac{C}{2}(\cos A+\cos B)=2\sum_{cyc}\cot\frac{A}{2}=$$

$$=2\prod_{cyc}\cot\frac{A}{2}=2\cdot\frac{s}{r}\leq 2\cdot\frac{3\sqrt{3}R}{2r}=3\sqrt{3}\cdot\frac{R}{r}$$

$$\because s^2\leq\frac{27}{4}R^2$$

718. **In ΔABC the following relationship holds:**

$$\frac{5R^2}{12r^2}\geq\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{abc}{2(a^3+b^3+c^3)}$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution by Alex Szoros-Romania

$$\frac{a^3+b^3+c^3}{3}\geq abc\Rightarrow\frac{1}{3}\geq\frac{abc}{a^3+b^3+c^3}\Rightarrow\frac{1}{6}\geq\frac{abc}{2(a^3+b^3+c^3)};(1)$$

$$R\geq 2r\ (Euler)\Rightarrow\frac{R}{12r}\geq\frac{1}{6}\overset{(1)}{\Rightarrow}\frac{R}{12r}\geq\frac{abc}{2(a^3+b^3+c^3)};(2)$$

$$\sum_{cyc}\left(\frac{a}{b}+\frac{b}{a}\right)=\sum_{cyc}\left(\frac{a}{b}+\frac{a}{c}\right)=\sum_{cyc}\frac{a(b+c)}{bc}\geq 4\sum_{cyc}\frac{a}{b+c}$$

$$\sum_{cyc}\left(\frac{a}{b}+\frac{b}{a}\right)\geq 4\sum_{cyc}\frac{a}{b+c};(3)$$

$$\frac{R}{r}.=\frac{a}{b}+\frac{b}{a}(Bandila)\Rightarrow\frac{3R}{r}\geq\sum_{cyc}\left(\frac{a}{b}+\frac{b}{a}\right)\overset{(3)}{\Rightarrow}$$

$$\frac{3R}{r}\geq 4\sum_{cyc}\frac{a}{b+c}\Rightarrow\frac{3R}{4r}\geq\sum_{cyc}\frac{a}{b+c};(4)$$

From (2),(4) we get:

$$\frac{R}{12r}+\frac{3R}{4r}\geq\sum_{cyc}\frac{a}{b+c}+\frac{abc}{2(a^3+b^3+c^3)};(5)$$

We must show:

$$\frac{5R^2}{12r^2}\geq\frac{R}{12r}+\frac{3R}{4r}\,\Big|:\frac{R}{4r};\quad(6)\Leftrightarrow\frac{5R}{3r}\geq\frac{1}{3}+3\Leftrightarrow$$

$$\frac{5R}{3r}\geq\frac{10}{3}\Leftrightarrow R\geq 2r\ (Euler)$$

From (5) and (6), we get:

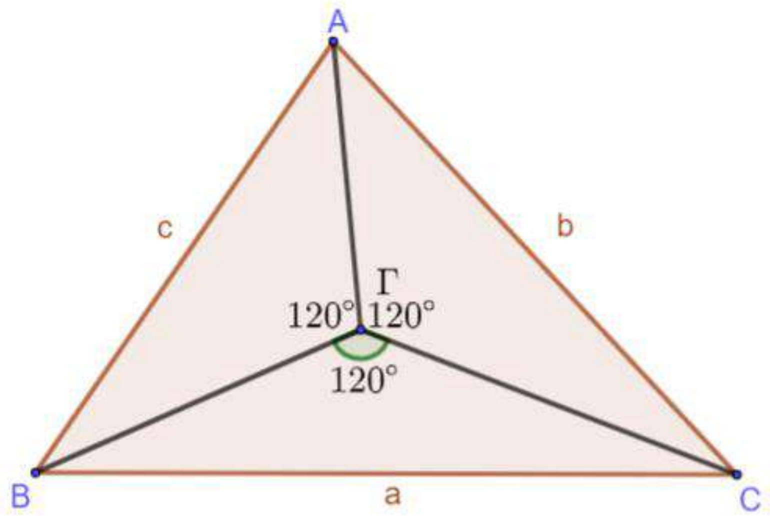
$$\frac{5R^2}{12r^2}\geq\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{abc}{2(a^3+b^3+c^3)}$$

719. In $\Delta ABC,\Gamma$ –Toricelli's point, holds:

$$(\Gamma A^3+\Gamma B^3+\Gamma C^3)\left(\frac{1}{\Gamma A}+\frac{1}{\Gamma B}+\frac{1}{\Gamma C}\right)\geq 36r^2$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Tapas Das-India



Γ –Toricelli's point for triangle ABC, then $\angle A\Gamma B=\angle B\Gamma C=\angle C\Gamma A$

$$(\Gamma A^3+\Gamma B^3+\Gamma C^3)\left(\frac{1}{\Gamma A}+\frac{1}{\Gamma B}+\frac{1}{\Gamma C}\right)=$$

$$=\left(\left(\Gamma A^{\frac{3}{2}}\right)^2+\left(\Gamma B^{\frac{3}{2}}\right)^2+\left(\Gamma C^{\frac{3}{2}}\right)^2\right)\left(\frac{1}{\Gamma A}+\frac{1}{\Gamma B}+\frac{1}{\Gamma C}\right)\geq$$

$$\geq (\Gamma A+\Gamma B+\Gamma C)^2 \geq 3(\Gamma A\cdot\Gamma B+\Gamma B\cdot\Gamma C+\Gamma C\cdot\Gamma A) \geq 3\cdot 12r^2 = 36r^2$$

$$[AB\Gamma]=\frac{1}{2}\Gamma A\cdot\Gamma B\cdot\sin 120^\circ=\frac{\sqrt{3}}{4}\cdot\Gamma A\cdot\Gamma B$$

Similarly,

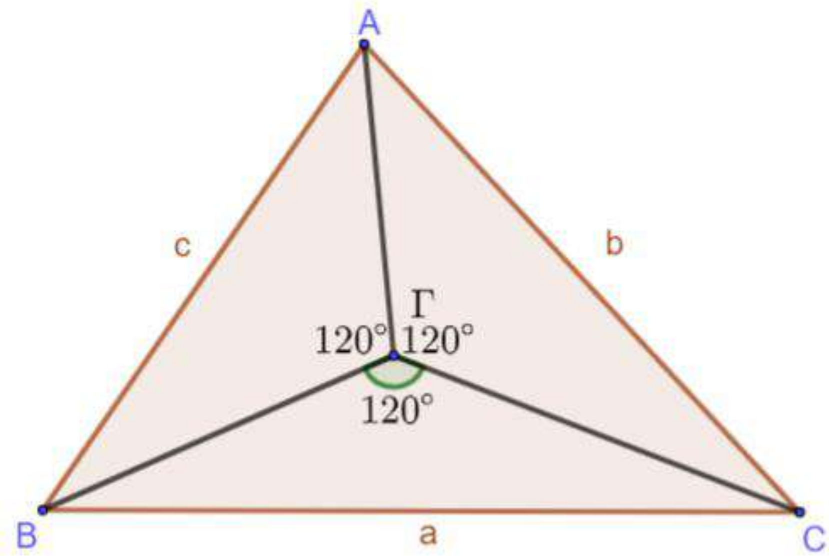
$$[BC\Gamma]=\frac{\sqrt{3}}{4}\cdot\Gamma B\cdot\Gamma C \text{ and } [CA\Gamma]=\frac{\sqrt{3}}{4}\cdot\Gamma A\cdot\Gamma C$$

$$[ABC]=[\Gamma AB]+[\Gamma BC]+[\Gamma CA]=\frac{\sqrt{3}}{4}(\Gamma A\cdot\Gamma B+\Gamma B\cdot\Gamma C+\Gamma C\cdot\Gamma A)$$

$$rs=\frac{\sqrt{3}}{4}(\Gamma A\cdot\Gamma B+\Gamma B\cdot\Gamma C+\Gamma C\cdot\Gamma A)$$

$$\Gamma A\cdot\Gamma B+\Gamma B\cdot\Gamma C+\Gamma C\cdot\Gamma A=\frac{4rs}{\sqrt{3}}\overset{Mitrinovic}{\geq}\frac{4r\cdot 3\sqrt{3}r}{\sqrt{3}}=12r^2$$

Solution 2 by Adrian Popa-Romania



$$(\Gamma A^3+\Gamma B^3+\Gamma C^3)\left(\frac{1}{\Gamma A}+\frac{1}{\Gamma B}+\frac{1}{\Gamma C}\right)\overset{CBS}{\geq}(\Gamma A+\Gamma B+\Gamma C)^2\overset{(?)}{\geq}36r^2$$

$$\because (x+y+z)^2=x^2+y^2+z^2+2(xy+yz+zx)\geq 3(xy+yz+zx)$$

$$[AB\Gamma]=\frac{1}{2}\Gamma A\cdot\Gamma B\cdot\sin 120^\circ=\frac{\sqrt{3}}{4}\cdot\Gamma A\cdot\Gamma B$$

Similarly,

$$[BC\Gamma] = \frac{\sqrt{3}}{4}\cdot \Gamma B \cdot \Gamma C \text{ and } [CA\Gamma] = \frac{\sqrt{3}}{4}\cdot \Gamma A \cdot \Gamma C$$

$$[ABC] = [\Gamma AB] + [\Gamma BC] + [\Gamma CA] = \frac{\sqrt{3}}{4}(\Gamma A\cdot\Gamma B + \Gamma B\cdot\Gamma C + \Gamma C\cdot\Gamma A)$$

$$rs = \frac{\sqrt{3}}{4}(\Gamma A\cdot\Gamma B + \Gamma B\cdot\Gamma C + \Gamma C\cdot\Gamma A)$$

$$\Gamma A\cdot\Gamma B + \Gamma B\cdot\Gamma C + \Gamma C\cdot\Gamma A = \frac{4rs}{\sqrt{3}} \overset{Mitrinovic}{\geq} \frac{4r\cdot 3\sqrt{3}r}{\sqrt{3}} = 12r^2$$

720. Let ω be the Brocard's angle in ΔABC. Prove that:

$$4 + \frac{2}{\sin^2\omega} > \sum_{cyc}(n_a+g_a)\sqrt{\frac{r_b+r_c}{h_a^{\ 3}}} + \sum_{cyc}\frac{n_a g_a}{h_a^{\ 2}}\left(2 - \frac{9\sqrt{2r_b r_c}}{n_a+g_a+\sqrt{2r_b r_c}}\right)$$

Proposed by Bogdan Fuştei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have: \frac{r_b+r_c}{r_b r_c} = \frac{1}{r_b}+\frac{1}{r_c} = \frac{s-b}{F}+\frac{s-c}{F} = \frac{a}{F} = \frac{2}{h_a},\ then:$$

$$\sqrt{\frac{r_b+r_c}{h_a^{\ 3}}} = \frac{\sqrt{2r_b r_c}}{h_a^{\ 2}}\ (and\ analogs)$$

$$Lemma:\ If\ x,y,z>0\ then:\ 2(xy+yz+zx) - \frac{9xyz}{x+y+z} \leq x^2+y^2+z^2\ (*).$$

$$Proof:\ (*) \Leftrightarrow 2(xy+yz+zx)(x+y+z) - 9xyz \leq (x^2+y^2+z^2)(x+y+z)$$

$$\Leftrightarrow xy(x+y)+yz(y+z)+zx(z+x) \leq x^3+y^3+z^3+3xyz,$$

which is Schur's inequality.

$$Setting\ x = n_a,\ y = g_a,\ z = \sqrt{2r_b r_c}\ we\ get:$$

$$2\left(n_a g_a + g_a\sqrt{2r_b r_c} + n_a\sqrt{2r_b r_c}\right) - \frac{9n_a g_a\sqrt{2r_b r_c}}{n_a+g_a+\sqrt{2r_b r_c}} \leq n_a^{\ 2}+g_a^{\ 2}+2r_b r_c\ (1)$$

Now, we have:

$$n_a{}^2 = r_b r_c + \frac{(b-c)^2 s}{a},\ \ g_a{}^2 = r_b r_c - \frac{(b-c)^2(s-a)}{a},\ \ r_b r_c = s(s-a).$$

Then :

$$n_a{}^2 + g_a{}^2 + 2r_b r_c = 4s(s-a) + (b-c)^2 = (b+c)^2 - a^2 + (b-c)^2$$
$$= 2b^2 + 2c^2 - a^2 = 4m_a{}^2.$$

Then :

$$(1) \Leftrightarrow 2(n_a + g_a)\sqrt{2r_b r_c} + n_a g_a\left(2 - \frac{9\sqrt{2r_b r_c}}{n_a + g_a + \sqrt{2r_b r_c}}\right) \le 4m_a{}^2 \ (and\ analogs)$$

Therefore,
$$\sum_{cyc}(n_a + g_a)\sqrt{\frac{r_b + r_c}{h_a{}^3}} + \sum_{cyc}\frac{n_a g_a}{h_a{}^2}\left(2 - \frac{9\sqrt{2r_b r_c}}{n_a + g_a + \sqrt{2r_b r_c}}\right) =$$

$$= \sum_{cyc}\frac{1}{h_a{}^2}\left((n_a + g_a)\sqrt{2r_b r_c} + n_a g_a\left(2 - \frac{9\sqrt{2r_b r_c}}{n_a + g_a + \sqrt{2r_b r_c}}\right)\right) \overset{(1)}{\lesssim} \sum_{cyc}\frac{4m_a{}^2}{h_a{}^2} =$$

$$= \sum_{cyc}\frac{a^2(2b^2 + 2c^2 - a^2)}{4F^2} = \frac{2\sum_{cyc} a^2 b^2 - \sum_{cyc} a^4}{4F^2} + \frac{\sum_{cyc} a^2 b^2}{2F^2}.$$

Using the following identities : $2\sum_{cyc} a^2 b^2 - \sum_{cyc} a^4 = 16F^2$ *and*

$$\sin\omega = \frac{2F}{\sqrt{\sum_{cyc} a^2 b^2}}, we\ get :$$

$$\sum_{cyc}(n_a + g_a)\sqrt{\frac{r_b + r_c}{h_a{}^3}} + \sum_{cyc}\frac{n_a g_a}{h_a{}^2}\left(2 - \frac{9\sqrt{2r_b r_c}}{n_a + g_a + \sqrt{2r_b r_c}}\right) < 4 + \frac{2}{\sin^2\omega}$$

721.

In ΔABC the following relationship holds:

$$\frac{R}{2r}\left(\sqrt{\frac{m_a + m_b}{m_b + m_c}} + \sqrt{\frac{m_b + m_c}{m_c + m_a}} + \sqrt{\frac{m_c + m_a}{m_a + m_b}}\right) \ge \sqrt{\frac{m_b + m_c}{m_a + m_b}} + \sqrt{\frac{m_c + m_a}{m_b + m_c}} + \sqrt{\frac{m_a + m_b}{m_c + m_a}}$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc}\sqrt{\frac{m_b+m_c}{m_a+m_b}} \overset{CBS}{\le} \sqrt{\sum_{cyc}(m_b+m_c).\sum_{cyc}\frac{1}{m_a+m_b}} \overset{CBS}{\le} \sqrt{2\sum_{cyc}m_a.\sum_{cyc}\frac{1}{4}\left(\frac{1}{m_a}+\frac{1}{m_b}\right)} \le$$

$$\overset{Leuenberger}{\le} \sqrt{(4R+r).\sum_{cyc}\frac{1}{m_a}} \overset{m_a\ge h_a}{\le} \sqrt{(4R+r).\sum_{cyc}\frac{1}{h_a}} = \sqrt{\frac{4R+r}{r}} \overset{Euler}{\le} \frac{3R}{2r}.$$

$$\sqrt{\frac{m_b+m_c}{m_a+m_b}}+\sqrt{\frac{m_c+m_a}{m_b+m_c}}+\sqrt{\frac{m_a+m_b}{m_c+m_a}} \le 3.\frac{R}{2r} \overset{AM-GM}{\le}$$

$$\frac{R}{2r}\left(\sqrt{\frac{m_a+m_b}{m_b+m_c}}+\sqrt{\frac{m_b+m_c}{m_c+m_a}}+\sqrt{\frac{m_c+m_a}{m_a+m_b}}\right)$$

722. In ΔABC the following relationship holds:

$$\sqrt{\frac{2}{\sin\omega}}\sum_{cyc}\sqrt{\frac{m_a}{h_a}} \ge \sum_{cyc}\frac{m_b+m_c}{m_a}$$

Proposed by Bogdan Fuştei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Lemma\ 1:\ \ In\ \Delta ABC\ we\ have:\ \frac{m_b}{m_c}+\frac{m_c}{m_b} \le \frac{2m_a}{h_a}\quad (1)$$

$$Proof:We\ have:\ \frac{2m_a}{h_a} \overset{Tereshin}{\ge} 2.\frac{b^2+c^2}{4R}.\frac{2R}{bc} = \frac{b}{c}+\frac{c}{b},\ then:\ \frac{b}{c}+\frac{c}{b} \le \frac{2m_a}{h_a}\quad (i)$$

$$m_a,m_b,m_c\ can\ be\ the\ sides\ of\ triangle\ with:$$

$$F_m = \frac{3}{4}F,\ \overline{m_a} = \frac{3}{4}a,\ \overline{h_a} = \frac{2F_m}{m_a} = \frac{3F}{2m_a}.$$

$$Using\ (i)\ in\ \Delta m_a m_b m_c, we\ get:\ \frac{m_b}{m_c}+\frac{m_c}{m_b} \le \frac{2\overline{m_a}}{\overline{h_a}} = \frac{3}{2}a.\frac{2m_a}{3F} = \frac{2m_a}{h_a},\ as\ desired.$$

$$Lemma\ 2:\ \ In\ \Delta ABC\ we\ have:\ \frac{b}{c}+\frac{c}{b} \le \frac{\sqrt{a^2b^2+b^2c^2+c^2a^2}}{2F}\quad (*)$$

Proof: ***Since*** $4F=\sqrt{2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)}$ ***then we have:***

$$(*)\Leftrightarrow\left(b^2+c^2\right)\sqrt{2\left(a^2b^2+b^2c^2+c^2a^2\right)-\left(a^4+b^4+c^4\right)}\le 2bc\sqrt{a^2b^2+b^2c^2+c^2a^2}$$

$$\overset{squaring}{\Leftrightarrow}\left(2b^2c^2+b^4+c^4\right)\left[2\left(a^2b^2+b^2c^2+c^2a^2\right)-\left(a^4+b^4+c^4\right)\right]\le 4b^2c^2\left(a^2b^2+b^2c^2+c^2a^2\right)$$

$$\Leftrightarrow -a^4(b^2+c^2)^2-2b^2c^2(b^4+c^4)+2(b^4+c^4)(a^2b^2+b^2c^2+c^2a^2)-(b^4+c^4)^2\le 0$$

$$\Leftrightarrow -a^4(b^2+c^2)^2+2(b^4+c^4)(a^2b^2+c^2a^2)-(b^4+c^4)^2=-[a^2(b^2+c^2)-(b^4+c^4)]^2\le 0$$

Which is true and the proof of the lemma 2 is completed.

Using the lemma 2 in $\Delta m_am_bm_c$***, we have:***

$$\frac{m_b}{m_c}+\frac{m_c}{m_b}\le\frac{\sqrt{{m_a}^2{m_b}^2+{m_b}^2{m_c}^2+{m_c}^2{m_a}^2}}{2F_m}=\frac{\sqrt{\frac{9}{16}\left(a^2b^2+b^2c^2+c^2a^2\right)}}{\frac{3F}{2}}=\frac{1}{\sin\omega}\quad(2)$$

From (1) ***and*** (2) ***we have:***

$$\sqrt{\frac{1}{\sin\omega}}.\sqrt{\frac{2m_a}{h_a}}\ge\sqrt{\frac{m_b}{m_c}+\frac{m_c}{m_b}}.\sqrt{\frac{m_b}{m_c}+\frac{m_c}{m_b}}=\frac{m_b}{m_c}+\frac{m_c}{m_b}\quad(\text{and analogs})$$

Therefore, $\sqrt{\frac{2}{\sin\omega}}\sum_{cyc}\sqrt{\frac{m_a}{h_a}}\ge\sum_{cyc}\left(\frac{m_b}{m_c}+\frac{m_c}{m_b}\right)=\sum_{cyc}\frac{m_b+m_c}{m_a}.$

723. In ΔABC the following relationship holds:

$$\frac{R}{2r}\left(\sqrt{\frac{a+b}{b+c}}+\sqrt{\frac{b+c}{c+a}}+\sqrt{\frac{c+a}{a+b}}\right)\ge\sqrt{\frac{b+c}{a+b}}+\sqrt{\frac{c+a}{b+c}}+\sqrt{\frac{a+b}{c+a}}$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:\ \sum_{cyc}\sqrt{\frac{b+c}{a+b}}\overset{CBS}{\leq}\sqrt{\sum_{cyc}(b+c).\sum_{cyc}\frac{1}{a+b}}\overset{CBS}{\leq}\sqrt{4s.\sum_{cyc}\frac{1}{4}\left(\frac{1}{a}+\frac{1}{b}\right)}$$

$$=\sqrt{2s\sum_{cyc}\frac{1}{a}}\overset{Leuenberger}{\leq}\sqrt{2s.\frac{\sqrt{3}}{2r}}\overset{Mitrinovic}{\leq}\sqrt{3\sqrt{3}R.\frac{\sqrt{3}}{2r}}=3\sqrt{\frac{R}{2r}}\overset{Euler}{\leq}\frac{3R}{2r}.$$

$$Therefore,$$

$$\sqrt{\frac{b+c}{a+b}}+\sqrt{\frac{c+a}{b+c}}+\sqrt{\frac{a+b}{c+a}}\leq 3.\frac{R}{2r}\overset{AM-GM}{\leq}\frac{R}{2r}\left(\sqrt{\frac{a+b}{b+c}}+\sqrt{\frac{b+c}{c+a}}+\sqrt{\frac{c+a}{a+b}}\right).$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\text{Let } s-a=x, s-b=y \text{ and } s-c=z \therefore s=x+y+z\Rightarrow a=y+z, b=z+x \text{ and } c=x+y$$

$$\text{Now, }\frac{s^2}{r^2}=\frac{s^4}{\Delta^2}=\frac{s^4}{s(s-a)(s-b)(s-c)}\overset{(\bullet)}{=}\frac{\left(\sum_{cyc}x\right)^3}{xyz}\text{ and } 1+\frac{4R}{r}$$

$$=1+\frac{4sabc}{4s(s-a)(s-b)(s-c)}=1+\frac{\prod_{cyc}(y+z)}{xyz}$$

$$\Rightarrow 1+\frac{4R}{r}\overset{(\bullet\bullet)}{=}\frac{xyz+\prod_{cyc}(y+z)}{xyz}$$

$$\text{Now, }\sum_{cyc}\frac{b}{a}=\sum\frac{z+x}{y+z}\Rightarrow\sum_{cyc}\frac{b}{a}\overset{(\bullet\bullet\bullet)}{=}\frac{\sum_{cyc}(x+y)^2(y+z)}{\prod_{cyc}(y+z)}\therefore(\bullet),(\bullet\bullet),(\bullet\bullet\bullet)\Rightarrow\frac{s^2}{r^2}$$

$$\geq\left(\sum_{cyc}\frac{b}{a}\right)\left(1+\frac{4R}{r}\right)\Leftrightarrow\frac{\left(\sum_{cyc}x\right)^3}{xyz}\geq\left(\frac{xyz+\prod_{cyc}(y+z)}{xyz}\right)\left(\frac{\sum_{cyc}(x+y)^2(y+z)}{\prod_{cyc}(y+z)}\right)$$

$$\Leftrightarrow\left(\prod_{cyc}(y+z)\right)\left(\sum_{cyc}x\right)^3\geq\left(xyz+\prod_{cyc}(y+z)\right)\left(\sum_{cyc}(x+y)^2(y+z)\right)$$

$$\Leftrightarrow\sum_{cyc}x^4y^2+\sum_{cyc}x^3y^3\overset{(i)}{\geq}xyz\left(\sum_{cyc}xy^2\right)+3x^2y^2z^2$$

$$\text{Now, }\forall u,v,w>0, u^3+u^3+v^3\overset{A-G}{\geq}3u^2v, v^3+v^3+w^3\overset{A-G}{\geq}3v^2w \text{ and } w^3+w^3$$
$$+u^3\overset{A-G}{\geq}3w^2u \text{ and summing up}: \sum_{cyc}u^3\geq\sum_{cyc}u^2v \text{ and choosing } u=xy, v$$
$$=yz \text{ and } w=zx,$$

$$\sum_{cyc} x^3y^3 \underbrace{\geq}^{(*)} xyz\left(\sum_{cyc} xy^2\right) \text{ and } \sum_{cyc} x^4y^2 \underbrace{\geq}_{(**)}^{A-G} 3x^2y^2z^2 \therefore (*) + (**) \Rightarrow \text{(i) is true} \Rightarrow \frac{s^2}{r^2}$$

$$\geq \left(\sum_{cyc} \frac{b}{a}\right)\left(1 + \frac{4R}{r}\right) \Rightarrow \boxed{\sum_{cyc} \frac{b}{a} \overset{(\bullet\bullet\bullet\bullet)}{\leq} \frac{s^2}{r(4R+r)}}$$

$$\text{Now, } \frac{a}{b} + \frac{b}{c} + \frac{c}{a} + \frac{R}{2r} \overset{?}{\geq} 1 + \frac{b}{a} + \frac{c}{b} + \frac{a}{c} \Leftrightarrow \sum_{cyc} \frac{a}{b} + \sum_{cyc} \frac{b}{a} + \frac{R-2r}{2r} \overset{?}{\geq} 2\sum_{cyc} \frac{b}{a}$$

$$\Leftrightarrow \frac{\sum_{cyc}\left(ab\left(\sum_{cyc} a - c\right)\right)}{4Rrs} + \frac{R-2r}{2r} \overset{?}{\geq} 2\sum_{cyc} \frac{b}{a}$$

$$\Leftrightarrow \frac{2s\left(s^2+4Rr+r^2\right) - 12Rrs}{4Rrs} + \frac{R-2r}{2r} \overset{?}{\geq} 2\sum_{cyc} \frac{b}{a}$$

$$\Leftrightarrow \frac{s^2 - 2Rr + r^2 + R(R-2r)}{2Rr} \underbrace{\overset{?}{\geq}}_{(***)} 2\sum_{cyc} \frac{b}{a} \text{ and } \because 2\sum_{cyc} \frac{b}{a} \overset{\text{via } (\bullet\bullet\bullet\bullet)}{\leq} \frac{2s^2}{r(4R+r)}$$

$$\therefore \text{ in order to prove } (***), \text{it suffices to prove : } \frac{s^2 - 2Rr + r^2 + R(R-2r)}{2Rr}$$

$$\geq \frac{2s^2}{r(4R+r)}$$

$$\Leftrightarrow rs^2 + R(R-2r)(4R+r) \overset{(****)}{\geq} r(2R-r)(4R+r)$$

$$\text{Now, LHS of } (****) \overset{\text{Gerretsen}}{\geq} r\left(16Rr - 5r^2\right) + R(R-2r)(4R+r) \overset{?}{\geq} r(2R-r)(4R+r)$$

$$\Leftrightarrow 4t^3 - 15t^2 + 16t - 4 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right) \Leftrightarrow (t-2)(4t(t-2)+t+2) \overset{?}{\geq} 0$$

$$\rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (****) \Rightarrow (***) \text{ is true} \Rightarrow \frac{a}{b} + \frac{b}{c} + \frac{c}{a} + \frac{R}{2r} \geq 1 + \frac{b}{a} + \frac{c}{b} + \frac{a}{c}$$

$$\Rightarrow \frac{a}{b} + \frac{b}{c} + \frac{c}{a} + \frac{abc(a+b+c)}{16F^2} \geq 1 + \frac{b}{a} + \frac{c}{b} + \frac{a}{c}$$

$$\Rightarrow \boxed{1 + \frac{b}{a} + \frac{c}{b} + \frac{a}{c} - \left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) \overset{(l)}{\leq} \frac{abc(a+b+c)}{2(a^2b^2+b^2c^2+c^2a^2) - (a^4+b^4+c^4)}}$$

$$\text{Now, } \frac{s^2}{s^2+4Rr+r^2} \leq \frac{3}{4}\cdot\sqrt{\frac{R}{2r}} \Leftrightarrow \frac{9R}{32r} \geq \frac{s^4}{(s^2+4Rr+r^2)^2}$$

$$\Leftrightarrow (9R-32r)s^4 + 18Rs^2\left(4Rr+r^2\right) + 9R\left(4Rr+r^2\right)^2 \overset{(*****)}{\geq} 0$$

$\boxed{\text{Case 1}}$ $9R - 32r \geq 0$ and then, LHS of $(*****) \geq 18Rs^2\left(4Rr+r^2\right) + 9R\left(4Rr+r^2\right)^2 > 0$

$\Rightarrow (*****)$ is true (strict inequality)

$\boxed{\text{Case 2}}$ $9R - 32r < 0$ and then, LHS of $(*****)$

$$\geq (9R-32r)\left(4R^2+4Rr+3r^2\right)s^2 + 18Rs^2\left(4Rr+r^2\right) + 9R\left(4Rr+r^2\right)^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow \left(36R^3 - 20R^2r - 83Rr^2 - 96r^3\right)s^2 + 9R\left(4Rr+r^2\right)^2 \underbrace{\overset{?}{\geq}}_{(******)} 0$$

Case 2i $36R^3-20R^2r-83Rr^2-96r^3\geq 0$ *and* then, LHS of $(******)\geq 9R(4Rr+r^2)^2$
$>0\Rightarrow (******)$ is true (strict inequality)

Case 2ii $36R^3-20R^2r-83Rr^2-96r^3<0$ *and* then, LHS of $(******)$

$$\geq(36R^3-20R^2r-83Rr^2-96r^3)(4R^2+4Rr+3r^2)+9R(4Rr+r^2)^2\overset{?}{\geq}0$$

$$\Leftrightarrow 9t^5+4t^4-10t^3-44t^2-39t-18\overset{?}{\geq}0\Leftrightarrow(t-2)(9t^4+22t^3+34t^2+24t+9)\overset{?}{\geq}0$$

$\rightarrow$ true $\because t\overset{\text{Euler}}{\geq}2\Rightarrow(******)$ is true

$\therefore$ combining cases 2i, 2ii, $(******)$ is true

for *all* triangles *and* subsequently, combining cases 1, 2, $(*****)$ is true for *all* triangles

$$\Rightarrow \text{in any }\Delta ABC,\ \boxed{\frac{s^2}{s^2+4Rr+r^2}\overset{(m)}{\leq}\frac{3}{4}\cdot\sqrt{\frac{R}{2r}}}$$

$\because \sqrt{a+b},\sqrt{b+c},\sqrt{c+a}$ form sides of *a* triangle,

$\therefore$ via (l) on triangle with sides $\sqrt{a+b},\sqrt{b+c},\sqrt{c+a}$, we arrive at :

$$1+\sqrt{\frac{b+c}{a+b}}+\sqrt{\frac{c+a}{b+c}}+\sqrt{\frac{a+b}{c+a}}-\left(\sqrt{\frac{a+b}{b+c}}+\sqrt{\frac{b+c}{c+a}}+\sqrt{\frac{c+a}{a+b}}\right)$$

$$\leq\frac{\sqrt{\prod_{cyc}(a+b)}.\left(\sum_{cyc}\sqrt{a+b}\right)}{2\sum_{cyc}(a+b)(b+c)-\sum_{cyc}(a+b)^2}\leq\frac{\sum_{cyc}(a+b)(b+c)}{2\sum_{cyc}(a+b)(b+c)-\sum_{cyc}(a+b)^2}$$

$$=\frac{\left(\sum_{cyc}a^2+2\sum_{cyc}ab\right)+\sum_{cyc}ab}{2\sum_{cyc}a^2+6\sum_{cyc}ab-2\sum_{cyc}a^2-2\sum_{cyc}ab}=\frac{4s^2+s^2+4Rr+r^2}{4(s^2+4Rr+r^2)}$$

$$=\frac{1}{4}+\frac{s^2}{s^2+4Rr+r^2}\overset{\text{via (m)}}{\leq}\frac{1}{4}+\frac{3}{4}\cdot\sqrt{\frac{R}{2r}}$$

$$\Rightarrow\sqrt{\frac{b+c}{a+b}}+\sqrt{\frac{c+a}{b+c}}+\sqrt{\frac{a+b}{c+a}}-\left(\sqrt{\frac{a+b}{b+c}}+\sqrt{\frac{b+c}{c+a}}+\sqrt{\frac{c+a}{a+b}}\right)\leq\frac{3}{4}\cdot\left(\sqrt{\frac{R}{2r}}-1\right)$$

$$=\frac{3}{4}\cdot\frac{\frac{R-2r}{2r}}{\sqrt{\frac{R}{2r}}+1}\overset{\text{Euler}}{\leq}\frac{3}{8}\cdot\frac{R-2r}{2r}\leq\sqrt{3}.\frac{R-2r}{2r}$$

$$\Rightarrow\boxed{\sqrt{\frac{b+c}{a+b}}+\sqrt{\frac{c+a}{b+c}}+\sqrt{\frac{a+b}{c+a}}-\left(\sqrt{\frac{a+b}{b+c}}+\sqrt{\frac{b+c}{c+a}}+\sqrt{\frac{c+a}{a+b}}\right)\overset{(n)}{\leq}\sqrt{3}.\frac{R-2r}{2r}}$$

$$\text{Now, }\frac{R}{2r}\left(\sqrt{\frac{a+b}{b+c}}+\sqrt{\frac{b+c}{c+a}}+\sqrt{\frac{c+a}{a+b}}\right)\geq\sqrt{\frac{b+c}{a+b}}+\sqrt{\frac{c+a}{b+c}}+\sqrt{\frac{a+b}{c+a}}$$

$$\Leftrightarrow \frac{R-2r}{2r}\cdot\left(\sqrt{\frac{a+b}{b+c}}+\sqrt{\frac{b+c}{c+a}}+\sqrt{\frac{c+a}{a+b}}\right)\overset{(******)}{\geq}\sqrt{\frac{b+c}{a+b}}+\sqrt{\frac{c+a}{b+c}}+\sqrt{\frac{a+b}{c+a}}$$

$$-\left(\sqrt{\frac{a+b}{b+c}}+\sqrt{\frac{b+c}{c+a}}+\sqrt{\frac{c+a}{a+b}}\right)$$

$$\text{Now, LHS of }(******)\overset{A-G}{\geq}\sqrt{3}.\frac{R-2r}{2r}\overset{\text{via }(n)}{\geq}\sqrt{\frac{b+c}{a+b}}+\sqrt{\frac{c+a}{b+c}}+\sqrt{\frac{a+b}{c+a}}$$

$$-\left(\sqrt{\frac{a+b}{b+c}}+\sqrt{\frac{b+c}{c+a}}+\sqrt{\frac{c+a}{a+b}}\right)\Rightarrow(******)\text{ is true}$$

$$\therefore \text{ in any }\Delta\, ABC, \frac{R}{2r}\left(\sqrt{\frac{a+b}{b+c}}+\sqrt{\frac{b+c}{c+a}}+\sqrt{\frac{c+a}{a+b}}\right)$$

$$\geq\sqrt{\frac{b+c}{a+b}}+\sqrt{\frac{c+a}{b+c}}+\sqrt{\frac{a+b}{c+a}}, \text{equality iff }\Delta\, ABC\text{ is equilateral (QED)}$$

Solution 3 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\frac{R}{2r}\sum_{cyc}\sqrt{\frac{a+b}{b+c}}\geq\frac{R}{2r}\cdot 3=\frac{3R}{2r};(1)$$

$$\sum_{cyc}\sqrt{\frac{b+c}{a+b}}\overset{CBS}{\leq}\sqrt{3\sum_{cyc}\frac{b+c}{a+b}}\overset{CBS}{\leq}\sqrt{\frac{3}{4}\sum_{cyc}(b+c)\left(\frac{1}{a}+\frac{1}{b}\right)}=$$

$$=\sqrt{\frac{3}{4}\sum_{cyc}\left(1+\frac{c}{a}+\frac{c}{b}+\frac{b}{a}\right)}=\sqrt{\frac{3}{4}\left(3+\sum_{cyc}\left(\frac{b}{a}+\frac{a}{b}\right)+\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)\right)}=$$

$$\sqrt{\frac{3}{4}(3+2\sum_{cyc}\left(\frac{a}{b}+\frac{b}{a}\right)-\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)}=$$

$$=\sqrt{\frac{3}{4}\left(3+\frac{6R}{r}-2\right)}=\sqrt{\frac{3}{4}\cdot\frac{6R}{r}}=\frac{3}{2}\sqrt{2\cdot\frac{R}{r}}\overset{Euler}{\leq}\frac{3}{2}\sqrt{\left(\frac{R}{r}\right)^2}=\frac{2R}{2r}$$

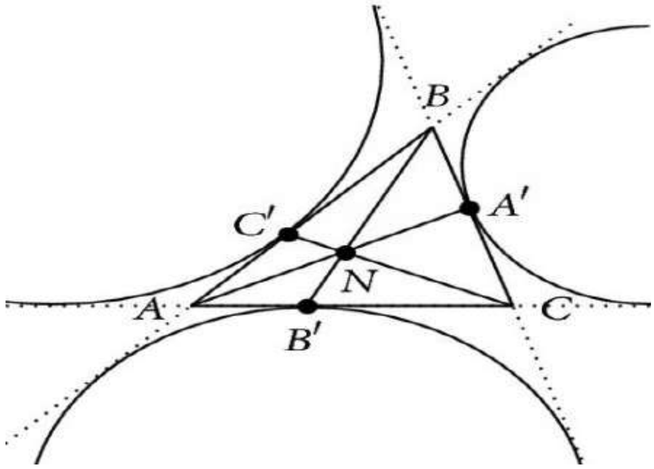
724. **In $\Delta ABC, n_a$ – Nagel's cevian. Prove that :**

$$n_an_bn_c\sum_{cyc}\left(\frac{n_ah_b}{n_bh_a}+\frac{n_bh_a}{n_ah_b}\right)\geq s^3$$

Proposed by Bogdan Fuştei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\text{Let } N \text{ be the Nagel's point of } \Delta ABC \text{ and } AA', BB', CC' \text{ be the Nagel's cevians.}$$



$$\text{We have}: AB' = s-c,\ \ AC' = s-b,\ \ BC' = CB' = s-a.$$

$$\text{From Van Aubel's theorm, we have}: \frac{AN}{NA'} = \frac{AC'}{C'B} + \frac{AB'}{B'C} = \frac{s-b}{s-a} + \frac{s-c}{s-a} = \frac{a}{s-a}$$

$$\text{Then}: \frac{n_a}{NA} = 1 + \frac{A'N}{NA} = 1 + \frac{s-a}{a} = \frac{s}{a} \Rightarrow n_a = \frac{s.NA}{a} = \frac{NA.h_a}{2r}\ (\text{and analogs}).$$

$$\text{The given inequality is successively equivalent to}$$

$$: \frac{s.NA}{a}.\frac{s.NB}{b}.\frac{s.NC}{c}.\sum_{cyc}\left(\frac{NA}{NB} + \frac{NB}{NA}\right) \ge s^3$$

$$\Leftrightarrow \sum_{cyc}(NA^2.NC + NB^2.NC) \ge abc \Leftrightarrow \sum_{cyc}(NB+NC)NB.NC \ge abc.$$

$$\text{In } \Delta BNC \text{ we have}: \ \ NB + NC \ge BC = a\ \ (\text{and analogs})$$

$$\text{Then}:$$

$$\sum_{cyc}(NB+NC)NB.NC \ge \sum_{cyc} a.NB.NC \overset{\text{Hayashi's inequality}}{\ge} abc,$$

$$\text{and the proof is complete.}$$

725.

$$\textbf{In } \Delta ABC, \omega - \textbf{Brocard's angle. Prove that}:$$

$$\frac{1}{2}\sum_{cyc}\left(\frac{h_b}{h_c} + \frac{h_c}{h_b}\right)\frac{{w_a}^2}{{h_a}^2} \le 1 + \frac{1}{2\sin^2\omega}$$

Proposed by Bogdan Fuştei-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have: \frac{1}{2}\left(\frac{h_b}{h_c}+\frac{h_c}{h_b}\right){w_a}^2=\frac{1}{2}\left(\frac{c}{b}+\frac{b}{c}\right)\left(\frac{2bc.\cos\frac{A}{2}}{b+c}\right)^2=\frac{2bc(b^2+c^2)}{(b+c)^2}.\cos^2\frac{A}{2}\le$$

$$\overset{AM-GM}{\le}\frac{\left[2bc+\left(b^2+c^2\right)\right]^2}{4(b+c)^2}.\cos^2\frac{A}{2}=\left(\frac{b+c}{2}.\cos\frac{A}{2}\right)^2\overset{Lascu}{\le}{m_a}^2.$$

$$Then: \frac{1}{2}\left(\frac{h_b}{h_c}+\frac{h_c}{h_b}\right)\frac{{w_a}^2}{{h_a}^2}\le\frac{{m_a}^2}{{h_a}^2}\ (and\ analogs)$$

$$Therefore,\ \frac{1}{2}\sum_{cyc}\left(\frac{h_b}{h_c}+\frac{h_c}{h_b}\right)\frac{{w_a}^2}{{h_a}^2}\le\sum_{cyc}\frac{{m_a}^2}{{h_a}^2}=\sum_{cyc}\frac{a^2(2b^2+2c^2-a^2)}{16F^2}$$
$$=\frac{2\sum_{cyc}a^2b^2-\sum_{cyc}a^4}{16F^2}+\frac{\sum_{cyc}a^2b^2}{8F^2}.$$

Using the following identities:

$$2\sum_{cyc}a^2b^2-\sum_{cyc}a^4=16F^2\ \ and\ \ \sin\omega=\frac{2F}{\sqrt{\sum_{cyc}a^2b^2}},we\ get:$$

$$\frac{1}{2}\sum_{cyc}\left(\frac{h_b}{h_c}+\frac{h_c}{h_b}\right)\frac{{w_a}^2}{{h_a}^2}\le 1+\frac{1}{2\sin^2\omega}.$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\frac{1}{2}\sum_{cyc}\left(\frac{h_b}{h_c}+\frac{h_c}{h_b}\right)\frac{w_a^2}{h_a^2}=\frac{1}{2}\sum_{cyc}\left(\left(\frac{c}{b}+\frac{b}{c}\right).\frac{4bc}{(b+c)^2}.\frac{s(s-a)}{h_a^2}\right)$$
$$=\sum_{cyc}\left(\left(\frac{2\left(b^2+c^2\right)}{(b+c)^2}-1+1\right).\frac{s(s-a)}{h_a^2}\right)$$
$$=\sum_{cyc}\left(\frac{(b-c)^2}{(b+c)^2}.\frac{s(s-a)}{h_a^2}\right)+\sum_{cyc}\frac{s(s-a)}{h_a^2}$$

$$=\frac{1}{4}\sum_{cyc}\left(\frac{(b-c)^2}{(b+c)^2}\cdot\frac{(b+c)^2-a^2}{h_a^2}\right)+\frac{s}{4r^2s^2}\sum_{cyc}a^2(s-a)$$

$$=\frac{1}{4}\sum_{cyc}\frac{(b-c)^2}{h_a^2}-\frac{1}{4}\sum_{cyc}\frac{a^2(b-c)^2}{(b+c)^2.h_a^2}$$

$$+\frac{s^2}{4r^2s^2}\left(2(s^2-4Rr-r^2)-2(s^2-6Rr-3r^2)\right)$$

$$\le\frac{1}{16r^2s^2}\sum_{cyc}a^2(b-c)^2+\frac{8Rrs^2+8r^2s^2}{8r^2s^2}\left(\because -\frac{a^2(b-c)^2}{4(b+c)^2.h_a^2}\le 0 \text{ and analogs}\right)$$

$$=\frac{\sum_{cyc}a^2b^2-abc(a+b+c)+8Rrs^2+8r^2s^2}{8r^2s^2}$$

$$=\frac{\sum_{cyc}a^2b^2-8Rrs^2++8Rrs^2+8r^2s^2}{8r^2s^2}$$

$$=1+\frac{\sum_{cyc}a^2b^2}{8r^2s^2}=1+\frac{1}{2\sin^2\omega}\ \text{(QED)}$$

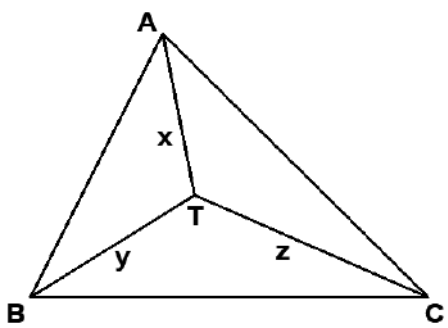
726.

Let ABC be a triangle with the measures of all its angles smaller than $\frac{2\pi}{3}$ and $T-$ its Torricelli point. Prove that :

$$6r\le TA+TB+TC\le\sqrt{p^2+S\sqrt{3}}$$

Proposed by Alex Szoros-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco



$$Let\ x=TA, y=TB, z=TC.$$

By the law of cosinus in ΔBTC, ΔCTA, ΔATB we have :

$$a = BC = \sqrt{y^2 + yz + z^2} \ (and\ analogs)$$

$$S = S_{\Delta ATB} + S_{\Delta BTC} + S_{\Delta CTA} = \frac{1}{2}xy.\sin\frac{2\pi}{3} + \frac{1}{2}yz.\sin\frac{2\pi}{3} + \frac{1}{2}zx.\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{4}(xy + yz + zx).$$

Now, we have :

$$TA + TB + TC = x + y + z \geq \sqrt{3(xy + yz + zx)} = \sqrt{3.\frac{4S}{\sqrt{3}}} = 2\sqrt{\sqrt{3}pr} \overset{Mitrinovic}{\geq} 6r.$$

Also we have :

$$p^2 + S\sqrt{3} = \frac{(a + b + c)^2 + 4\sqrt{3}S}{4} = \frac{1}{4}\left[\left(\sum_{cyc}\sqrt{y^2 + yz + z^2}\right)^2 + 3\sum_{cyc}xy\right] =$$

$$= \frac{1}{4}\left(2\sum_{cyc}x^2 + 4\sum_{cyc}xy + 2\sum_{cyc}\sqrt{(x^2 + xy + y^2)(x^2 + xz + z^2)}\right) =$$

$$= \frac{1}{2}\left(\sum_{cyc}x^2 + 2\sum_{cyc}xy + \sum_{cyc}\sqrt{\left[\left(x + \frac{y}{2}\right)^2 + \frac{3y^2}{4}\right]\left[\left(x + \frac{z}{2}\right)^2 + \frac{3z^2}{4}\right]}\right) \geq$$

$$\overset{CBS}{\geq} \frac{1}{2}\left(\sum_{cyc}x^2 + 2\sum_{cyc}xy + \sum_{cyc}\left[\left(x + \frac{y}{2}\right)\left(x + \frac{z}{2}\right) + \frac{3yz}{4}\right]\right) = (x + y + z)^2 = (TA + TB + TC)^2.$$

$$Therefore,\ \ 6r \leq TA + TB + TC \leq \sqrt{p^2 + S\sqrt{3}}.$$

727. **If $x, y, z > 0$ and $A_1B_1C_2, A_2B_2C_2$ are two triangles with the areas F_1, respectively F_2, then holds the following inequality:**

$$\frac{x + y}{z\sqrt{a_1a_2}} + \frac{y + z}{x\sqrt{b_1b_2}} + \frac{z + x}{y\sqrt{c_1c_2}} \geq \frac{2\sqrt{3}}{\sqrt{R_1R_2}}$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu-Romania

Solution by Tapas Das-India

$$\frac{x+y}{z\sqrt{a_1a_2}}+\frac{y+z}{x\sqrt{b_1b_2}}+\frac{z+x}{y\sqrt{c_1c_2}}\geq\frac{2\sqrt{xy}}{z\sqrt{a_1a_2}}+\frac{2\sqrt{yz}}{x\sqrt{b_1b_2}}+\frac{2\sqrt{zx}}{y\sqrt{c_1c_2}}=$$

$$=2\left(\frac{\sqrt{xy}}{z}\cdot\frac{1}{\sqrt{a_1a_2}}+\frac{\sqrt{yz}}{x}\cdot\frac{1}{\sqrt{b_1b_2}}+\frac{\sqrt{zx}}{y}\cdot\frac{1}{\sqrt{c_1c_2}}\right)\geq$$

$$\geq 2\cdot 3\left[\frac{\sqrt{xy}\cdot\sqrt{yz}\cdot\sqrt{zx}}{xyz}\cdot\frac{1}{(a_1b_1c_2)^{\frac{1}{2}}(a_2b_2c_2)^{\frac{1}{2}}}\right]^{\frac{1}{3}}=$$

$$=6\left(\frac{1}{\sqrt{4R_1F_1}\cdot\sqrt{4R_2F_2}}\right)^{\frac{1}{3}}\geq 6\left(\frac{1}{\sqrt{4\cdot 4\cdot R_1R_2\cdot\frac{3\sqrt{3}}{2}R_1\frac{R_1}{2}\cdot\frac{3\sqrt{3}}{2}R_2\frac{R_2}{2}}}\right)^{\frac{1}{3}}=$$

$$=\frac{6}{\sqrt{3}\sqrt{R_1R_2}}=\frac{2\sqrt{3}}{\sqrt{R_1R_2}}$$

$$\because 4R_1F_1=4R_1r_1s_1\leq 4R_1\frac{R_1}{2}\cdot\frac{3\sqrt{3}}{2}R_1\ (and\ analogs)$$

728. If $x,y,z>0$ and $A_1B_1C_2, A_2B_2C_2$ are two triangles with the areas F_1, respectively F_2, then holds the following inequality:

$$\frac{x+y}{z}a_1b_2+\frac{y+z}{x}b_1c_2+\frac{z+x}{y}c_1a_2\geq 8\sqrt{3F_1F_2}$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu-Romania

Solution 1 by Tapas Das-India

$$\frac{x+y}{z}a_1b_2+\frac{y+z}{x}b_1c_2+\frac{z+x}{y}c_1a_2\geq\frac{2\sqrt{xy}}{z}a_1b_2+\frac{2\sqrt{yz}}{x}b_1c_2+\frac{2\sqrt{zx}}{y}c_1a_2\geq$$

$$\geq 6\left(\frac{\sqrt{xy}\cdot\sqrt{yz}\cdot\sqrt{zx}}{xyz}\cdot a_1b_1c_1\cdot a_2b_2c_2\right)^{\frac{1}{3}}=6(a_1b_1c_1)^{\frac{1}{3}}\cdot(a_2b_2c_2)^{\frac{1}{3}}\geq$$

$$\geq 6\cdot\left(\frac{4F_1}{\sqrt{3}}\right)^{\frac{3}{2}\cdot\frac{1}{3}}\cdot\left(\frac{4F_2}{\sqrt{3}}\right)^{\frac{3}{2}\cdot\frac{1}{3}}=6\cdot\frac{2\sqrt{F_1}}{(\sqrt{3})^{\frac{1}{2}}}\cdot\frac{2\sqrt{F_2}}{(\sqrt{3})^{\frac{1}{2}}}=24\cdot\frac{\sqrt{F_1F_2}}{\sqrt{3}}=8\cdot\frac{3\sqrt{F_1F_2}}{\sqrt{3}}=8\sqrt{3F_1F_2}$$

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$T_1 = a_1b_2, T_2 = b_1c_2, T_3 = c_1a_2 \Rightarrow T_1T_2T_3 = (a_1b_1c_1)(a_2b_2c_2) =$$

$$= (4F_1R_1)(4F_2R_2) = 16F_1F_2\sqrt{(R_1R_1)(R_2R_2)} \geq$$

$$\geq 16F_1F_2\sqrt{\left(\frac{2}{3\sqrt{3}}s_1 \cdot 2r_1\right)\left(\frac{2}{3\sqrt{3}}s_2 \cdot 2r_2\right)} = \frac{16 \cdot 4}{3\sqrt{3}} \cdot F_1F_2\sqrt{F_1F_2} = \frac{64}{3\sqrt{3}}F_1F_2\sqrt{F_{1F_2}}$$

$$T_1T_2T_3 \geq \left(\frac{4}{\sqrt{3}\sqrt{F_1F_2}}\right)^3 ; (*)$$

$$\sum_{cyc}\frac{x+y}{z} \cdot T_1 = \sum_{cyc}\frac{x+y+z}{z} \cdot T_1 - (T_1 + T_2 + T_3) =$$

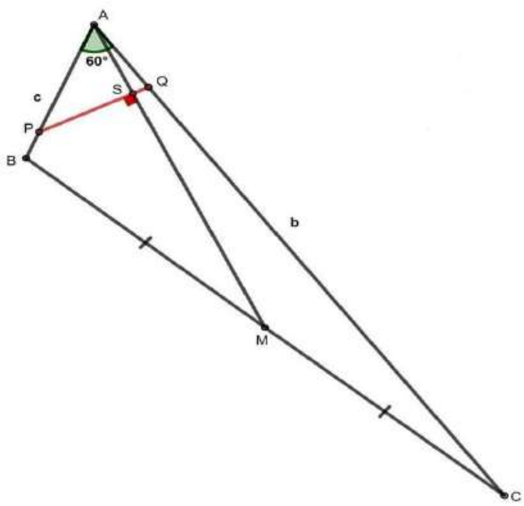
$$= (x+y+z)\left(\frac{T_2}{x} + \frac{T_3}{y} + \frac{T_1}{z}\right) - (T_1 + T_2 + T_3) \overset{CBS}{\geq}$$

$$\geq \left(\sqrt{T_1} + \sqrt{T_2} + \sqrt{T_3}\right)^2 - (T_1 + T_2 + T_3) =$$

$$= 2\sum_{cyc}\sqrt{T_1T_2} \overset{AGM}{\geq} 6 \cdot \sqrt[3]{\sqrt{(T_1T_2T_3)^2}} = 6 \cdot \sqrt[3]{T_1T_2T_3} \overset{(*)}{\geq} 6 \cdot 4\sqrt{F_1F_2} = 8\sqrt{3F_1F_2}$$

729.

$$\frac{PS}{SQ} = m, \frac{b}{c} = n \neq 1, m, n, b, c \in \mathbb{N}. \text{Find: } \min[ABC] = ?, ([*] - \text{area})$$



Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by proposer

Plagiogonal system: $AB \equiv Ax, AC \equiv Ay$

Let $AP = p, AQ = q, A(0,0), B(c,0), Q(0,q), C(0,b)$

$$\frac{PS}{SQ} = \frac{BM}{MC} \cdot \frac{AP}{AB} \cdot \frac{AC}{AQ} \Rightarrow m = \frac{p}{q} \cdot \frac{b}{c} \Rightarrow \frac{q}{p} = \frac{n}{m}; (1)$$

$$\lambda_1 = \lambda_{PQ} = -\frac{q}{p}, \lambda_2 = \lambda_{AM} = \frac{b}{c} = n$$

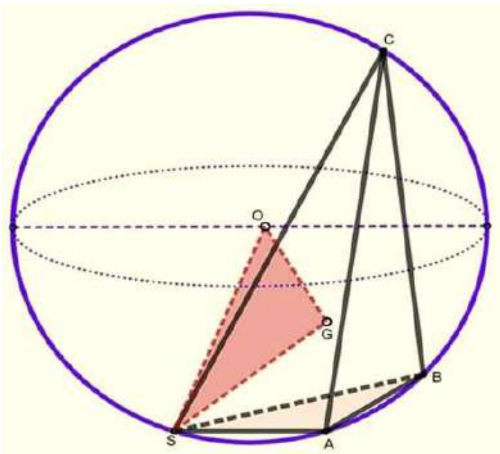
$$PQ \perp BC \Rightarrow (\lambda_1 + \lambda_2)\cos A + \lambda_1\lambda_2 + 1 = 0 \overset{(1)}{\Rightarrow} \left(-\frac{n}{m} + n\right) \cdot \frac{1}{2} - \frac{n^2}{m} + 1 = 0$$

$$2n^2 + n - 2m - mn = 0 \Rightarrow m = 6, n = 4$$

$$n = 4 \Rightarrow \frac{b}{c} = 4 \Rightarrow b_{min} = 4, c_{min} = 1$$

$$min\{[ABC]\} = \frac{AB \cdot AC \cdot \sin A}{2} = \frac{1 \cdot 4 \cdot \frac{\sqrt{3}}{2}}{2} = \sqrt{3}.$$

730. $SABC$ –tetrahedron, $SA = 4, SB = 8, SC = 12, \measuredangle BSC = \theta_1 = 60°, \measuredangle CSA = \theta_2 = 60°, \measuredangle ASB = \theta_3 = 60°$, O –circumspherecenter, G –centroid of $SABC$. Find: $[SOG] =?$ (area) and $d_{(OG,SA)} =?$ (distance).



Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by proposer

Plagiogonal 3D system: $SA \equiv Sx, SB \equiv Sy, SC \equiv Sz$

$$S(0,0,0), A(4,0,0), B(0,8,0), C(0,0,12), G\left(\frac{a}{4},\frac{b}{4},\frac{c}{4}\right)$$

$$G(1,2,3), O(o_1,o_2,o_3), o_1=\frac{3a-b-c}{4}=-2; o_2=\frac{3b-c-a}{4}=2, o_3=\frac{3c-a-b}{4}=6$$

$$O(-2,2,6), \overrightarrow{SG}=(1,2,3), \left|\overrightarrow{SG}\right|^2=1^2+2^2+1\cdot 2+2\cdot 3+3\cdot 1=25$$

$$\overrightarrow{SO}=(-2,2,6), \left|\overrightarrow{SO}\right|^2=(-2)^2+2^2+6^2-2\cdot 2+2\cdot 6-2\cdot 6=40$$

$$\overrightarrow{SG}\cdot\overrightarrow{SO}=-2+4+18+(2-4+12+6+6-6)\cdot\frac{1}{2}=28$$

$$[SOG]=\frac{1}{2}\sqrt{\left|\overrightarrow{SG}\right|^2\cdot\left|\overrightarrow{SO}\right|^2-\left(\overrightarrow{SG}\cdot\overrightarrow{SO}\right)^2}=\frac{1}{2}\sqrt{25\cdot 40-28^2},\qquad [SOG]=3\sqrt{6}$$

$$OG: \frac{x-1}{-2-1}=\frac{z-3}{6-3}\Rightarrow x-1=3-z$$

Let $K\in OG, K(k_1,0,k_3)$. Is $k_1-1=3-k_3\Rightarrow k_3=4-k_1, K(k_1,2,4-k_1)$

Let $L\in SA\Rightarrow L(l_1,0,0)$

$$KL^2=f(k_1,l_1)=$$

$$=(k_1-l_1)^2+2^2+(4-k_1)^2+2(k_1-l_1)+2(4-k_1)+(k_1-l_1)(4-k_1)$$

$$\begin{cases}\dfrac{df}{dv_n}=0\\ \dfrac{df}{dl_n}=0\end{cases}\Rightarrow\begin{cases}k_1=\dfrac{14}{3}\\ l_1=\dfrac{16}{3}\end{cases}\Rightarrow\begin{cases}K\left(\dfrac{14}{3},2,-\dfrac{2}{3}\right)\\ L\left(\dfrac{16}{3},0,0\right)\end{cases}$$

$$\min KL^2=\frac{8}{3}\Rightarrow (KL)_{min}=2\sqrt{\frac{2}{3}}\Rightarrow d_{(OG,SA)}=2\sqrt{\frac{2}{3}}.$$

731. If $x,y,z>0$ then in ΔABC the following relationship holds:

$$\frac{a^4e^{x^2}}{y+z}+\frac{b^4e^{y^2}}{z+x}+\frac{c^4e^{z^2}}{x+y}>16F^2$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

Solution by Tapas Das-India

$$a^2x+b^2y+c^2z\geq 4F\sqrt{xy+yz+zx};\ \ (Oppenheim);(1)$$

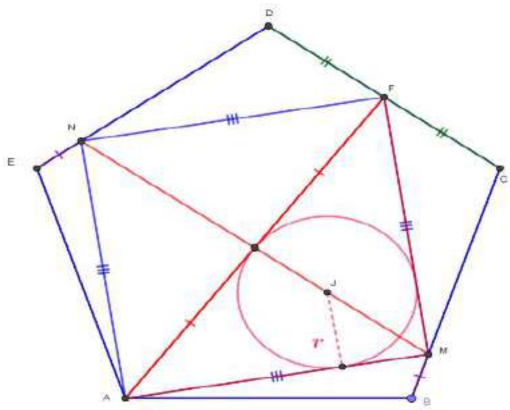
$$e^{t^2}>1+t^2\geq 2t;(2)$$

$$\frac{a^4e^{x^2}}{y+z}+\frac{b^4e^{y^2}}{z+x}+\frac{c^4e^{z^2}}{x+y}\geq\frac{a^4\cdot 2x}{y+z}+\frac{b^4\cdot 2y}{z+x}+\frac{c^4\cdot 2z}{x+y}=$$

$$= 2\left(\frac{a^4x^2}{xy+xz} + \frac{b^4y^2}{yz+yx} + \frac{c^4z^2}{zx+zy}\right) = 2\left(\frac{(a^2x)^2}{xy+xz} + \frac{(b^2y)^2}{yz+yx} + \frac{(c^2z)^2}{zx+zy}\right) \geq$$

$$\geq \frac{2(a^2x+b^2y+c^2z)^2}{2(xy+yz+zx)} \geq \frac{\left(4F\sqrt{xy+yz+zx}\right)^2}{xy+yz+zx} = \frac{16F^2(xy+yz+zx)}{xy+yz+zx} = 16F^2$$

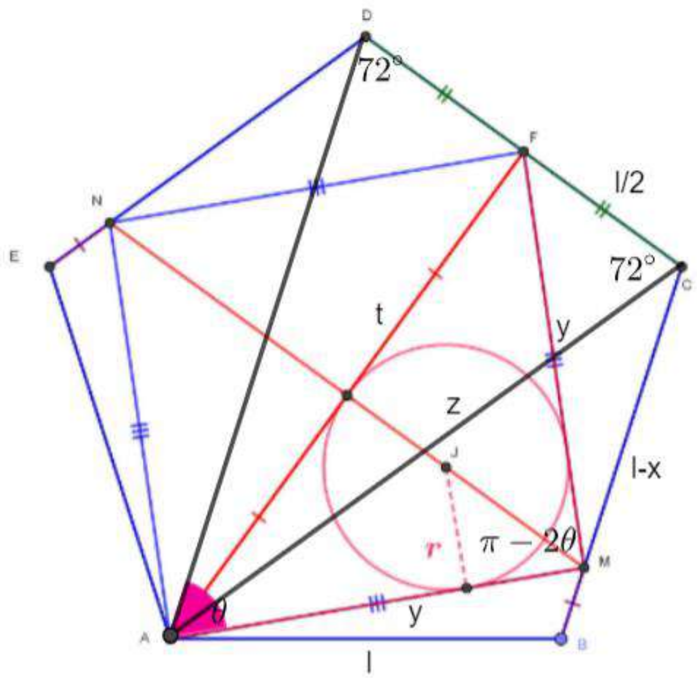
732.



Find r.

Proposed by Juan Jose Isach-Mayo-Valencia-Spain

Solution by Jose Ferreira Queiroz-Olinda-Brazil



$$AB = BC = CD = DE = EA = l$$

$$BM = x, MC = l - x, AC = z, AM = y, AF = t$$

$$\angle ABC = 108°, \cos 108° = \frac{1}{4}(1 - \sqrt{5})$$

Applying cosine law in the ΔABM and ΔMCF:

$$y^2 = l^2 + x^2 - 2lx \cdot \cos 108° = \frac{l^2}{4} + (l - x)^2 - 2\frac{l}{2}(l - x)\cos 108°$$

$$x = \frac{l(3 - \sqrt{5})}{4}, \qquad y^2 = l^2 + \frac{l^2}{16}\left(3 - \sqrt{5}\right)^2 - 2l \cdot \frac{l(3 - \sqrt{5})}{4} \cdot \frac{1}{4}(1 - \sqrt{5})$$

$$16y^2 = 14l^2 + 2\sqrt{5}l^2 \Rightarrow y = \frac{l}{4}\sqrt{14 + 2\sqrt{5}}$$

In the ΔABC: $z^2 = 2l^2 - 2l^2 \cdot \cos 108° \Rightarrow z = \frac{l}{2}\sqrt{6 + 2\sqrt{5}}$

Now, in the ΔAFC: $t^2 = z^2 - \frac{l^2}{4} = \frac{l^2}{4}\left(6 + 2\sqrt{5}\right) - \frac{l^2}{4} \Rightarrow t = \frac{l}{2}\sqrt{5 + 2\sqrt{5}}$

In the ΔAFM applying sine law:

$$\frac{t}{\sin(\pi - 2\theta)} = \frac{y}{\sin\theta} \Rightarrow \cos\theta = \frac{t}{2y}$$

$$\begin{cases} \cos\theta = \dfrac{\frac{l}{2}\sqrt{5 + 2\sqrt{5}}}{\frac{l}{2}\sqrt{14 + 2\sqrt{5}}} = \dfrac{1}{44}\sqrt{22\left(25 + 9\sqrt{5}\right)} \\ \sin^2\theta = 1 - \cos^2\theta \Rightarrow \sin\theta = \dfrac{3}{44}\sqrt{22\left(7 - \sqrt{5}\right)} \end{cases}$$

$$[AFM] = \frac{1}{2}yt \cdot \sin\theta = \frac{r}{2}(yz + t)$$

$$\frac{l}{4}\sqrt{14 + 2\sqrt{5}} \cdot \frac{l}{2}\sqrt{5 + 2\sqrt{5}} \cdot \frac{3}{44}\sqrt{22\left(7 - \sqrt{5}\right)} = r\left(\frac{l}{2}\sqrt{14 + 2\sqrt{5}} + \frac{l}{2}\sqrt{5 + 2\sqrt{5}}\right)$$

$$\frac{3l}{4}\sqrt{5 + 2\sqrt{5}} = r\left(\sqrt{14 + 2\sqrt{5}} + \sqrt{5 + 2\sqrt{5}}\right)$$

$$r = \frac{3l}{4} \cdot \frac{\sqrt{5 + 2\sqrt{5}}}{\sqrt{14 + 2\sqrt{5}} + \sqrt{5 + 2\sqrt{5}}}, r = \frac{l}{4} \cdot \frac{-5 - 2\sqrt{5} + \sqrt{90 + 38\sqrt{5}}}{3}$$

$$r = \frac{l\left(-5 - 2\sqrt{5} + \sqrt{90 + 38\sqrt{5}}\right)}{12}$$

733. In ΔABC the following relationship holds:

$$\prod_{cyc}\left(1+\frac{1}{w_a}\right)\ge\left(1+\frac{2}{3R}\right)^3$$

Proposed by Marin Chirciu-Romania

Solution 1 by Marian Ursărescu-Romania

We must show that:

$$\sqrt[3]{\prod_{cyc}\left(1+\frac{1}{w_a}\right)}\ge 1+\frac{2}{3R};(1)$$

From Huygens's inequality:

$$\sqrt[3]{\prod_{cyc}\left(1+\frac{1}{w_a}\right)}\ge 1+\sqrt[3]{\frac{1}{w_aw_bw_c}};(2)$$

From (1) and (2) we must show:

$$1+\sqrt[3]{\frac{1}{w_aw_bw_c}}\ge 1+\frac{2}{3R}\Leftrightarrow\sqrt[3]{w_aw_bw_c}\le\frac{3R}{2}\Leftrightarrow w_aw_bw_c\le\frac{27R^3}{8};(3)$$

But $w_a\le\sqrt{s(s-a)}\Rightarrow w_aw_bw_c\le sF=s^2r;(4)$

From (3) and (4) we must show:

$$s^2r\le\frac{27R^3}{8};(5)$$

But $s^2\le\frac{27R^2}{4}(Mitrinovic)$ and $r\le\frac{R}{2}\ (Euler)\Rightarrow s^2\le\frac{27R^3}{8}\Rightarrow(5)$ true.

Solution 2 by Tapas Das-India

$$\sqrt[3]{\prod_{cyc}\left(1+\frac{1}{w_a}\right)}\overset{Holder}{\ge}1+\sqrt[3]{\frac{1}{w_aw_bw_c}}$$

$$\prod_{cyc}\left(1+\frac{1}{w_a}\right)\ge\left[1+\sqrt[3]{\frac{1}{w_aw_bw_c}}\right]^3\ge\left[1+\sqrt[3]{\frac{1}{rs^2}}\right]^3\ge\left[1+\sqrt[3]{\frac{1}{\frac{27R^3}{8}}}\right]^3=\left(1+\frac{2}{3R}\right)^3$$

But $w_a \le \sqrt{s(s-a)} \Rightarrow w_a w_b w_c \le sF = s^2 r$;

$$s^2 \le \frac{27R^3}{4}(Mitrinovic) \text{ and } r \le \frac{R}{2}\ (Euler) \Rightarrow s^2 \le \frac{27R^3}{8}$$

734. If $x, y, z > 0$ then in ΔABC the following relationship holds:

$$\frac{x^2a^3}{(y+z)^2h_a}+\frac{y^2b^3}{(z+x)^2h_b}+\frac{z^2c^3}{(x+y)^2h_c}\ge 2F$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

Solution by Tapas Das-India

$$\frac{x^2a^3}{(y+z)^2h_a}+\frac{y^2b^3}{(z+x)^2h_b}+\frac{z^2c^3}{(x+y)^2h_c}=\frac{x^2a^4}{(y+z)^2 2F}+\frac{y^2b^4}{(z+x)^2 2F}+\frac{z^2c^4}{(x+y)^2 2F}=$$

$$=\frac{1}{2F}\left[\frac{\left(\frac{x}{y+z}a^2\right)^2}{1}+\frac{\left(\frac{y}{z+X}b^2\right)^2}{1}+\frac{\left(\frac{z}{x+y}c^2\right)^2}{1}\right]\ge$$

$$\ge\frac{1}{2F}\cdot\frac{\left(\frac{x}{y+z}a^2+\frac{y}{z+x}b^2+\frac{z}{x+y}c^2\right)^2}{3}\ge\frac{1}{2F}\cdot\frac{\left(2\sqrt{3}F\right)^2}{3}=\frac{1}{6F}\cdot 12F^2=2F$$

$$\because \frac{x}{y+z}a^2+\frac{y}{z+x}b^2+\frac{z}{x+y}c^2\ge 2\sqrt{3}F\ (Tsintsifas)$$

735. In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{h_a}{\sqrt{{h_b}^2-h_bh_c+{h_c}^2}}\ge\frac{12r^2}{R^2}$$

Proposed by Marian Ursărescu-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{h_a}{\sqrt{h_b^2-h_bh_c+h_c^2}}=\sum_{cyc}\frac{h_a^2}{h_a\cdot\sqrt{h_b^2-h_bh_c+h_c^2}}\overset{\text{Bergstrom}}{\ge}\frac{\left(\sum_{cyc}h_a\right)^2}{\sum_{cyc}\left(h_a\cdot\sqrt{h_b^2-h_bh_c+h_c^2}\right)}$$

$$=\frac{\left(\sum_{cyc}h_a\right)^2}{\sum_{cyc}\left(\sqrt{h_a}\cdot\sqrt{h_ah_b^2-h_ah_bh_c+h_ah_c^2}\right)}\overset{\text{CBS}}{\ge}\frac{\left(\sum_{cyc}h_a\right)^2}{\sqrt{\sum_{cyc}h_a}\cdot\sqrt{\sum_{cyc}(h_ah_b^2-h_ah_bh_c+h_ah_c^2)}}$$

$$= \frac{\left(\sum_{cyc} h_a\right)^2}{\sqrt{\sum_{cyc} h_a \cdot \sqrt{\left(\sum_{cyc} h_a\right)\left(\sum_{cyc} h_b h_c\right) - 6h_a h_b h_c}}}$$

$$= \frac{\left(\sum_{cyc} h_a\right)^2}{\sqrt{\sum_{cyc} h_a \cdot \sqrt{\left(\sum_{cyc} h_a\right)\left(\sum_{cyc}\left(\frac{ca}{2R}\cdot\frac{ab}{2R}\right)\right) - 6.\frac{8r^3s^3}{4Rrs}}}}$$

$$= \frac{\left(\sum_{cyc} h_a\right)^2}{\sqrt{\sum_{cyc} h_a \cdot \sqrt{\left(\sum_{cyc} h_a\right)\left(\frac{2rs^2}{R}\right) - \frac{12r^2s^2}{R}}}} \overset{?}{\geq} \frac{12r^2}{R^2}$$

$$\Leftrightarrow \frac{\left(\sum_{cyc} h_a\right)^3}{\left(\sum_{cyc} h_a\right)\left(\frac{2rs^2}{R}\right) - \frac{12r^2s^2}{R}} \overset{?}{\geq} \frac{144r^4}{R^4} \Leftrightarrow \frac{R^5\left(\sum_{cyc} h_a\right)^3}{2rs^2\left(\frac{s^2+4Rr+r^2}{2R} - 6r\right)} \overset{?}{\geq} 144r^4$$

$$\Leftrightarrow R^6\left(\sum_{cyc} h_a\right)^3 \underset{(*)}{\overset{?}{\geq}} 144r^5s^2\left(s^2 - 8Rr + r^2\right)$$

Now, LHS of $(*)$

$\geq 729R^6r^3$ *and* RHS of $(*)$ $\overset{\text{Gerretsen + Mitrinovic}}{\leq}$ $144r^5.\frac{27R^2}{4}.\left(4R^2 - 4Rr + 4r^2\right)$ $\therefore$ in order to prove $(*)$, it suffices to prove :

$$729R^6r^3 \geq 144r^5.\frac{27R^2}{4}.\left(4R^2 - 4Rr + 4r^2\right) \Leftrightarrow 3R^4 \geq 4r^2\left(4R^2 - 4Rr + 4r^2\right)$$

$$\Leftrightarrow 3t^4 - 16t^2 + 16t - 16 \geq 0 \left(t = \frac{R}{r}\right) \Leftrightarrow (t-2)\left(3t^3 + 4t^2 + 2t(t-2) + 8\right)$$

$\geq 0 \rightarrow$ true $\because t \overset{\text{Euler}}{\geq} 2$

$\Rightarrow (*)$ is true $\therefore$ in *any* Δ ABC, $\sum_{cyc} \frac{h_a}{\sqrt{h_b^2 - h_b h_c + h_c^2}}$

$\geq \frac{12r^2}{R^2}$ (QED),"=" iff Δ ABC is equi*lateral* (QED)

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Since $h_a = \frac{bc}{2R}$ (*and analogs*) *then we have* :

$$\sum_{cyc} \frac{h_a}{\sqrt{{h_b}^2 - h_b h_c + {h_c}^2}} = \sum_{cyc} \frac{bc}{\sqrt{(ca)^2 - ca.ab + (ab)^2}} = \sum_{cyc} \frac{\sqrt{bc}}{a\sqrt{\frac{c}{b} + \frac{b}{c} - 1}} \geq$$

$$\overset{Bandila}{\geq} \sum_{cyc} \frac{\sqrt{bc}}{a\sqrt{\frac{R}{r} - 1}} = \frac{r}{\sqrt{r(R-r)}} \sum_{cyc} \frac{\sqrt{bc}}{a} \overset{AM-GM}{\geq} \frac{2r}{r + (R-r)}.3 = \frac{6r}{R} \overset{Euler}{\geq}$$

$$\geq \frac{6r}{R}\cdot\frac{2r}{R}=\frac{12r^2}{R^2}.$$

$$Therefore,\ \sum_{cyc}\frac{h_a}{\sqrt{{h_b}^2-h_bh_c+{h_c}^2}}\geq\frac{12r^2}{R^2}.$$

Equality holds iff ΔABC is equilateral

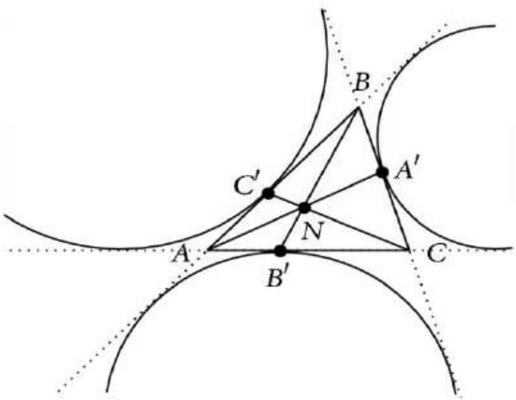
736. In $\Delta ABC, n_a$ – Nagel's cevian. Prove that:

$$\sum_{cyc}\left(\frac{2n_ah_b}{n_bh_a}-\frac{n_ah_c}{n_ch_a}\right)\geq 3$$

Proposed by Bogdan Fuştei-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\text{Stewart's theorem}\Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$$
$$\Rightarrow s(b^2+c^2)-bc(2s-a)=an_a^2+a(s^2-s(2s-a)+bc)$$
$$\Rightarrow s(b^2+c^2)-2sbc=an_a^2+a(as-s^2)\Rightarrow s(b^2+c^2-a^2-2bc)=an_a^2-as^2\Rightarrow an_a^2$$
$$=as^2+s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}$$
$$=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$
$$=as^2-\frac{4\Delta^2}{s-a}=as^2-2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right)=as^2-2ah_ar_a\Rightarrow n_a^2+2h_ar_a\overset{(*)}{=}s^2\text{ and analogs}$$



Let the Nagel point be N. Now, $A'B=s-c, A'C=s-b, B'A=s-c, B'C=s-a, C'A=s-b, C'B=s-a\ \therefore\ \frac{AN}{A'N}\overset{\text{Van Aubel}}{=}\frac{s-b}{s-a}+\frac{s-c}{s-a}=\frac{a}{s-a}\Rightarrow\frac{A'N}{AN}+1=\frac{s-a}{a}+1$

$\Rightarrow \frac{n_a}{AN} = \frac{s}{a} \Rightarrow AN = \frac{an_a}{s}$ and similarly, $BN = \frac{bn_b}{s}$ and CN

$= \frac{cn_c}{s}$ and via triangle − inequality, $BN + CN > BC \Rightarrow \frac{bn_b}{s} + \frac{cn_c}{s} > a$

$\Rightarrow bn_b + cn_c > sa \overset{?}{>} an_a \Leftrightarrow s^2 \overset{?}{>} n_a^2$

$\overset{\text{via }(*)}{\Leftrightarrow} n_a^2 + 2h_a r_a \overset{?}{>} n_a^2 \Leftrightarrow 2h_a r_a \overset{?}{>} 0 \to$ true $\therefore bn_b + cn_c > an_a$ and analogously, $cn_c + an_a > bn_b$ and $an_a + bn_b > cn_c \Rightarrow an_a, bn_b, cn_c$ form sides of a triangle with semi − perimeter s' (say) and assuming $s' - an_a = x, s' - bn_b = y, s' - cn_c = z$ $(x, y, z > 0)$, we arrive at : $3s' - 2s' = s' = \sum_{cyc} x$

$\Rightarrow an_a = y + z, bn_b = z + x, cn_c = x + y \to (1)$

Now, $\sum_{cyc}\left(\frac{2n_a h_b}{n_b h_a} - \frac{n_a h_c}{n_c h_a}\right) \geq 3 \Leftrightarrow 2\sum_{cyc}\frac{an_a}{bn_b} - \sum_{cyc}\frac{bn_b}{an_a} \geq 3 \overset{\text{via }(1)}{\Leftrightarrow} 2\sum_{cyc}\frac{y+z}{z+x} \geq 3 + \sum_{cyc}\frac{z+x}{y+z}$

$\Leftrightarrow 2\sum_{cyc}(x+y)(y+z)^2 \geq 3\prod_{cyc}(x+y) + \sum_{cyc}(x+y)^2(y+z)$

$\Leftrightarrow \sum_{cyc} x^3 + \sum_{cyc} x^2 y \overset{(\bullet)}{\geq} 2\sum_{cyc} xy^2$

Now, $x^3 + z^2 x \overset{A-G}{\geq} 2zx^2, y^3 + x^2 y \overset{A-G}{\geq} 2xy^2, z^3 + y^2 z \overset{A-G}{\geq} 2yz^2$ and summing up, $\sum_{cyc} x^3 + \sum_{cyc} x^2 y \geq 2\sum_{cyc} xy^2 \Rightarrow (\bullet)$ is true

$\therefore$ in any Δ ABC, $\sum_{cyc}\left(\frac{2n_a h_b}{n_b h_a} - \frac{n_a h_c}{n_c h_a}\right) \geq 3$ (QED)

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:\ {n_a}^2 = s(s-a) + \frac{s(b-c)^2}{a}\quad (and\ analogs)$$

$$Then:\ \left(\frac{n_a}{h_a}\right)^2 = \frac{a^2}{4s^2r^2}\left(s(s-a) + \frac{s(b-c)^2}{a}\right) = \frac{1}{4r^2}\left(\frac{a^2(s-a)}{s} + \frac{a(b-c)^2}{s}\right) =$$

$$= \frac{1}{4r^2}\left(\frac{(s-a)[a^2-(b-c)^2]}{s} + (b-c)^2\right) =$$

$$\frac{1}{4r^2}\left(\frac{4(s-a)(s-b)(s-c)}{s} + (b-c)^2\right) = \frac{4r^2+(b-c)^2}{4r^2}.$$

$$Then:\ \frac{n_a}{h_a} = \sqrt{1 + \frac{(b-c)^2}{4r^2}}\quad (and\ analogs)$$

$$Let\ \vec{u}=\left(1,\frac{b-c}{2r}\right)\ and\ \vec{v}=\left(1,\frac{c-a}{2r}\right).$$

$$By\ the\ triangle\ inequality\ we\ have:\ \ \|\vec{u}\|+\|\vec{v}\|\geq\|\vec{u}+\vec{v}\|$$

$$\Leftrightarrow\sqrt{1^2+\left(\frac{b-c}{2r}\right)^2}+\sqrt{1^2+\left(\frac{c-a}{2r}\right)^2}\geq\sqrt{2^2+\left(\frac{b-a}{2r}\right)^2}>\sqrt{1+\frac{(a-b)^2}{4r^2}}$$

$$Then:\ \frac{n_a}{h_a}+\frac{n_b}{h_b}>\frac{n_c}{h_c}\ \ (and\ analogs)$$

$$Now\ let$$

$$x=\frac{n_a}{h_a},\ \ y=\frac{n_b}{h_b},\ \ z=\frac{n_c}{h_c}.\ The\ given\ inequality\ is\ successively\ equivalent\ to:$$

$$\sum_{cyc}\left(\frac{2x}{y}-\frac{x}{z}\right)\geq 3\ \Leftrightarrow\ 2\sum_{cyc}(2x^2z-x^2y)\geq 6xyz$$

$$\Leftrightarrow\sum_{cyc}[(xy^2+2xyz+xz^2)+(y^3-z^3+3yz^2-3y^2z)]\geq 0$$

$$\Leftrightarrow\sum_{cyc}[x(y-z)^2+(y-z)^3]\geq 0\ \Leftrightarrow\sum_{cyc}(x+y-z)(y-z)^2\geq 0.$$

$$Which\ is\ true\ because\ x+y>z\ (and\ analogs).\ So\ the\ proof\ is\ complete.$$

737. In ΔABC the following relationship holds:

$$11\leq\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}+\frac{8(ab+bc+ca)}{a^2+b^2+c^2}\leq\frac{11R^2}{4r^2}$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$By\ Oppenheim's\ inequality\ we\ have$$

$$:\ xa^2+yb^2+zc^2\geq 4F\sqrt{xy+yz+zx},\ \ \forall x,y,z>0.$$

$$For\ \ x=\frac{b^2}{a^2},\ \ y=\frac{c^2}{b^2},\ \ z=\frac{a^2}{c^2}\ we\ have:\ \ a^2+b^2+c^2\ge 4F\sqrt{\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}}$$

$$Then:\ \frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\le\frac{(a^2+b^2+c^2)^2}{16F^2}=\frac{(s^2-r^2-4Rr)^2}{4s^2r^2}$$
$$=\frac{s^2-r^2-4Rr}{4r^2}.\left(1-\frac{r(4R+r)}{s^2}\right)\le$$

$$\overset{Gerretsen}{\le}\frac{4R^2+2r^2}{4r^2}.\left(1-\frac{r(4R+r)}{4R^2+4Rr+3r^2}\right)=\frac{(2R^2+r^2)^2}{r^2(4R^2+4Rr+3r^2)}.$$

$$Then:\ \frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}+\frac{8(ab+bc+ca)}{a^2+b^2+c^2}\le\frac{(2R^2+r^2)^2}{r^2(4R^2+4Rr+3r^2)}+8\overset{?}{\le}\frac{11R^2}{4r^2}$$

$$\Leftrightarrow 8\le\frac{28R^4+44R^3r+17R^2r^2-4r^4}{4r^2(4R^2+4Rr+3r^2)}$$
$$\Leftrightarrow\ 28R^4+44R^3r-111R^2r^2-128Rr^3-100r^4\ge 0$$

$$\Leftrightarrow\ (R-2r)(28R^3+100R^2r+89Rr^2+50r^3)\ge 0$$

which is true by Euler's inequality $R\ge 2r$.

$$Then:\ \frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}+\frac{8(ab+bc+ca)}{a^2+b^2+c^2}\le\frac{11R^2}{4r^2}$$

Assume now that $c=\min\{a,b,c\}$. *We have:* $\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}+\frac{8(ab+bc+ca)}{a^2+b^2+c^2}\ge 11$

$$\Leftrightarrow\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}-2\right)+\left(\frac{b^2}{c^2}+\frac{c^2}{a^2}-\frac{b^2}{a^2}-1\right)-8\left(1-\frac{ab+bc+ca}{a^2+b^2+c^2}\right)\ge 0$$

$$\Leftrightarrow\frac{(a^2-b^2)^2}{a^2b^2}+\frac{(a^2-c^2)(b^2-c^2)}{c^2a^2}-8.\frac{(a-b)^2+(a-c)(b-c)}{a^2+b^2+c^2}\ge 0$$

$$\Leftrightarrow\left(\frac{(a+b)^2}{a^2b^2}-\frac{8}{a^2+b^2+c^2}\right)(a-b)^2+\left(\frac{(a+c)(b+c)}{c^2a^2}-\frac{8}{a^2+b^2+c^2}\right)(a-c)(b-c)\ge 0$$

$$Which\ is\ true\ because:\ \frac{(a+b)^2}{a^2b^2}-\frac{8}{a^2+b^2+c^2}\ge\frac{4ab}{a^2b^2}-\frac{8}{a^2+b^2}\ge\frac{4}{ab}-\frac{8}{2ab}=0$$

$$\& \ \frac{(a+c)(b+c)}{c^2a^2}-\frac{8}{a^2+b^2+c^2}\overset{b\geq c}{\geq}\frac{(a+c).2c}{c^2a^2}-\frac{8}{a^2+2c^2}$$
$$=\frac{2(a^3-3a^2c+2ac^2+2c^3)}{ca^2(a^2+2c^2)}=$$

$$=\frac{2[(a-2c)^2(a+c)+2(a-c)c^2]}{ca^2(a^2+2c^2)}\overset{a\geq c}{\geq}0 \ \textit{and} \ (a-c)(b-c)\geq 0.$$

$$\textit{Therefore,} \ \frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}+\frac{8(ab+bc+ca)}{a^2+b^2+c^2}\geq 11 \ \textit{and the proof is completed.}$$

738. In any Δ ABC the following relationship holds:

$$\frac{n_a-g_a}{2}<m_a\leq\frac{2x+y+z-\sqrt{yz}}{\sqrt{3}}; x=s-a, y=s-b, z=s-c$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$an_a^2.ag_a^2\geq a^2s^2(s-a)^2$$
$$\Leftrightarrow\{b^2(s-c)+c^2(s-b)-a(s-b)(s-c)\}\{b^2(s-b)+c^2(s-c)$$
$$-a(s-b)(s-c)\}\overset{(1)}{\geq}a^2s^2(s-a)^2$$
$$\because s-a=x, s-b=y \text{ and } s-c=z \therefore s=x+y+z\Rightarrow a=y+z, b=z+x \text{ and } c$$
$$=x+y \text{ and via such substitutions,}$$
$$(1)\Leftrightarrow\{z(z+x)^2+y(x+y)^2-yz(y+z)\}\{y(z+x)^2+z(x+y)^2-yz(y+z)\}$$
$$\geq x^2(y+z)^2(x+y+z)^2\Leftrightarrow xy^2+xz^2+y^3+z^3\geq 2xyz+yz(y+z)$$
$$\Leftrightarrow x(y-z)^2+(y+z)(y-z)^2\geq 0\to \text{true}\Rightarrow(1)\text{ is true}\Rightarrow n_ag_a\overset{(*)}{\geq}s(s-a)$$
$$\text{Now, Stewart's theorem}$$
$$\Rightarrow b^2(s-c)+c^2(s-b)\overset{(i)}{=}an_a^2+a(s-b)(s-c) \text{ and } b^2(s-b)$$
$$+c^2(s-c)\overset{(ii)}{=}ag_a^2+a(s-b)(s-c) \text{ and } (i)+(i)\Rightarrow$$
$$(b^2+c^2)(2s-b-c)=an_a^2+ag_a^2+2a(s-b)(s-c)\Rightarrow 2a(b^2+c^2)$$
$$=2a(n_a^2+g_a^2)+a(a+b-c)(c+a-b)\Rightarrow 2(b^2+c^2)$$
$$=2(n_a^2+g_a^2)+a^2-(b-c)^2$$
$$\Rightarrow 2(b^2+c^2)-a^2+(b-c)^2=2(n_a^2+g_a^2)\Rightarrow 4m_a^2+(b-c)^2=2(n_a^2+g_a^2)$$
$$\Rightarrow(b-c)^2+4s(s-a)+(b-c)^2=2(n_a^2+g_a^2)\Rightarrow n_a^2+g_a^2$$
$$=(b-c)^2+2s(s-a)$$
$$\Rightarrow n_a^2+g_a^2-2n_ag_a=(b-c)^2+2s(s-a)-2n_ag_a\overset{\text{via }(*)}{\leq}(b-c)^2+2s(s-a)-2s(s-a)$$
$$\Rightarrow(n_a-g_a)^2\leq(b-c)^2=(b-c)^2+4s(s-a)-4s(s-a)$$

$$< (b-c)^2 + 4s(s-a) = 4m_a^2 \Rightarrow \frac{(n_a - g_a)^2}{4} < m_a^2 \Rightarrow \frac{|n_a - g_a|}{2} \overset{(\bullet)}{<} m_a$$

$$\text{Again, (i)} - \text{(ii)} \Rightarrow an_a^2 + a(s-b)(s-c) - ag_a^2 - a(s-b)(s-c)$$
$$= (s-c)(b^2-c^2) - (s-b)(b^2-c^2) \Rightarrow a(n_a^2 - g_a^2) = (b^2-c^2)(b-c)$$
$$= (b+c)(b-c)^2 \geq 0$$

$$\Rightarrow n_a - g_a \overset{(\bullet\bullet)}{>} 0 \therefore (\bullet), (\bullet\bullet) \Rightarrow \boxed{\frac{n_a - g_a}{2} < m_a}$$

$$\text{Also, } \sqrt{3}m_a + \sqrt{yz} \overset{CBS}{\leq} \sqrt{3+1}\sqrt{m_a^2 + (s-b)(s-c)} = \sqrt{(b-c)^2 + 4s(s-a) + a^2 - (b-c)^2}$$

$$= \sqrt{(2s-a)^2} = 2s - a = b + c \therefore \sqrt{3}m_a + \sqrt{yz} \overset{(\bullet\bullet\bullet)}{\leq} b + c$$

$$\text{and } 2x + y + z = b + c - a + (2s - b - c) = b + c - a + a = b + c \overset{\text{via } (\bullet\bullet\bullet)}{\geq} \sqrt{3}m_a + \sqrt{yz}$$

$$\Rightarrow 2x + y + z - \sqrt{yz} \geq \sqrt{3}m_a \Rightarrow \boxed{m_a \leq \frac{2x + y + z - \sqrt{yz}}{\sqrt{3}}}$$

$$\therefore \text{in any } \Delta ABC \frac{n_a - g_a}{2} < m_a \leq \frac{2x + y + z - \sqrt{yz}}{\sqrt{3}}; x = s - a, y = s - b, z = s - c \text{ (QED)}$$

739. If $x, y, z > 0$, then in any ΔABC with usual notations $a = BC, b = CA$, $c = AB, s$ –semiperimeter and F the area of triangle, then:

$$\frac{y+z}{x} \cdot a^3 + \frac{z+x}{y} \cdot b^3 + \frac{x+y}{z} \cdot c^3 \geq 16 \cdot \sqrt[4]{3} \cdot F\sqrt{F}$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu-Romania

Solution by Tapas Das-India

$$\frac{y+z}{x} \cdot a^3 + \frac{z+x}{y} \cdot b^3 + \frac{x+y}{z} \cdot c^3 = \left(\frac{y}{x}a^3 + \frac{z}{y}b^3 + \frac{x}{z}c^3\right) + \left(\frac{z}{x}a^3 + \frac{x}{y}b^3 + \frac{y}{z}c^3\right) \geq$$

$$\geq 3 \cdot \sqrt[3]{\frac{x}{z} \cdot \frac{z}{y} \cdot \frac{y}{x}(abc)^3} + 3 \cdot \sqrt[3]{\frac{z}{x} \cdot \frac{x}{y} \cdot \frac{y}{z}(abc)^3} =$$

$$= 3abc + 3abc = 6abc \geq 6\left(\frac{4F}{\sqrt{3}}\right)^{\frac{3}{2}} = \frac{6 \cdot (2^2)^{\frac{3}{2}} \cdot F\sqrt{F}}{3^{\frac{1}{4}}} = 3^{\frac{3}{4}} \cdot 16F\sqrt{F} \geq 16 \cdot \sqrt[4]{3} \cdot F\sqrt{F}$$

740. In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{r_a}{\sqrt{{r_b}^2 - r_b r_c + {r_c}^2}} \geq \frac{4R + r}{4R - 5r}$$

Proposed by Marian Ursărescu-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{r_a}{\sqrt{r_b^2-r_br_c+r_c^2}}=\sum_{cyc}\frac{(r_a)^{\frac{3}{2}}}{\sqrt{r_ar_b^2-r_ar_br_c+r_ar_c^2}}\overset{Radon}{\geq}\frac{(4R+r)^{\frac{3}{2}}}{\sqrt{(\sum_{cyc}r_a)(\sum_{cyc}r_br_c)-6r_ar_br_c}}$$

$$=\frac{(4R+r)^{\frac{3}{2}}}{\sqrt{s^2(4R+r)-6rs^2}}=\frac{(4R+r)^{\frac{3}{2}}}{\sqrt{s^2(4R-5r)}}\overset{Trucht}{\geq}\frac{(4R+r).\sqrt{3(4R+r)}}{\sqrt{(4R+r)^2(4R-5r)}}\overset{?}{\geq}\frac{4R+r}{4R-5r}$$

$$\Leftrightarrow\frac{3}{(4R+r)(4R-5r)}\overset{?}{\geq}\frac{1}{(4R-5r)^2}\Leftrightarrow 3(4R-5r)\overset{?}{\geq}4R+r\Leftrightarrow 8(R-2r)\overset{?}{\geq}0$$

$\rightarrow$ true via Euler $\therefore$ in any ΔABC, $\sum_{cyc}\frac{r_a}{\sqrt{r_b^2-r_br_c+r_c^2}}$

$\geq\frac{4R+r}{4R-5r}$,"=" iff ΔABC is equilateral (QED)

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\sum_{cyc}\frac{r_a}{\sqrt{r_b{}^2-r_br_c+r_c{}^2}}\cdot\sum_{cyc}r_a\sqrt{r_b^2-r_br_c+r_c^2}\overset{CBS}{\geq}(4R+r)^2;(1)$$

$$\frac{4R+r}{4R-5r}\cdot\sum_{cyc}\sqrt{r_a}\cdot\sqrt{r_a(r_b^2+r_c^2)-r_ar_br_c}\overset{CBS}{\leq}$$

$$\leq\frac{4R+r}{4R-5r}\cdot\sqrt{\sum_{cyc}r_a\left(\sum_{cyc}r_a(r_b^2+r_c^2)-3r_ar_br_c\right)}=$$

$$\leq\frac{4R+r}{4R-5r}\cdot\sqrt{(4R+r)\left(\sum_{cyc}r_a\sum_{cyc}r_ar_b-6r_ar_br_c\right)}=$$

$$=\frac{4R+r}{4R-5r}\cdot\sqrt{(4R+r)\big((4R+r)s^2-6rs^2\big)}=\frac{4R+r}{4R-5r}\cdot\sqrt{(4R+r)(4R-5r)s^2}\leq$$

$$\leq\frac{4R+r}{4R-5r}\sqrt{(4R+r)(4R-5r)\cdot\frac{4R+r}{\sqrt{3}}\cdot\sqrt{3}(4R-5r)}=$$

$$=\frac{4R+r}{4R-5r}\cdot(4R+r)(4R-5r)=(4R+r)^2$$

$$\because s\leq\frac{4R+r}{\sqrt{3}}\Leftrightarrow s\leq\sqrt{3}(4R-5r)\Leftrightarrow s\leq\frac{4R+r}{\sqrt{3}}\leq\sqrt{3}(4R-5r)\Leftrightarrow$$

$$4R + r \le 3(4R - 5r) \Leftrightarrow 2r \le R \ (Euler).$$

Solution 3 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By Hölder's inequality we have :

$$\left(\sum_{cyc} \frac{r_a}{\sqrt{r_b{}^2 - r_b r_c + r_c{}^2}}\right)^2 \left(\sum_{cyc} r_a(r_b{}^2 - r_b r_c + r_c{}^2)\right) \ge \left(\sum_{cyc} r_a\right)^3$$

Then we have :

$$\sum_{cyc} \frac{r_a}{\sqrt{r_b{}^2 - r_b r_c + r_c{}^2}} \ge \sqrt{\frac{\left(\sum_{cyc} r_a\right)^3}{\left(\sum_{cyc} r_a\right)\left(\sum_{cyc} r_b r_c\right) - 6r_a r_b r_c}} = \sqrt{\frac{(4R + r)^3}{(4R + r)s^2 - 6rs^2}} =$$

$$= \sqrt{\frac{(4R + r)^3}{(4R - 5r)s^2}} \overset{?}{\ge} \frac{4R + r}{4R - 5r} \Leftrightarrow (4R + r)(4R - 5r) \ge s^2$$

$$\Leftrightarrow (4R^2 + 4Rr + 3r^2 - s^2) + 4(R - 2r)(3R + r) \ge 0, \text{ which is true by Gerretsen's}$$

inequality $4R^2 + 4Rr + 3r^2 \ge s^2$ *and Euler's inequality* $R \ge 2r$.

So the proof is completed. Equality holds iff ΔABC *is equilateral.*

741. In ΔABC the following relationship holds:

$$\frac{(2 + \sqrt{2})R^2}{r^2} \ge \frac{(a + b)(b + c)(c + a)}{abc} + \frac{4\sqrt{2}(ab + bc + ca)}{a^2 + b^2 + c^2} \ge 8 + 4\sqrt{2}$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\frac{(2 + \sqrt{2})R^2}{r^2} \overset{(2)}{\ge} \frac{(a + b)(b + c)(c + a)}{abc} + \frac{4\sqrt{2}(ab + bc + ca)}{a^2 + b^2 + c^2} \overset{(1)}{\ge} 8 + 4\sqrt{2}$$

$$We\ have: \ (1) \Leftrightarrow \frac{(a + b)(b + c)(c + a)}{abc} - 8 \ge 4\sqrt{2}\left(1 - \frac{ab + bc + ca}{a^2 + b^2 + c^2}\right)$$

$$\Leftrightarrow \frac{a(b-c)^2+b(c-a)^2+c(a-b)^2}{abc} \geq 2\sqrt{2}.\frac{(b-c)^2+(c-a)^2+(a-b)^2}{a^2+b^2+c^2}$$

$$\Leftrightarrow \left(\frac{1}{bc}-\frac{2\sqrt{2}}{a^2+b^2+c^2}\right)(b-c)^2+\left(\frac{1}{ca}-\frac{2\sqrt{2}}{a^2+b^2+c^2}\right)(c-a)^2$$
$$+\left(\frac{1}{ab}-\frac{2\sqrt{2}}{a^2+b^2+c^2}\right)(a-b)^2 \geq 0$$

$$\Leftrightarrow S_a(b-c)^2+S_b(c-a)^2+S_c(a-b)^2 \geq 0, \ \ where$$

$$S_a=\frac{1}{bc}-\frac{2\sqrt{2}}{a^2+b^2+c^2} \ \ (and\ analogs)$$

Assume now that $a \geq b \geq c$. *We have*: $S_a \geq S_b \geq S_c$.

Also,

$$S_b+S_c=\frac{1}{ca}+\frac{1}{ab}-\frac{4\sqrt{2}}{a^2+b^2+c^2} \overset{AM-GM}{\geq} \frac{2}{a\sqrt{bc}}-\frac{4\sqrt{2}}{2a\sqrt{b^2+c^2}} \overset{AM-GM}{\geq}$$

$$\geq \frac{2}{a\sqrt{bc}}-\frac{2\sqrt{2}}{a\sqrt{2bc}}=0.$$

Then: $S_a \geq S_b \geq 0$ *and since* $(c-a)^2 \geq (a-b)^2$ *then we have*:

$$S_a(b-c)^2+S_b(c-a)^2+S_c(a-b)^2 \geq 0+S_b(a-b)^2+S_c(a-b)^2$$
$$=(S_b+S_c)(a-b)^2 \geq 0$$

The proof of (1) *is completed. Now, we have*:

$$\frac{(a+b)(b+c)(c+a)}{abc}+\frac{4\sqrt{2}(ab+bc+ca)}{a^2+b^2+c^2} \leq \frac{2s(s^2+r^2+2Rr)}{4Rsr}+4\sqrt{2}=$$

$$=\frac{s^2}{2Rr}+\frac{r}{2R}+1+4\sqrt{2} \overset{Mitrinovic\ \&\ Euler}{\leq}$$

$$\leq \frac{27R^2}{4.2Rr}\cdot\frac{R}{2r}+\frac{r}{2R}\cdot\left(\frac{R}{2r}\right)^3+\left(1+4\sqrt{2}\right)\left(\frac{R}{2r}\right)^2=\frac{(2+\sqrt{2})R^2}{r^2}.$$

$$Therefore,\ \ \frac{(2+\sqrt{2})R^2}{r^2} \geq \frac{(a+b)(b+c)(c+a)}{abc}+\frac{4\sqrt{2}(ab+bc+ca)}{a^2+b^2+c^2} \geq 8+4\sqrt{2}.$$

742. In ΔABC the following relationship holds:

$$\frac{\sum\sqrt{(r_b+h_a)(h_a+r_c)}}{ab+bc+ca}\geq\frac{1}{R}$$

Proposed by Ertan Yildirim-Izmir-Turkiye

Solution 1 by Marian Ursărescu-Romania

$$\frac{\sum\sqrt{(r_b+h_a)(h_a+r_c)}}{ab+bc+ca}\geq\frac{1}{R}\Leftrightarrow$$

$$\sum_{cyc}\sqrt{(r_b+h_a)(h_a+r_c)}\geq\frac{ab+bc+ca}{R};\quad (1)$$

$$But: ab+bc+ca=s^2+r^2+4Rr;\quad (2)$$

$$From\ (1)\ and\ (2)\ we\ must\ show:$$

$$\sum_{cyc}\sqrt{(h_a+r_b)(h_a+r_c)}\geq\frac{s^2+r^2+4Rr}{R};\quad (3)$$

$$From\ Huygens'\ inequality, we\ have:$$

$$\sqrt{(h_a+r_b)(h_a+r_c)}\geq h_a+\sqrt{r_br_c}\Rightarrow$$

$$\sum_{cyc}\sqrt{(h_a+r_b)(h_a+r_c)}\geq\sum_{cyc}h_a+\sum_{cyc}\sqrt{r_br_c};\quad (4)$$

$$\sum_{cyc}h_a=\frac{s^2+r^2+4Rr}{2R};\quad (5)$$

$$\sqrt{r_br_c}\geq\frac{2r_br_c}{r_b+r_c}\Rightarrow\sum_{cyc}\sqrt{r_br_c}\geq 2\sum_{cyc}\frac{r_br_c}{r_b+r_c}=$$

$$=2\cdot\frac{s^2+r^2+4Rr}{4R}=\frac{s^2+r^2+4Rr}{2R};\quad (6)$$

$$From\ (4),(5)\ and\ (6):$$

$$\sum_{cyc}\sqrt{(h_a+r_b)(h_a+r_c)}\geq\frac{s^2+r^2+4Rr+s^2+r^2+4Rr}{2R}=$$

$$=\frac{2s^2+2r^2+8Rr}{2R}=\frac{s^2+r^2+4Rr}{R}\Rightarrow (3)\ it's\ true.$$

Solution 2 by Tapas Das-India

$$\sqrt{s(s-a)} = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{\sqrt{(s-b)(s-c)}} = \frac{F}{\sqrt{(s-b)(s-c)}} \overset{AM-GM}{\geq}$$

$$\geq \frac{F}{\frac{s-b+s-c}{2}} = \frac{2F}{2s-(b+c)} = \frac{2F}{a} = h_a$$

$$\Rightarrow \sum_{cyc} \sqrt{s(s-a)} \geq \sum_{cyc} h_a$$

$$\sqrt{(h_a+r_b)(h_a+r_c)} \overset{Holder}{\geq} h_a + \sqrt{r_b r_c}$$

$$\Rightarrow \sum_{cyc} \sqrt{(h_a+r_b)(h_a+r_c)} \geq \sum_{cyc} h_a + \sum_{cyc} \sqrt{r_b r_c} =$$

$$= \sum_{cyc} h_a + \sum_{cyc} \sqrt{\frac{F^2}{(s-b)(s-c)}} = \sum_{cyc} h_a + \sum_{cyc} \sqrt{s(s-a)} \geq$$

$$\geq \sum_{cyc} h_a + \sum_{cyc} h_a = 2\sum_{cyc} h_a = 2\left(\frac{2F}{a} + \frac{2F}{b} + \frac{2F}{c}\right) = 4F \cdot \frac{ab+bc+ca}{abc} =$$

$$= \frac{ab+bc+ca}{4RF} \cdot 4F = \frac{1}{R}(ab+bc+ca)$$

743. In ΔABC the following relationship holds:

$$\frac{3R^2}{4r^2} \geq \sum_{cyc} \frac{a(b+c)}{b^2+c^2} \geq 2 + \frac{8(abc)^2}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{a(b+c)}{b^2+c^2} + \frac{b(c+a)}{c^2+a^2} + \frac{c(a+b)}{a^2+b^2} = \sum_{cyc} \frac{\sum_{cyc} ab - bc}{b^2+c^2}$$

$$= \frac{(\sum_{cyc} ab)}{\prod_{cyc}(b^2+c^2)} \sum_{cyc} (c^2+a^2)(a^2+b^2) - \frac{1}{\prod_{cyc}(b^2+c^2)} \sum_{cyc} bc(c^2+a^2)(a^2+b^2)$$

$$=\frac{(\sum_{cyc}ab)}{\prod_{cyc}(b^2+c^2)}\left(\sum_{cyc}a^4+3\sum_{cyc}a^2b^2\right)-\frac{1}{\prod_{cyc}(b^2+c^2)}\sum_{cyc}\left(bc\left(a^4+\sum_{cyc}a^2b^2\right)\right)$$

$$=\frac{1}{\prod_{cyc}(b^2+c^2)}\left(\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^4+3\sum_{cyc}a^2b^2\right)-abc\sum_{cyc}a^3\right.$$

$$\left.-\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2b^2\right)\right)$$

$$=\frac{1}{\prod_{cyc}(b^2+c^2)}\left(\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^4+2\sum_{cyc}a^2b^2\right)-8Rrs^2(s^2-6Rr-3r^2)\right)$$

$$=\frac{4(s^2+4Rr+r^2)(s^2-4Rr-r^2)^2-8Rrs^2(s^2-6Rr-3r^2)}{\prod_{cyc}(b^2+c^2)}\overset{(*)}{=}\sum_{cyc}\frac{a(b+c)}{b^2+c^2}$$

$$\text{Now, }2+\frac{8(abc)^2}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}=\frac{8(abc)^2+2\left((\sum_{cyc}a^2b^2)(\sum_{cyc}a^2)-(abc)^2\right)}{\prod_{cyc}(b^2+c^2)}$$

$$=\frac{4(s^2-4Rr-r^2)((s^2+4Rr+r^2)^2-16Rrs^2)+96R^2r^2s^2}{\prod_{cyc}(b^2+c^2)}\le\sum_{cyc}\frac{a(b+c)}{b^2+c^2}$$

$$\overset{\text{via }(*)}{\Leftrightarrow}(s^2+4Rr+r^2)(s^2-4Rr-r^2)^2-2Rrs^2(s^2-6Rr-3r^2)$$

$$\ge(s^2-4Rr-r^2)((s^2+4Rr+r^2)^2-16Rrs^2)+24R^2r^2s^2$$

$$\Leftrightarrow\boxed{(3R-r)s^4-rs^2(38R^2+5Rr)+r^2(4R+r)^3\overset{(\bullet)}{\ge}0}$$

$$\text{Now, LHS of }(\bullet)\overset{\text{Gerretsen}}{\ge}\left((3R-r)(16Rr-5r^2)-r(38R^2+5Rr)\right)s^2+r^2(4R+r)^3\overset{?}{\ge}0$$

$$\Leftrightarrow(10R^2-36Rr+5r^2)s^2+r(4R+r)^3\underset{(\bullet\bullet)}{\overset{?}{\ge}}0$$

$\boxed{Case\ 1}$ $10R^2-36Rr+5r^2\ge0$ and then, LHS of $(\bullet\bullet)\ge r(4R+r)^3>0$

$\Rightarrow(\bullet\bullet)$ is true (strict inequality)

$\boxed{Case\ 2}$ $10R^2-36Rr+5r^2<0$ and then, LHS of $(\bullet\bullet)$

$$=-\left(-(10R^2-36Rr+5r^2)\right)s^2+r(4R+r)^3\overset{\text{Gerretsen}}{\ge}$$

$$-\left(-(10R^2-36Rr+5r^2)\right)(4R^2+4Rr+3r^2)+r(4R+r)^3\overset{?}{\ge}0$$

$$\Leftrightarrow20t^4-20t^3-23t^2-38t+8\overset{?}{\ge}0\ \left(t=\frac{R}{r}\right)\Leftrightarrow(t-2)\left(20t^3+20t^2+15t+2(t-2)\right)\overset{?}{\ge}0\to\text{true}$$

$\because t\overset{\text{Euler}}{\ge}2\Rightarrow(\bullet\bullet)$ is true $\therefore$ combining cases 1, 2, $(\bullet\bullet)$ is true $\forall$ triangles

$$\therefore\text{ in any }\Delta ABC,\ \boxed{\boxed{\frac{a(b+c)}{b^2+c^2}+\frac{b(c+a)}{c^2+a^2}+\frac{c(a+b)}{a^2+b^2}\ge2+\frac{8(abc)^2}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}}},$$

equality iff Δ ABC is equilateral

$$\text{Again, }\sum_{cyc}\frac{a(b+c)}{b^2+c^2}\overset{A-G}{\ge}\frac{1}{2abc}\sum_{cyc}a^2(2s-a)=\frac{4s(s^2-4Rr-r^2)-2s(s^2-6Rr-3r^2)}{8Rrs}$$

$$=\frac{s^2-2Rr+r^2}{4Rr}\overset{\text{Gerretsen}}{\le}\frac{4R^2+2Rr+4r^2}{4Rr}\overset{?}{\le}\frac{3R^2}{4r^2}\Leftrightarrow3t^3-4t^2-2t-4\overset{?}{\ge}0$$

$$\Leftrightarrow (t-2)(3t^2+2t+2) \overset{?}{\geq} 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$$\therefore \text{ in any } \Delta\, ABC, \boxed{\boxed{\frac{3R^2}{4r^2} \geq \frac{a(b+c)}{b^2+c^2} + \frac{b(c+a)}{c^2+a^2} + \frac{c(a+b)}{a^2+b^2}}}, \text{equality iff } \Delta\, ABC \text{ is equilateral}$$

$$\therefore \text{ in any } \Delta\, ABC, \frac{3R^2}{4r^2} \geq \frac{a(b+c)}{b^2+c^2} + \frac{b(c+a)}{c^2+a^2} + \frac{c(a+b)}{a^2+b^2}$$

$$\geq 2 + \frac{8(abc)^2}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}, \text{equalities iff } \Delta\, ABC \text{ is equilateral (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have :

$$\sum_{cyc} \frac{a(b+c)}{b^2+c^2} \overset{CBS}{\leq} \sum_{cyc} \frac{2a}{b+c} \overset{CBS}{\leq} \sum_{cyc} \frac{2a}{4}\left(\frac{1}{b}+\frac{1}{c}\right) = \frac{1}{2}\sum_{cyc}\left(\frac{a}{b}+\frac{b}{a}\right) \overset{Bandila}{\leq}$$

$$\leq \frac{1}{2}\sum_{cyc} \frac{R}{r} = \frac{3R}{2r} \overset{Euler}{\leq} \frac{3R^2}{4r^2}.$$

Also :

$$3 - \sum_{cyc} \frac{a(b+c)}{b^2+c^2} = \sum_{cyc}\left(1 - \frac{a(b+c)}{b^2+c^2}\right) = \sum_{cyc}\left(\frac{c(c-a)}{b^2+c^2} - \frac{b(a-b)}{b^2+c^2}\right)$$

$$= \sum_{cyc}\left(\frac{a(a-b)}{c^2+a^2} - \frac{b(a-b)}{b^2+c^2}\right) =$$

$$= \sum_{cyc} \frac{(a-b)[a(b^2+c^2) - b(c^2+a^2)]}{(b^2+c^2)(c^2+a^2)} = \sum_{cyc} \frac{(a-b)^2(c^2-ab)}{(b^2+c^2)(c^2+a^2)}$$

$$\leq \sum_{cyc} \frac{(a-b)^2c^2}{(b^2+c^2)(c^2+a^2)} \leq$$

$$\leq \sum_{cyc} \frac{(a+b)^2(a-b)^2c^2}{(a^2+b^2)(b^2+c^2)(c^2+a^2)} = \frac{\sum_{cyc}(a^4 - 2a^2b^2 + b^4)\, c^2}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}$$

$$= 1 - \frac{8(abc)^2}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}.$$

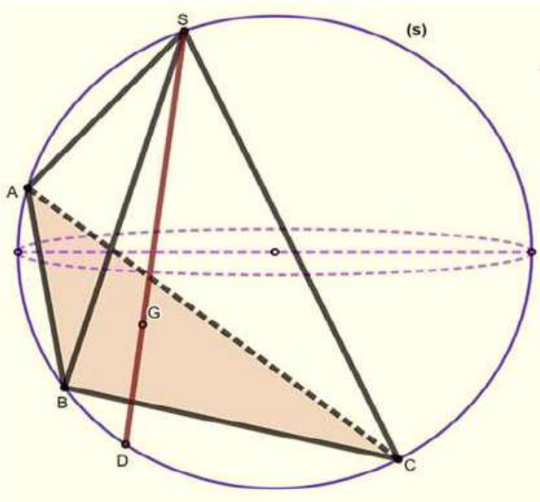
Then :

$$\sum_{cyc} \frac{a(b+c)}{b^2+c^2} \geq 2 + \frac{8(abc)^2}{(a^2+b^2)(b^2+c^2)(c^2+a^2)} \text{ and the proof is completed.}$$

744. $SABC$ –tetrahedron, G –centroid, (s) –circumsphere, $(SAB)=(P)$

$$SA=2, SB=3, SC=4, \angle BSC=\angle CSA=\angle ASB=60^\circ$$

$$SG\cap(s)=D, CD\cap(P)=F. \text{ Find: } SF=? \text{ and } \angle\big(CD,(P)\big)=?$$



Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by proposer

Plagiogonal 3D system: $SA\equiv Sx; SB\equiv Sy; SC\equiv Sz$

$$S(0,0,0), A(2,0,0), B(0,3,0), C(0,0,4), D(d_1,d_2,d_3), F(f_1,f_2,f_3)$$

$$d_1=\frac{58}{55}, d_2=\frac{87}{55}, d_3=\frac{116}{55}$$

$$CD: \frac{x}{\frac{58}{55}}=\frac{y}{\frac{87}{55}}=\frac{z-4}{\frac{116}{55}-4}$$

$$(SAB): z=0\Rightarrow f_1=\frac{58}{26}; f_2=\frac{87}{26}; f_3=0$$

$$SF^2=f_1^2+f_2^2+f_1f_2\Rightarrow SF=\frac{29\sqrt{19}}{26}$$

$$\overrightarrow{CD}\left(\frac{58}{55},\frac{87}{55},-\frac{104}{55}\right)$$

$$\vec{v}\parallel\overrightarrow{CD}, \vec{v}(58,87,-104)$$

$$(SAB): z=0, \overrightarrow{u'}=(u'_1,u'_2,u'_3), \overrightarrow{u'}\perp(SAB)$$

$$u'_1=-\frac{1}{4}; u'_2=-\frac{1}{4}; u'_3=\frac{3}{4}; \vec{u}\parallel\overrightarrow{u'}, \vec{u}(-1,-1,3)$$

$$|\vec{v}|^2 = 1+1+9+1-3-3 = 6 \Rightarrow |\vec{u}| = \sqrt{6}$$

$$|\vec{v}|^2 = 58^2 + 87^2 + (-104)^2 + 58\cdot 87 - 87\cdot 104 - 58\cdot 104 = 11715$$

$$|\vec{v}| = \sqrt{11715}$$

$$\vec{u}\cdot\vec{v} = 58 - 87 - 3\cdot 104 + (-58-87) + (3\cdot 87 + 104) + (3\cdot 58 + 104) = 41$$

$$\cos\rho = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|\cdot|\vec{v}|} = \frac{41}{\sqrt{6}\cdot\sqrt{11715}} = \frac{41}{\sqrt{70290}}$$

$$\sin\theta = \cos\rho = \frac{41}{\sqrt{70290}} \Rightarrow \theta = \sin^{-1}\left(\frac{41}{\sqrt{70290}}\right) \approx 8,9^\circ$$

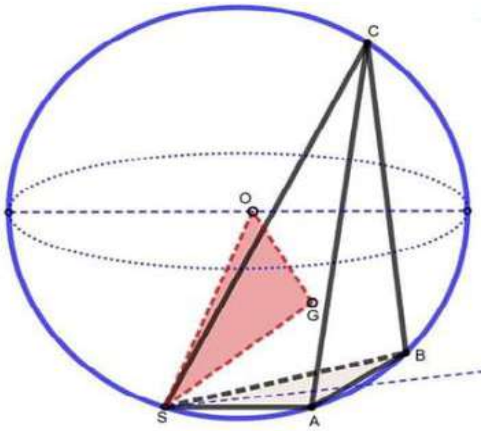
745. $SABC$ **−tetrahedron,** G **−centroid,**

$$SA = 4, SB = 8, SC = 12, \angle BSC = \theta_1 = 60^\circ, \angle CSA = \theta_2 = 60^\circ,$$

$\angle ASB = \theta_3 = 60^\circ, O$ **−circumspherecenter,** G **−centroid of** $SABC$**,**

SW **−bisector of** $\angle ASB, P = OG \cap (ASB)$**. Find:**

$$[SOG] =?, d_{(OG,SA)} =?, \sphericalangle(SG,(ASC)) =?, \sphericalangle(OG,SW) =?, SP =?$$



Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by proposer

Plagiogonal 3D system: $SA \equiv Sx; SB \equiv Sy; SC \equiv Sz$

$$S(0,0,0), A(4,0,0), B(0,8,0), C(0,0,12), G\left(\frac{a}{4},\frac{b}{4},\frac{c}{4}\right) \Rightarrow G(1,2,3)$$

$$O(o_1,o_2,o_3), o_1 = \frac{3a-b-c}{4} = -2; o_2 = \frac{3b-c-a}{4} = 2; o_3 = \frac{3c-a-b}{4} = 6$$

$O(-2,2,6), \overrightarrow{SG}=(1,2,3), |\overrightarrow{SG}|^2=25; \overrightarrow{SO}=(-2,2,6); |\overrightarrow{SO}|^2=40, \quad \overrightarrow{SG}\cdot\overrightarrow{SO}=28$

$$[SOG]=\frac{1}{2}\sqrt{|\overrightarrow{SG}|^2\cdot|\overrightarrow{SO}|^2-(\overrightarrow{SG}\cdot\overrightarrow{SO})^2}=3\sqrt{6}$$

$$OG: \frac{x-1}{-2-1}=\frac{z-3}{6-3}\Rightarrow x-1=3-z$$

Let $K\in OG, K(k_1,0,k_3)$. Is $k_1-1=3-k_3\Rightarrow k_3=4-k_1, \Rightarrow K(k_1,2,4-k_1)$.

Let $L\in SA\Rightarrow L(l_1,0,0)$. $KL^2=f(k_1,l_1)=$

$=(k_1-l_1)^2+2^2+(4-k_1)^2+2(k_1-l_1)+2(4-k_1)+(k_1-l_1)(4-k_1)$

$$\begin{cases}\frac{df}{dk_n}=0\\ \frac{df}{dl_n}=0\end{cases}\Rightarrow\begin{cases}k_1=\frac{14}{3}\\ l_1=\frac{16}{3}\end{cases}\Rightarrow K\left(\frac{14}{3},2,-\frac{2}{3}\right), L\left(\frac{16}{3},0,0\right)$$

$$\min\{KL^2\}=\frac{8}{3}\Rightarrow KL_{min}=2\sqrt{\frac{2}{3}}\Rightarrow d(OG,SA)=2\sqrt{\frac{2}{3}}$$

$$\overrightarrow{SG}=(-3,0,3)\Rightarrow\overrightarrow{SG}\parallel(xSz)\Rightarrow SG\parallel(ACS)$$

$\overrightarrow{SG}=(-3,0,3)\Rightarrow\overrightarrow{SG}\parallel SW'(SW'$ –bisector of $\sphericalangle zSx)$

$SW\perp SW'($ SW bisector of $\sphericalangle xSy)$ $\Rightarrow OG\perp OW$

Is $OG: x-1=3-z\Rightarrow x=4$. Let $SG\cap(ASB)=P$. So,

$P(4,2,0)$. Is $SP^2=4^2+2^2+2\cdot4\cdot2\cdot\frac{1}{2}\Rightarrow SP=2\sqrt{7}$.

746. Prove that :

$$\cos^7\left(\frac{\pi}{18}\right)+\cos^9\left(\frac{5\pi}{18}\right)+\cos^9\left(\frac{7\pi}{18}\right)=\cos^9\left(\frac{\pi}{18}\right)+\cos^7\left(\frac{5\pi}{18}\right)+\cos^7\left(\frac{7\pi}{18}\right)$$

Proposed by Carlos Paiva-Brazil

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Let\ a=-\cos\left(\frac{\pi}{18}\right),\ b=\cos\left(\frac{5\pi}{18}\right),\ c=\cos\left(\frac{7\pi}{18}\right).$$

$$The\ problem\ becomes\ to\ prove:\ a^9+b^9+c^9=a^7+b^7+c^7$$

$$We\ have: \ -\cos\left(3\times\frac{\pi}{18}\right)=\cos\left(3\times\frac{5\pi}{18}\right)=\cos\left(3\times\frac{7\pi}{18}\right)$$

$$=-\frac{\sqrt{3}}{2}\ and\ since\ \cos 3\alpha=4\cos^3\alpha-3\cos\alpha,$$

$$then\ a,b,c\ are\ the\ roots\ of\ the\ equation\ 4x^3-3x+\frac{\sqrt{3}}{2}=0.$$

$$From\ Vieta's\ formulas, we\ have:$$

$$a+b+c=0,\ \ ab+bc+ca=-\frac{3}{4},\ \ abc=-\frac{\sqrt{3}}{8}.$$

$$Since:$$

$$x^9-x^7=\frac{1}{128}\left(4x^3-3x+\frac{\sqrt{3}}{2}\right)\left(32x^6-8x^4-4\sqrt{3}x^3-6x^2-2\sqrt{3}x-3\right)-\frac{3\sqrt{3}}{128}x^2$$
$$-\frac{3}{64}x+\frac{3\sqrt{3}}{256},$$

$$then\ we\ have:\ \ a^9-a^7=-\frac{3\sqrt{3}}{128}a^2-\frac{3}{64}a+\frac{3\sqrt{3}}{256}\ \ (and\ analogs)$$

$$Therefore,$$

$$(a^9+b^9+c^9)-(a^7+b^7+c^7)=-\frac{3\sqrt{3}}{128}(a^2+b^2+c^2)-\frac{3}{64}(a+b+c)+3.\frac{3\sqrt{3}}{256}=$$

$$=-\frac{3\sqrt{3}}{128}[(a+b+c)^2-2(ab+bc+ca)]+\frac{9\sqrt{3}}{256}=$$

$$=-\frac{3\sqrt{3}}{128}.\frac{3}{2}+\frac{9\sqrt{3}}{256}=0, and\ the\ proof\ is\ complete.$$

747. In ΔABC the following relationship holds:

$$\sqrt{\sum_{cyc}\frac{a}{b+c}}+\sqrt{\sum_{cyc}\frac{a^2}{b^2+c^2}}+\sqrt{\sum_{cyc}\frac{b+c}{a}}+\sqrt{\sum_{cyc}\frac{b^2+c^2}{a^2}}\le\frac{3\sqrt{6}R}{2r}$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:\ \sum_{cyc}\frac{b+c}{a}=\sum_{cyc}\left(\frac{2s}{a}-1\right)=\frac{s^2+r^2+4Rr}{2Rr}-3$$

$$=\frac{s^2+r^2-2Rr}{2Rr}\overset{Gerretsen}{\leq}\frac{2R^2+Rr+2r^2}{Rr}\ (1)$$

$$And,\ \sum_{cyc}\frac{a}{b+c}\overset{CBS}{\leq}\sum_{cyc}\frac{a}{4}\left(\frac{1}{b}+\frac{1}{c}\right)=\frac{1}{4}\sum_{cyc}\frac{b+c}{a}\overset{(1)}{\leq}\frac{2R^2+Rr+2r^2}{4Rr}\ \ (2)$$

$$Now,$$

$$\sum_{cyc}\frac{a^2}{b^2+c^2}\overset{AM-GM}{\leq}\sum_{cyc}\frac{a^2}{2bc}=\frac{\sum_{cyc}a^3}{2abc}=\frac{s^2-3r^2-6Rr}{4Rr}\overset{Gerretsen}{\leq}$$

$$\leq\frac{4R^2-2Rr}{4Rr}=\frac{2R-r}{2r}\ (3)$$

$$\sum_{cyc}\frac{b^2+c^2}{a^2}=\sum_{cyc}\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right)=\sum_{cyc}\left(\left(\frac{a}{b}+\frac{b}{a}\right)^2-2\right)\overset{Bandila}{\leq}$$

$$\leq\sum_{cyc}\left(\left(\frac{R}{r}\right)^2-2\right)=3\left(\frac{R}{r}\right)^2-6\ \ (4)$$

$$Using\ (1),(2),(3)\ and\ (4), it\ suffices\ to\ prove$$

$$:\ \frac{3}{2}\sqrt{\frac{2R^2+Rr+2r^2}{Rr}}+\sqrt{\frac{2R-r}{2r}}+\sqrt{3\left(\frac{R}{r}\right)^2-6}\leq\frac{3\sqrt{6}R}{2r}\ (*)$$

$$We\ have:\ LHS_{(*)}\overset{CBS}{\leq}\sqrt{(3+1+2)\left(\frac{3(2R^2+Rr+2r^2)}{4Rr}+\frac{2R-r}{2r}+\frac{3}{2}\left(\frac{R^2}{r^2}-2\right)\right)}=$$

$$= \sqrt{6\left(\frac{3R^2}{2r^2}+\frac{5R}{2r}-\frac{11}{4}+\frac{3r}{2R}\right)} \overset{?}{\underset{(3)}{\leq}} \frac{3\sqrt{6}R}{2r} \Leftrightarrow \frac{3R^2}{4r^2}-\frac{5R}{2r}+\frac{11}{4}-\frac{3r}{2R} \geq 0$$

$$\Leftrightarrow \left(\frac{R}{2r}-1\right)\left(\frac{3R}{2r}-2+\frac{3r}{2R}\right) \geq 0$$

Which is true by Euler's inequality $R \geq 2r$. The proof is complete.

Solution 2 by Soumava Chakraborty-Kolkata-India

$$4\sum_{cyc}\frac{a}{b+c} = 4\sum_{cyc}\frac{2s-(b+c)}{b+c} = 4\left(\frac{2s\sum_{cyc}(c+a)(a+b)}{\prod_{cyc}(b+c)}-3\right)$$

$$= 4\left(\frac{2s(5s^2+4Rr+r^2)}{2s(s^2+2Rr+r^2)}-3\right) = \frac{8(s^2-Rr-r^2)}{s^2+2Rr+r^2} \overset{(\bullet)}{=} 4\sum_{cyc}\frac{a}{b+c} \text{ and,}$$

$$4\sum_{cyc}\frac{a^2}{b^2+c^2} \overset{A-G}{\leq} \frac{4}{2abc}\cdot\sum_{cyc}a^3 \Rightarrow 4\sum_{cyc}\frac{a^2}{b^2+c^2} \overset{(\bullet\bullet)}{\leq} \frac{s^2-6Rr-3r^2}{Rr} \text{ and, } \sum_{cyc}\frac{b+c}{a} = \sum_{cyc}\frac{2s-a}{a}$$

$$= \frac{2s(s^2+4Rr+r^2)}{4Rrs}-3 \Rightarrow \sum_{cyc}\frac{b+c}{a} \overset{(\bullet\bullet\bullet)}{=} \frac{s^2-2Rr+r^2}{2Rr} \text{ and,}$$

$$\sum_{cyc}\frac{b^2+c^2}{a^2} = \sum_{cyc}\left(\frac{b^2}{c^2}+\frac{c^2}{b^2}\right) = \sum_{cyc}\left(\left(\frac{b}{c}+\frac{c}{b}\right)^2-2\right) \overset{Bandila}{\leq} \sum_{cyc}\left(\frac{R^2}{r^2}-2\right)$$

$$\therefore \sum_{cyc}\frac{b^2+c^2}{a^2} \overset{(\bullet\bullet)}{\leq} \frac{3(R^2-2r^2)}{r^2}$$

$$\text{Now, } \sqrt{\sum_{cyc}\frac{a}{b+c}}+\sqrt{\sum_{cyc}\frac{a^2}{b^2+c^2}}+\sqrt{\sum_{cyc}\frac{b+c}{a}}+\sqrt{\sum_{cyc}\frac{b^2+c^2}{a^2}}$$

$$= \frac{1}{2}\cdot\sqrt{\sum_{cyc}\frac{4a}{b+c}}+\frac{1}{2}\cdot\sqrt{\sum_{cyc}\frac{4a^2}{b^2+c^2}}+\frac{1}{2}\cdot\sqrt{\sum_{cyc}\frac{b+c}{a}}+\frac{1}{2}\cdot\sqrt{\sum_{cyc}\frac{b+c}{a}}$$

$$+\frac{1}{2}\cdot\sqrt{\sum_{cyc}\frac{b^2+c^2}{a^2}}+\frac{1}{2}\cdot\sqrt{\sum_{cyc}\frac{b^2+c^2}{a^2}}$$

$$\overset{CBS}{\leq} \sqrt{6\cdot\frac{1}{4}}\cdot\sqrt{\sum_{cyc}\frac{4a}{b+c}+\sum_{cyc}\frac{4a^2}{b^2+c^2}+\sum_{cyc}\frac{b+c}{a}+\sum_{cyc}\frac{b+c}{a}+\sum_{cyc}\frac{b^2+c^2}{a^2}+\sum_{cyc}\frac{b^2+c^2}{a^2}}$$

$$\overset{via\,(\bullet),(\bullet\bullet),(\bullet\bullet\bullet),(\bullet\bullet\bullet\bullet)}{\leq} \sqrt{\frac{6}{4}} \cdot \sqrt{\frac{8(s^2-Rr-r^2)}{s^2+2Rr+r^2}+\frac{s^2-6Rr-3r^2}{Rr}+\frac{s^2-2Rr+r^2}{Rr}+\frac{6(R^2-2r^2)}{r^2}} \overset{?}{\leq} \frac{3\sqrt{6}R}{2r}$$

$$\Leftrightarrow \frac{6}{4} \cdot \frac{8Rr^2(s^2-Rr-r^2)+r(s^2+2Rr+r^2)(2s^2-8Rr-2r^2)+6R(R^2-2r^2)(s^2+2Rr+r^2)}{Rr^2(s^2+2Rr+r^2)} \overset{?}{\leq} \frac{9.6R^2}{4r^2}$$

$$\Leftrightarrow \boxed{(3R^3-8R^2r+12Rr^2-4r^3)(s^2+2Rr+r^2) \underset{(*)}{\overset{?}{\geq}} 8Rr^2(s^2-Rr-r^2)}$$

$$\because 3R^3-8R^2r+12Rr^2-4r^3 = (R-2r)\left(2R^2+R(R-2r)+8r^2\right)+12r^3 \overset{Euler}{\geq} 12r^3 > 0$$

$$\therefore \text{LHS of } (*) \underset{(i)}{\overset{Gerretsen}{\geq}} (3R^3-8R^2r+12Rr^2-4r^3)(18Rr-4r^2)\ and,$$

$$\text{RHS of } (*) \underset{(ii)}{\overset{Gerretsen}{\leq}} 8Rr^2(4R^2+3Rr+2r^2) \therefore (i),(ii)$$

$\Rightarrow$ in order to prove $(*)$, it suffices to prove

$$: (3R^3-8R^2r+12Rr^2-4r^3)(18Rr-4r^2) \geq 8Rr^2(4R^2+3Rr+2r^2)$$

$$\Leftrightarrow 27t^4-94t^3+112t^2-68t+8 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((t-2)(27t^2+14t+60)+116\right) \overset{?}{\geq} 0 \to \text{true} \because t \overset{Euler}{\geq} 2$$

$\Rightarrow (*)$ is true

$$\therefore \text{in any } \Delta\,ABC, \sqrt{\sum_{cyc}\frac{a}{b+c}}+\sqrt{\sum_{cyc}\frac{a^2}{b^2+c^2}}+\sqrt{\sum_{cyc}\frac{b+c}{a}}+\sqrt{\sum_{cyc}\frac{b^2+c^2}{a^2}}$$

$$\leq \frac{3\sqrt{6}R}{2r}, \text{with equality iff } \Delta\,ABC \text{ is equilateral (QED)}$$

748. In ΔABC the following relationship holds:

$$\sqrt{\sum_{cyc}\frac{m_a}{m_b+m_c}}+\sqrt{\sum_{cyc}\frac{m_b+m_c}{m_a}}+\sqrt{\sum_{cyc}\frac{{m_a}^2}{{m_b}^2+{m_c}^2}}+\sqrt{\sum_{cyc}\frac{{m_b}^2+{m_c}^2}{{m_a}^2}} \leq \frac{3\sqrt{6}R}{2r}$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By Oppenheim's inequality in $\Delta m_a m_b m_c$ we have:

$$x.{m_a}^2+y.{m_b}^2+z.{m_c}^2 \geq 4F_m\sqrt{xy+yz+zx}, \quad \forall x,y,z>0. \quad \therefore F_m = \frac{3F}{4}.$$

$$For\ \ x=\frac{{m_b}^2}{{m_a}^2},\ \ y=\frac{{m_c}^2}{{m_b}^2},\ \ z=\frac{{m_a}^2}{{m_c}^2},\ \ we\ have:$$

$${m_a}^2+{m_b}^2+{m_c}^2\geq 3F\sqrt{\frac{{m_a}^2}{{m_b}^2}+\frac{{m_b}^2}{{m_c}^2}+\frac{{m_c}^2}{{m_a}^2}}$$

$$Then:\frac{{m_a}^2}{{m_b}^2}+\frac{{m_b}^2}{{m_c}^2}+\frac{{m_c}^2}{{m_a}^2}\leq\frac{(a^2+b^2+c^2)^2}{16F^2}=\frac{(s^2-r^2-4Rr)^2}{4s^2r^2}$$
$$=\frac{s^2-r^2-4Rr}{4r^2}.\left(1-\frac{r(4R+r)}{s^2}\right)\leq$$

$$\overset{Gerretsen}{\leq}\frac{4R^2+2r^2}{4r^2}.\left(1-\frac{r(4R+r)}{4R^2+4Rr+3r^2}\right)=\frac{(2R^2+r^2)^2}{r^2(4R^2+4Rr+3r^2)}.$$

$$Similarly:$$

$$\frac{{m_b}^2}{{m_a}^2}+\frac{{m_c}^2}{{m_b}^2}+\frac{{m_a}^2}{{m_c}^2}\leq\frac{(2R^2+r^2)^2}{r^2(4R^2+4Rr+3r^2)},\ \ then:\sum_{cyc}\frac{{m_b}^2+{m_c}^2}{{m_a}^2}$$
$$\leq\frac{2(2R^2+r^2)^2}{r^2(4R^2+4Rr+3r^2)}\ (1)$$

$$Also,\ \sum_{cyc}\frac{{m_a}^2}{{m_b}^2+{m_c}^2}\overset{CBS}{\leq}\sum_{cyc}\frac{{m_a}^2}{4}\left(\frac{1}{{m_b}^2}+\frac{1}{{m_c}^2}\right)$$
$$=\frac{1}{4}\sum_{cyc}\frac{{m_b}^2+{m_c}^2}{{m_a}^2}\overset{(1)}{\leq}\frac{(2R^2+r^2)^2}{2r^2(4R^2+4Rr+3r^2)}\ (2)$$

$$Now\ we\ have:\sum_{cyc}\frac{m_b+m_c}{m_a}=\sum_{cyc}\left(\frac{m_b}{m_a}+\frac{m_c}{m_a}\right)\overset{CBS}{\leq}2\sqrt{\left(\sum{m_a}^2\right)\left(\sum\frac{1}{{m_a}^2}\right)}\leq$$

$$\overset{m_a\geq\sqrt{s(s-a)}}{\leq}2\sqrt{\left(\frac{3}{4}\sum a^2\right)\left(\sum\frac{1}{s(s-a)}\right)}\overset{Leibniz}{\leq}2\sqrt{\frac{3}{4}.9R^2.\frac{4R+r}{s^2r}}\overset{Doucet}{\leq}\frac{3R}{r}\ (3)$$

$$Also\ we\ have:\sum_{cyc}\frac{m_a}{m_b+m_c}\overset{CBS}{\leq}\sum_{cyc}\frac{m_a}{4}\left(\frac{1}{m_b}+\frac{1}{m_c}\right)=\frac{1}{4}\sum_{cyc}\frac{m_b+m_c}{m_a}\overset{(3)}{\leq}\frac{3R}{4r}\ (4)$$

$$Using\ (1),(2),(3)\ and\ (4),it\ suffices\ to\ prove:$$

$$\frac{3}{2}\sqrt{\frac{3R}{r}}+\frac{3}{2}\sqrt{\frac{2(2R^2+r^2)^2}{r^2(4R^2+4Rr+3r^2)}}\le\frac{3\sqrt{6}R}{2r}\ (*)$$

$$LHS_{(*)}\overset{CBS}{\le}\frac{3}{2}\sqrt{2\left(\frac{3R}{r}+\frac{2(2R^2+r^2)^2}{r^2(4R^2+4Rr+3r^2)}\right)}\overset{?}{\le}\frac{3\sqrt{6}R}{2r}\Leftrightarrow$$

$$\frac{3R}{r}+\frac{2(2R^2+r^2)^2}{r^2(4R^2+4Rr+3r^2)}\le\frac{3R^2}{r^2}$$

$$\Leftrightarrow\ 2(2R^2+r^2)^2\le 3(R^2-Rr)(4R^2+4Rr+3r^2)\Leftrightarrow 4R^4-11R^2r^2-9Rr^3-2r^4\ge 0$$

$$\Leftrightarrow\ (R-2r)(4R^3+8R^2r+5Rr^2+r^3)\ge 0$$

which is true by Euler's inequality $R\ge 2r$.

Therefore,

$$\sqrt{\sum_{cyc}\frac{m_a}{m_b+m_c}}+\sqrt{\sum_{cyc}\frac{m_b+m_c}{m_a}}+\sqrt{\sum_{cyc}\frac{m_a^2}{m_b^2+m_c^2}}+\sqrt{\sum_{cyc}\frac{m_b^2+m_c^2}{m_a^2}}\le\frac{3\sqrt{6}R}{2r}.$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{m_a}{m_b+m_c}=\sum_{cyc}\frac{\sum_{cyc}m_a-(m_b+m_c)}{m_b+m_c}$$

$$=\left(\sum_{cyc}m_a\right)\left(\sum_{cyc}\frac{1}{m_b+m_c}\right)-3\overset{A-G}{\le}\frac{1}{2}\left(\sum_{cyc}m_a\right)\left(\sum_{cyc}\frac{1}{\sqrt{m_bm_c}}\right)$$

$$-3\overset{CBS}{\le}\frac{1}{2}\left(\sum_{cyc}m_a\right).\sqrt{\sum_{cyc}\frac{1}{m_b}}.\sqrt{\sum_{cyc}\frac{1}{m_c}}-3$$

$$=\frac{1}{2}\left(\sum_{cyc}m_a\right)\left(\sum_{cyc}\frac{1}{m_a}\right)-3\overset{Bager+m_a\ge h_a\text{ and analogs}}{\le}\frac{4R+r}{2}\left(\sum_{cyc}\frac{1}{h_a}\right)-3=\frac{4R+r}{2r}-3$$

$$\Rightarrow\sum_{cyc}\frac{4m_a}{m_b+m_c}\overset{(\cdot)}{\le}\frac{8R-10r}{r}\ \text{and,}$$

$$\sum_{cyc}\frac{m_b+m_c}{m_a}=\sum_{cyc}\frac{\sum_{cyc}m_a-m_a}{m_a}$$

$$=\left(\sum_{cyc}m_a\right)\left(\sum_{cyc}\frac{1}{m_a}\right)-3\overset{Bager+m_a\ge h_a\text{ and analogs}}{\le}(4R+r)\left(\sum_{cyc}\frac{1}{h_a}\right)-3=\frac{4R+r}{r}-3$$

$$\Rightarrow 2\sum_{cyc}\frac{m_b+m_c}{m_a}\overset{(\cdot\cdot)}{\le}\frac{8R-4r}{r}\ \text{and,}$$

$$\sum_{cyc}\frac{4m_a^2}{m_b^2+m_c^2}\overset{\text{Reverse Bergstrom}}{\leq}\sum_{cyc}\left(m_a^2\left(\frac{1}{m_b^2}+\frac{1}{m_c^2}\right)\right)=\sum_{cyc}\frac{m_b^2+m_c^2}{m_a^2}\Rightarrow\sum_{cyc}\frac{4m_a^2}{m_b^2+m_c^2}\overset{(\bullet\bullet\bullet)}{\leq}\sum_{cyc}\frac{m_b^2+m_c^2}{m_a^2}$$

$$\text{Now, }\sqrt{\sum_{cyc}\frac{m_a}{m_b+m_c}}+\sqrt{\sum_{cyc}\frac{m_b+m_c}{m_a}}+\sqrt{\sum_{cyc}\frac{m_a^2}{m_b^2+m_c^2}}+\sqrt{\sum_{cyc}\frac{m_b^2+m_c^2}{m_a^2}}$$

$$=\frac{1}{2}.\sqrt{\sum_{cyc}\frac{4m_a}{m_b+m_c}}+\frac{1}{2}.\sqrt{\sum_{cyc}\frac{m_b+m_c}{m_a}}+\frac{1}{2}.\sqrt{\sum_{cyc}\frac{m_b+m_c}{m_a}}+\frac{1}{2}.\sqrt{\sum_{cyc}\frac{4m_a^2}{m_b^2+m_c^2}}+\frac{1}{2}.\sqrt{\sum_{cyc}\frac{m_b^2+m_c^2}{m_a^2}}$$

$$+\frac{1}{2}.\sqrt{\sum_{cyc}\frac{m_b^2+m_c^2}{m_a^2}}$$

$$\overset{CBS}{\leq}\sqrt{6.\frac{1}{4}}.\sqrt{\sum_{cyc}\frac{4m_a}{m_b+m_c}+2\sum_{cyc}\frac{m_b+m_c}{m_a}+\sum_{cyc}\frac{4m_a^2}{m_b^2+m_c^2}+2\sum_{cyc}\frac{m_b^2+m_c^2}{m_a^2}}\overset{\text{via }(\bullet),(\bullet\bullet),(\bullet\bullet\bullet)}{\leq}$$

$$\sqrt{\frac{6}{4}}.\sqrt{\frac{8R-10r+8R-4r}{r}+3\sum_{cyc}\frac{m_b^2+m_c^2}{m_a^2}}\overset{?}{\leq}\frac{3\sqrt{6}R}{2r}$$

$$\Leftrightarrow 3\sum_{cyc}\frac{m_b^2+m_c^2}{m_a^2}\overset{?}{\leq}\frac{9R^2}{r^2}-\frac{16R-14r}{r}\Leftrightarrow\boxed{\sum_{cyc}\frac{m_b^2+m_c^2}{m_a^2}\underset{(*)}{\overset{?}{\leq}}\frac{9R^2-16Rr+14r^2}{3r^2}}$$

$$\text{Now, }\sum_{cyc}\frac{m_b^2+m_c^2}{m_a^2}=\left(\sum_{cyc}m_a^2\right)\left(\sum_{cyc}\frac{1}{m_a^2}\right)-3\overset{\text{Lascu + A-G}}{\leq}\frac{3(s^2-4Rr-r^2)}{2s}.\sum_{cyc}\frac{1}{s-a}-3$$

$$=\frac{3(s^2-4Rr-r^2)(4Rr+r^2)}{2s.r^2s}-3\overset{?}{\leq}\frac{9R^2-16Rr+14r^2}{3r^2}$$

$$\Leftrightarrow 3(s^2-4Rr-r^2)(4Rr+r^2)\overset{?}{\leq}2s^2(9R^2-16Rr+23r^2)$$

$$\Leftrightarrow(18R^2-68Rr+37r^2)s^2+9(4Rr+r^2)^2\underset{(**)}{\overset{?}{\geq}}0$$

$\boxed{Case\ 1}$ $18R^2-68Rr+37r^2\geq 0$ and then, LHS of $(**)\geq 9(4Rr+r^2)^2>0$

$\Rightarrow(**)$ is true (strict inequality)

$\boxed{Case\ 2}$ $18R^2-68Rr+37r^2<0$ and then, LHS of $(**)$

$$=-\left(-(18R^2-68Rr+37r^2)\right)s^2+9(4Rr+r^2)^2$$

$$\overset{\text{Gerretsen}}{\geq}-\left(-(18R^2-68Rr+37r^2)\right)(4R^2+4Rr+3r^2)+9(4Rr+r^2)^2\overset{?}{\geq}0$$

$$\Leftrightarrow 36t^4-100t^3+37t^2+8t+60\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)\left((t-2)(36t^2+44t+69)+108\right)\overset{?}{\geq}0\to\text{true}\because t\overset{\text{Euler}}{\geq}2$$

$\Rightarrow(**)$ is true and combining cases 1 and 2, $(**)\Rightarrow(*)$ is true $\forall$ triangles

$$\therefore\text{ in any }\Delta ABC,\ \sqrt{\sum_{cyc}\frac{m_a}{m_b+m_c}}+\sqrt{\sum_{cyc}\frac{m_b+m_c}{m_a}}+\sqrt{\sum_{cyc}\frac{m_a^2}{m_b^2+m_c^2}}+\sqrt{\sum_{cyc}\frac{m_b^2+m_c^2}{m_a^2}}$$

$\leq\frac{3\sqrt{6}R}{2r}$, with equality iff Δ ABC is equilateral (QED)

749. In ΔABC the following relationship holds:

$$\frac{3R}{2r}\geq\frac{a+b+c}{\sqrt[3]{abc}}\geq\sqrt[2022]{\frac{a+b}{a+c}}+\sqrt[2022]{\frac{b+c}{b+a}}+\sqrt[2022]{\frac{c+a}{c+b}}\geq 3$$

Proposed Nguyen Van Canh-Ben Tre-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Let\ x:=\frac{a+b}{a+c},\ \ y:=\frac{b+c}{b+a},\ \ z:=\frac{c+a}{c+b}.\ \ \therefore xyz=1.$$

$$By\ AM-GM\ inequality\ we\ have:\ \sqrt[2022]{x}+\sqrt[2022]{y}+\sqrt[2022]{z}\geq 3\sqrt[3]{\sqrt[2022]{xyz}}=3.$$

$$By\ Chebyshev's\ inequality, for\ any\ n\geq m>0, we\ have:$$

$$x^n+y^n+z^n\geq(x^m+y^m+z^m)\left(\frac{x^{n-m}+y^{n-m}+z^{n-m}}{3}\right)\overset{AM-GM}{\geq}$$

$$(x^m+y^m+z^m).\sqrt[3]{(xyz)^{n-m}}$$

$$Then:\ \ x^n+y^n+z^n\geq x^m+y^m+z^m,\ \ \forall n\geq m>0.$$

$$For\ m=\frac{1}{2022}\leq\frac{2}{3}=n,\ \ we\ have:$$

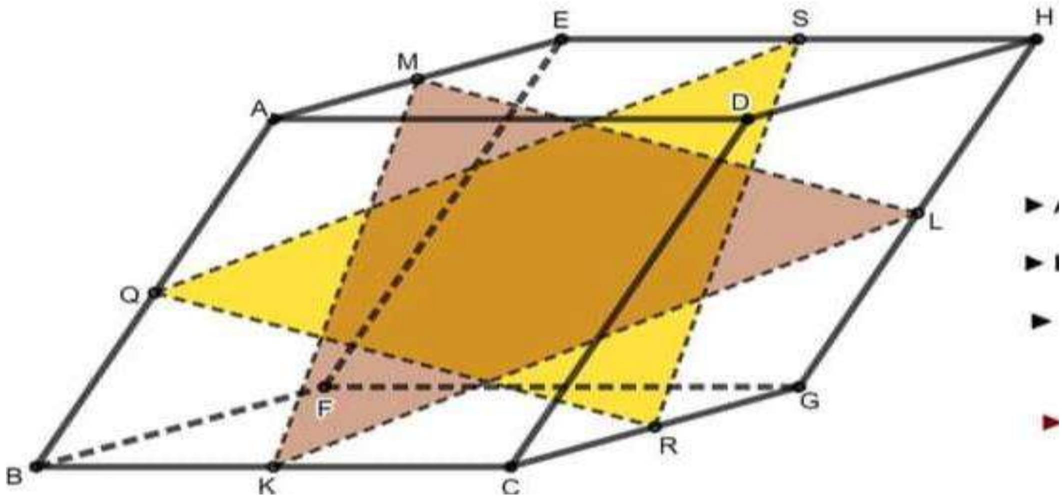
$$\sum_{cyc}\sqrt[2022]{\frac{a+b}{a+c}}\leq\sum_{cyc}\sqrt[3]{\left(\frac{a+b}{a+c}\right)^2}\overset{AM-GM}{\leq}\sum_{cyc}\sqrt[3]{\frac{(a+b)^2}{4ac}}=$$

$$=\frac{1}{\sqrt[3]{abc}}\sum_{cyc}\sqrt[3]{b.\left(\frac{a+b}{2}\right)^2}\overset{AM-GM}{\leq}\frac{1}{\sqrt[3]{abc}}\sum_{cyc}\frac{1}{3}\left(b+2\left(\frac{a+b}{2}\right)\right)=$$

$$=\frac{a+b+c}{\sqrt[3]{abc}}=\frac{2s}{\sqrt[3]{4Rsr}}\overset{Mitrinovic\ \&\ Euler}{\leq}\frac{3\sqrt{3}R}{\sqrt[3]{4.2r.3\sqrt{3}r.r}}=\frac{3R}{2r}.$$

$$Therefore,\ \frac{3R}{2r}\geq\frac{a+b+c}{\sqrt[3]{abc}}\geq\sqrt[2022]{\frac{a+b}{a+c}}+\sqrt[2022]{\frac{b+c}{b+a}}+\sqrt[2022]{\frac{c+a}{c+b}}\geq 3.$$

750.



$ABCDEFGH$ – parallelipiped, K, L, M, Q, R, S – midpoints of

BC, GH, AE, AB, CG, EH, $(K, L, M) = (P_1), (Q, R, S) = (P_2)$

Find: $\sphericalangle((P_1), (P_2))$

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by proposer

Let $BC = 2a, BF = 2b, BA = 2c$

Plagiogonal 3D system: $BC \equiv Bx; BF \equiv By; BA \equiv Bz$

$$B(0,0,0), K(a,0,0), L(2a,2b,c), M(0,b,2c), Q(0,0,c), R(2a,b,0), S(a,2b,2c)$$

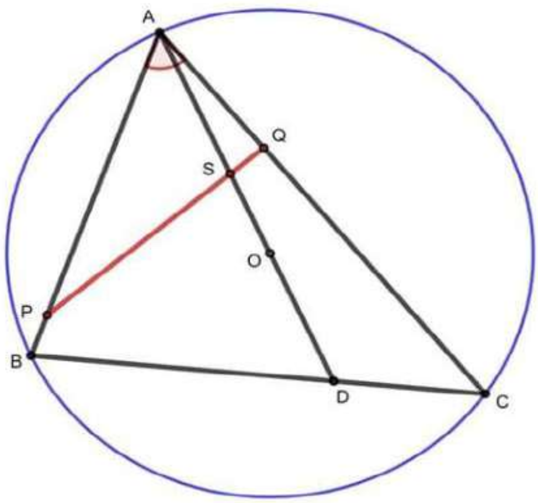
$$(P_1): \begin{vmatrix} 1 & 1 & 1 & 1 \\ x & a & 2a & 0 \\ y & 0 & 2b & b \\ z & 0 & c & 2c \end{vmatrix} = 0, \qquad (P_1): bcx - acy + abz = abc$$

$$(P_1): \frac{x}{a} - \frac{y}{b} + \frac{z}{c} = 1$$

$$\begin{cases} \frac{0}{a} - \frac{0}{b} + \frac{z}{c} = 1 \to Q \in (P_1) \\ \frac{2a}{a} - \frac{b}{b} + \frac{0}{c} = 1 \to R \in (P_1) \\ \frac{a}{a} - \frac{2b}{b} + \frac{2c}{c} = 1 \to S \in (P_1) \end{cases} \Longrightarrow \sphericalangle((P_1), (P_2)) = 0$$

751. If O −circumcenter then:

$$\frac{SP}{SQ}=\frac{AP}{AQ}\cdot\frac{(b-c\cos A)}{(c-b\cos A)},\ \cos A=\frac{\frac{SP}{SQ}\cdot\frac{AQ}{AP}-\frac{b}{c}}{\frac{SP}{SQ}\cdot\frac{AQ}{AP}\cdot\frac{b}{c}-1}$$



Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by Jose Ferreira Queiroz-Brazil

We know that:

$$\frac{BD}{DC}=\frac{c(b-c\cos A)}{b(c-b\cos A)}$$

By Gakopoulos' lemma:

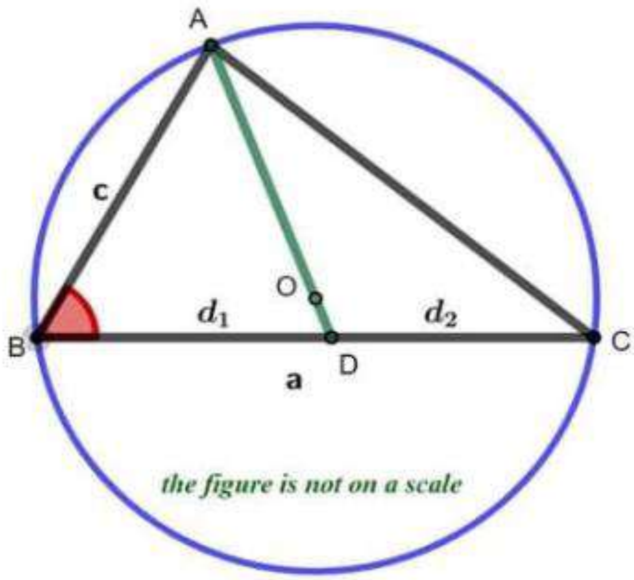
$$\frac{SQ}{SP}=\frac{DC}{DB}\cdot\frac{AQ}{AC}\cdot\frac{AB}{AP}\Rightarrow\frac{SP}{SQ}=\frac{AP}{AQ}\cdot\frac{AC}{AB}\cdot\frac{BD}{DC}\Rightarrow$$

$$\frac{SP}{SQ}=\frac{AP}{AQ}\cdot\frac{b}{c}\cdot\frac{c(b-c\cos A)}{b(c-b\cos A)}=\frac{AP}{AQ}\cdot\frac{(b-c\cos A)}{(c-b\cos A)}\Rightarrow$$

$$\cos A=\frac{\frac{SP}{SQ}\cdot\frac{AQ}{AP}-\frac{b}{c}}{\frac{SP}{SQ}\cdot\frac{AQ}{AP}\cdot\frac{b}{c}-1}$$

752. If O −circumcenter, O, A, D −collinears then:

$$\cos B=\frac{a^2+c^2\left(1+\frac{d_2}{d_1}\right)\pm\sqrt{\left(a^2+c^2\left(1+\frac{d_2}{d_1}\right)\right)^2-8a^2c^2\frac{d_2}{d_1}}}{4ac}$$



Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by Jose Ferreira Queiroz-Brazil

$$\frac{d_1}{d_2}=\frac{\sin 2C}{\sin 2B}$$

$$c^2=a^2+b^2-2ab\cos C,\qquad b^2=a^2+c^2-2ac\cos B$$

$$\frac{d_1}{d_2}=\frac{c}{2R}\cdot\frac{a^2+b^2-c^2}{2ab}\cdot\frac{2R}{b\cos B}=\frac{c(a^2+a^2+c^2-2ac\cos B-c^2)}{2ab^2\cos B}$$

$$2ab^2\cos B\cdot\frac{d_1}{d_2}=c(2a^2-2ac\cos B)$$

$$(a^2+c^2-2ac\cos B)\cos B\cdot\frac{d_1}{d_2}=ac-c^2\cos B$$

$$2ac\cos^2 B-\left(a^2+c^2\left(1+\frac{d_2}{d_1}\right)\right)\cos B+ac\frac{d_2}{d_1}=0$$

$$cosB=\frac{a^2+c^2\left(1+\frac{d_2}{d_1}\right)\pm\sqrt{\left(a^2+c^2\left(1+\frac{d_2}{d_1}\right)\right)^2-8a^2c^2\frac{d_2}{d_1}}}{4ac}$$

753. In ΔABC the following relationship holds:

$$\sum_{cyc}b^2c^2=16F^2+2F^2\sum_{cyc}\left(\frac{a}{h_a}-\frac{b}{h_b}\right)^2$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum_{cyc}b^2c^2=16F^2+2F^2\sum_{cyc}\left(\frac{a}{h_a}-\frac{b}{h_b}\right)^2=\sum_{cyc}b^2c^2=16F^2+2F^2\sum_{cyc}\left(\frac{a^2}{2F}-\frac{b^2}{2F}\right)^2=$$

$$=16F^2+\frac{2F^2}{4F^2}\sum_{cyc}(a^2-b^2)^2=16F^2+\frac{1}{2}\left(2\sum_{cyc}a^4-2\sum_{cyc}a^2b^2\right)=$$

$$=16F^2-\left(16F^2-\sum_{cyc}a^2b^2\right)=\sum_{cyc}a^2b^2$$

754. In ΔABC the following relationship holds:

$$\frac{9R}{4r}\ge\left(\sum\sqrt{\frac{a}{b+c}}\right)^2\ge\frac{36abc}{(a+b)(b+c)(c+a)}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by Tapas Das-India

$$\left(\sum\sqrt{\frac{a}{b+c}}\right)^2=\left(\sqrt{\frac{a}{b+c}}+\sqrt{\frac{b}{c+a}}+\sqrt{\frac{c}{a+b}}\right)^2$$

$$\le(a+b+c)\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right)\le\frac{1}{4}(a+b+c)\left(\frac{1}{b}+\frac{1}{c}+\frac{1}{c}+\frac{1}{a}+\frac{1}{a}+\frac{1}{b}\right)$$

$$(AM\ge HM)$$

$$=\frac{1}{4}(a+b+c)\cdot 2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=\frac{1}{2}\cdot 2s\cdot\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

$$\leq s\cdot\frac{9R}{4F}\left[Note:\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\leq\frac{9R}{4F}\right]=s\cdot\frac{9R}{4rs}=\frac{9R}{4r}$$

2nd part

$$\sqrt{\frac{a}{b+c}}+\sqrt{\frac{b}{c+a}}+\sqrt{\frac{c}{a+b}}\geq 3\cdot\left(\sqrt{\frac{a}{b+c}}\cdot\sqrt{\frac{b}{c+a}}\cdot\sqrt{\frac{c}{a+b}}\right)^{\frac{1}{3}}$$

$$=3\left[\frac{abc}{(a+b)(b+c)(c+a)}\right]^{\frac{1}{6}}$$

$$AM\geq GM$$

$$\left(\sqrt{\frac{a}{b+c}}+\sqrt{\frac{b}{c+a}}+\sqrt{\frac{c}{a+b}}\right)^2\geq 9\frac{(abc)^{\frac{1}{3}}}{[(a+b)(b+c)(c+a)]^{\frac{1}{3}}}$$

We need to show:

$$9\frac{(abc)^{\frac{1}{3}}}{[(a+b)(b+c)(c+a)]^{\frac{1}{3}}}\geq\frac{36abc}{(a+b)(b+c)(c+a)}$$

or

$$(a+b)(b+c)(c+a)^{\frac{2}{3}}\geq 4(abc)^{\frac{2}{3}}$$

or

$$(a+b)(b+c)(c+a)\geq 8abc$$

This is true

$$\therefore\left(\sum\sqrt{\frac{a}{b+c}}\right)^2\geq\frac{36abc}{(a+b)(b+c)(c+a)}$$

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{ab+bc+ca}{abc}=\frac{s^2+r^2+4Rr}{4RF}\leq\frac{\frac{27R^2}{4}+\frac{R^2}{4}+2R^2}{4RF}$$

$$\left(s^2\leq\frac{27}{4}R^2,R\geq 2r\right)$$

$$=\frac{9R^2}{4RF}=\frac{9R}{4F}$$

755. *Let $\omega-$ be the Brocard's angle of ΔABC. Prove that:*

$$\sum_{cyc}\frac{m_a}{w_a}\cdot\sqrt[4]{\frac{m_a}{h_a}}\ge\sqrt[4]{\frac{\sin\omega}{8}}\cdot\sum_{cyc}\frac{b+c}{a}$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Lemma: In any ΔABC we have:

$$\frac{bc}{b^2+c^2}\ge\sin\omega\quad(*)$$

Proof: Since $\sin\omega=\dfrac{2F}{\sqrt{a^2b^2+b^2c^2+c^2a^2}}$ *and*

$$4F=\sqrt{2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)},$$

then:

$$(*)\Leftrightarrow 2bc\sqrt{a^2b^2+b^2c^2+c^2a^2}\ge(b^2+c^2)\sqrt{2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)}$$

$$\overset{squaring}{\Leftrightarrow}\quad 4b^2c^2(a^2b^2+b^2c^2+c^2a^2)\ge(2b^2c^2+b^4+c^4)[2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)]$$

$$\Leftrightarrow 0\ge-a^4(b^2+c^2)^2+2(b^4+c^4)(a^2b^2+c^2a^2)-(b^4+c^4)^2$$
$$=-[a^2(b^2+c^2)-(b^4+c^4)]^2$$

Which is true and the proof of lemma is complete.

$$Now:\ \frac{m_a}{w_a}\overset{Lascu}{\ge}\frac{\frac{b+c}{2}.\cos\frac{A}{2}}{\frac{2bc}{b+c}.\cos\frac{A}{2}}=\frac{(b+c)^2}{4bc}=\frac{(b^2+c^2)+2bc}{4bc}\overset{AM-GM}{\ge}$$

$$\ge\frac{2\sqrt{2bc(b^2+c^2)}}{4bc}=\sqrt{\frac{1}{2}\left(\frac{b}{c}+\frac{c}{b}\right)}.$$

$$Also,\ \frac{m_a}{h_a}\overset{Tereshin}{\ge}\frac{b^2+c^2}{4R.h_a}=\frac{b^2+c^2}{2bc}=\frac{1}{2}\left(\frac{b}{c}+\frac{c}{b}\right).$$

Then:

$$\frac{m_a}{w_a}\cdot\sqrt[4]{\frac{m_a}{h_a}}\ge\sqrt{\frac{1}{2}\left(\frac{b}{c}+\frac{c}{b}\right)}\cdot\sqrt[4]{\frac{1}{2}\left(\frac{b}{c}+\frac{c}{b}\right)}=\sqrt[4]{\frac{1}{8}\cdot\frac{bc}{b^2+c^2}\cdot\left(\frac{b}{c}+\frac{c}{b}\right)}\overset{Lemma}{\ge}\sqrt[4]{\frac{\sin\omega}{8}}\cdot\left(\frac{b}{c}+\frac{c}{b}\right).$$

$$Therefore,\ \sum_{cyc}\frac{m_a}{w_a}\cdot\sqrt[4]{\frac{m_a}{h_a}}\ge\sum_{cyc}\sqrt[4]{\frac{\sin\omega}{8}}\cdot\left(\frac{b}{c}+\frac{c}{b}\right)=\sqrt[4]{\frac{\sin\omega}{8}}\cdot\sum_{cyc}\frac{b+c}{a}.$$

756. *In ΔABC the following relationship holds:*

$$\sqrt{\sum_{cyc}\frac{b+c}{a}}+\sqrt{\sum_{cyc}\frac{m_b+m_c}{m_a}}+\sqrt{\sum_{cyc}\frac{b^2+c^2}{a^2}}+\sqrt{\sum_{cyc}\frac{{m_b}^2+{m_c}^2}{{m_a}^2}}\overset{(*)}{\le}\frac{2\sqrt{6}R}{r}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have :

$$\sum_{cyc}\frac{b+c}{a}=\sum_{cyc}\left(\frac{2s}{a}-1\right)=\frac{2s(s^2+r^2+4Rr)}{4Rsr}-3\overset{Gerretsen}{\le}$$

$$\frac{4R^2+8Rr+4r^2}{2Rr}-3=\frac{2R}{r}+1+\frac{2r}{R}.$$

$$And:\ \sum_{cyc}\frac{b^2+c^2}{a^2}=\sum_{cyc}a^2\cdot\sum_{cyc}\frac{1}{a^2}-3\overset{Leibniz\ \&\ Steining}{\le}\frac{9R^2}{4r^2}-3.$$

$$Then:\ \sum_{cyc}\frac{b+c}{a}+\sum_{cyc}\frac{b^2+c^2}{a^2}\le\frac{9R^2}{4r^2}+\frac{2R}{r}-2+\frac{2r}{R}$$

$$=\frac{3R^2}{r^2}-\left(\frac{R}{2r}-1\right)\left(\frac{3R}{2r}-1+\frac{2r}{R}\right)\overset{Euler}{\le}\frac{3R^2}{r^2}\quad(1)$$

Now we have :

$$\sum_{cyc}\frac{m_b+m_c}{m_a}=\sum_{cyc}\left(\frac{m_b}{m_a}+\frac{m_c}{m_a}\right)\overset{CBS}{\le}2\sqrt{\left(\sum{m_a}^2\right)\left(\sum\frac{1}{{m_a}^2}\right)}\le$$

$$\overset{m_a\ge\sqrt{s(s-a)}}{\le}2\sqrt{\left(\frac{3}{4}\sum a^2\right)\left(\sum\frac{1}{s(s-a)}\right)}\overset{Leibniz}{\le}2\sqrt{\frac{3}{4}\cdot9R^2\cdot\frac{4R+r}{s^2r}}\overset{Doucet}{\le}\frac{3R}{r}\overset{Euler}{\le}\frac{3R^2}{2r^2}\quad(2)$$

Also,

$$\sum_{cyc}\frac{m_b^2+m_c^2}{m_a^2}=\sum_{cyc}m_a^2\cdot\sum_{cyc}\frac{1}{m_a^2}-3\overset{m_a\ge\sqrt{s(s-a)}}{\lesssim}\frac{3(s^2-r^2-4Rr)}{2}\cdot\sum_{cyc}\frac{1}{s(s-a)}-3$$

$$=\frac{3(s^2-r^2-4Rr)}{2}\cdot\frac{4R+r}{s^2r}-3$$

$$=\frac{3(4R+r)}{2r}\left(1-\frac{r(4R+r)}{s^2}\right)-3\overset{Doucet}{\lesssim}\frac{3(4R+r)}{2r}\left(1-\frac{3r}{4R+r}\right)-3=$$

$$=\frac{3(2R-r)}{r}-3=\frac{3R^2}{2r^2}-3\left(\frac{R}{2r}-1\right)\left(\frac{R}{r}-2\right)\overset{Euler}{\lesssim}\frac{3R^2}{2r^2}\ (3)$$

Therefore,

$$LHS_{(*)}\overset{CBS,(2)\,\&\,(3)}{\lesssim}\sqrt{2\left(\sum\frac{b+c}{a}+\sum\frac{b^2+c^2}{a^2}\right)}+2\sqrt{\frac{3R^2}{2r^2}}\overset{(1)}{\lesssim}\sqrt{2.\frac{3R^2}{r^2}}.+\frac{\sqrt{6}R}{r}=\frac{2\sqrt{6}R}{r}.$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{b+c}{a}=\sum_{cyc}\frac{2s-a}{a}=\frac{2s(s^2+4Rr+r^2)}{4Rrs}-3\Rightarrow\sum_{cyc}\frac{b+c}{a}\overset{(\bullet)}{=}\frac{s^2-2Rr+r^2}{2Rr}\ \textit{and,}$$

$$\sum_{cyc}\frac{m_b+m_c}{m_a}=\sum_{cyc}\frac{\sum_{cyc}m_a-m_a}{m_a}$$

$$=\left(\sum_{cyc}m_a\right)\left(\sum_{cyc}\frac{1}{m_a}\right)-3\overset{Bager+m_a\ge h_a\text{ and analogs}}{\le}(4R+r)\left(\sum_{cyc}\frac{1}{h_a}\right)-3$$

$$=\frac{4R+r}{r}-3\Rightarrow\sum_{cyc}\frac{m_b+m_c}{m_a}\overset{(\bullet\bullet)}{\le}\frac{4R-2r}{r}\ \textit{and,}$$

$$\sum_{cyc}\frac{b^2+c^2}{a^2}=\sum_{cyc}\left(\frac{b^2}{c^2}+\frac{c^2}{b^2}\right)=\sum_{cyc}\left(\left(\frac{b}{c}+\frac{c}{b}\right)^2-2\right)\overset{Bandila}{\le}\sum_{cyc}\left(\frac{R^2}{r^2}-2\right)$$

$$\therefore\sum_{cyc}\frac{b^2+c^2}{a^2}\overset{(\bullet\bullet\bullet)}{\le}\frac{3(R^2-2r^2)}{r^2}\ \textit{and,}$$

$$\sum_{cyc}\frac{m_b^2+m_c^2}{m_a^2}=\left(\sum_{cyc}m_a^2\right)\left(\sum_{cyc}\frac{1}{m_a^2}\right)-3\overset{Lascu+A-G}{\le}\frac{3(s^2-4Rr-r^2)}{2s}\cdot\sum_{cyc}\frac{1}{s-a}-3$$

$$=\frac{3(s^2-4Rr-r^2)(4Rr+r^2)}{2s.r^2s}-3$$

$$\Rightarrow \sum_{cyc} \frac{m_b^2 + m_c^2}{m_a^2} \overset{(\bullet\bullet\bullet\bullet)}{\leq} \frac{3\left((s^2 - 4Rr - r^2)(4R + r) - 2s^2r\right)}{2s^2r}$$

Now, $\sqrt{\sum_{cyc} \frac{b+c}{a}} + \sqrt{\sum_{cyc} \frac{m_b + m_c}{m_a}} + \sqrt{\sum_{cyc} \frac{b^2 + c^2}{a^2}}$

$$+ \sqrt{\sum_{cyc} \frac{m_b^2 + m_c^2}{m_a^2}} \overset{CBS}{\leq} \sqrt{4}. \sqrt{\sum_{cyc} \frac{b+c}{a} + \sum_{cyc} \frac{m_b + m_c}{m_a} + \sum_{cyc} \frac{b^2 + c^2}{a^2} + \sum_{cyc} \frac{m_b^2 + m_c^2}{m_a^2}}$$

$$\overset{via\ (\bullet),(\bullet\bullet),(\bullet\bullet\bullet),(\bullet\bullet\bullet\bullet)}{\leq} 2. \sqrt{\frac{s^2 - 2Rr + r^2}{2Rr} + \frac{4R - 2r}{r} + \frac{3(R^2 - 2r^2)}{r^2} + \frac{3\left((s^2 - 4Rr - r^2)(4R + r) - 2s^2r\right)}{2s^2r}} \overset{?}{\leq} \frac{2\sqrt{6}R}{r}$$

$$\Leftrightarrow \frac{6R(R^2 - 2r^2)s^2 + rs^2(s^2 - 2Rr + r^2) + 2Rr(4R - 2r)s^2 + 3Rr\left((s^2 - 4Rr - r^2)(4R + r) - 2s^2r\right)}{2Rr^2s^2} \overset{?}{\leq} \frac{6R^2}{r^2}$$

$$\Leftrightarrow \boxed{(6R^3 - 20R^2r + 21Rr^2 - r^3)s^2 + 3Rr^2(4R + r)^2 - rs^4 \underset{(*)}{\overset{?}{\geq}} 0}$$

Now, LHS of $(*) \overset{Gerretsen}{\geq} \left(6R^3 - 20R^2r + 21Rr^2 - r^3 - r(4R^2 + 4Rr + 3r^2)\right)s^2$

$$+ 3Rr^2(4R + r)^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow \boxed{(6R^3 - 24R^2r + 17Rr^2 - 4r^3)s^2 + 3Rr^2(4R + r)^2 \underset{(**)}{\overset{?}{\geq}} 0}$$

$\boxed{Case\ 1}$ $6R^3 - 24R^2r + 17Rr^2 - 4r^3 \geq 0$ *and* then, LHS of $(**) \geq 3Rr^2(4R + r)^2 > 0$

$\Rightarrow (**)$ is true (strict inequ*a*lity)

$\boxed{Case\ 2}$ $6R^3 - 24R^2r + 17Rr^2 - 4r^3 < 0$ *and* then, LHS of $(**)$

$$= -\left(-(6R^3 - 24R^2r + 17Rr^2 - 4r^3)\right)s^2 + 3Rr^2(4R + r)^2$$

$$\overset{Gerretsen}{\geq} -\left(-(6R^3 - 24R^2r + 17Rr^2 - 4r^3)\right)(4R^2 + 4Rr + 3r^2) + 3Rr^2(4R + r)^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow 12t^5 - 36t^4 + 19t^3 + 2t^2 + 19t - 6 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t - 2)\left((t - 2)(12t^3 + 12t^2 + 19t + 30) + 63\right) \overset{?}{\geq} 0 \rightarrow \text{true} \because t \overset{Euler}{\geq} 2$$

$\Rightarrow (**)$ is true *and* combining *cases* 1 *and* 2, $(**) \Rightarrow (*)$ is true $\forall$ tri*a*ngles

$\therefore$ in any Δ ABC, $\sqrt{\sum_{cyc}\frac{b+c}{a}}+\sqrt{\sum_{cyc}\frac{m_b+m_c}{m_a}}+\sqrt{\sum_{cyc}\frac{b^2+c^2}{a^2}}+\sqrt{\sum_{cyc}\frac{m_b^2+m_c^2}{m_a^2}}$

$\leq\frac{2\sqrt{6}R}{r}$, with equality iff Δ ABC is equilateral (QED)

757. *In ΔABC the following relationship holds:*

$$\sqrt{\sum_{cyc}\frac{w_a}{w_b+w_c}}+\sqrt{\sum_{cyc}\frac{m_a}{m_b+m_c}}+\sqrt{\sum_{cyc}\frac{w_b+w_c}{w_a}}+\sqrt{\sum_{cyc}\frac{m_a^2}{m_b^2+m_c^2}}\overset{(*)}{\leq}\frac{5\sqrt{6}R}{4r}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc}\frac{w_b+w_c}{w_a}=\sum_{cyc}\left(\frac{w_b}{w_a}+\frac{w_c}{w_a}\right)\overset{CBS}{\leq}2\sqrt{\left(\sum w_a^2\right)\left(\sum\frac{1}{w_a^2}\right)}\overset{h_a\leq w_a\leq\sqrt{s(s-a)}}{\leq}2\sqrt{\left(\sum s(s-a)\right)\left(\sum\frac{1}{h_a^2}\right)}$$

$$=\frac{\sqrt{a^2+b^2+c^2}}{r}\overset{Leibniz}{\leq}\frac{3R}{r}\overset{Euler}{\leq}\frac{3R^2}{2r^2}\quad(1)$$

$$\blacksquare\sum_{cyc}\frac{w_a}{w_b+w_c}\overset{CBS}{\leq}\sum_{cyc}\frac{w_a}{4}\left(\frac{1}{w_b}+\frac{1}{w_c}\right)=\frac{1}{4}\sum_{cyc}\frac{w_b+w_c}{w_a}\overset{(1)}{\leq}\frac{3R^2}{8r^2}\quad(2)$$

$$\blacksquare\sum_{cyc}\frac{m_a}{m_b+m_c}\overset{AM-GM}{\leq}\sum_{cyc}\frac{m_a}{2\sqrt{m_bm_c}}\overset{CBS}{\leq}\frac{1}{2}\sqrt{\left(\sum m_a^2\right)\left(\sum\frac{1}{m_bm_c}\right)}\overset{Leibniz}{\leq}$$

$$\frac{1}{2}\sqrt{\frac{3.9R^2}{4}.\frac{m_a+m_b+m_c}{m_am_bm_c}}\overset{Leuenberger\ \&\ m_a\geq\sqrt{s(s-a)}}{\leq}\frac{3R}{4}\sqrt{\frac{3(4R+r)}{s^2r}}\overset{Doucet}{\leq}$$

$$\leq\frac{3R}{4}\sqrt{\frac{3(4R+r)}{3r(4R+r).r}}=\frac{3R}{4r}\overset{Euler}{\leq}\frac{3R^2}{8r^2}\quad(3)$$

Also,

$$\sum_{cyc}\frac{m_a^2}{m_b^2+m_c^2}=\sum m_a^2.\sum_{cyc}\frac{1}{m_b^2+m_c^2}-3\overset{m_a\geq\sqrt{s(s-a)}}{\leq}\frac{3(s^2-r^2-4Rr)}{2}.\sum_{cyc}\frac{1}{s(s-b)+s(s-c)}-3$$

$$=\frac{3(s^2-r^2-4Rr)}{2s^2}.\sum_{cyc}\frac{s}{a}-3=\frac{3}{2}\left(1-\frac{r(4R+r)}{s^2}\right).\frac{s^2+r^2+4Rr}{4Rr}-3\leq$$

$$\overset{Gerretsen}{\leq}\frac{3}{2}\left(1-\frac{r(4R+r)}{4R^2+4Rr+3r^2}\right)\cdot\frac{R^2+2Rr+r^2}{Rr}-3$$

$$=\frac{3(2R^2+r^2)(R^2+2Rr+r^2)}{Rr(4R^2+4Rr+3r^2)}-3=$$

$$=\frac{3R^2}{8r^2}-\frac{3(R-2r)[(R-2r)(4R^3+4R^2r+3Rr^2+4r^3)+12r^4]}{8Rr^2(4R^2+4Rr+3r^2)}\overset{Euler}{\leq}\frac{3R^2}{8r^2}\quad(4)$$

Therefore,

$$LHS_{(*)}\overset{(1),(2),(3),(4)}{\leq}3\sqrt{\frac{3R^2}{8r^2}}+\sqrt{\frac{3R^2}{2r^2}}=\frac{5\sqrt{6}R}{4r}$$ *and the proof is complete.*

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{m_a}{m_b+m_c}=\sum_{cyc}\frac{\sum_{cyc}m_a-(m_b+m_c)}{m_b+m_c}$$

$$=\left(\sum_{cyc}m_a\right)\left(\sum_{cyc}\frac{1}{m_b+m_c}\right)-3\overset{A-G}{\leq}\frac{1}{2}\left(\sum_{cyc}m_a\right)\left(\sum_{cyc}\frac{1}{\sqrt{m_bm_c}}\right)$$

$$-3\overset{CBS}{\leq}\frac{1}{2}\left(\sum_{cyc}m_a\right)\cdot\sqrt{\sum_{cyc}\frac{1}{m_b}}\cdot\sqrt{\sum_{cyc}\frac{1}{m_c}}-3$$

$$=\frac{1}{2}\left(\sum_{cyc}m_a\right)\left(\sum_{cyc}\frac{1}{m_a}\right)-3\overset{Bager\,+\,m_a\geq h_a\text{ and analogs}}{\leq}\frac{4R+r}{2}\left(\sum_{cyc}\frac{1}{h_a}\right)-3=\frac{4R+r}{2r}-3$$

$$\Rightarrow\sum_{cyc}\frac{m_a}{m_b+m_c}\overset{(\bullet)}{\leq}\frac{4R-5r}{2r}\text{ and,}$$

$$\sum_{cyc}\frac{m_a^2}{m_b^2+m_c^2}\overset{Reverse\ Bergstrom}{\leq}\frac{1}{4}\sum_{cyc}\left(m_a^2\left(\frac{1}{m_b^2}+\frac{1}{m_c^2}\right)\right)=\frac{1}{4}\sum_{cyc}\frac{m_b^2+m_c^2}{m_a^2}$$

$$=\frac{1}{4}\left(\left(\sum_{cyc}m_a^2\right)\left(\sum_{cyc}\frac{1}{m_a^2}\right)-3\right)\overset{Lascu\,+\,A-G}{\leq}\frac{1}{4}\left(\frac{3(s^2-4Rr-r^2)}{2s}\cdot\sum_{cyc}\frac{1}{s-a}-3\right)$$

$$=\frac{1}{4}\left(\frac{3(s^2-4Rr-r^2)(4Rr+r^2)}{2s.r^2s}-3\right)$$

$$\Rightarrow\sum_{cyc}\frac{m_a^2}{m_b^2+m_c^2}\overset{(\bullet\bullet)}{\leq}\frac{3}{8s^2r}\left((s^2-4Rr-r^2)(4R+r)-2s^2r\right)\text{ and,}$$

$$\sum_{cyc}\frac{w_b+w_c}{w_a}=\sum_{cyc}\frac{\sum_{cyc}w_a-w_a}{w_a}=\left(\sum_{cyc}w_a\right)\left(\sum_{cyc}\frac{1}{w_a}\right)-3$$

$$\le\left(\sum_{cyc}m_a\right)\left(\sum_{cyc}\frac{1}{h_a}\right)-3\overset{Bager}{\le}\frac{4R+r}{r}-3\Rightarrow\sum_{cyc}\frac{w_b+w_c}{w_a}\overset{(\bullet\bullet\bullet)}{\le}\frac{4R-2r}{r}$$

Now, $\sqrt{\sum_{cyc}\frac{w_a}{w_b+w_c}}+\sqrt{\sum_{cyc}\frac{m_a}{m_b+m_c}}+\sqrt{\sum_{cyc}\frac{w_b+w_c}{w_a}}$

$$+\sqrt{\sum_{cyc}\frac{m_a^2}{m_b^2+m_c^2}}\overset{Reverse\ Bergstrom}{\le}\sqrt{\frac{1}{4}\sum_{cyc}\left(w_a\left(\frac{1}{w_b}+\frac{1}{w_c}\right)\right)}+\sqrt{\sum_{cyc}\frac{m_a}{m_b+m_c}}$$

$$+\sqrt{\sum_{cyc}\frac{w_b+w_c}{w_a}}+\sqrt{\sum_{cyc}\frac{m_a^2}{m_b^2+m_c^2}}$$

$$=\sqrt{\frac{1}{4}\sum_{cyc}\frac{w_b+w_c}{w_a}}+\sqrt{\sum_{cyc}\frac{m_a}{m_b+m_c}}+\sqrt{\sum_{cyc}\frac{w_b+w_c}{w_a}}+\sqrt{\sum_{cyc}\frac{m_a^2}{m_b^2+m_c^2}}$$

$$=\sqrt{\frac{1}{4}\sum_{cyc}\frac{w_b+w_c}{w_a}}+\sqrt{\frac{1}{4}\sum_{cyc}\frac{w_b+w_c}{w_a}}+\sqrt{\frac{1}{4}\sum_{cyc}\frac{w_b+w_c}{w_a}}+\sqrt{\sum_{cyc}\frac{m_a}{m_b+m_c}}$$

$$+\sqrt{\sum_{cyc}\frac{m_a^2}{m_b^2+m_c^2}}$$

$$\overset{CBS}{\le}\sqrt{1+1+1+1+1}.\sqrt{\frac{3}{4}\sum_{cyc}\frac{w_b+w_c}{w_a}+\sum_{cyc}\frac{m_a}{m_b+m_c}+\sum_{cyc}\frac{m_a^2}{m_b^2+m_c^2}}\overset{via\ (\bullet),(\bullet\bullet),(\bullet\bullet\bullet)}{\le}\sqrt{5}.\sqrt{\frac{3(4R-2r)}{4r}+\frac{4R-5r}{2r}+\frac{3}{8s^2r}\left((s^2-4Rr-r^2)(4R+r)-2s^2r\right.}$$

$$\Leftrightarrow\frac{5}{8s^2r}.\left(3\left((s^2-4Rr-r^2)(4R+r)-2s^2r\right)+6(4R-2r)s^2+4(4R-5r)s^2\right)\overset{?}{\le}25R^2.\frac{3}{8}$$

$$\Leftrightarrow\boxed{(15R^2-52Rr+35r^2)s^2+3r^2(4R+r)^2\underset{(*)}{\overset{?}{\ge}}0}$$

$\boxed{Case\ 1}$ $15R^2-52Rr+35r^2\ge 0$ and then, LHS of $(*)\ge 3r^2(4R+r)^2>0$

$\Rightarrow (*)$ is true (strict inequality)

$\boxed{Case\ 2}$ $15R^2-52Rr+35r^2<0$ and then, LHS of $(*)$

$$=-\left(-(15R^2-52Rr+35r^2)\right)s^2+3r^2(4R+r)^2$$

$$\overset{Gerretsen}{\ge}-\left(-(15R^2-52Rr+35r^2)\right)\left(4R^2+4Rr+3r^2\right)+3r^2(4R+r)^2\overset{?}{\ge}0$$

$$\Leftrightarrow 60t^4-148t^3+25t^2+8t+108\overset{?}{\ge}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((t-2)(60t^2+92t+153)+252\right)\overset{?}{\ge}0\to\text{true}\because t\overset{Euler}{\ge}2\Rightarrow(*)\text{ is true}$$

$\therefore$ combining cases 1 and 2, $(*)$ is true $\forall$ triangles $\Rightarrow (**)$ is true

$\therefore$ in any Δ ABC, $\sqrt{\sum_{cyc}\frac{w_a}{w_b+w_c}}+\sqrt{\sum_{cyc}\frac{m_a}{m_b+m_c}}+\sqrt{\sum_{cyc}\frac{w_b+w_c}{w_a}}+\sqrt{\sum_{cyc}\frac{m_a^2}{m_b^2+m_c^2}}$

$\leq\frac{5R\sqrt{6}}{4r}$, equality iff Δ ABC is equilateral (QED)

758. *In ΔABC prove that :*

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq\frac{2}{3}\sum_{cyc}a\,.\sum_{cyc}\frac{a}{b+c}+\sum_{cyc}\frac{(a-b)^2}{3b}$$

Proposed by Neculai Stanciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

The given inequality is successively equivalent to :

$$\sum_{cyc}\left(\frac{a^2}{b}-2a+b\right)-\frac{1}{3}\sum_{cyc}\frac{(a-b)^2}{b}\geq\frac{2}{3}\sum_{cyc}a\,.\left(\sum_{cyc}\frac{a}{b+c}-\frac{3}{2}\right)$$

$$\Leftrightarrow\sum_{cyc}\frac{(a-b)^2}{b}-\frac{1}{3}\sum_{cyc}\frac{(a-b)^2}{b}\geq\frac{2}{3}\sum_{cyc}a\,.\sum_{cyc}\left(\frac{a}{b+c}-\frac{1}{2}\right)$$

$$\Leftrightarrow\sum_{cyc}\frac{(a-b)^2}{b}\geq\sum_{cyc}a\,.\sum_{cyc}\left(\frac{a-b}{2(b+c)}-\frac{c-a}{2(b+c)}\right)$$

$$\Leftrightarrow\sum_{cyc}\frac{(a-b)^2}{b}\geq\sum_{cyc}a\,.\sum_{cyc}\left(\frac{a-b}{2(b+c)}-\frac{a-b}{2(c+a)}\right)$$

$$\Leftrightarrow\sum_{cyc}\frac{(a-b)^2}{b}\geq\sum_{cyc}a\,.\sum_{cyc}\frac{(a-b)^2}{2(b+c)(c+a)}\Leftrightarrow\sum_{cyc}\left(\frac{1}{b}-\frac{a+b+c}{2(b+c)(c+a)}\right)(a-b)^2\geq 0$$

$$\Leftrightarrow\sum_{cyc}\frac{2c^2+b(a-b+c)+2ca}{2b(b+c)(c+a)}.(a-b)^2\geq 0,\ \ \textit{which is true in any }\Delta ABC.$$

So the proof is completed. Equality holds iff ΔABC is equilateral.

759. *In ΔABC holds*:

$$\sqrt{\sum_{cyc}\frac{m_a}{m_b+m_c}}+\sqrt{\sum_{cyc}\frac{a}{b+c}}+\sqrt{\sum_{cyc}\frac{{m_a}^2}{{m_b}^2+{m_c}^2}}+\sqrt{\sum_{cyc}\frac{a^2}{b^2+c^2}}\le\frac{\sqrt{6}R}{r}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have :

$$\sum_{cyc}\frac{m_a}{m_b+m_c}\overset{AM-GM}{\le}\sum_{cyc}\frac{m_a}{2\sqrt{m_bm_c}}\overset{CBS}{\le}\frac{1}{2}\sqrt{\left(\sum{m_a}^2\right)\left(\sum\frac{1}{m_bm_c}\right)}\overset{Leibniz}{\le}$$

$$\frac{1}{2}\sqrt{\frac{3.9R^2}{4}\cdot\frac{m_a+m_b+m_c}{m_am_bm_c}}\le$$

$$\overset{Leuenberger\ \&\ m_a\ge\sqrt{s(s-a)}}{\le}\frac{3R}{4}\sqrt{\frac{3(4R+r)}{s^2r}}\overset{Doucet}{\le}\frac{3R}{4}\sqrt{\frac{3(4R+r)}{3r(4R+r).r}}=\frac{3R}{4r}\overset{Euler}{\le}\frac{3R^2}{8r^2}\quad(1)$$

And :

$$\sum_{cyc}\frac{a}{b+c}\overset{CBS}{\le}\sum_{cyc}\frac{a}{4}\left(\frac{1}{b}+\frac{1}{c}\right)=\frac{1}{4}\sum_{cyc}\left(\frac{a}{b}+\frac{b}{a}\right)\overset{Bandila}{\le}\frac{1}{4}\sum_{cyc}\frac{R}{r}=\frac{3R}{4r}\overset{Euler}{\le}\frac{3R^2}{8r^2}\quad(2)$$

Also,

$$\sum_{cyc}\frac{{m_a}^2}{{m_b}^2+{m_c}^2}=\sum_{cyc}{m_a}^2\cdot\sum_{cyc}\frac{1}{{m_b}^2+{m_c}^2}-3\overset{m_a\ge\sqrt{s(s-a)}}{\le}\frac{3(s^2-r^2-4Rr)}{2}\cdot\sum_{cyc}\frac{1}{s(s-b)+s(s-c)}-3$$

$$=$$

$$=\frac{3(s^2-r^2-4Rr)}{2s^2}\cdot\sum_{cyc}\frac{s}{a}-3=\frac{3}{2}\left(1-\frac{r(4R+r)}{s^2}\right)\cdot\frac{s^2+r^2+4Rr}{4Rr}-3\le$$

$$\overset{Gerretsen}{\le}\frac{3}{2}\left(1-\frac{r(4R+r)}{4R^2+4Rr+3r^2}\right)\cdot\frac{R^2+2Rr+r^2}{Rr}-3$$

$$=\frac{3(2R^2+r^2)(R^2+2Rr+r^2)}{Rr(4R^2+4Rr+3r^2)}-3=$$

$$=\frac{3R^2}{8r^2}-\frac{3(R-2r)[(R-2r)(4R^3+4R^2r+3Rr^2+4r^3)+12r^4]}{8Rr^2(4R^2+4Rr+3r^2)}\overset{Euler}{\leq}\frac{3R^2}{8r^2}\quad(3),$$

and :

$$\sum_{cyc}\frac{a^2}{b^2+c^2}\overset{AM-GM}{\leq}\sum_{cyc}\frac{a^3}{2abc}=\frac{2s(s^2-3r^2-6Rr)}{2.4Rsr}\overset{Gerretsen}{\leq}\frac{4R^2-2Rr}{4Rr}$$

$$=\frac{3R^2}{8r^2}-\left(\frac{R}{2r}-1\right)\left(\frac{3R}{4r}-\frac{1}{2}\right)\overset{Euler}{\leq}\frac{3R^2}{8r^2}\quad(4)$$

$$Therefore,\ LHS_{(*)}\overset{(1),(2),(3),(4)}{\leq}4\sqrt{\frac{3R^2}{8r^2}}=\frac{\sqrt{6}R}{r}\ \text{and the proof is complete.}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{m_a}{m_b+m_c}=\sum_{cyc}\frac{\sum_{cyc}m_a-(m_b+m_c)}{m_b+m_c}$$

$$=\left(\sum_{cyc}m_a\right)\left(\sum_{cyc}\frac{1}{m_b+m_c}\right)-3\overset{A-G}{\leq}\frac{1}{2}\left(\sum_{cyc}m_a\right)\left(\sum_{cyc}\frac{1}{\sqrt{m_bm_c}}\right)$$

$$-3\overset{CBS}{\leq}\frac{1}{2}\left(\sum_{cyc}m_a\right).\sqrt{\sum_{cyc}\frac{1}{m_b}}.\sqrt{\sum_{cyc}\frac{1}{m_c}}-3$$

$$=\frac{1}{2}\left(\sum_{cyc}m_a\right)\left(\sum_{cyc}\frac{1}{m_a}\right)-3\overset{Bager+m_a\geq h_a\text{ and analogs}}{\leq}\frac{4R+r}{2}\left(\sum_{cyc}\frac{1}{h_a}\right)-3=\frac{4R+r}{2r}-3$$

$$\Rightarrow\sum_{cyc}\frac{m_a}{m_b+m_c}\overset{(\bullet)}{\leq}\frac{4R-5r}{2r}\ \text{and,}$$

$$\sum_{cyc}\frac{m_a^2}{m_b^2+m_c^2}\overset{\text{Reverse Bergstrom}}{\leq}\frac{1}{4}\sum_{cyc}\left(m_a^2\left(\frac{1}{m_b^2}+\frac{1}{m_c^2}\right)\right)=\frac{1}{4}\sum_{cyc}\frac{m_b^2+m_c^2}{m_a^2}$$

$$=\frac{1}{4}\left(\left(\sum_{cyc}m_a^2\right)\left(\sum_{cyc}\frac{1}{m_a^2}\right)-3\right)\overset{Lascu+A-G}{\leq}\frac{1}{4}\left(\frac{3(s^2-4Rr-r^2)}{2s}.\sum_{cyc}\frac{1}{s-a}-3\right)$$

$$=\frac{1}{4}\left(\frac{3(s^2-4Rr-r^2)(4Rr+r^2)}{2s.r^2s}-3\right)$$

$$\Rightarrow\sum_{cyc}\frac{m_a^2}{m_b^2+m_c^2}\overset{(\bullet\bullet)}{\leq}\frac{3}{8s^2r}\left((s^2-4Rr-r^2)(4R+r)-2s^2r\right)\ \text{and,}$$

$$\sum_{cyc}\frac{a}{b+c}=\sum_{cyc}\frac{2s-(b+c)}{b+c}=\left(\frac{2s\sum_{cyc}(c+a)(a+b)}{\prod_{cyc}(b+c)}-3\right)=\left(\frac{2s\left(5s^2+4Rr+r^2\right)}{2s(s^2+2Rr+r^2)}-3\right)$$

$$\Rightarrow\sum_{cyc}\frac{a}{b+c}\overset{(\bullet\bullet\bullet)}{=}\frac{2\left(s^2-Rr-r^2\right)}{s^2+2Rr+r^2}\ and,$$

$$\sum_{cyc}\frac{a^2}{b^2+c^2}\overset{A-G}{\leq}\frac{1}{2abc}.\sum_{cyc}a^3\Rightarrow\sum_{cyc}\frac{a^2}{b^2+c^2}\overset{(\bullet\bullet\bullet\bullet)}{\leq}\frac{s^2-6Rr-3r^2}{4Rr}$$

$$\text{Now, }\sqrt{\sum_{cyc}\frac{m_a}{m_b+m_c}}+\sqrt{\sum_{cyc}\frac{a}{b+c}}+\sqrt{\sum_{cyc}\frac{m_a^2}{m_b^2+m_c^2}}$$

$$+\sqrt{\sum_{cyc}\frac{a^2}{b^2+c^2}}\overset{CBS}{\leq}\sqrt{4}.\sqrt{\sum_{cyc}\frac{m_a}{m_b+m_c}+\sum_{cyc}\frac{a}{b+c}+\sum_{cyc}\frac{m_a^2}{m_b^2+m_c^2}+\sum_{cyc}\frac{a^2}{b^2+c^2}}$$

$$\overset{via\ (\bullet),(\bullet\bullet),(\bullet\bullet\bullet),(\bullet\bullet\bullet\bullet)}{\leq}2.\sqrt{\frac{4R-5r}{2r}+\frac{2(s^2-Rr-r^2)}{s^2+2Rr+r^2}+\frac{3}{8s^2r}\left((s^2-4Rr-r^2)(4R+r)-2s^2r\right)+\frac{s^2-6Rr-3r^2}{4Rr}}$$

$$=2.\sqrt{\frac{3R(s^2+2Rr+r^2)\left((s^2-4Rr-r^2)(4R+r)-2s^2r\right)+4R(4R-5r)s^2(s^2+2Rr+r^2)+2s^2(s^2+2Rr+r^2)(s^2-6Rr-3r^2)+16Rrs^2(s^2-Rr}{8Rs^2r(s^2+2Rr+r^2)}}$$

$$\Leftrightarrow\boxed{(12R^3-28R^2r+15Rr^2+4r^3)s^4+rs^2(24R^4+4R^3r+82R^2r^2+66Rr^3+6r^4)+Rr^3(96R^3+96R^2r+30Rr^2+3r^3)-2rs^6\underset{(*)}{\overset{?}{\geq}}0}$$

$$\text{Now, LHS of }(*)\overset{Gerretsen}{\geq}\left(12R^3-28R^2r+15Rr^2+4r^3-2r\left(4R^2+4Rr+3r^2\right)\right)s^4$$

$$+rs^2\left(24R^4+4R^3r+82R^2r^2+66Rr^3+6r^4\right)$$

$$+Rr^3\left(96R^3+96R^2r+30Rr^2+3r^3\right)\overset{?}{\geq}0$$

$$\Leftrightarrow\boxed{(12R^3-36R^2r+7Rr^2-2r^3)s^4+rs^2\left(24R^4+4R^3r+82R^2r^2+66Rr^3+6r^4\right)+Rr^3\left(96R^3+96R^2r+30Rr^2+3r^3\right)\underset{(**)}{\overset{?}{\geq}}0}$$

$\boxed{Case\ 1}$ $12R^3-36R^2r+7Rr^2-2r^3\geq 0$ and then, LHS of $(**)$

$$\geq rs^2\left(24R^4+4R^3r+82R^2r^2+66Rr^3+6r^4\right)$$

$$+Rr^3\left(96R^3+96R^2r+30Rr^2+3r^3\right)>0$$

$\Rightarrow (**)$ is true (strict inequality)

$\boxed{Case\ 2}$ $12R^3-36R^2r+7Rr^2-2r^3<0$ and then, LHS of $(**)$

$$=-\left(-(12R^3-36R^2r+7Rr^2-2r^3)\right)s^4+rs^2\left(24R^4+4R^3r+82R^2r^2+66Rr^3+6r^4\right)$$

$$+Rr^3\left(96R^3+96R^2r+30Rr^2+3r^3\right)$$

$$\overset{Gerretsen}{\geq}\left(-\left(-(12R^3-36R^2r+7Rr^2-2r^3)\right)\left(4R^2+4Rr+3r^2\right)\right.$$

$$\left.+r\left(24R^4+4R^3r+82R^2r^2+66Rr^3+6r^4\right)\right)s^2$$

$$+Rr^3\left(96R^3+96R^2r+30Rr^2+3r^3\right)\overset{?}{\geq}0$$

$$\Leftrightarrow \boxed{(48R^4-72R^3r-76R^2r^2-6Rr^3+79r^4)s^2+r^3(96R^3+96R^2r+30Rr^2+3r^3)\underset{(***)}{\overset{?}{\geq}}0}$$

$\boxed{\text{Case 2i}}$ $48R^4-72R^3r-76R^2r^2-6Rr^3+79r^4\geq 0$ *and* then, LHS of $(***)$

$$\geq Rr^3(96R^3+96R^2r+30Rr^2+3r^3)>0\Rightarrow(***)\text{ is true (strict inequality)}$$

$\boxed{\text{Case 2ii}}$ $48R^4-72R^3r-76R^2r^2-6Rr^3+79r^4<0$ *and* then, LHS of $(***)$

$$=-\left(-(48R^4-72R^3r-76R^2r^2-6Rr^3+79r^4)\right)s^2$$

$$+r^3(96R^3+96R^2r+30Rr^2+3r^3)$$

$$\overset{\text{Gerretsen}}{\geq}-\left(-(48R^4-72R^3r-76R^2r^2-6Rr^3+79r^4)\right)(4R^2+4Rr+3r^2)$$

$$+r^3(96R^3+96R^2r+30Rr^2+3r^3)\overset{?}{\geq}0$$

$$\Leftrightarrow 24t^6-12t^5-56t^4-56t^3+20t^2+41t+30\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)\left((t-2)(24t^4+84t^3+184t^2+344t+660)+1305\right)\overset{?}{\geq}0\to\text{true}$$

$$\because t\overset{\text{Euler}}{\geq}2\Rightarrow(***)\text{ is true and}$$

combining cases 2i and 2ii, $(***)$ is true $\forall$ triangles

$$\Rightarrow(**)\text{ is true and combining cases 1 and 2, }(**)\Rightarrow(*)\text{ is true }\forall\text{ triangles}$$

$$\therefore\text{ in any }\Delta\,ABC,\ \sqrt{\sum_{cyc}\frac{m_a}{m_b+m_c}}+\sqrt{\sum_{cyc}\frac{a}{b+c}}+\sqrt{\sum_{cyc}\frac{m_a^2}{m_b^2+m_c^2}}+\sqrt{\sum_{cyc}\frac{a^2}{b^2+c^2}}\leq\frac{R\sqrt{6}}{r}\ \text{(QED}$$

760. ***In ΔABC holds:***

$$\frac{3R}{4r\sqrt[3]{abc}}\geq\frac{a}{c(a+b)}+\frac{b}{a(b+c)}+\frac{c}{b(c+a)}\geq\frac{3}{2\sqrt[3]{abc}}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc}\frac{a}{c(a+b)}\overset{CBS}{\geq}\frac{\left(\sum_{cyc}\sqrt{\frac{a}{c}}\right)^2}{\sum_{cyc}(a+b)}=\frac{\sum_{cyc}\left(\frac{a}{c}+2\sqrt{\frac{a}{b}}\right)}{2\sum_{cyc}a}\overset{AM-GM}{\geq}\frac{\sum_{cyc}3\sqrt[3]{\frac{a}{c}\cdot\sqrt{\frac{a}{b}}^{\,2}}}{2\sum_{cyc}a}=$$

$$=\frac{3\sum_{cyc}a}{\sqrt[3]{abc}.2\sum_{cyc}a}=\frac{3}{2\sqrt[3]{abc}}.$$

Now, we have :

$$\sum_{cyc}\frac{a}{c(a+b)}\overset{CBS}{\leq}\sum_{cyc}\frac{a}{4c}\left(\frac{1}{a}+\frac{1}{b}\right)=\frac{1}{4}\sum_{cyc}\frac{a^2+ab}{abc}=\frac{3s^2-r^2-4Rr}{16Rsr}\leq$$

$$\overset{Gerretsen}{\leq}\frac{12R^2+8Rr+8r^2}{8\sqrt[3]{2s^2R^2r^2}.\sqrt[3]{abc}}\overset{Cosnita\ and\ Turtoiu}{\leq}\frac{6R^2+2R.2r+(2r)^2}{4\sqrt[3]{27Rr.R^2r^2}.\sqrt[3]{abc}}\overset{Euler}{\leq}$$

$$\leq\frac{6R^2+2R.R+R^2}{4.3Rr.\sqrt[3]{abc}}=\frac{3R}{4r\sqrt[3]{abc}}.$$

$$Therefore,\quad\frac{3R}{4r\sqrt[3]{abc}}\geq\frac{a}{c(a+b)}+\frac{b}{a(b+c)}+\frac{c}{b(c+a)}\geq\frac{3}{2\sqrt[3]{abc}}.$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\frac{a}{c(a+b)}+\frac{b}{a(b+c)}+\frac{c}{b(c+a)}=\frac{\sum_{cyc}\left(a^2b\left(\sum_{cyc}ab+c^2\right)\right)}{abc(a+b)(b+c)(c+a)}$$

$$=\frac{\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2b\right)+abc\left(\sum_{cyc}ab\right)}{8Rrs^2(s^2+2Rr+r^2)}\overset{A-G}{\leq}\frac{(s^2+4Rr+r^2)\left(\sum_{cyc}a^3+4Rrs\right)}{8Rrs^2(s^2+2Rr+r^2)}$$

$$=\frac{(s^2+4Rr+r^2)\left(2s(s^2-6Rr-3r^2)+4Rrs\right)}{8Rrs^2(s^2+2Rr+r^2)}\overset{Gerretsen}{\leq}\frac{(4R^2+8Rr+4r^2)(s^2-4Rr-3r^2)}{4Rrs(s^2+2Rr+r^2)}$$

$$\therefore\frac{a}{c(a+b)}+\frac{b}{a(b+c)}+\frac{c}{b(c+a)}\overset{(*)}{\leq}\frac{(R^2+2Rr+r^2)(s^2-4Rr-3r^2)}{Rrs(s^2+2Rr+r^2)}$$

$$\text{Also,}\ \frac{3R}{4r.\sqrt[3]{abc}}\overset{A-G}{\geq}\frac{9R}{4r.2s}\overset{?}{\geq}\frac{(R^2+2Rr+r^2)(s^2-4Rr-3r^2)}{Rrs(s^2+2Rr+r^2)}$$

$$\Leftrightarrow 9R^2(s^2+2Rr+r^2)\overset{?}{\geq}8(R^2+2Rr+r^2)(s^2-4Rr-3r^2)$$

$$\Leftrightarrow(R^2-16Rr-8r^2)s^2+r(50R^3+97R^2r+80Rr^2+24r^3)\underset{(\bullet)}{\overset{?}{\geq}}0$$

Case 1 $R^2-16Rr-8r^2\geq 0$ and then, LHS of $(\bullet)\geq r(50R^3+97R^2r+80Rr^2+24r^3)>0$
$\Rightarrow(\bullet)$ is true (strict inequality)

Case 2 $R^2-16Rr-8r^2<0$ and then, LHS of $(\bullet)$

$$=-\left(-(R^2-16Rr-8r^2)\right)s^2+r(50R^3+97R^2r+80Rr^2+24r^3)$$

$$\overset{Gerretsen}{\geq}-\left(-(R^2-16Rr-8r^2)\right)(4R^2+4Rr+3r^2)+r(50R^3+97R^2r+80Rr^2+24r^3)$$

$$=2R^2(2R^2-5Rr+2r^2)=2R^2(R-2r)(2R-r)\overset{Euler}{\geq}0\Rightarrow(\bullet)\text{ is true and,}$$

combining cases 1 and 2, $(\bullet)$ is true $\forall$ triangles $\therefore\frac{3R}{4r.\sqrt[3]{abc}}$

$$\geq\frac{(R^2+2Rr+r^2)(s^2-4Rr-3r^2)}{Rrs(s^2+2Rr+r^2)}\overset{via\ (*)}{\geq}\frac{a}{c(a+b)}+\frac{b}{a(b+c)}+\frac{c}{b(c+a)}$$

$$\therefore\frac{3R}{4r.\sqrt[3]{abc}}\geq\frac{a}{c(a+b)}+\frac{b}{a(b+c)}+\frac{c}{b(c+a)}$$

$$\text{Now, } \frac{a}{c(a+b)}+\frac{b}{a(b+c)}+\frac{c}{b(c+a)}=\sum_{cyc}\frac{\left(\sqrt{\frac{a}{c}}\right)^2}{a+b}\overset{\text{Bergstrom}}{\geq}\frac{\left(\sum_{cyc}\sqrt{\frac{a}{c}}\right)^2}{4s}$$

$$=\frac{\sum_{cyc}\frac{b}{a}+2\sum_{cyc}\sqrt{\frac{a}{b}}}{4s}\overset{?}{\geq}\frac{3}{2.\sqrt[3]{abc}}=\frac{3\sum_{cyc}a}{4s.\sqrt[3]{abc}}\Leftrightarrow\sum_{cyc}\frac{b}{a}+2\sum_{cyc}\sqrt{\frac{a}{b}}\underset{(\bullet\bullet)}{\overset{?}{\geq}}\frac{3\sum_{cyc}a}{\sqrt[3]{abc}}$$

$$\frac{b}{a}+\sqrt{\frac{b}{c}}+\sqrt{\frac{b}{c}}\overset{A-G}{\geq}3\sqrt[3]{\frac{b^2}{ac}}=3\sqrt[3]{\frac{b^3}{abc}}\Rightarrow\frac{b}{a}+\sqrt{\frac{b}{c}}+\sqrt{\frac{b}{c}}\geq\frac{3b}{\sqrt[3]{abc}}\text{ and analogs}$$

$$\Rightarrow\sum_{cyc}\left(\frac{b}{a}+\sqrt{\frac{b}{c}}+\sqrt{\frac{b}{c}}\right)\geq\sum_{cyc}\frac{3b}{\sqrt[3]{abc}}\Rightarrow\sum_{cyc}\frac{b}{a}+2\sum_{cyc}\sqrt{\frac{a}{b}}\geq\frac{3\sum_{cyc}a}{\sqrt[3]{abc}}\Rightarrow(\bullet\bullet)\text{ is true}$$

$$\therefore\frac{a}{c(a+b)}+\frac{b}{a(b+c)}+\frac{c}{b(c+a)}\geq\frac{3}{2.\sqrt[3]{abc}}\text{ and hence, in any }\Delta\,ABC,\frac{3R}{4r.\sqrt[3]{abc}}$$

$$\geq\frac{a}{c(a+b)}+\frac{b}{a(b+c)}+\frac{c}{b(c+a)}$$

$$\geq\frac{3}{2.\sqrt[3]{abc}},\text{ equalities iff }\Delta\,ABC\text{ is equilateral (QED)}$$

761. *In ΔABC, prove or disprove :*

$$1)\ 2\sum_{cyc}w_a^2+\sum_{cyc}m_a^2\leq 3s^2.\quad 2)\ \sum_{cyc}w_a^2+2\sum_{cyc}m_a^2\geq 3s^2$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

1) *Let ABC be a triangle such that :* $a=1$ *and* $b=c=5$. *We have :* $s=\frac{11}{2}$.

$$\text{We have : } m_a^2=\frac{2b^2+2c^2-a^2}{4}=\frac{99}{4},\quad m_b^2=m_c^2=\frac{2c^2+2a^2-b^2}{4}=\frac{25}{4}$$

And :

$$w_a^2=bc\left(1-\left(\frac{a}{b+c}\right)^2\right)=\frac{99}{4},\quad w_b^2=w_c^2=ca\left(1-\left(\frac{b}{c+a}\right)^2\right)=\frac{55}{36},$$

then :

$$2\sum_{cyc}w_a^2+\sum_{cyc}m_a^2=2\left(\frac{99}{4}+2\times\frac{55}{36}\right)+\left(\frac{99}{4}+2\times\frac{25}{4}\right)=$$

$$=\frac{3343}{36}>\frac{363}{4}=3\times\left(\frac{11}{2}\right)^2=3s^2.$$

So the inequality $2\sum_{cyc}w_a^2+\sum_{cyc}m_a^2\leq 3s^2$ *is not always true.*

2) *Now we have* :

$$m_a{}^2+w_a{}^2\overset{AM-GM}{\geq}2m_aw_a\overset{Lascu}{\geq}2.\frac{b+c}{2}\cos\frac{A}{2}.\frac{2bc}{b+c}\cos\frac{A}{2}=2s(s-a).$$

Then :

$$\sum_{cyc}w_a{}^2+2\sum_{cyc}m_a{}^2=\sum_{cyc}(w_a{}^2+m_a{}^2)+\sum_{cyc}m_a{}^2\geq\sum_{cyc}2s(s-a)+\sum_{cyc}s(s-a)$$
$$=3s^2.$$

$$So\sum_{cyc}w_a{}^2+2\sum_{cyc}m_a{}^2\geq 3s^2$$

is true in any ΔABC. *Equality holds iff* ΔABC *is equilateral.*

762. In acute $\Delta ABC, h_a>h_b, h_a>h_c, h_a=m_b$. Prove that:

$$m(\sphericalangle ABC)<60^\circ$$

Proposed by Iulia Sanda, Ramona Nălbaru – Romania

Solution 1 by Adrian Popa-Romania

$$h_a>h_b\Rightarrow a<b$$
$$h_a>h_c\Rightarrow a<c$$
$$m_b=h_a\Rightarrow m_b^2=h_a^2$$

$$\left.\begin{array}{c} m_b^2=\dfrac{2(a^2+c^2)-b^2}{4} \\ Cosine\ Theorem: b^2=a^2+c^2-2a\cos B\end{array}\right\}\Rightarrow$$

$$m_b^2=\frac{2(a^2+c^2)-a^2-c^2+2a\cos B}{4}=\frac{a^2+c^2+2ac\cos B}{4}$$

$$S=\frac{ah_a}{2}\Rightarrow h_a=\frac{2S}{a}=\frac{2ac\sin B}{2a}=c\sin B\Rightarrow h_a^2=c^2\sin^2 B=c^2(1-\cos^2 B)$$

$$\text{So, } a^2+c^2+2ac\cos B=4c^2-4c^2\cos^2 B$$

$$4c^2\cos^2 B+2ac\cos B+a^2-3c^2=0$$

$$\Delta=4a^2c^2-16c^2(a^2-3c^2)=4c^2(a^2-4a^2+12c^2)=4c^2(12c^2-3a^2)=$$
$$=12c^2(4c^2-a^2)$$

$$\left.\begin{array}{c}\cos B=\dfrac{-2ac\pm 2c\sqrt{3(4c^2-a^2)}}{8c^2}=\dfrac{-a\pm\sqrt{3(4c^2-a^2)}}{4c} \\ \Delta ABC\rightarrow acute\ angled\Rightarrow\cos B>0\end{array}\right\}\Rightarrow$$

$$\Rightarrow \text{we must show that } \frac{-a+\sqrt{3(4c^2-a^2)}}{4c} > \frac{1}{2} \Leftrightarrow$$

$$\Leftrightarrow -2a + 2\sqrt{(4c^2-a^2)\cdot 3} > 4c \,\Big|:2 \Leftrightarrow \sqrt{3}\sqrt{4c^2-a^2} > 2c + a\Big|^2 \Leftrightarrow$$

$$\Leftrightarrow 12c^2 - 3a^2 > 4c^2 + 4ac + a^2$$

$$8c^2 > 4a^2 + 4ac|:4 \Rightarrow 2c^2 > a^2 + ac$$

$$\left.\begin{matrix} a < c \Rightarrow a^2 < c^2 \\ ac < c^2 \end{matrix}\right\} \Rightarrow ac + a^2 < 2c^2 \text{ (True)} \Rightarrow \cos B > \frac{1}{2} \Rightarrow \widehat{B} < 60^\circ$$

Solution 2 by Tapas Das-India

$$h_a > h_b \Rightarrow \frac{2F}{a} > \frac{2F}{b} \Rightarrow b > a$$

$$h_a > h_c \Rightarrow \frac{2F}{a} > \frac{2F}{c} \Rightarrow c > a$$

$$h_a = m_b, \qquad \therefore h_a^2 = m_b^2 \Rightarrow \frac{4F^2}{a^2} = \frac{1}{4}(2a^2+2c^2-b^2)$$

$$\Rightarrow \frac{4}{a^2}\cdot\left(\frac{1}{2}ac\sin B\right)^2 = \frac{1}{4}(2a^2+2c^2-b^2)$$

$$\Rightarrow \frac{4}{a^2}\cdot\frac{1}{4}\cdot a^2c^2\sin^2 B = \frac{1}{4}(2a^2+2c^2-b^2)$$

$$\Rightarrow c^2\sin^2 B = \frac{2a^2+2c^2-b^2}{4} \Rightarrow \sin^2 B = \frac{2a^2+2c^2-b^2}{4c^2} \quad \textbf{(1)}$$

$$\text{Now } \frac{2a^2+2c^2-b^2}{4c^2} - \frac{3}{4} = \frac{2a^2+2c^2-b^2-3c^2}{4c^2} = \frac{2a^2-b^2-c^2}{4c^2} =$$

$$= \frac{(a^2-b^2)+(a^2-c^2)}{4c^2} = \frac{(a+b)(a-b)+(a+c)(a-c)}{4c^2} < 0$$

$$(\because a < b, a < c)$$

$$\therefore \frac{2a^2+2c^2-b^2}{4c^2} < \frac{3}{4} \Rightarrow \sin^2 B < \frac{3}{4} \quad \text{(From (1))}$$

$$\therefore \sin B < \frac{\sqrt{3}}{2}, \qquad B < 60^\circ$$

We know that $\sin x$ is an increasing function when $x > 0$

$$\therefore f(x) = \sin x < \frac{\sqrt{3}}{2} \Rightarrow \sin x < \sin 60^\circ, \qquad x < 60^\circ$$

Solution 3 by Hikmat Mammadov-Azerbaijan

$$h_a > h_c \Rightarrow a < c \rightarrow (a \cdot h_a = c \cdot h_c \Rightarrow a < c)$$

$$m_b = h_a \Leftrightarrow h_a = \sqrt{\frac{2\cdot(a^2+c^2)-b^2}{4}} = \sqrt{\frac{a^2+c^2+a^2+c^2-b^2}{4}}$$

$$= \sqrt{\frac{a^2+c^2+2ac\cos B}{4}} < \sqrt{\frac{2c^2+2c^2\cos B}{4}}$$

$$\Leftrightarrow \frac{h_a}{c} < \sqrt{\frac{1+\cos B}{2}} \Leftrightarrow \sin B < \cos\frac{B}{2} \rightarrow \left(1+\cos B = 2\cos^2\frac{B}{2}\right)$$

$$\Leftrightarrow 2\sin\frac{B}{2}\cos\frac{B}{2} < \cos\frac{B}{2} \Leftrightarrow \sin\frac{B}{2} < \frac{1}{2} \Leftrightarrow \frac{B}{2} < \frac{\pi}{6} \Leftrightarrow B < \frac{\pi}{3} \Rightarrow m(\sphericalangle ABC) < 60^\circ$$

763. ***In ΔABC holds:***

$$3 + \sum_{cyc}\left(\frac{n_a}{r_a}\right)^2 \overset{(*)}{\geq} 2(\sqrt{2}-1)\sum_{cyc}\sqrt{\frac{2(2m_a+n_a-h_a)}{r_a}}$$

Proposed by Bogdan Fuștei-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:\ {n_a}^2 = s(s-a) + \frac{s(b-c)^2}{a} = s^2 - \frac{s[a^2-(b-c)^2]}{a}$$
$$= s^2 - \frac{4s(s-b)(s-c)}{a} = s^2 - \frac{4s.sr^2}{a(s-a)} = s^2 - 2h_a r_a\ \ (i)$$

$$Also,\ 2r_a(n_a+h_a) \overset{AM-GM}{\leq} {r_a}^2 + {n_a}^2 + 2h_a r_a \overset{(i)}{=} {r_a}^2 + s^2 = s^2\left(\tan^2\frac{A}{2}+1\right) = \frac{s^2}{\cos^2\frac{A}{2}}$$
$$= \frac{s.bc}{s-a} = 2r_a.\frac{2Rh_a}{r}.$$

$$Then:\ n_a + h_a \leq \frac{Rh_a}{r} \Rightarrow 2m_a + n_a - h_a \overset{Panaitopol}{\leq} 2.\frac{Rh_a}{2r} + \frac{Rh_a}{r} - 2h_a$$
$$= 2\left(\frac{R}{r}-1\right)h_a\ \ (and\ analogs)$$

$$Then:\ RHS_{(*)} \le 4(\sqrt{2}-1)\sqrt{\frac{R}{r}-1}.\sum_{cyc}\sqrt{\frac{h_a}{r_a}} \overset{2(\sqrt{2}-1)<1}{\overbrace{\le}} 2\sqrt{\frac{R-r}{r}}.\sum_{cyc}\sqrt{\frac{2(s-a)}{a}} \le$$

$$\overset{CBS}{\overbrace{\le}} 2\sqrt{\frac{R-r}{r}}.\sqrt{2\sum_{cyc}(s-a).\sum_{cyc}\frac{1}{a}} \overset{Leuenberger}{\overbrace{\le}} 2\sqrt{\frac{R-r}{r}.2s.\frac{\sqrt{3}}{2r}} \overset{Mitrinovic}{\overbrace{\le}} 6\sqrt{\frac{R(R-r)}{2r^2}}\quad (1)$$

$$Also\ we\ have:\ LHS_{(*)} \overset{(i)}{\overbrace{=}} 3+\sum_{cyc}\frac{s^2-2h_ar_a}{{r_a}^2} = 3+s^2.\sum_{cyc}\frac{1}{{r_a}^2}-4\sum_{cyc}\frac{s-a}{a} =$$

$$= 3+s^2.\frac{s^2-2r(4R+r)}{s^2r^2}-4.\frac{s^2+r^2-8Rr}{4Rr} = \frac{(R-r)s^2-8R^2r+9Rr^2-r^3}{Rr^2} \ge$$

$$\overset{Gerretsen}{\overbrace{\ge}} \frac{(R-r)(16Rr-5r^2)-8R^2r+9Rr^2-r^3}{Rr^2} = \frac{4(R-r)(2R-r)}{Rr} \ge$$

$$\overset{Euler}{\overbrace{\ge}} \frac{4\sqrt{\frac{R}{2}(R-r).\frac{3R}{2}}}{Rr} = 6\sqrt{\frac{R(R-r)}{2r^2}} \overset{(1)}{\overbrace{\ge}} RHS_{(*)},\ \ as\ desired.$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\text{Proof}: \text{Stewart's theorem} \Rightarrow b^2(s-c)+c^2(s-b) = an_a^2 + a(s-b)(s-c)$$
$$\Rightarrow s(b^2+c^2)-bc(2s-a) = an_a^2 + a(s^2-s(2s-a)+bc) \Rightarrow s(b^2+c^2)-2sbc$$
$$= an_a^2 + a(as-s^2) \Rightarrow s(b^2+c^2-a^2-2bc) = an_a^2 - as^2$$
$$\Rightarrow an_a^2 = as^2 + s(2bc\cos A - 2bc) = as^2 - 4sbc\sin^2\frac{A}{2}$$
$$= as^2 - \frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)} = as^2 - \frac{4\Delta^2}{s-a}$$
$$= as^2 - 2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right) = as^2 - 2ah_ar_a \therefore n_a^2 \overset{(*)}{=} s^2 - 2r_ah_a$$
$$\Rightarrow (n_a+r_a)^2 - s^2 - r_a^2 = s^2 - 2r_ah_a + r_a^2 + 2n_ar_a - s^2 - r_a^2 = 2r_a(n_a-h_a)$$
$$\Rightarrow (n_a+r_a)^2 = s^2 + s^2\tan^2\frac{A}{2} + 2r_a(n_a-h_a) \ge s^2 + s^2\tan^2\frac{A}{2} = s^2\sec^2\frac{A}{2}$$
$$\Rightarrow n_a^2 + r_a^2 \ge \frac{1}{2}(n_a+r_a)^2 \ge \frac{1}{2}s^2\sec^2\frac{A}{2}$$
$$\Rightarrow \frac{n_a^2+r_a^2}{r_a^2} \ge \frac{1}{2}.\frac{s^2\left(1+\tan^2\frac{A}{2}\right)}{s^2\tan^2\frac{A}{2}} = \frac{1}{2}.\left(1+\cot^2\frac{A}{2}\right)$$
$$\Rightarrow 1+\left(\frac{n_a}{r_a}\right)^2 \ge \frac{1}{2}\text{cosec}^2\frac{A}{2}\ \ and\ analogs$$

$$\overset{\text{via summation}}{\Rightarrow} \boxed{3+\sum_{cyc}\left(\frac{n_a}{r_a}\right)^2 \overset{(\bullet)}{\geq} \frac{1}{2}\sum_{cyc}\text{cosec}^2\frac{A}{2}}$$

Now, $a^2n_a^2 \leq 4(R-r)^2s^2 \overset{\text{via }(*)}{\Leftrightarrow} a^2(s^2-2h_ar_a) \leq 4(R-r)^2s^2$

$$\Leftrightarrow (4R^2\sin^2A)s^2 - 4rs\left(4R\sin\frac{A}{2}\cos\frac{A}{2}\right)\left(s\tan\frac{A}{2}\right) \leq 4(R^2-2Rr+r^2)s^2$$

$$\Leftrightarrow R^2(1-\sin^2A) - 2Rr\left(1-2\sin^2\frac{A}{2}\right) + r^2 \geq 0 \Leftrightarrow R^2\cos^2A - 2Rr\cos A + r^2 \geq 0$$

$$\Leftrightarrow (R\cos A - r)^2 \geq 0 \to \text{true} \therefore an_a \leq 2Rs - 2rs \Rightarrow \frac{n_a}{h_a} \leq \frac{2Rs}{a\left(\frac{2rs}{a}\right)} - \frac{2rs}{a\left(\frac{2rs}{a}\right)}$$

$$\Rightarrow \frac{n_a}{h_a} \leq \frac{R}{r} - 1 \text{ and } \because \text{via Panaitopol}, \frac{2m_a}{h_a} \leq \frac{R}{r} \therefore \frac{2m_a}{h_a} + \frac{n_a}{h_a} - 1 \leq \frac{R}{r} + \frac{R}{r} - 1 - 1$$

$$\Rightarrow \frac{2m_a+n_a-h_a}{h_a} \leq 2\left(\frac{R}{r}-1\right) \Rightarrow \sqrt{\frac{2(2m_a+n_a-h_a)}{r_a}} \leq 2.\sqrt{\frac{R-r}{r}}.\sqrt{\frac{h_a}{r_a}}$$

$$\Rightarrow 2(\sqrt{2}-1).\sqrt{\frac{2(2m_a+n_a-h_a)}{r_a}} \leq \frac{2(2-1)}{\sqrt{2}+1}.2.\sqrt{\frac{R-r}{r}}.\sqrt{\frac{h_a}{r_a}}$$

$$\overset{\sqrt{2}>1}{<} \frac{4}{2}.\sqrt{\frac{R-r}{r}}.\sqrt{\frac{h_a}{r_a}}$$

$$\therefore 2(\sqrt{2}-1).\sqrt{\frac{2(2m_a+n_a-h_a)}{r_a}} < \frac{4}{2}.\sqrt{\frac{R-r}{r}}.\sqrt{\frac{h_a}{r_a}} \text{ and analogs}$$

$$\overset{\text{via summation}}{\Rightarrow} 2(\sqrt{2}-1)\sum_{cyc}\sqrt{\frac{2(2m_a+n_a-h_a)}{r_a}} < \frac{4}{2}.\sqrt{\frac{R-r}{r}}.\sum_{cyc}\sqrt{\frac{h_a}{r_a}}$$

$$\overset{CBS}{\leq} \frac{4}{2}.\sqrt{\frac{3(R-r)}{r}}.\sqrt{\sum_{cyc}\frac{h_a}{r_a}} = \frac{1}{2}.\sqrt{\frac{48(R-r)}{r}}.\sqrt{\sum_{cyc}\frac{2rs}{4Rs\left(\cos^2\frac{A}{2}\right)\left(\tan^2\frac{A}{2}\right)}}$$

$$= \frac{1}{2}.\sqrt{\frac{24(R-r)}{R}}.\sqrt{\sum_{cyc}\text{cosec}^2\frac{A}{2}}$$

$$\therefore \boxed{2(\sqrt{2}-1)\sum_{cyc}\sqrt{\frac{2(2m_a+n_a-h_a)}{r_a}} \overset{(\bullet\bullet)}{<} \frac{1}{2}.\sqrt{\frac{24(R-r)}{R}}.\sqrt{\sum_{cyc}\text{cosec}^2\frac{A}{2}}}$$

$\therefore (\bullet), (\bullet\bullet) \Rightarrow$ **it suffices to prove :**

$$\sum_{cyc}\text{cosec}^2\frac{A}{2} \geq \sqrt{\frac{24(R-r)}{R}}.\sqrt{\sum_{cyc}\text{cosec}^2\frac{A}{2}} \Leftrightarrow \sum_{cyc}\text{cosec}^2\frac{A}{2} \geq \frac{24(R-r)}{R}$$

$$\Leftrightarrow \frac{\sum_{cyc}bc(s-a)}{(s-a)(s-b)(s-c)} \geq \frac{24(R-r)}{R} \Leftrightarrow \frac{s(s^2+4Rr+r^2)-12Rrs}{r^2s} \geq \frac{24(R-r)}{R}$$

$$\Leftrightarrow R(s^2-8Rr+r^2) \geq 24r^2(R-r) \Leftrightarrow Rs^2 \overset{(i)}{\geq} 8R^2r + 23Rr^2 - 24r^3$$

$$\text{Now}, Rs^2 \overset{\text{Gerretsen}}{\geq} R(16Rr-5r^2) \overset{?}{\geq} 8R^2r+23Rr^2-24r^3 \Leftrightarrow 2R^2-7Rr+6r^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow (R-2r)(2R-3r) \overset{?}{\geq} 0 \to \text{true} \because R \overset{\text{Euler}}{\geq} 2r \Rightarrow (i) \text{ is true}$$

$$\therefore \text{in any } \Delta\ ABC, 3+\sum_{cyc}\left(\frac{n_a}{r_a}\right)^2 \geq 2(\sqrt{2}-1)\sum_{cyc}\sqrt{\frac{2(2m_a+n_a-h_a)}{r_a}} \text{ (QED)}$$

764. *In ΔABC, prove or disprove :*

$$1)\sum_{cyc} g_a{}^2 + 2\sum_{cyc} w_a{}^2 + 3\sum_{cyc} m_a{}^2 \geq 6s^2. \quad 2)\ 3\sum_{cyc} g_a{}^2 + 2\sum_{cyc} w_a{}^2 + \sum_{cyc} m_a{}^2 \leq 6s^2.$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

1) *Let ABC be a triangle such that :* $a = 3$ *and* $b = c = 2$. *We have :* $s = \frac{7}{2}$.

$$\text{We have}: \ m_a{}^2 = \frac{2b^2+2c^2-a^2}{4} = \frac{7}{4}, \ m_b{}^2 = m_c{}^2 = \frac{2c^2+2a^2-b^2}{4} = \frac{11}{2},$$

$$\text{And}: \ w_a{}^2 = bc\left(1-\left(\frac{a}{b+c}\right)^2\right) = \frac{7}{4}, \ w_b{}^2 = w_c{}^2 = ca\left(1-\left(\frac{b}{c+a}\right)^2\right) = \frac{126}{25},$$

$$\text{And}: \ g_a{}^2 = (s-a)\left(s-\frac{(b-c)^2}{a}\right) = \frac{7}{4},$$

$$g_b{}^2 = g_c{}^2 = (s-b)\left(s-\frac{(c-a)^2}{b}\right) = \frac{9}{2},$$

$$\sum_{cyc} g_a{}^2 + 2\sum_{cyc} w_a{}^2 + 3\sum_{cyc} m_a{}^2 =$$

$$= \left(\frac{7}{4}+2\times\frac{9}{2}\right) + 2\left(\frac{7}{4}+2\times\frac{126}{25}\right) + 3\left(\frac{7}{4}+2\times\frac{11}{2}\right) = \frac{3633}{50} <$$

$$< \frac{147}{2} = 6\times\left(\frac{7}{2}\right)^2 = 6s^2. \text{ So the inequality (1) is not always true.}$$

1) *Similarly to the first part, if we take a triangle such that :*

$$a = 1 \text{ and } b = c = 10,$$

$$we\ find:\ 3\sum_{cyc} g_a{}^2+2\sum_{cyc} w_a{}^2+\sum_{cyc} m_a{}^2>663>6\left(\frac{21}{2}\right)^2=6s^2.$$

$$So\ the\ inequality\ (2)\ is\ not\ always\ true.$$

765. *If* $x,y,z>0$ *then in* ΔABC :

$$\frac{1}{\sqrt{xy+yz+zx}}\cdot\sum_{cyc}\frac{xr_a}{s-n_a}\ge\sqrt{2\sum_{cyc}\frac{s(s-c)}{(s-n_a)(s-n_b)}-\sum_{cyc}\left(\frac{r_a}{s-n_a}\right)^2}$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:\ s^2-n_a{}^2=s^2-\left(s(s-a)+\frac{s(b-c)^2}{a}\right)=\frac{s[a^2-(b-c)^2]}{a}$$
$$=\frac{4s(s-b)(s-c)}{a}=\frac{4s.sr^2}{a(s-a)}=2r_ah_a.$$

$$Then:\ \frac{r_a}{s-n_a}=\frac{s+n_a}{2h_a}=\frac{a(s+n_a)}{4F}\quad (and\ analogs)$$

$$Also\ we\ have:\ (an_a)^2=a^2.s(s-a)+a.s(b-c)^2$$
$$=s(s-a)[a^2-(b-c)^2]+s^2(b-c)^2=$$

$$=4s(s-a)(s-b)(s-c)+s^2(b-c)^2=4s^2r^2+s^2(b-c)^2.\ \ Then:\ an_a$$
$$=s\sqrt{4r^2+(b-c)^2}\ \ (and\ analogs)$$

$$Now,$$

$$an_a+bn_b=s\left(\sqrt{(2r)^2+(b-c)^2}+\sqrt{(2r)^2+(c-a)^2}\right)\overset{Minkowski}{\ge}$$

$$\ge s\sqrt{(2r+2r)^2+[(b-c)+(c-a)]^2}=$$

$$=s\sqrt{16r^2+(a-b)^2}>s\sqrt{4r^2+(a-b)^2}=cn_c.\ \ Also\ we\ have:\ a+b>c,$$

$$Then:\ a(s+n_a)+b(s+n_b)>c(s+n_c)\ \ or\ \ \frac{r_a}{s-n_a}+\frac{r_b}{s-n_b}>\frac{r_c}{s-n_c}$$

$$Thus,\ \sqrt{\frac{r_a}{s-n_a}}+\sqrt{\frac{r_b}{s-n_b}}>\sqrt{\frac{r_a}{s-n_a}+\frac{r_b}{s-n_b}}>\sqrt{\frac{r_c}{s-n_c}}\ (and\ analogs)$$

$$Then\ \sqrt{\frac{r_a}{s-n_a}},\sqrt{\frac{r_b}{s-n_b}},\sqrt{\frac{r_c}{s-n_c}}\ can\ be\ the\ sides\ of\ a\ triangle\ \Delta\ with\ area\ S.$$

Using Oppenheim's inequality in triangle Δ we have :

$$\frac{1}{\sqrt{xy+yz+zx}}\cdot\sum_{cyc}x\sqrt{\frac{r_a}{s-n_a}}^{\,2}\ge 4S=\sqrt{2\sum_{cyc}\sqrt{\frac{r_a}{s-n_a}}^{\,2}\sqrt{\frac{r_b}{s-n_b}}^{\,2}-\sum_{cyc}\sqrt{\frac{r_a}{s-n_a}}^{\,4}}=$$

$$\Leftrightarrow\frac{1}{\sqrt{xy+yz+zx}}\cdot\sum_{cyc}\frac{xr_a}{s-n_a}\ge\sqrt{2\sum_{cyc}\frac{s(s-c)}{(s-n_a)(s-n_b)}-\sum_{cyc}\left(\frac{r_a}{s-n_a}\right)^2}$$

766. ***In ΔABC holds:***

$$\left(\sum\frac{g_a}{g_b+g_c}\right)\left(\sum\frac{g_b+g_c}{g_a}\right)\left(\sum\frac{w_a}{w_b+w_c}\right)\left(\sum\frac{w_b+w_c}{w_a}\right)\left(\sum\frac{m_a}{m_b+m_c}\right)\left(\sum\frac{m_b+m_c}{m_a}\right)\le\frac{729R^6}{64r^6}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\blacksquare\sum\frac{m_b+m_c}{m_a}=\sum\left(\frac{m_b}{m_a}+\frac{m_c}{m_a}\right)\overset{CBS}{\le}2\sqrt{\left(\sum{m_b}^2\right)\left(\sum\frac{1}{{m_a}^2}\right)}\overset{m_a\ge\sqrt{s(s-a)}}{\le}2\sqrt{\left(\frac{3}{4}\sum a^2\right)\left(\sum\frac{1}{s(s-a)}\right)}\le$$

$$\overset{Leibniz}{\le}\sqrt{3.9R^2.\frac{4R+r}{s^2r}}\overset{Doucet}{\le}3R\sqrt{\frac{3(4R+r)}{3r(4R+r).r}}=\frac{3R}{r}\quad(1)$$

$$\blacksquare\sum\frac{w_b+w_c}{w_a}=\sum\left(\frac{w_b}{w_a}+\frac{w_c}{w_a}\right)\overset{CBS}{\le}2\sqrt{\left(\sum{w_b}^2\right)\left(\sum\frac{1}{{w_a}^2}\right)}\overset{h_a\le w_a\le\sqrt{s(s-a)}}{\le}2\sqrt{\left(\sum s(s-a)\right)\left(\sum\frac{1}{{h_a}^2}\right)}=$$

$$=\frac{\sqrt{a^2+b^2+c^2}}{r}\overset{Leibniz}{\le}\frac{3R}{r}\quad(2)$$

$$\blacksquare\sum\frac{g_b+g_c}{g_a}=\sum\left(\frac{g_b}{g_a}+\frac{g_c}{g_a}\right)\overset{CBS}{\le}2\sqrt{\left(\sum{g_b}^2\right)\left(\sum\frac{1}{{g_a}^2}\right)}\overset{h_a\le g_a\le\sqrt{s(s-a)}}{\le}2\sqrt{\left(\sum s(s-a)\right)\left(\sum\frac{1}{{h_a}^2}\right)}=$$

$$=\frac{\sqrt{a^2+b^2+c^2}}{r}\overset{Leibniz}{\lesssim}\frac{3R}{r}\ (3)$$

$$\textit{Now we have}:\sum\frac{m_a}{m_b+m_c}\overset{CBS}{\lesssim}\sum\frac{m_a}{4}\left(\frac{1}{m_b}+\frac{1}{m_c}\right)=\frac{1}{4}\sum\frac{m_b+m_c}{m_a}\overset{(1)}{\lesssim}\frac{3R}{4r}\ (4)$$

$$\textit{Similarly we have}:\sum\frac{w_a}{w_b+w_c}\le\frac{3R}{4r}\ (5)\ \textit{and}\ \sum\frac{g_b+g_c}{g_a}\le\frac{3R}{4r}\ (6)$$

Multiplying the inequalities (1), (2), (3), (4), (5), (6) yields the desired inequality.

767. ***In ΔABC holds:***

$$5(a+b+c).\sum_{cyc}\frac{a}{b^2+c^2}\ge4(a+b+c).\sum_{cyc}\frac{1}{b+c}+\frac{(6abc)^2}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Via Holder},\left(\sum_{cyc}a(b^2+c^2)\right)\left(\sum_{cyc}\frac{a}{b^2+c^2}\right)\ge\left(\sum_{cyc}a\right)^2\Rightarrow\sum_{cyc}\frac{a}{b^2+c^2}\ge\frac{4s^2}{\sum_{cyc}a(b^2+c^2)}$$

$$=\frac{4s^2}{\sum_{cyc}ab(2s-c)}=\frac{4s^2}{2s(s^2+4Rr+r^2)-12Rrs}$$

$$\Rightarrow5(a+b+c).\sum_{cyc}\frac{a}{b^2+c^2}\overset{(i)}{\ge}\frac{20s^2}{s^2-2Rr+r^2}\ \text{and},\sum_{cyc}\frac{1}{b+c}=\frac{\sum_{cyc}(c+a)(a+b)}{(a+b)(b+c)(c+a)}$$

$$=\frac{\left(\sum_{cyc}a^2+2\sum_{cyc}ab\right)+\sum_{cyc}ab}{2s(s^2+2Rr+r^2)}=\frac{4s^2+s^2+4Rr+r^2}{2s(s^2+2Rr+r^2)}$$

$$\Rightarrow4(a+b+c).\sum_{cyc}\frac{1}{b+c}=\frac{20s^2+16Rr+4r^2}{s^2+2Rr+r^2}$$

$$\therefore5(a+b+c).\sum_{cyc}\frac{a}{b^2+c^2}-4(a+b+c).\sum_{cyc}\frac{1}{b+c}\overset{\text{via (i)}}{\ge}\frac{20s^2}{s^2-2Rr+r^2}-\frac{20s^2+16Rr+4r^2}{s^2+2Rr+r^2}$$

$$=\frac{4r\left((16R-r)s^2+r(8R^2-2Rr-r^2)\right)}{(s^2+2Rr+r^2)(s^2-2Rr+r^2)}\overset{?}{\ge}\frac{8(6abc)^2}{\left((a+b)(b+c)(c+a)\right)^2}=\frac{144.8R^2r^2}{(s^2+2Rr+r^2)^2}$$

$$\Leftrightarrow(s^2+2Rr+r^2)\left((16R-r)s^2+r(8R^2-2Rr-r^2)\right)\overset{?}{\ge}288R^2r(s^2-2Rr+r^2)$$

$\Leftrightarrow (16R-r)s^4 - rs^2(248R^2 - 12Rr + 2r^2) + r^2(592R^3 - 284R^2r - 4Rr^2 - r^3) \underset{(\bullet)}{\overset{?}{\geq}} 0$

Now, LHS of $(\bullet) \overset{\text{Gerretsen}}{\geq} \left((16R-r)(16Rr-5r^2) - r(248R^2 - 12Rr + 2r^2)\right)s^2$

$+ r^2(592R^3 - 284R^2r - 4Rr^2 - r^3) \overset{?}{\geq} 0$

$\Leftrightarrow (8R^2 - 84Rr + 3r^2)s^2 + r(592R^3 - 284R^2r - 4Rr^2 - r^3) \underset{(\bullet\bullet)}{\overset{?}{\geq}} 0$

Case 1 $8R^2 - 84Rr + 3r^2 \geq 0$ and then, LHS of $(\bullet\bullet) \geq r(592R^3 - 284R^2r - 4Rr^2 - r^3)$

$= r(R-2r)(592R^2 + 900Rr + 1796r^2) + 3591r^4 \overset{\text{Euler}}{\geq} 3591r^4 > 0$

$\Rightarrow (\bullet)$ is true

(strict inequality)

Case 2 $8R^2 - 84Rr + 3r^2 < 0$ and then, LHS of $(\bullet\bullet)$

$= -\left(-(8R^2 - 84Rr + 3r^2)\right)s^2 + r(592R^3 - 284R^2r - 4Rr^2 - r^3)$

$\overset{\text{Gerretsen}}{\geq} -\left(-(8R^2 - 84Rr + 3r^2)\right)(4R^2 + 4Rr + 3r^2)$

$+ r(592R^3 - 284R^2r - 4Rr^2 - r^3) \overset{?}{\geq} 0$

$\Leftrightarrow 8t^4 + 72t^3 - 146t^2 - 61t + 2 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right)$

$\Leftrightarrow (t-2)(8t^3 + 88t^2 + 29t + (t-2) + 1) \overset{?}{\geq} 0 \rightarrow$ true $\because t \overset{\text{Euler}}{\geq} 2$

$\Rightarrow (\bullet\bullet)$ is true and, combining cases 1 and 2, $(\bullet\bullet) \Rightarrow (\bullet)$

$\Rightarrow$ is true $\forall$ triangles

$\therefore \forall$ triangles, $5(a+b+c).\sum_{cyc}\frac{a}{b^2+c^2} - 4(a+b+c).\sum_{cyc}\frac{1}{b+c} \geq \frac{8(6abc)^2}{\left((a+b)(b+c)(c+a)\right)^2}$

$\geq \frac{(6abc)^2}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}$

$\Rightarrow$ in any Δ ABC, $5(a+b+c).\sum_{cyc}\frac{a}{b^2+c^2}$

$\geq 4(a+b+c).\sum_{cyc}\frac{1}{b+c}$

$+\frac{(6abc)^2}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}$, with equality iff Δ ABC is equilateral (QED)

768. *In acute ΔABC holds :*

$$max\left(\frac{h_a}{r_a},\frac{h_b}{r_b},\frac{h_c}{r_c}\right)\ge\sum_{cyc}\sqrt{\frac{n_bn_c}{rr_a}}\cdot\left(\sum_{cyc}\frac{w_b+w_c}{a}\right)^{-1}$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have :

$$n_a{}^2=s(s-a)+\frac{s(b-c)^2}{a}=s^2-\frac{s[a^2-(b-c)^2]}{a}=s^2-\frac{4s(s-b)(s-c)}{a}=$$

$$=s^2-\frac{4s.sr^2}{a(s-a)}=s^2-2h_ar_a,\ \ then:\ n_a{}^2+2h_ar_a=s^2\ \ (i)$$

$$Also,\ \ 2r_a(n_a+h_a)\overset{AM-GM}{\le}r_a{}^2+n_a{}^2+2h_ar_a\overset{(i)}{=}r_a{}^2+s^2=s^2\left(\tan^2\frac{A}{2}+1\right)=$$

$$=\frac{s^2}{\cos^2\frac{A}{2}}=\frac{s.bc}{s-a}=2r_a.\frac{Rh_a}{r}.$$

$$Then:\ \ n_a+h_a\le\frac{Rh_a}{r}\ \ or\ \ n_a\le\left(\frac{R}{r}-1\right)h_a\ \ (and\ analogs)$$

$$\Rightarrow\sqrt{\frac{n_bn_c}{rr_a}}\le\left(\frac{R}{r}-1\right)\sqrt{\frac{h_bh_c}{rr_a}}=\left(\frac{R}{r}-1\right)\sqrt{\frac{ca.ab}{4R^2rr_a}}=$$

$$=\left(\frac{R}{r}-1\right)\sqrt{\frac{4Rsr.4R\sin\frac{A}{2}\cos\frac{A}{2}}{4R^2r.s\tan\frac{A}{2}}}=2\left(\frac{R}{r}-1\right)\cos\frac{A}{2}$$

$$Then:\ \sum_{cyc}\sqrt{\frac{n_bn_c}{rr_a}}\le2\left(\frac{R}{r}-1\right)\sum_{cyc}\cos\frac{A}{2}\ \ (1)$$

$$Also,\ \sum_{cyc}\frac{w_b+w_c}{a}=\sum_{cyc}w_a\left(\frac{1}{b}+\frac{1}{c}\right)=\sum_{cyc}\frac{2bc.\cos\frac{A}{2}}{b+c}.\frac{b+c}{bc}=2\sum_{cyc}\cos\frac{A}{2}\ (2)$$

$$From\ (1)\ and\ (2)\ we\ get: \sum_{cyc}\sqrt{\frac{n_bn_c}{rr_a}}.\left(\sum_{cyc}\frac{w_b+w_c}{a}\right)^{-1}\le\frac{R}{r}-1\ (*)$$

$$Now\ we\ assume\ that\ a\ge b\ge c.\ We\ have:\ h_a\le h_b\le h_c\ and\ r_a\ge r_b\ge r_c,$$

$$then: \frac{h_a}{r_a}\le\frac{h_b}{r_b}\le\frac{h_c}{r_c}\Rightarrow max\left(\frac{h_a}{r_a},\frac{h_b}{r_b},\frac{h_c}{r_c}\right)=\frac{h_c}{r_c}=\frac{2(s-c)}{c}=\frac{a+b}{c}-1\ (**)$$

$$From\ (*)\ and\ (**)\ it\ suffices\ to\ prove: \frac{a+b}{c}\ge\frac{R}{r}=\frac{2abc}{(-a+b+c)(a-b+c)(a+b-c)}$$

$$\Leftrightarrow (a+b)(-a+b+c)\left[2bc-\left(b^2+c^2-a^2\right)\right]\ge 2abc^2$$

$$\Leftrightarrow \left(-a^2+b^2+ca+bc\right).2bc(1-\cos A)\ge 2abc^2$$

$$\Leftrightarrow \left(-a^2+b^2+ca+bc\right)+\left(a^2-b^2-ca-bc\right)\cos A\ge ca$$

$$\Leftrightarrow \left(2bc.\cos A-c^2\right)+bc+\left(a^2-b^2-ca-bc\right)\cos A\ge 0$$

$$\Leftrightarrow c(b-c)+\left(a^2-b^2-ca+bc\right)\cos A\ge 0$$

$$\Leftrightarrow c(b-c)+(a-b)(a+b-c)\cos A\ge 0$$

Which is true because $a\ge b\ge c$ *and* ΔABC *is acute.*

So the proof is complete. Equality holds iff ΔABC *is equilateral.*

769. ***In acute*** ΔABC ***holds:***

$$\frac{a^4b^4}{c^5}+\frac{b^4c^4}{a^5}+\frac{c^4a^4}{b^5}\ge 9\sqrt{3}R^3$$

Proposed by Daniel Sitaru-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Since ΔABC *is acute then we have:*

$$a^2+b^2\ge c^2\ \ (and\ analogs)$$

$$Let\ 2x:=-a^2+b^2+c^2,\ 2y:=a^2-b^2+c^2,\ 2z:=a^2+b^2-c^2,\ \therefore x,y,z\ge 0.$$

$$then:\ a=\sqrt{y+z},\ b=\sqrt{z+x},\ c=\sqrt{x+y}.$$

Now:

$$16F^2=2\sum_{cyc}a^2b^2-\sum_{cyc}a^4=2\sum_{cyc}(y+z)(z+x)-\sum_{cyc}(y+z)^2=4(xy+yz+zx).$$

By Iran 96 inequality we have :

$$\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\geq\frac{9}{4(xy+yz+zx)}$$

$$\Leftrightarrow\frac{1}{a^4}+\frac{1}{b^4}+\frac{1}{c^4}\geq\frac{9}{16F^2}\ \textit{in any acute}\ \Delta ABC.$$

$$\textit{Now by Power Mean inequality we have}\left(\because\frac{9}{4}>1\right):$$

$$\frac{1}{a^9}+\frac{1}{b^9}+\frac{1}{c^9}=\left(\frac{1}{a^4}\right)^{\frac{9}{4}}+\left(\frac{1}{b^4}\right)^{\frac{9}{4}}+\left(\frac{1}{c^4}\right)^{\frac{9}{4}}\geq3\left[\frac{1}{3}\left(\frac{1}{a^4}+\frac{1}{b^4}+\frac{1}{c^4}\right)\right]^{\frac{9}{4}}\geq3\left(\frac{1}{3}.\frac{9}{16F^2}\right)^{\frac{9}{4}}=3\sqrt{\left(\frac{\sqrt{3}}{4F}\right)^9}.$$

$$\textit{Also by Carlitz's inequality we have :}\quad abc\geq\sqrt{\left(\frac{4F}{\sqrt{3}}\right)^3}.$$

Therefore,

$$\frac{a^4b^4}{c^5}+\frac{b^4c^4}{a^5}+\frac{c^4a^4}{b^5}=(abc)^4.\left(\frac{1}{a^9}+\frac{1}{b^9}+\frac{1}{c^9}\right)\geq(4RF)^3.\sqrt{\left(\frac{4F}{\sqrt{3}}\right)^3}.3\sqrt{\left(\frac{\sqrt{3}}{4F}\right)^9}=9\sqrt{3}R^3.$$

Equality holds iff ΔABC is equilateral.

770. Prove that in any triangle ABC holds:

$$a^2+b^2+c^2\geq4\sqrt{3}F\sin2x+\sum_{cyc}(a\sin x-b\cos x)^2$$

$$\text{for any } x\in\left(0,\frac{\pi}{2}\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution by Tapas Das – India

$$ab+bc+ca\geq4F\sqrt{3}$$

$$(\sin A)^{-1}+(\sin B)^{-1}+(\sin C)^{-1}\geq3\sqrt[3]{(\sin A\sin B\sin C)^{-1}}$$

We know that

$$\sin A \sin B \sin C \le \frac{3\sqrt{3}}{8}$$

$$\therefore (\sin A)^{-1} + (\sin B)^{-1} + (\sin C)^{-1} \ge 2\sqrt{3}$$

$$bc + ca + ab = 2F[(\sin A)^{-1} + (\sin B)^{-1}(\sin C)^{-1}] \ge 2F \cdot 2\sqrt{3} = 4F\sqrt{3}$$

$$\sum (a \sin x - b \cos x)^2 =$$

$$= (a^2 + b^2 + c^2)\sin^2 x + (a^2 + b^2 + c^2)\cos^2 x - (ab + bc + ca)\sin 2x$$

$$= (a^2 + b^2 + c^2) - (ab + bc + ca)\sin 2x$$

Now,

$$4\sqrt{3}F \sin 2x + \sum (a \sin x - b \cos x)^2 =$$

$$= 4\sqrt{3}F \sin 2x + (a^2 + b^2 + c^2) - (ab + bc + ca)\sin 2x$$

Now,

$$(a^2 + b^2 + c^2) - \left[4\sqrt{3}F \sin x + \sum (a \sin x + b \cos x)^2\right] =$$

$$= (a^2 + b^2 + c^2) - (a^2 + b^2 + c^2) - 4\sqrt{3}F \sin 2x + (ab + bc + ca)\sin 2x$$

$$= \sin 2x\left[(ab + bc + ca) - 4\sqrt{3}F\right] \ge 0$$

$$\therefore a^2 + b^2 + c^2 \ge 4\sqrt{3}F \sin 2x - \sum (a \sin x - b \cos x)^2$$

Equality occurs when $a = b = c$. (Equilateral triangle)

771. In any Δ ABC, prove or disprove :

$$(1) \sum_{cyc} g_a^2 + 2\sum_{cyc} w_a^2 + 3\sum_{cyc} n_a^2 \ge 6s^2,$$

$$(2)\ 3\sum_{cyc} g_a^2 + 2\sum_{cyc} w_a^2 + \sum_{cyc} n_a^2 \le 6s^2,$$

$$(3)\ g_a + 2m_a > n_a$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$an_a^2 . ag_a^2 \ge a^2 s^2 (s-a)^2$$
$$\Leftrightarrow \{b^2(s-c) + c^2(s-b) - a(s-b)(s-c)\}\{b^2(s-b) + c^2(s-c) - a(s-b)(s-c)\}$$

$$\overset{(1)}{\geq} a^2s^2(s-a)^2$$

$\because s-a=x, s-b=y$ and $s-c=z \therefore s=x+y+z \Rightarrow a=y+z, b=z+x$ and $c=x+y$ and via such substitutions,

$$(1) \Leftrightarrow \{z(z+x)^2+y(x+y)^2-yz(y+z)\}\{y(z+x)^2+z(x+y)^2-yz(y+z)\}$$
$$\geq x^2(y+z)^2(x+y+z)^2 \Leftrightarrow xy^2+xz^2+y^3+z^3 \geq 2xyz+yz(y+z)$$
$$\Leftrightarrow x(y-z)^2+(y+z)(y-z)^2 \geq 0 \to \text{true} \Rightarrow (1) \text{ is true} \Rightarrow n_a g_a \overset{(*)}{\geq} s(s-a)$$

Now, Stewart's theorem

$$\Rightarrow b^2(s-c)+c^2(s-b) \overset{(i)}{=} an_a^2+a(s-b)(s-c) \text{ and}$$
$$b^2(s-b)+c^2(s-c) \overset{(ii)}{=} ag_a^2+a(s-b)(s-c) \text{ and (i)} + \text{(i)} \Rightarrow$$
$$(b^2+c^2)(2s-b-c)=an_a^2+ag_a^2+2a(s-b)(s-c)$$
$$\Rightarrow 2a(b^2+c^2)=2a(n_a^2+g_a^2)+a(a+b-c)(c+a-b) \Rightarrow 2(b^2+c^2)$$
$$=2(n_a^2+g_a^2)+a^2-(b-c)^2$$
$$\Rightarrow 2(b^2+c^2)-a^2+(b-c)^2=2(n_a^2+g_a^2) \Rightarrow 4m_a^2+(b-c)^2=2(n_a^2+g_a^2)$$
$$\Rightarrow (b-c)^2+4s(s-a)+(b-c)^2=2(n_a^2+g_a^2)$$
$$\Rightarrow n_a^2+g_a^2 \overset{(**)}{=} (b-c)^2+2s(s-a)$$
$$\Rightarrow n_a^2+g_a^2-2n_ag_a=(b-c)^2+2s(s-a)-2n_ag_a \overset{\text{via } (*)}{\leq} (b-c)^2+2s(s-a)$$
$$-2s(s-a) \Rightarrow (n_a-g_a)^2 \leq (b-c)^2=(b-c)^2+4s(s-a)-4s(s-a)$$
$$<(b-c)^2+4s(s-a)=4m_a^2 \Rightarrow \frac{(n_a-g_a)^2}{4}<m_a^2 \Rightarrow \frac{|n_a-g_a|}{2} \overset{(\bullet)}{<} m_a$$

$$\text{Again, (i)} - \text{(ii)} \Rightarrow an_a^2+a(s-b)(s-c)-ag_a^2-a(s-b)(s-c)$$
$$=(s-c)(b^2-c^2)-(s-b)(b^2-c^2) \Rightarrow a(n_a^2-g_a^2)=(b^2-c^2)(b-c)$$
$$=(b+c)(b-c)^2 \geq 0$$
$$\Rightarrow n_a-g_a \overset{(\bullet\bullet)}{>} 0 \therefore (\bullet),(\bullet\bullet) \Rightarrow \boxed{\frac{n_a-g_a}{2}<m_a} \Rightarrow g_a+2m_a>n_a \Rightarrow (3) \textit{ is true}$$

$$\sum_{cyc} g_a^2+2\sum_{cyc} w_a^2+3\sum_{cyc} n_a^2$$
$$=\sum_{cyc}(n_a^2+g_a^2)+2\sum_{cyc}(w_a^2+n_a^2) \overset{\text{via } (**)}{=} \sum_{cyc}(b-c)^2+2s^2+2\sum_{cyc}(w_a^2+n_a^2)$$
$$\geq \sum_{cyc}(b-c)^2+2s^2+4\sum_{cyc} w_an_a$$
$$\geq \sum_{cyc}(b-c)^2+2s^2+4\sum_{cyc} w_am_a \overset{\text{Lascu}+\text{A-G}}{\geq} \sum_{cyc}(b-c)^2+2s^2+4\sum_{cyc} s(s-a)$$
$$=\sum_{cyc}(b-c)^2+6s^2 \geq 6s^2 \Rightarrow (1) \textit{ is true}$$

$$\text{Now}, \sum_{cyc}\frac{a}{(b+c)^2}=\sum_{cyc}\frac{(a-2s)+2s}{(b+c)^2}=2s\frac{\sum_{cyc}(c+a)^2(a+b)^2}{\prod_{cyc}(b+c)^2}-\sum_{cyc}\frac{1}{b+c}$$
$$=\frac{\left(\sum_{cyc}(c+a)(a+b)\right)^2-2.2s(s^2+2Rr+r^2)(4s)}{2s(s^2+2Rr+r^2)^2}-\frac{\sum_{cyc}(c+a)(a+b)}{2s(s^2+2Rr+r^2)}$$

$$=\frac{\left(\left(\sum_{\text{cyc}} a^2+2\sum_{\text{cyc}} ab\right)+\sum_{\text{cyc}} ab\right)^2-16s^2(s^2+2Rr+r^2)-(s^2+2Rr+r^2)(5s^2+4Rr+r^2)}{2s(s^2+2Rr+r^2)^2}$$

$$\Rightarrow \sum_{\text{cyc}}\frac{a}{(b+c)^2}$$

$$\overset{(***)}{=}\frac{(5s^2+4Rr+r^2)^2-16s^2(s^2+2Rr+r^2)-(s^2+2Rr+r^2)(5s^2+4Rr+r^2)}{2s(s^2+2Rr+r^2)^2}$$

Again, Stewart's theorem $\Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$

$$\Rightarrow s(b^2+c^2)-bc(2s-a)=an_a^2+a(s^2-s(2s-a)+bc)\Rightarrow s(b^2+c^2)-2sbc$$

$$=an_a^2+a(as-s^2)\Rightarrow s(b^2+c^2-a^2-2bc)=an_a^2-as^2\Rightarrow an_a^2$$

$$=as^2+s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}=as^2-\frac{4as(s-b)(s-c)}{a}$$

$$\Rightarrow n_a^2\overset{(iii)}{=}s^2-\frac{4s(s-b)(s-c)}{a}$$

$$\therefore (iii),(**)\Rightarrow g_a^2=(b-c)^2+2s(s-a)-s^2+\frac{4s(s-b)(s-c)}{a}$$

$$=s^2-2sa+a^2+(b-c)^2-a^2+\frac{4s(s-b)(s-c)}{a}$$

$$=(s-a)^2+(b-c+a)(b-c-a)+\frac{4s(s-b)(s-c)}{a}$$

$$=(s-a)^2-4(s-b)(s-c)+\frac{4s(s-b)(s-c)}{a}$$

$$=(s-a)^2+4(s-b)(s-c)\left(\frac{s}{a}-1\right)$$

$$=(s-a)^2+\frac{4(s-a)(s-b)(s-c)}{a}=(s-a)^2+\frac{4r^2s}{a}=(s-a)^2+2rh_a$$

$$\Rightarrow g_a^2=(s-a)^2+2rh_a \text{ and analogs}\rightarrow(2)$$

$$\text{Also, }2\sum_{\text{cyc}}w_a^2=2\sum_{\text{cyc}}\left(bc-\frac{a^2bc}{(b+c)^2}\right)\overset{\text{via }(***)}{=}2\sum_{\text{cyc}}ab$$

$$+4Rr.\frac{(s^2+2Rr+r^2)(5s^2+4Rr+r^2)+16s^2(s^2+2Rr+r^2)-(5s^2+4Rr+r^2)^2}{(s^2+2Rr+r^2)^2}$$

$$\therefore 3\sum_{\text{cyc}}g_a^2+2\sum_{\text{cyc}}w_a^2+\sum_{\text{cyc}}n_a^2$$

$$=\sum_{\text{cyc}}(n_a^2+g_a^2)+2\sum_{\text{cyc}}g_a^2+2\sum_{\text{cyc}}ab$$

$$+4Rr.\frac{(s^2+2Rr+r^2)(5s^2+4Rr+r^2)+16s^2(s^2+2Rr+r^2)-(5s^2+4Rr+r^2)^2}{(s^2+2Rr+r^2)^2}$$

$$\overset{\text{via }(**)\text{ and }(2)}{=}\sum_{\text{cyc}}(b-c)^2+2s^2+2\sum_{\text{cyc}}(s-a)^2+4r\sum_{\text{cyc}}h_a+2\sum_{\text{cyc}}ab$$

$$+4Rr.\frac{(s^2+2Rr+r^2)(5s^2+4Rr+r^2)+16s^2(s^2+2Rr+r^2)-(5s^2+4Rr+r^2)^2}{(s^2+2Rr+r^2)^2}$$

$$=4\sum_{\text{cyc}}a^2+2s^2-2s^2+\frac{2r(s^2+4Rr+r^2)}{R}$$

$$+4Rr.\frac{-4s^4+s^2(6Rr+12r^2)-2r^2(4R^2+Rr)}{(s^2+2Rr+r^2)^2}\overset{?}{\geq}6s^2$$

$$\Leftrightarrow 8(s^2-4Rr-r^2)-6s^2+\frac{2r(s^2+4Rr+r^2)}{R}$$

$$+4Rr.\frac{-4s^4+s^2(6Rr+12r^2)-2r^2(4R^2+Rr)}{(s^2+2Rr+r^2)^2}\overset{?}{\geq}0$$

$$\Leftrightarrow\frac{(R+r)s^6-rs^4(20R^2-6Rr-3r^2)-r^2s^2(28R^3-9Rr^2-3r^3)}{R(s^2+2Rr+r^2)^2}$$

$$\frac{-r^3(80R^4+68R^3r+12R^2r^2-4Rr^3-r^4)}{R(s^2+2Rr+r^2)^2}\overset{?}{\geq}0$$

$$\Leftrightarrow (R+r)s^6-rs^4(20R^2-6Rr-3r^2)-r^2s^2(48R^3-9Rr^2-3r^3)$$

$$-r^3(80R^4+68R^3r+12R^2r^2-4Rr^3-r^4)\underset{(\bullet\bullet\bullet)}{\overset{?}{\geq}}0$$

$$\because (R+r)(s^2-16Rr+5r^2)^3\overset{Gerretsen}{\geq}0,$$

$\therefore$ in orde to prove $(\bullet\bullet\bullet)$, it suffices to prove :

$$\text{LHS of }(\bullet\bullet\bullet)\geq(R+r)(s^2-16Rr+5r^2)^3$$

$$\Leftrightarrow(28R^2+39Rr-12r^2)s^4-rs^2(816R^3+288R^2r-414Rr^2+72r^3)$$

$$+r^2(4016R^4+188R^3r-2652R^2r^2+1079Rr^3-124r^4)\overset{(\bullet\bullet\bullet\bullet)}{\geq}0$$

$$\because (28R^2+39Rr-12r^2)(s^2-16Rr+5r^2)^2\overset{Gerretsen}{\geq}0,$$

$\therefore$ in orde to prove $(\bullet\bullet\bullet\bullet)$, it suffices to prove : LHS of $(\bullet\bullet\bullet\bullet)$

$$\geq(28R^2+39Rr-12r^2)(s^2-16Rr+5r^2)^2$$

$$\Leftrightarrow(20R^3+170R^2r-90Rr^2+12r^3)s^2\overset{(\bullet\bullet\bullet\bullet\bullet)}{\geq}$$

$$r(788R^4+1329R^3r-1490R^2r^2+454Rr^3-44r^4)$$

If Δ ABC is $a$cute, LHS of $(\bullet\bullet\bullet\bullet\bullet)\overset{Walker}{\geq}$

$$(20R^3+170R^2r-90Rr^2+12r^3)(2R^2+8Rr+3r^2)$$

$$\overset{?}{\geq}r(788R^4+1329R^3r-1490R^2r^2+454Rr^3-44r^4)$$

$$\Leftrightarrow 40t^5-288t^4-89t^3+1304t^2-628t+80\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)\left((t-7)(40t^3+71t^2+t(t-2)+t+287)+1969\right)\overset{?}{\geq}0$$

$\rightarrow$ true for $t\geq 7$

$\Rightarrow(\bullet\bullet\bullet\bullet\bullet)\Rightarrow(\bullet\bullet\bullet\bullet)\Rightarrow(\bullet\bullet\bullet)$ is true for $a$cute tri$a$ngles with $\frac{R}{r}\geq 7$

$\therefore$ reverse of (2) is true for $a$cute tri$a$ngles with $\frac{R}{r}\geq 7$

$\Rightarrow$ (2) *is not always true* (Done)

772. In ΔABC the following relationship holds:

$$4R+r\geq\frac{p}{\sqrt{3}}+\sqrt[3]{4p^2R}\geq 3\sqrt[3]{p^2r}$$

Proposed by Alex Szoros-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\text{We have :}$$

$$\frac{p}{\sqrt{3}}+\sqrt[3]{4p^2R}=\frac{p}{\sqrt{3}}+\frac{\sqrt[3]{2p.2p.3\sqrt{3}R}}{\sqrt{3}}\overset{AM-GM}{\leq}\frac{p}{\sqrt{3}}+\frac{2p+2p+3\sqrt{3}R}{3\sqrt{3}}=\frac{7p}{3\sqrt{3}}+R\leq$$

$$\overset{Blundon}{\leq}\frac{7\left[2R+\left(3\sqrt{3}-4\right)r\right]}{3\sqrt{3}}+R=4R+r-\frac{9\sqrt{3}-14}{3\sqrt{3}}(R-2r)\overset{Euler}{\leq}4R+r.$$

$$\text{Also we have :}$$

$$\frac{p}{\sqrt{3}}+\sqrt[3]{4p^2R}\overset{Euler}{\geq}\frac{\sqrt[3]{p^2.p}}{\sqrt{3}}+\sqrt[3]{4p^2.2r}\overset{Mitrinovic}{\geq}\frac{\sqrt[3]{p^2.3\sqrt{3}r}}{\sqrt{3}}+2\sqrt[3]{p^2r}=3\sqrt[3]{p^2r}.$$

$$\text{Therefore,}$$

$$4R+r\geq\frac{p}{\sqrt{3}}+\sqrt[3]{4p^2R}\geq3\sqrt[3]{p^2r}.\ \text{Equality holds iff }\Delta ABC\text{ is equilateral.}$$

Solution 2 by Tapas Das-India

$$4p^2R\geq4p^2\cdot2r=8p^2r\quad(R\geq2r)$$

$$\sqrt[3]{4p^2R}\geq2\sqrt[3]{p^2r}$$

Now, we need to show

$$\frac{p}{\sqrt{3}}\geq\sqrt[3]{p^2r}\ \text{ or }\ \frac{p^3}{3\sqrt{3}}\geq p^2r$$

or $p\geq3\sqrt{3}r$ This is true $(p^2\geq27r^2)$

$$\therefore\frac{p}{\sqrt{3}}\geq\sqrt[3]{p^2r}$$

$$\therefore\frac{p}{\sqrt{3}}+\sqrt[3]{4p^2R}\geq\sqrt[3]{p^2r}+2\sqrt[3]{p^2r}=3\sqrt[3]{p^2r}$$

Note: $p^2\leq\frac{(4R+r)^2}{3}$

Now, $\frac{p}{\sqrt{3}}\leq\frac{4R+r}{\sqrt{3}\cdot\sqrt{3}}=\frac{4R+r}{3}$ (1)

Now, $(4R+r)-\frac{(4R+r)}{3}=\frac{2}{3}(4R+r)$

We need to show

$$\frac{2}{3}(4R+r)\ge\sqrt[3]{4p^2R}\text{ or }\frac{8}{27}(4R+r)^3\ge 4p^2R\text{ or }8(4R+r)^3\ge 27\times 4p^2R$$

$$\text{or }8(4R+r)^3\ge 108p^2R\text{ or }8(64R^3+48R^2r+12Rr^2+r^3)\ge 108p^2R$$

$$\text{or }512R^3+384R^2r+96Rr^2+8r^3\ge 108p^2R$$

$$\text{Now, }108p^2R\le 108R(4R^2+4Rr+3r^2)$$

(Gerretsen's)

$$\Rightarrow 108p^2R\le 432R^3+432R^2r+324r^2R$$

Now,

$$(512R^3+384R^2r+96Rr^2+8r^3)-(432R^3+432R^2r+324r^2R)=$$

$$=80R^3-48R^2r-228Rr^2+8r^3=4[20R^3-12R^2r-57Rr^2+2r^3]$$

$$=4(R-2r)(20R^2+28Rr-r^2)\ge 0$$

$$\therefore\frac{2}{3}(4R+r)\ge\sqrt[3]{4p^2R}$$

$$\frac{p}{\sqrt{3}}+\sqrt[3]{4p^2R}\le\frac{4R+r}{3}+\frac{2}{3}(4R+r)=4R+r$$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$4R+r\ge\frac{p}{\sqrt{3}}+\sqrt[3]{4p^2R}\Leftrightarrow\left(4R+r-\frac{p}{\sqrt{3}}\right)^3\ge\left(\sqrt[3]{4p^2R}\right)^3$$

$$\Leftrightarrow(4R+r)^3-\frac{p^3}{3\sqrt{3}}-\frac{3p(4R+r)}{\sqrt{3}}.\left(4R+r-\frac{p}{\sqrt{3}}\right)\ge 4p^2R$$

$$\Leftrightarrow(4R+r)^3-4p^2R+p^2(4R+r)\ge\sqrt{3}p\left((4R+r)^2+\frac{p^2}{9}\right)\Leftrightarrow\left((4R+r)^3+p^2r\right)^2$$

$$\ge\left(\sqrt{3}p\left((4R+r)^2+\frac{p^2}{9}\right)\right)^2\Leftrightarrow\left((4R+r)^3+p^2r\right)^2\ge\frac{3p^2}{81}\left(p^2+9(4R+r)^2\right)^2$$

$$\Leftrightarrow 27\left((4R+r)^3+p^2r\right)^2\overset{(*)}{\ge}p^2\left(p^2+9(4R+r)^2\right)^2$$

$$\text{Now, LHS of }(*)\overset{\text{Gerretsen}}{\ge}27\left((4R+r)^3+r\left(16Rr-5r^2\right)\right)^2$$

$$\text{and RHS of }(*)\overset{\text{Gerretsen}}{\le}(4R^2+4Rr+3r^2)\left(4R^2+4Rr+3r^2+9(4R+r)^2\right)^2$$

$$\therefore\text{ in order to prove }(*)\text{, it suffices to prove :}$$

$$27\left((4R+r)^3+r\left(16Rr-5r^2\right)\right)^2$$

$$\geq (4R^2+4Rr+3r^2)(4R^2+4Rr+3r^2+9(4R+r)^2)^2$$

$$\Leftrightarrow 1436t^5-732t^4-2127t^3-3334t^2-1566t-756\geq 0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(1436t^4+2140t^3+2153t^2+972t+378)\geq 0\to \text{true} \because t\overset{\text{Euler}}{\geq} 2$$

$$\Rightarrow (*) \text{ is true } \therefore \boxed{4R+r\geq \frac{p}{\sqrt{3}}+\sqrt[3]{4p^2R}}$$

$$\text{Again}, \frac{p}{\sqrt{3}}+\sqrt[3]{4p^2R}\overset{\text{Euler}}{\geq}\frac{p}{\sqrt{3}}+\sqrt[3]{8p^2r}\overset{?}{\geq}3.\sqrt[3]{p^2r}\Leftrightarrow\frac{p}{\sqrt{3}}\overset{?}{\geq}\sqrt[3]{p^2r}\Leftrightarrow\frac{p^3}{3\sqrt{3}}\overset{?}{\geq}p^2r$$

$$\Leftrightarrow p\overset{?}{\geq}3\sqrt{3}r\to \text{true via Mitrinovic} \therefore \boxed{\frac{p}{\sqrt{3}}+\sqrt[3]{4p^2R}\geq 3.\sqrt[3]{p^2r}}$$

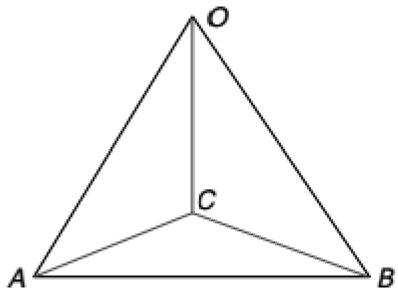
$$\therefore \text{in any } \Delta ABC, 4R+r\geq\frac{p}{\sqrt{3}}+\sqrt[3]{4p^2R}\geq 3.\sqrt[3]{p^2r},$$

equalities iff Δ ABC is equilateral (QED)

773. $OA=a, OB=b, OC=c, \sphericalangle BOC=\theta_1, \sphericalangle COA=\theta_2, \sphericalangle AOB=\theta_3$

$S_1=[OBC], S_2=[OCA], S_3=[OAB], S_4=[ABC]$. *Prove that*:

$$S_4^2-\sum_{k=1}^{3}S_k^2=\frac{abc}{2}\cdot\sum_{cyc}a(-\cos\theta_1+\cos\theta_2\cdot\cos\theta_3)$$



Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by proposer

Plagiogonal 3D system:

$$O(0,0,0), A(a,0,0), B(0,b,0), C(0,0,c), \overrightarrow{AB}(-a,b,0), \overrightarrow{AC}(-a,0,c)$$

$$|\overrightarrow{AB}|^2=a^2+b^2-2ab\cos\theta_1, |\overrightarrow{AC}|^2=a^2+c^2-2ac\cos\theta_2$$

$$S_4^2=\frac{1}{4}\left(|\overrightarrow{AB}|^2\cdot|\overrightarrow{AC}|^2-(\overrightarrow{AB}\cdot\overrightarrow{AC})^2\right)$$

$$S_4^2 = \sum_{cyc}\left(\frac{absin\theta_3}{2}\right)^2 + \frac{abc}{2}\cdot\sum_{cyc} a(-cos\theta_1 + cos\theta_2\cdot cos\theta_3)$$

$$S_4^2 - \sum_{k=1}^{3} S_k^2 = \frac{abc}{2}\cdot\sum_{cyc} a(-cos\theta_1 + cos\theta_2\cdot cos\theta_3)$$

$$\text{If } \theta_1 = \theta_2 = \theta_3 = 90° \Rightarrow \quad S_4^2 = \sum_{k=1}^{3} S_k^2$$

$$\text{If } \theta_1 = \theta_2 = \theta_3 = 60° \Rightarrow \sum_{k=1}^{3} S_k^2 - S_4^2 = \frac{abc(a+b+c)}{8}$$

774. *If ω – Brocard's angle in ΔABC then :*

$$\frac{m_a}{h_b} + \frac{m_b}{h_c} + \frac{m_c}{h_a} \geq \frac{3}{2\sin\omega}$$

Proposed by Bogdan Fuștei-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Lemma : For any triangles ΔABC and $\Delta A_1B_1C_1$ of area F and F_1 respectively and $M \in Int(\Delta ABC)$, we have :

$$a_1.AM + b_1.BM + c_1.CM \geq \sqrt{\frac{1}{2}\sum_{cyc} a_1{}^2.(b^2+c^2-a^2) + 8FF_1}, \quad (Bottema's\ inequality)$$

(*See : O.BOTTEMA, R.Ž.DJORDEVIĆ, R.R.JANIĆ, D.S.MITRINOVIĆ AND P.M.VASIĆ, Geometric inequalities, Wolters – Noordhoff Publishing, Groningen, The Netherlands, (1969), pp. 118, th. 12.56.*)

Choosing $a_1 = b,\ b_1 = c,\ c_1 = a$ and $M \equiv G$ we get :

$$b.AG + c.BG + a.CG \geq \sqrt{\frac{1}{2}\sum_{cyc} b^2.(b^2+c^2-a^2) + 8F^2}. \quad (\because F_1 = F)$$

$$\Leftrightarrow \frac{2F}{h_b}.\frac{2}{3}m_a + \frac{2F}{h_c}.\frac{2}{3}m_b + \frac{2F}{h_a}.\frac{2}{3}m_c \geq \sqrt{\frac{1}{2}\sum_{cyc} a^4 + \frac{1}{2}\left(2\sum_{cyc} a^2b^2 - \sum_{cyc} a^4\right)} = \sqrt{\sum_{cyc} a^2b^2}$$

$$\Leftrightarrow \frac{m_a}{h_b}+\frac{m_b}{h_c}+\frac{m_c}{h_a} \geq \frac{3\sqrt{a^2b^2+b^2c^2+c^2a^2}}{4F} = \frac{3}{2\sin\omega}, \text{ as desired.}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\text{Lemma}: \text{If } P, M \in Int(\Delta ABC), \text{then}:$$

$$a.AP.AM + b.BP.BM + c.CP.CM \geq abc \ \ (G.Bennett's\ inequality)$$

$$(Reference: 584\ G.Bennett - Multiple\ Triangle\ Inequalities)$$

$$For\ P \equiv \Omega\ and\ M = G, where\ \Omega\ is\ the\ first\ Brocard's\ point$$

$$and\ G\ is\ the\ centroid\ of\ \Delta ABC,$$

$$We\ have:\ a.A\Omega.AG + b.B\Omega.BG + c.C\Omega.CG \geq abc,\ \ with:$$

$$A\Omega = b.\frac{\sin\omega}{\sin A} = bc.\frac{\sin\omega}{h_b}\ and\ AG = \frac{2}{3}m_a\ \ (and\ analogs)$$

$$Then:\ \ a.bc.\frac{\sin\omega}{h_b}.\frac{2}{3}m_a + b.ca.\frac{\sin\omega}{h_c}.\frac{2}{3}m_b + a.bc.\frac{\sin\omega}{h_a}.\frac{2}{3}m_c \geq abc$$

$$\Leftrightarrow \frac{m_a}{h_b}+\frac{m_b}{h_c}+\frac{m_c}{h_a} \geq \frac{3}{2\sin\omega}, \text{ as desired.}$$

775. In ΔABC the following relationship holds:

$$\frac{a^5}{m_b+m_c}+\frac{b^5}{m_c+m_a}+\frac{c^5}{m_a+m_b} \geq \frac{16\sqrt{3}}{3}F^2$$

Proposed by D.M.Bătinețu-Giurgiu, Claudia Nănuți-Romania

Solution by Marin Chirciu-Romania

Lemma

In ΔABC

$$\sum\frac{a^5}{m_b+m_c} \geq \frac{16s^5}{27(4R+r)}.$$

By Hőlder:

$$\sum\frac{a^5}{m_b+m_c} \overset{Holder}{\geq} \frac{\left(\sum a\right)^5}{3^3\sum(m_b+m_c)} = \frac{(2s)^5}{27\cdot 2\sum m_a} \overset{Leuenberger}{\geq} \frac{32s^5}{27\cdot 2(4R+r)} = \frac{16s^5}{27(4R+r)}.$$

Equality holds for an equilateral triangle

By Lemma remains to prove: $\frac{16s^5}{27(4R+r)} \geq 3\sqrt{3}F^2$.

$$\frac{16s^5}{27(4R+r)} \geq \frac{16\sqrt{3}}{3}F^2 \Leftrightarrow \frac{16s^5}{27(4R+r)} \geq \frac{16\sqrt{3}}{3}s^2r^2 \Leftrightarrow \frac{s^3}{9(4R+r)} \geq \sqrt{3}r^2,$$

(by Mitrinovic $p \geq 3\sqrt{3}r$ **.) Remains to prove:**

$$\frac{s^2 \cdot 3\sqrt{3}}{9(4R+r)} \geq \sqrt{3}r^2 \Leftrightarrow \frac{s^2}{3(4R+r)} \geq r^2 \Leftrightarrow s^2 \geq 3r^2(4R+r),$$

(by Gerretsen $s^2 \geq 16Rr - 5r^2$ **.)**

$$16Rr - 5r^2 \geq 3r^2(4R+r) \Leftrightarrow R \geq 2r, \text{ (Euler).}$$

Equality holds for an equilateral triangle

776.

In Δ ABC $(acute)$, let $\theta_1 = \angle(w_a, n_a), \theta_2 = \angle(w_b, n_b), \theta_3 = \angle(w_c, n_c)$.

Prove that : $0 \leq \theta_1 + \theta_2 + \theta_3 < \pi$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{If } b > c, \theta_1 = \angle(w_a, n_a) = \angle(b, w_a) - \angle(b, n_a) < \angle(b, w_a) = \frac{A}{2} \overset{A<\frac{\pi}{2}}{<} \frac{\pi}{4}$$

$$\therefore \theta_1 < \frac{\pi}{4} \text{ and,}$$

$$\text{if } c > b, \theta_1 = \angle(w_a, n_a) = \angle(c, w_a) - \angle(c, n_a) < \angle(c, w_a) = \frac{A}{2} \overset{A<\frac{\pi}{2}}{<} \frac{\pi}{4} \therefore \theta_1 < \frac{\pi}{4}$$

$$\therefore \forall\ acute \text{ triangles ABC}, \theta_1 \overset{(*)}{<} \frac{\pi}{4} \text{ and similarly}, \forall\ acute \text{ triangles ABC}, \theta_2, \theta_3 \overset{(**)}{<} \frac{\pi}{4}$$

$$\therefore (*) + (**) \Rightarrow \theta_1 + \theta_2 + \theta_3 < \frac{\pi}{4} + \frac{\pi}{4} + \frac{\pi}{4} = \frac{3\pi}{4} < \pi \therefore \theta_1 + \theta_2 + \theta_3 < \pi$$

$$Also, \theta_1 \geq 0, \text{with equality iff } w_a = n_a \Rightarrow \text{iff } \Delta \text{ ABC is equilateral}$$

$$\text{and analogously}, \theta_2, \theta_3 \geq 0 \Rightarrow \theta_1 + \theta_2 + \theta_3 \geq 0$$

$$\therefore 0 \leq \theta_1 + \theta_2 + \theta_3 < \pi, with\ equality \text{ iff } \Delta \text{ ABC is equilateral (QED)}$$

777. *In $\Delta ABC, AD, BE, CF$ – internal bisectors. Prove that :*

$$\left(\frac{EF}{BC}\right)^4+\left(\frac{FD}{CA}\right)^4+\left(\frac{DE}{AB}\right)^4+\frac{3}{16}\le\frac{3}{8}\cdot\left(\frac{R}{2r}\right)^2$$

Proposed by George Apostolopoulos-Messolonghi-Greece
Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $a=BC, b=CA, c=AB$ be the side lengths of ΔABC.

We know that : $AE=\frac{bc}{a+c}$ *and* $AF=\frac{bc}{a+b}$.

By the Law of cosines in ΔAEF :

$$EF^2=AE^2+AF^2-2.AE.AF.\cos A=\left(\frac{bc}{a+c}\right)^2+\left(\frac{bc}{a+b}\right)^2-2\left(\frac{bc}{a+c}\right)\left(\frac{bc}{a+b}\right).\frac{b^2+c^2-a^2}{2bc}$$

$$=\frac{b^2c^2}{(a+c)^2}+\frac{b^2c^2}{(a+b)^2}-\frac{bc[(b-c)^2+2bc-a^2]}{(a+b)(a+c)}$$

$$=b^2c^2\left(\frac{1}{a+c}-\frac{1}{a+b}\right)^2-\frac{bc(b-c)^2}{(a+b)(a+c)}+\frac{a^2bc}{(a+b)(a+c)}=$$

$$=\frac{b^2c^2(b-c)^2}{(a+b)^2(a+c)^2}-\frac{bc(b-c)^2}{(a+b)(a+c)}+\frac{a^2bc}{(a+b)(a+c)}$$

$$=\frac{a^2bc}{(a+b)(a+c)}-\frac{bc(b-c)^2[(a+b)(a+c)-bc]}{(a+b)^2(a+c)^2}\le$$

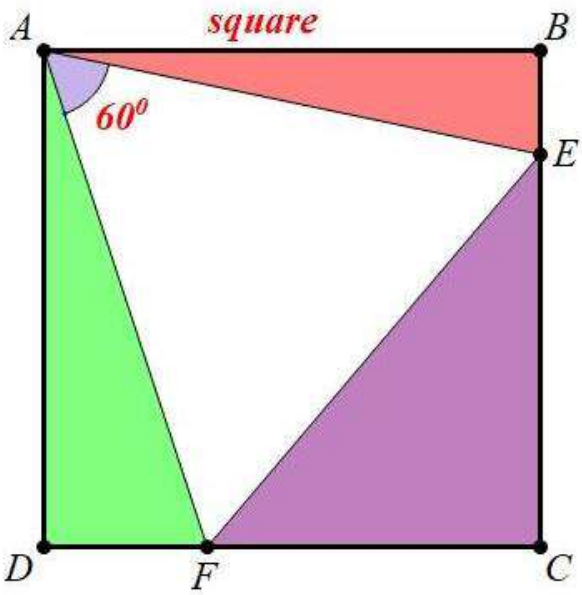
$$\le\frac{a^2bc}{(a+b)(a+c)}\overset{AM-GM}{\le}\frac{a^2bc}{2\sqrt{ab}.2\sqrt{ac}}=\frac{a\sqrt{bc}}{4}.$$

Then : $\left(\frac{EF}{BC}\right)^4\le\frac{bc}{16a^2}\overset{AM-GM}{\le}\frac{b^2+c^2}{32a^2}$ *(And analogs)*

Therefore, $\left(\frac{EF}{BC}\right)^4+\left(\frac{FD}{CA}\right)^4+\left(\frac{DE}{AB}\right)^4+\frac{3}{16}\le\sum_{cyc}\frac{b^2+c^2}{32a^2}+\frac{3}{16}$

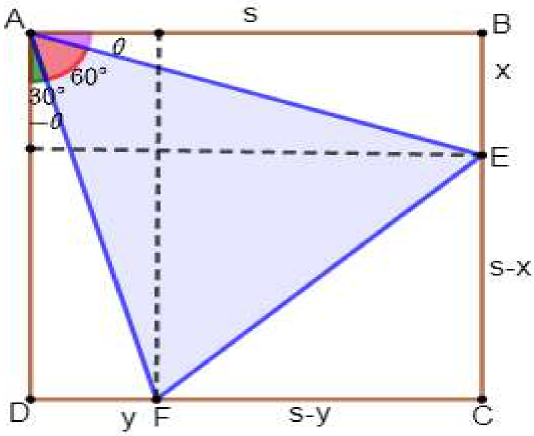
$$=\frac{1}{32}\sum_{cyc}\left(\frac{a}{b}+\frac{b}{a}\right)^2\overset{2\ Bandila}{\le}\frac{1}{32}\sum_{cyc}\left(\frac{R}{r}\right)^2=\frac{3}{8}\cdot\left(\frac{R}{2r}\right)^2.$$

778. Prove that: $[ABE]+[ADF]\geq[CEF]$.



Proposed by Binh Luc-Vietnam

Solution by Rajarshi Chakraborty-India



$$[ABE]=\frac{1}{2}sx;[ADF]=\frac{1}{2}sy;[CEF]=\frac{1}{2}(s-x)(s-y)$$

$$s^2+x^2=AE^2;s^2+y^2=AF^2;(s-x)^2+(s-y)^2=FE^2$$

$$\frac{s^2+x^2+s^2+y^2-(s-x)^2-(s-y)^2}{2(s^2+x^2)^{\frac{1}{2}}\cdot(s^2+y^2)^{\frac{1}{2}}}=\frac{1}{2}$$

$$2sx+2sy=[(s^2+x^2)(s^2+y^2)]^{\frac{1}{2}}$$

$$[ABE]+[ADF]=\frac{1}{4}[(s^2+x^2)(s^2+y^2)]^{\frac{1}{2}}=\frac{1}{\sqrt{3}}[AEF]$$

$$[FEC]=s^2-\left(1+\frac{1}{\sqrt{3}}\right)[AEF]$$

$$[ABE]+[ADF]-[FEC]=\left(7+\frac{2}{\sqrt{3}}\right)[AEF]-s^2$$

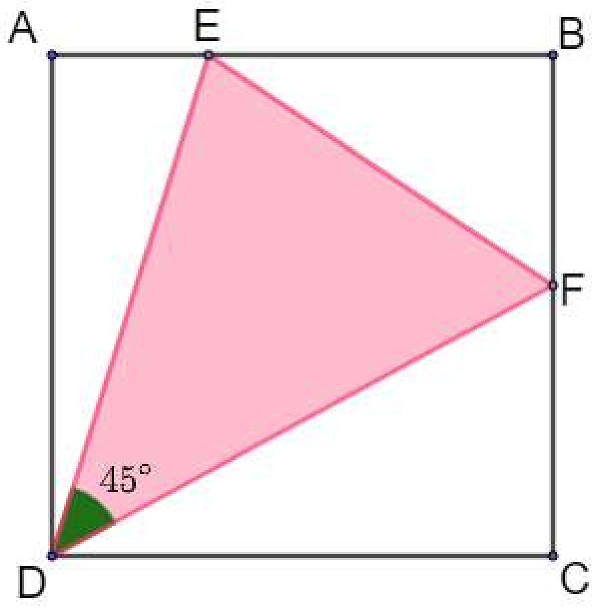
$$\min\{[AEF]\}=\min\left\{\frac{1}{2}\cdot\frac{s}{\cos\theta}\cdot\frac{s}{\cos(30^\circ-\theta)}\cdot\frac{\sqrt{3}}{2}\right\}=$$

$$=\frac{s^2\sqrt{3}}{2}\min\left\{\frac{1}{\cos 30^\circ+\cos(30^\circ-2\theta)}\right\}=\frac{s^2\sqrt{3}}{2}\cdot\frac{1}{1+\frac{\sqrt{3}}{2}}=\frac{s^2}{1+\frac{2}{\sqrt{3}}}$$

Therefore,

$$[ABE]+[ADF]-[FEC]\geq\left(1+\frac{2}{\sqrt{3}}\right)\cdot s^2\left(1+\frac{2}{\sqrt{3}}\right)-s^2=0$$

779. Prove: $[DEF]\geq\left(\sqrt{2}-1\right)[ABCD]$



Proposed by Luc Binh-Vietnam

Solution 1 by Daniel Sitaru-Romania

$$AB=a, AE=x, x\in[0,a], tan(\sphericalangle ADE)=\frac{x}{a}, \mu(\sphericalangle FDC)=\frac{\pi}{4}-arctan\frac{x}{a}$$

$$FC = atan\left(\frac{\pi}{4} - arctan\frac{x}{a}\right) = \frac{a(a-x)}{a+x}, BF = a - FC = \frac{2ax}{a+x}$$

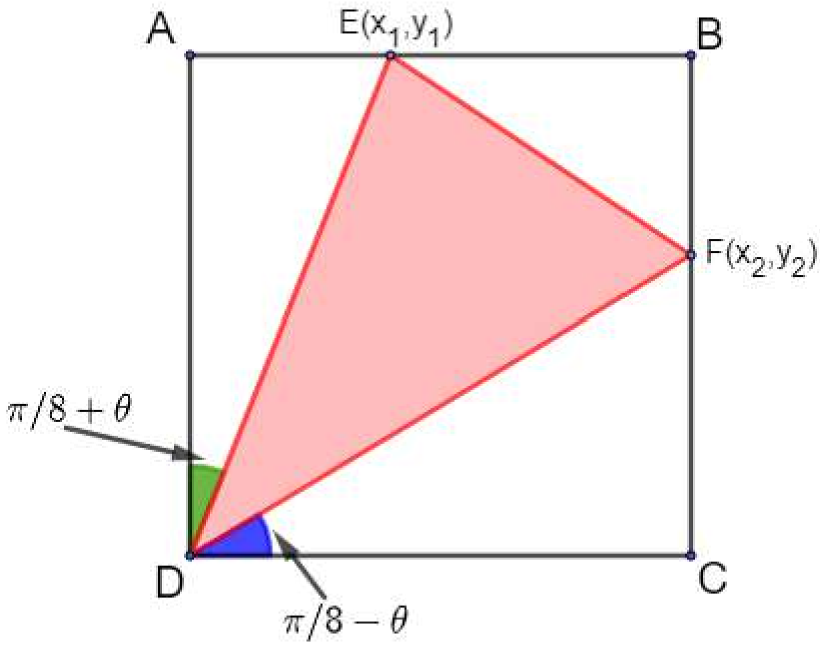
$$f: [0,a] \to \mathbb{R}, f(x) = [DEF] = a^2 - [ADE] - [DFC] - [BEF]$$

$$f(x) = a^2 - \frac{ax}{2} - \frac{a}{2}\cdot\frac{a(a-x)}{a+x} - \frac{a-x}{2}\cdot\frac{2ax}{a+x} = \frac{a^3 + ax^2}{2(a+x)}$$

$$f'(x) = \frac{2ax^2 + 2ax - 2ax^3}{4(a+x)^2}, f'(x) = 0 \Rightarrow x = (\sqrt{2}-1)a$$

$$[DEF] = f(x) \geq f\left((\sqrt{2}-1)a\right) = (\sqrt{2}-1)a^2 = (\sqrt{2}-1)[ABCD]$$

Solution 2 by Ravi Prakash-New Delhi-India



$$\text{Let: } AB = a, \frac{CD}{DC} = \tan\left(\frac{\pi}{8} - \theta\right); -\frac{\pi}{8} < \theta < \frac{\pi}{8} \Rightarrow CF = a\tan\left(\frac{\pi}{8} - \theta\right) = y_2$$

Similarly:

$$AE = a\tan\left(\frac{\pi}{8} + \theta\right) = x_1$$

$$[DEF] = \frac{1}{2}|x_1y_2 - x_2y_1| = \frac{1}{2}a^2\left(1 - \tan\left(\frac{\pi}{8} - \theta\right)\tan\left(\frac{\pi}{8} + \theta\right)\right) =$$

$$=\frac{1}{2}a^2\cdot\left(1-\frac{\sin^2\left(\frac{\pi}{8}\right)-\sin^2\theta}{\cos^2\frac{\pi}{8}-\sin^2\theta}\right)=\frac{a^2}{2}\cdot\frac{\cos^2\left(\frac{\pi}{8}\right)-\sin^2\left(\frac{\pi}{8}\right)}{\cos^2\left(\frac{\pi}{8}\right)-\sin^2\theta}=$$

$$=\frac{a^2}{2}\cos\left(\frac{\pi}{4}\right)\cdot\frac{1}{\cos^2\left(\frac{\pi}{8}\right)-\sin^2\theta}\geq\frac{a^2}{2\sqrt{2}}\sec^2\left(\frac{\pi}{8}\right)=$$

$$=\frac{a^2}{2\sqrt{2}}\left(1+\left(\sqrt{2}-1\right)^2\right)=\frac{a^2}{2\sqrt{2}}\left(4-2\sqrt{2}\right)=a^2\left(\sqrt{2}-1\right)=\left(\sqrt{2}-1\right)[ABCD].$$

780. In ΔABC:

$$\sum\frac{r_a^3}{r_b+\sqrt{r_br_c}+\sqrt[3]{r_ar_br_c}}\geq\sqrt{3}F$$

Proposed by Marin Chirciu – Romania

Solution by Tapas Das-India

$$\sqrt{r_br_c}\leq\frac{r_b+r_c}{2}\quad\text{(AM-GM)},\quad\sqrt[3]{r_ar_br_c}\leq\frac{r_a+r_b+r_c}{3}$$

$$\therefore r_b+\sqrt{r_br_c}+\sqrt[3]{r_ar_br_c}\leq r_b+\frac{r_b+r_c}{2}+\frac{r_a+r_b+r_c}{3}$$

$$=\frac{1}{6}(6r_b+3r_b+3r_c+2r_a+2r_b+2r_c)=\frac{1}{6}(2r_a+11r_b+5r_c)$$

$$\therefore\sum\frac{r_a^3}{r_b+\sqrt{r_br_c}+\sqrt[3]{r_br_ar_c}}\geq\sum\frac{r_a^3}{\frac{1}{6}(2r_a+11r_b+5r_c)}=6\sum\frac{r_a^3}{2r_a+11r_b+5r_c}$$

$$\overset{Holder}{\geq}6\frac{(r_a+r_b+r_c)^3}{3(2r_a+11r_b+5r_c+2r_b+11r_c+5r_a+2r_c+11r_a+5r_b)}$$

$$=6\cdot\frac{(r_a+r_b+r_c)^3}{3\times18(r_a+r_b+r_c)}=\frac{1}{9}(r_a+r_b+r_c)^2=\frac{1}{9}(4R+r)^2\geq\frac{1}{9}3s^2=\frac{1}{3}s^2$$

Note: $s^2\leq\frac{(4R+r)^2}{3}$

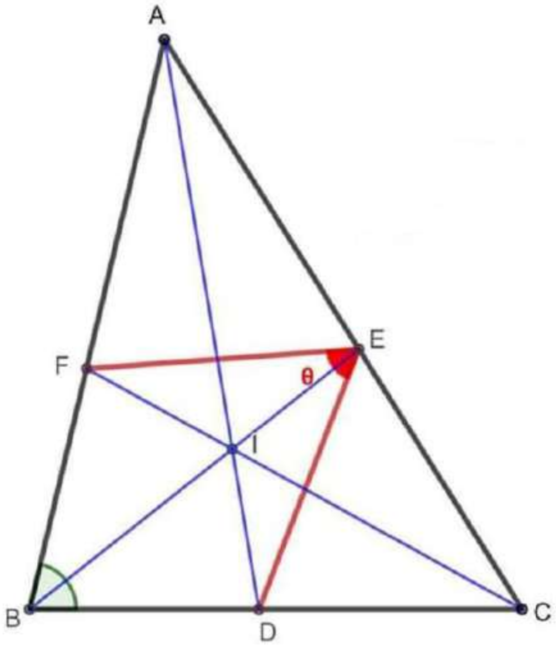
$$\therefore\sum\frac{r_a^3}{r_b+\sqrt{r_br_c}+\sqrt[3]{r_ar_br_c}}\geq\frac{1}{3}s^2=\frac{1}{3}\frac{F^2}{r^2},F=r\cdot s$$

$$\therefore s=\frac{F}{r}=\frac{1}{3}\cdot F\cdot\frac{F}{r^2}=\frac{1}{3}\cdot F\cdot\frac{r\cdot s}{r^2}\geq\frac{1}{3}F\cdot\frac{r}{r^2}\cdot3\sqrt{3}r=\sqrt{3}F$$

Note: $s^2\geq27r^2$ (Mitrinovic) $\therefore s\geq3\sqrt{3}r$

781. In $\Delta ABC, I$ −incenter, $\hat{B}$ −constant, a, b, c −variable, $\sphericalangle DEF = \theta$.

Find: $\theta_{min} = f_1(\hat{B}), \theta_{max} = f_2(\hat{B})$.



Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by proposer

Plagiogonal system: $BC \equiv Bx, BA \equiv By$

$$B(0,0), D(d,0), d = \frac{ac}{b+c}, F(0,f), f = \frac{ac}{a+b}, E(e,e), e = \frac{ac}{a+c}$$

$$\lambda_{ED} = \lambda_2 = \frac{b+c}{b-a}, \lambda_{EF} = \lambda_1 = \frac{f-e}{0-e} = \frac{b-c}{b+a}$$

$$\tan\theta = \frac{(\lambda_2 - \lambda_1)\sin B}{(\lambda_2 + \lambda_1)\cos B + \lambda_1\lambda_2 + 1} = \frac{-2b(a+c)\sin B}{a^2 + c^2 - 2ac \cdot \cos B - 2b^2 - 2b^2\cos B} =$$

$$= \frac{-2b(a+c)\sin B}{b^2 - 2b^2 - 2b^2\cos B} = \frac{2(a+c)\sin B}{b(1+2\cos B)} = \frac{2(a+c)}{\sqrt{a^2+c^2-2ac\cos B}} \cdot \frac{\sin B}{1+2\cos B}$$

$$\tan\theta = \frac{2\left(\frac{a}{c}+1\right)}{\sqrt{\left(\frac{a}{c}\right)^2 + 1 - 2\left(\frac{a}{c}\right)\cos B}} \cdot \frac{\sin B}{1+2\cos B} = \frac{2(x+1)}{\sqrt{x^2+1-2x\cos B}} \cdot \frac{\sin B}{1+2\cos B}$$

where, $x = \frac{a}{b}, y = \tan\theta$. If $x \to 0 \Rightarrow y \to \frac{2\sin B}{1+2\cos B}$

$$\text{If } x = 1; (a = b) \Rightarrow y = 2\sqrt{2} \cdot \frac{\sqrt{1+\cos B}}{1+2\cos B}$$

Finally,

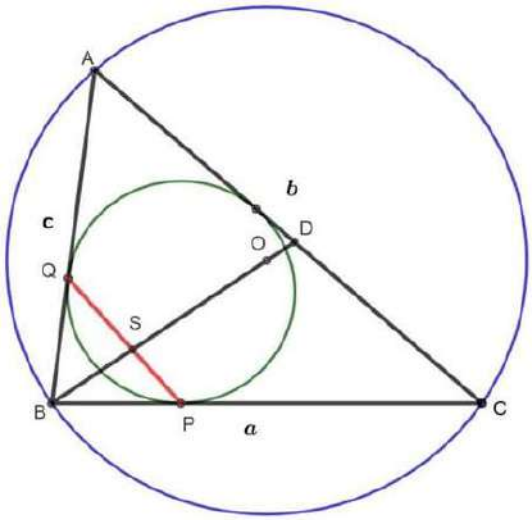
$$\frac{2\sin B}{1+2\cos B} < \tan\theta \le 2\sqrt{2}\cdot\frac{\sqrt{1+\cos B}}{1+2\cos B}$$

$$\tan^{-1}\left(\frac{2\sin B}{1+2\cos B}\right) < \theta < \tan^{-1}\left(2\sqrt{2}\cdot\frac{\sqrt{1+\cos B}}{1+2\cos B}\right)$$

$1+2\cos B \ne 0 \Rightarrow \cos B \ne -\frac{1}{2} \Rightarrow B \ne 120°$. If $B = 120° \Rightarrow \theta = 90°$.

782. ΔABC isn't right, a, b, c –are integers, $\frac{PS}{SQ} = \frac{3}{4}$.

Find $[ABC]$ minimum (area).



Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by Jose Ferreira Queiroz-Olinda-Brazil

$BP = BQ$. Using Gakopoulos' Lemma:

$$\frac{PS}{SQ} = \frac{BP}{BQ}\cdot\frac{c-a\cos B}{a-c\cos B} \Rightarrow \frac{3}{4} = \frac{c-a\cos B}{a-c\cos B} \Rightarrow \cos B = \frac{4c-3a}{4a-3c}$$

$$[ABC] = F = \frac{1}{2}ac\sin B,\ \ 4F^2 = a^2c^2(1-\cos^2 B)$$

$$F^2 = \frac{a^2c^2}{4}\cdot\frac{7(a^2-c^2)}{(4a-3c)^2},\qquad F = \frac{ac\sqrt{7}}{2}\cdot\frac{\sqrt{a^2-c^2}}{4a-3c}$$

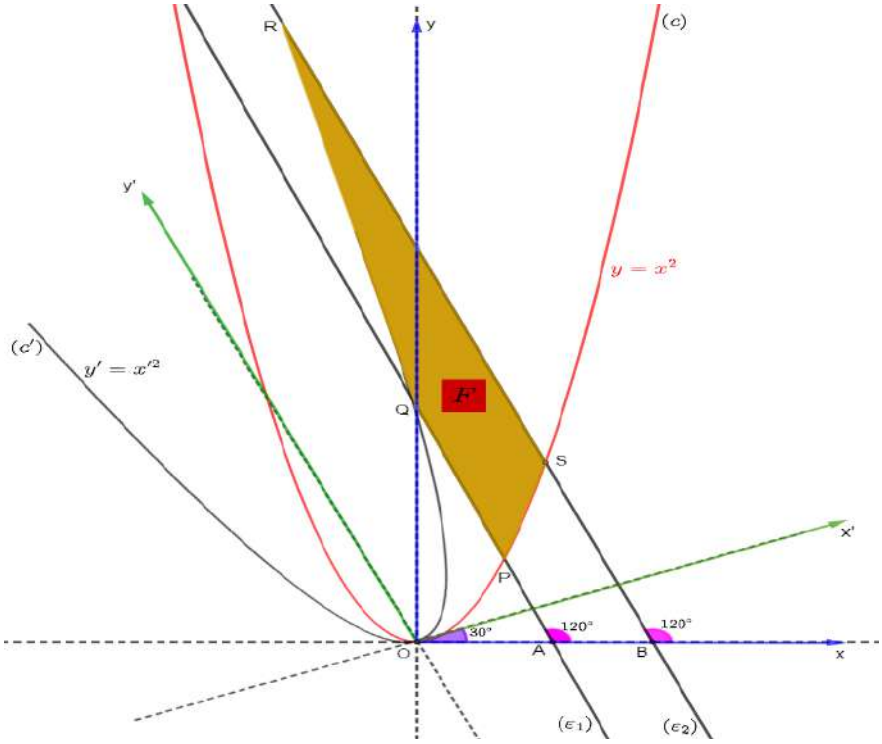
F –minimum, a, c integers: $a = 5, c = 4$, then:

$$\cos B = \frac{4\cdot 4 - 3\cdot 5}{4\cdot 5 - 3\cdot 4} = \frac{1}{8},\qquad b^2 = a^2 + c^2 - 2ac\cos\theta$$

$$b^2 = 5^2 + 4^2 - 2 \cdot 4 \cdot 5 \cdot \frac{1}{8}, \qquad b = 6, F = \frac{15\sqrt{7}}{2}.$$

783. $\sphericalangle xOy = \sphericalangle x'Oy' = 90°, \sphericalangle xOx' = 30°, OA = 2, OB = 3$

Find $F = [PQRS]$ −area.



Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by proposer

xOy −orthogonal system, $(c): y = x^2, A(2,0), B(3,0)$, $x'Oy$ −orthogonal system

$$(c'): y' = x'^2, \qquad \sphericalangle xOx' = 30°$$

$$A \in (\varepsilon_1), B \in (\varepsilon_2), (\varepsilon_1) \parallel (\varepsilon_2) \parallel Oy'$$

$$\begin{cases} (\varepsilon_1) \cap (c) = \{P\}, & (\varepsilon_1) \cap (c') = \{Q\} \\ (\varepsilon_2) \cap (c') = \{R\}, & (\varepsilon_2) \cap (c) = \{S\} \end{cases}$$

$$\{(xOy), y = x^2\} \to \{(xOy'), y = f_1(x)\}, \qquad y \to y_1 \sin 120°, x = x_1 + y_1 \cos 120°$$

$$y = x^2 \Rightarrow y_1 \cdot \left(\frac{\sqrt{3}}{2}\right) = \left(x_1 + y_1 \cdot \left(-\frac{1}{2}\right)\right)^2 \Rightarrow y_1 = 2x + \sqrt{3} - \sqrt{3 + 4\sqrt{3}x_1}$$

$$f_1(x) = 2x + \sqrt{3} - \sqrt{3 + 4\sqrt{3}x}$$

$$\{(x'Oy'), y' = x'^2\} \to \{(xOy'), y = f_2(x)\}, y' \to y_2 \cos 120°, x = x_2 + y_2 \sin 120°$$

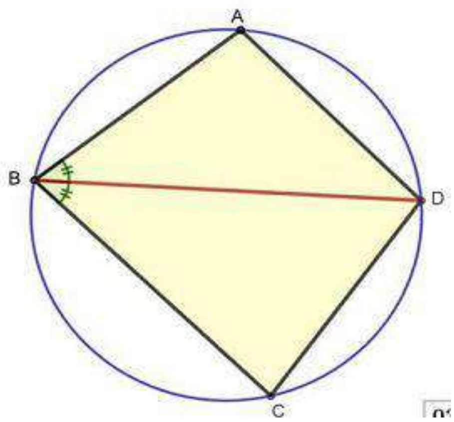
$$y' = x'^2 \Rightarrow y_2 + x_2 \cdot \left(-\frac{1}{2}\right) = \left(x_2 \cdot \frac{\sqrt{3}}{2}\right)^2 \Rightarrow y_2 = \frac{3x^2}{4} + \frac{x}{4}, f_2(x) = \frac{3x^2}{4} + \frac{x}{2}$$

$$F = [PQRS] = \sin 120° \cdot \int_2^3 [f_2(x) - f_1(x)]\, dx =$$

$$= \frac{6 - 6\sqrt{3} + 3(1 + 4\sqrt{3})^{\frac{3}{2}} - \sqrt{585 + 584\sqrt{3}}}{4\sqrt{3}} \cong 3.2651652815937$$

784. Prove that:

$$[ABCD] = BD^2 \cdot \frac{sinB}{2}$$



Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution 1 by Daniel Sitaru-Romania

$$AB = 2Rsin\left(\pi - A - \frac{B}{2}\right) = 2Rsin\left(A + \frac{B}{2}\right) = 2Rsin\left(\pi - C + \frac{B}{2}\right) = 2Rsin\left(C - \frac{B}{2}\right)$$

$$BC = 2Rsin\left(\pi - C - \frac{B}{2}\right) = 2Rsin\left(C + \frac{B}{2}\right)$$

$$[ABCD] = [ABD] + [BCD] = \frac{1}{2}AB \cdot BDsin\frac{B}{2} + \frac{1}{2}BC \cdot BDsin\frac{B}{2} =$$

$$= \frac{1}{2}2Rsin\left(C - \frac{B}{2}\right) \cdot BDsin\frac{B}{2} + \frac{1}{2}2Rsin\left(C + \frac{B}{2}\right) \cdot BDsin\frac{B}{2} =$$

$$= R \cdot BDsin\frac{B}{2}\left(sin\left(C - \frac{B}{2}\right) + sin\left(C + \frac{B}{2}\right)\right) =$$

$$= R \cdot BDsin\frac{B}{2}\left(sinCcos\frac{B}{2} - sin\frac{B}{2}cosC + sinCcos\frac{B}{2} + sin\frac{B}{2}cosC\right) =$$

$$= R \cdot BDsin\frac{B}{2} \cdot 2sinCcos\frac{B}{2} = R \cdot BDsinBsinC =$$

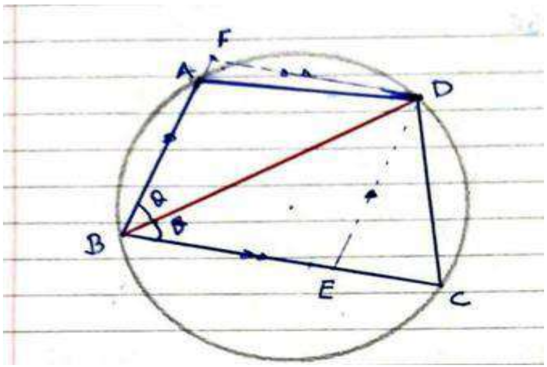
$$= \frac{BD}{2sinC} \cdot BDsinBsinC = BD^2 \cdot \frac{sinB}{2}$$

Solution 2 by proposer

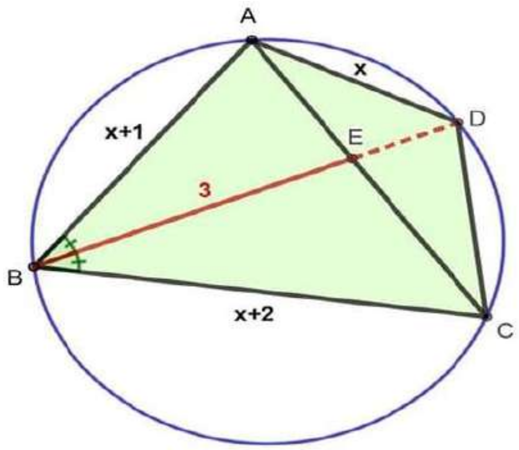
$$BD^2 = BE \cdot BC + BF \cdot BA\ (NCCQ_1 - Gakopoulos\ theorem)$$

$$[ABCD] = \frac{sinB}{2}(BC \cdot BF + BA \cdot BE)\ (Gakopoulos - Blatsis\ formula)$$

$$BE = BF \Rightarrow \frac{[ABCD]}{BD^2} = \frac{sinB}{2} \Rightarrow [ABCD] = BD^2 \cdot \frac{sinB}{2}$$



785. $DA = x, AB = x + 1, BC = x + 2, BE = 2. [ABCD] = ?$



Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by proposer

$$\begin{cases} BD \cdot BE = BA \cdot BC \\ BD^2 = BA \cdot BC + DA \cdot DC \end{cases};\ (Gakopoulos'th)$$

$$\Rightarrow BD^2 \cdot BE^2 = BA^2 \cdot BC^2, \qquad (BA \cdot BC + DA \cdot DC) \cdot BE^2 = BA^2 \cdot BC^2$$

$$\frac{1}{BE^2} = \frac{1}{BA \cdot BC} + \frac{DA \cdot DC}{(BA \cdot BC)^2} \Rightarrow \frac{1}{3^2} = \frac{1}{(x+1)(x+2)} + \frac{x^2}{(x+1)^2(x+2)^2}$$

$$x = 2, s = \frac{2+2+3+4}{2} = \frac{11}{2}$$

$$[ABCD] = (s-2)\sqrt{(s-3)(s-4)} \Rightarrow [ABCD] = \frac{7\sqrt{15}}{4}$$

786. ***If*** $M \in Int(\Delta ABC)$ ***then :***

$$\frac{AM}{h_b} + \frac{BM}{h_c} + \frac{CM}{h_a} \geq \frac{1}{3}\left(\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c}\right)$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Lemma : *If* $P, M \in Int(\Delta ABC)$, *then :*

$$a.AP.AM + b.BP.BM + c.CP.CM \geq abc \ \ (G.Bennett's\ inequality)$$

$$(Reference : 584\ G.Bennett - Multiple\ Triangle\ Inequalities)$$

For $P \equiv \Omega$, *where* Ω *is the first Brocard's point, we have :*

$$a.A\Omega.AM + b.B\Omega.BM + c.C\Omega.CM \geq abc, \ \ with :$$

$$A\Omega = b.\frac{\sin\omega}{\sin A} = bc.\frac{\sin\omega}{h_b} \ (and\ analogs)\ and\ \sin\omega = \frac{2F}{\sqrt{a^2b^2 + b^2c^2 + c^2a^2}}.$$

$$Then : \frac{AM}{h_b} + \frac{BM}{h_c} + \frac{CM}{h_a} \geq \frac{\sqrt{a^2b^2 + b^2c^2 + c^2a^2}}{2F} = \frac{2R\sqrt{a^2b^2 + b^2c^2 + c^2a^2}}{abc} \ (1)$$

$$Now\ we\ have : 3R\sqrt{a^2b^2 + b^2c^2 + c^2a^2} \overset{Leibniz}{\geq} \sqrt{(a^2+b^2+c^2)(a^2b^2 + b^2c^2 + c^2a^2)} \geq$$

$$\overset{CBS}{\geq} a^2b + b^2c + c^2a.\ \ Similarly\ we\ get : 3R\sqrt{a^2b^2 + b^2c^2 + c^2a^2} \geq ab^2 + bc^2 + ca^2.$$

$$Then : \ 6R\sqrt{a^2b^2 + b^2c^2 + c^2a^2} \geq ab(a+b) + bc(b+c) + ca(c+a) \ (2)$$

$$From\ (1)\ and\ (2)\ we\ have : \frac{AM}{h_b} + \frac{BM}{h_c} + \frac{CM}{h_a} \geq \frac{1}{3}\left(\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c}\right).$$

787.

In any Δ ABC, let $\varphi_1 = \angle(g_a, n_a), \varphi_2 = \angle(g_b, n_b), \varphi_3 = \angle(g_c, n_c)$.

Prove that : $0 \le \varphi_1 + \varphi_2 + \varphi_3 < \pi$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Triangle inequality} \Rightarrow g_a \le AI + r \overset{?}{\le} w_a \Leftrightarrow \frac{r}{\sin\frac{A}{2}} + r \overset{?}{\le} \frac{2abc\cos\frac{A}{2}}{a(b+c)}$$

$$\Leftrightarrow \frac{r}{\sin\frac{A}{2}} + r \overset{?}{\le} \frac{8Rrs\cos\frac{A}{2}}{4R(b+c)\sin\frac{A}{2}\cos\frac{A}{2}} \Leftrightarrow \frac{1}{\sin\frac{A}{2}} + 1 \overset{?}{\le} \frac{a+b+c}{(b+c)\sin\frac{A}{2}}$$

$$\Leftrightarrow \frac{1}{\sin\frac{A}{2}} + 1 \overset{?}{\le} + \frac{a}{(b+c)\sin\frac{A}{2}} + \frac{1}{\sin\frac{A}{2}} \Leftrightarrow (b+c)\sin\frac{A}{2} \overset{?}{\le} a$$

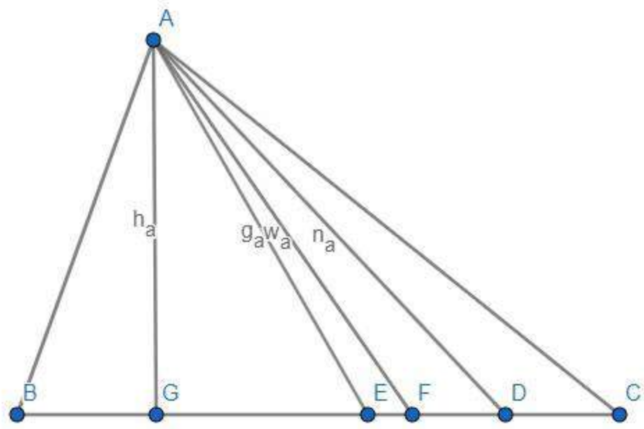
$$\Leftrightarrow 4R\cos\frac{A}{2}\cos\frac{B-C}{2}\sin\frac{A}{2} \overset{?}{\le} 4R\sin\frac{A}{2}\cos\frac{A}{2} \Leftrightarrow \cos\frac{B-C}{2} \overset{?}{\le} 1 \rightarrow \text{true}$$

$$\therefore g_a \le w_a \le \sqrt{s(s-a)} \le m_a \le n_a \therefore \varphi_1 = \angle(g_a, n_a) \ge 0,$$

$$\text{with equality iff } n_a = g_a \text{and analogs} \overset{\text{summing up}}{\Rightarrow} \varphi_1 + \varphi_2 + \varphi_3 \ge 0,$$

$$\text{with equality iff } n_a = g_a, n = g_b, n_c = g_c \Rightarrow \text{iff } \Delta\, ABC \text{ is equilateral}$$

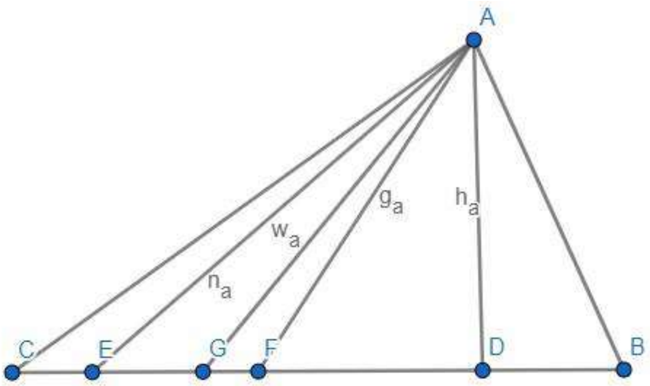
Case 1 $b > c$



$$\angle(g_a, n_a) = \angle(c, w_a) - \angle(c, g_a) + \angle(b, w_a) - \angle(b, n_a)$$

$$= \frac{A}{2} + \frac{A}{2} - \left(\angle(c, g_a) + \angle(b, n_a)\right) < A \; (\because \angle(c, g_a) + \angle(b, n_a) > 0) \therefore \angle(g_a, n_a) < A$$

Case 2 $c > b$



$$\angle(g_a, n_a) = \angle(b, w_a) - \angle(b, g_a) + \angle(c, w_a) - \angle(c, n_a)$$
$$= \frac{A}{2} + \frac{A}{2} - \big(\angle(b, g_a) + \angle(c, n_a)\big) < A \ (\because \angle(b, g_a) + \angle(c, n_a) > 0) \therefore \angle(g_a, n_a) < A$$

Combining cases 1, 2, in any Δ ABC, $\varphi_1 = \angle(g_a, n_a) < A$ and analogously,
$\varphi_2 = \angle(g_b, n_b) < B$ and $\varphi_3 = \angle(g_c, n_c) < C$ and via summation,
$$\varphi_1 + \varphi_2 + \varphi_3 < A + B + C = \pi,$$
$\therefore$ in any Δ ABC, $0 \le \varphi_1 + \varphi_2 + \varphi_3 < \pi$,
with equality iff Δ ABC is equilateral (QED)

788. ***In any ΔABC, prove that :***

$$\frac{3R^2}{2r^2} \overset{(1)}{\geq} \sum_{cyc} \left(\frac{a}{b}\right)^2 + 3 \overset{(2)}{\geq} \sum_{cyc} \left(\frac{a}{b} + \frac{b}{a}\right)$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Lemma :

In any ΔABC we have : $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \le \frac{R}{r} + 1.$

Proof : We assume that $c = \min\{a, b, c\}$. If $a \ge b \ge c$ then we have :

$$\frac{R}{r} + 1 \overset{Bandila}{\geq} \frac{a}{c} + \frac{c}{a} + 1 = \frac{a}{b} + \frac{b}{c} + \frac{c}{a} + \frac{(a-b)(b-c)}{bc} \ge \frac{a}{b} + \frac{b}{c} + \frac{c}{a}.$$

If $b \ge a \ge c$ we have : $$\frac{R}{r} + 1 \overset{Bandila}{\geq} \frac{b}{c} + \frac{c}{b} + 1 = \frac{a}{b} + \frac{b}{c} + \frac{c}{a} + \frac{(b-a)(a-c)}{ab} \ge \frac{a}{b} + \frac{b}{c} + \frac{c}{a}.$$

So the proof of the lemma is complete.

$$We\ have: \sum_{cyc}\left(\frac{a}{b}\right)^2+3=\left(\sum\frac{a}{b}\right)^2-2\sum_{cyc}\frac{b}{a}+3\overset{Lemma\ \&\ AM-GM}{\leq}\left(\frac{R}{r}+1\right)^2-2.3+3=$$

$$=\frac{R^2}{r^2}+\frac{2R}{r}-2=\frac{3R^2}{2r^2}-2\left(\frac{R}{2r}-1\right)^2\leq\frac{3R^2}{2r^2}.$$

$$Now\ let\ x:=\frac{a}{b},\ \ y:=\frac{b}{c},\ \ z:=\frac{c}{a},\ \ p:=x+y+z,\ \ q:=xy+yz+zx,\ \ r:=xyz=1.$$

$$The\ inequality\ (2)\ becomes:\ p^2+3\geq p+3q.$$

$$By\ Schur's\ inequality\ we\ have: q\leq\frac{p^3+9r}{4p}=\frac{p^3+9}{4p},$$

$$so\ it\ suffices\ to\ prove:\ p^2+3\geq p+3.\frac{p^3+9}{4p}\ \Leftrightarrow\ \frac{(p-3)(p^2-p+9)}{4p}\geq 0,$$

$$which\ is\ true\ because\ p\overset{AM-GM}{\geq}3\sqrt[3]{r}=3\ and\ the\ proof\ is\ completed.$$

789. In acute ΔABC the following relationship holds:

$$\sum_{cyc}a\tan B\tan C\leq\frac{2s}{3}\cdot\frac{\frac{r^2}{R^2}+2}{\frac{r^2}{R^2}+2\frac{r}{R}-1}$$

Proposed by Marin Chirciu-Romania

Solution by Marian Ursărescu-Romania

From Law of Sine:

$$2R\sum_{cyc}\frac{\sin A\sin B\sin C}{\cos B\cos C}\leq\frac{2s}{3}\cdot\frac{2R^2+r^2}{r^2+2Rr-R^2}\Leftrightarrow$$

$$R\sin A\sin B\sin C\cdot\sum_{cyc}\frac{1}{\cos B\cos C}\leq\frac{s}{3}\cdot\frac{2R^2+r^2}{r^2+2Rr-R^2};\qquad(1)$$

$$\text{But: }\sin A\sin B\sin C=\frac{sr}{2R^2}\text{ and }\sum_{cyc}\frac{1}{\cos B\cos C}=\frac{4R(R+r)}{s^2-(2R+r)^2};\qquad(2)$$

From (1) and (2), we must show that:

$$R\cdot\frac{sr}{2R^2}\cdot\frac{4R(R+r)}{s^2-(2R+r)^2}\leq\frac{s}{3}\cdot\frac{2R^2+r^2}{r^2+2Rr-R^2}\Leftrightarrow$$

$$\frac{6r(R+r)}{s^2-4R^2-4Rr-r^2} \le \frac{2R^2+r^2}{r^2+2Rr-R^2}; \qquad (3)$$

From Walker's inequality: $s^2 \ge 2R^2+8Rr+3r^2$

$$s^2-4R^2-4Rr-r^2 \ge -2R^2+4Rr+2r^2 = 2(r^2+2Rr-R^2); \qquad (4)$$

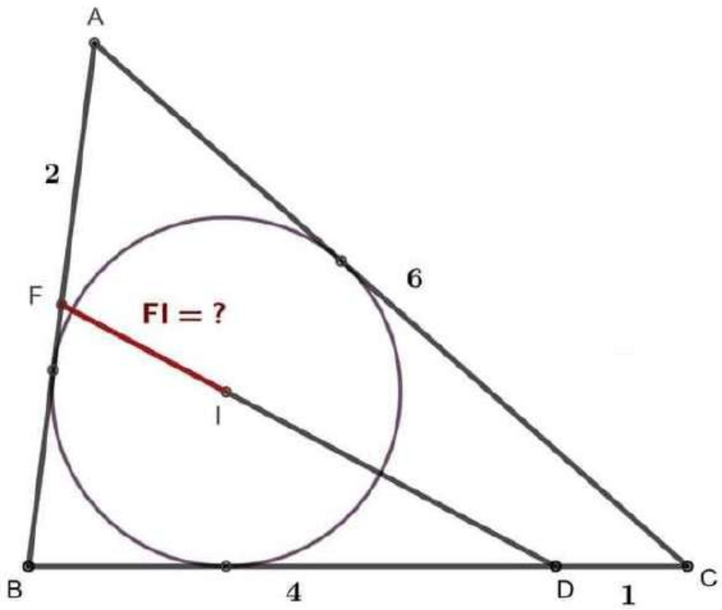
From (3) and (4), we must show:

$$\frac{3r(R+r)}{r^2+2Rr-R^2} \le \frac{2R^2+r^2}{r^2+2Rr-R^2} \Leftrightarrow 3Rr+3r^2 \le 2R^2+r^2 \Leftrightarrow$$

$2R^2-3Rr-2r^2 \ge 0 \Leftrightarrow (R-2r)(2R+r) \ge 0$ which is true from

Euler's inequality: $R \ge 2r$.

790.



Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by Jose Ferreira Queiroz-Olinda-Brazil

$$a=5, b=6, 3r=c-2$$

Using Gakopoulos' Lemmas:

$$\left(\frac{1}{BD}-\frac{1}{BC}\right)+\left(\frac{1}{BF}-\frac{1}{BA}\right)=\frac{AC}{BC\cdot BA}$$

We have:

$$\left(\frac{1}{4}-\frac{1}{5}\right)+\left(\frac{1}{c-2}-\frac{1}{c}\right)=\frac{6}{5c} \Rightarrow \begin{cases} c_1=22 \\ c_2=4 \end{cases} \Rightarrow c=4$$

Angle bisector theorem in the ΔBDF:

$$\frac{FI}{BF}=\frac{ID}{BD}\Rightarrow\frac{FI}{2}=\frac{FD-FI}{4}\Rightarrow FI=\frac{FD}{3}$$

In the ΔABC: $6^2=5^2+4^2-2\cdot 5\cdot 4\cos B\Rightarrow \cos B=\frac{1}{8}$

In the ΔBDF: $FD^2=2^2+4^2-2\cdot 2\cdot 4\cdot\frac{1}{8}\Rightarrow FD=3\sqrt{2}$

Hence: $FI=\frac{3\sqrt{2}}{3}$, $FI=\sqrt{2}$

791. In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{(sinA+sinB-sinC)(sinA+sinC-sinB)}{sinBsinC}=4-\frac{2r}{R}$$

Proposed by Ertan Yildirim-Izmir-Turkiye

Solution by Daniel Sitaru-Romania

$$\sum_{cyc}\frac{(sinA+sinB-sinC)(sinA+sinC-sinB)}{sinBsinC}=$$

$$=\sum_{cyc}\frac{\left(\frac{a}{2R}+\frac{b}{2R}-\frac{c}{2R}\right)\left(\frac{a}{2R}+\frac{c}{2R}-\frac{b}{2R}\right)}{\frac{b}{2R}\cdot\frac{c}{2R}}=$$

$$=\sum_{cyc}\frac{(a+b-c)(a+c-b)}{bc}=\sum_{cyc}\frac{(2s-2c)(2s-2b)}{bc}=$$

$$=4\sum_{cyc}\frac{(s-c)(s-b)}{bc}=4\sum_{cyc}sin^2\frac{A}{2}=4\left(1-\frac{r}{2R}\right)=4-\frac{2r}{R}$$

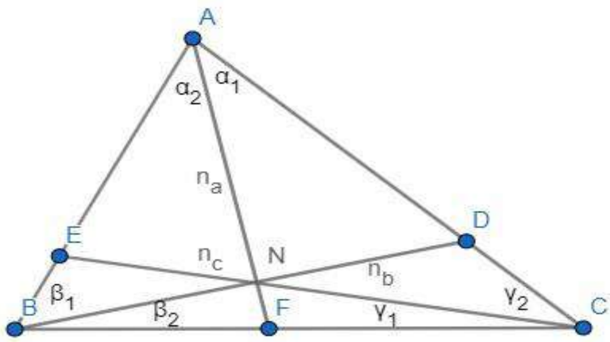
792.

In any Δ ABC (acute), prove that :

$$\min\{\angle(n_a,n_b),\angle(n_b,n_c),\angle(n_c,n_a)\}\le\frac{3\pi}{4}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by Soumava Chakraborty-Kolkata-India



$$\angle(n_a, n_b) + \angle(n_b, n_c) + \angle(n_c, n_a) = m\measuredangle(ANB) + m\measuredangle(BNC) + m\measuredangle(CNA)$$

$$= \big(\pi - (\alpha_2 + \beta_1)\big) + \big(\pi - (\beta_2 + \gamma_1)\big) + \big(\pi - (\gamma_2 + \alpha_1)\big)$$

$$= 3\pi - (\alpha_1 + \alpha_2) - (\beta_1 + \beta_2) - (\gamma_1 + \gamma_2) = 3\pi - (A + B + C) = 3\pi - \pi = 2\pi$$

$$\Rightarrow \frac{\angle(n_a, n_b) + \angle(n_b, n_c) + \angle(n_c, n_a)}{3} = \frac{2\pi}{3}$$

$$\therefore \min\{\angle(n_a, n_b), \angle(n_b, n_c), \angle(n_c, n_a)\} \le \frac{\angle(n_a, n_b) + \angle(n_b, n_c) + \angle(n_c, n_a)}{3}$$

$$= \frac{2\pi}{3} \le \frac{3\pi}{4} \text{ (QED)}$$

793. In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{\sqrt{b + \lambda c - a}}{b + \lambda c} \le \frac{3\sqrt{\lambda^2 + 2\lambda}}{(\lambda + 1)\sqrt{a + b + \lambda c}}; \quad \lambda \ge 1$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

First, we will show that the result is true for $\lambda = 1$.

$$\sum_{cyc} \frac{\sqrt{b + c - a}}{b + c} \le \frac{3\sqrt{3}}{2\sqrt{a + b + c}}; (a + b + c = 2s)$$

$$\text{Let } f(x)=\frac{\sqrt{2s-2x}}{2s-x}\Rightarrow f'(x)=\frac{-3x}{(2s-x)^2\sqrt{2s-2x}}$$

$$\Rightarrow f''(x)=-3\left[\frac{(2s-x)^2\sqrt{2s-2x}+x[(2s-x)\sqrt{2s-2x}+\frac{1}{2}\cdot\frac{(2s-x)^2}{\sqrt{2s-2x}}}{(2s-x)^4(2s-2x)}\right]<0$$

$\Rightarrow f$ –concave function, use Jensen's inequality:

$$f(a)+f(b)+f(c)\le f\left(\frac{a+b+c}{3}\right)$$

$$\sum_{cyc}\frac{\sqrt{2s-2x}}{2s-a}\le\frac{3\sqrt{2s-\frac{2\sum a}{3}}}{2s-\frac{\sum a}{3}}\Leftrightarrow\sum_{cyc}\frac{\sqrt{2s-2x}}{2s-a}\le\frac{3\sqrt{3}\cdot\sqrt{2s}}{4s}$$

$$\sum_{cyc}\frac{\sqrt{b+c-a}}{b+c}\le\frac{3\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2s}},\qquad\sum_{cyc}\frac{\sqrt{b+c-a}}{b+c}\le\frac{3\sqrt{3}}{2\sqrt{a+b+c}}$$

794. In acute $\Delta ABC, H$ –orthocenter, holds:

$$a^2+b^2+c^2\ge 4\sqrt{[ABC]}\left(\sqrt{[HBC]}+\sqrt{[HCA]}+\sqrt{[HAB]}\right)$$

$[*]$ –area.

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution by Tapas Das-India

$$[ABC]=[HAB]+[HBC]+[HCA]$$

$$\sqrt{[HBC]}+\sqrt{[HCA]}+\sqrt{[HAB]}\overset{CBS}{\le}\sqrt{3[ABC]}$$

We need to show:

$$a^2+b^{22}+c^2\ge 4\sqrt{[ABC]}\cdot\sqrt{3[ABC]}$$

$$a^2+b^2+c^2\ge 4\sqrt{3}\cdot[ABC];\quad (1)$$

Let $a=x+y, b=y+z, c=z+x; x,y,z>0$, then

$$(1)\Leftrightarrow((x+y)^2+(y+z)^2+(z+x)^2)^2\ge(4xy+4yz+4zx)^2$$

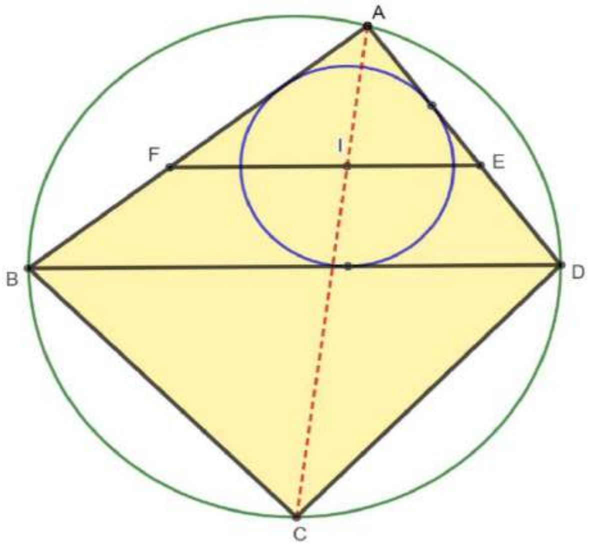
$$\Leftrightarrow((x+y)^2+(y+z)^2+(z+x)^2)^2\ge 16(xy+yz+zx)^2;\quad (2)$$

$$\because (p+q+r)^2\ge 3(pq+qr+rp)$$

$$(2)\Rightarrow((x+y)^2+(y+z)^2+(z+x)^2)^2\ge 16\cdot 3(x^2yz+xy^2z+xyz^2)$$

$$((x+y)^2+(y+z)^2+(z+x)^2)^2 \geq 48xyz(x+y+z)$$

Equality holds for $x=y=z \Leftrightarrow a=b=c$.



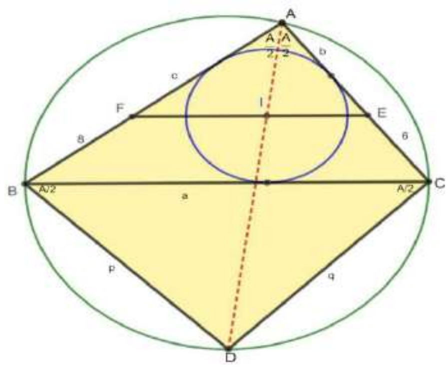
795.

In $\Delta ABC, I$ —incenter, $FE \parallel BD, BF=8, DE=6, P_{BDEF}=52$ perimeter.

Find $[ABCD], [*]$ —area of $*$

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by Jose Ferreira Queiroz-Olinda-Brazil



$$AB=c, AF=c-8, DC=q$$

$$AC=b, AE=b-6$$

$$BC=a, BD=p$$

$$a+8+6+FE=52 \Rightarrow a+EF=38$$

$$\Delta AFE \sim \Delta ABC: \frac{c-8}{c}=\frac{b-6}{b} \Rightarrow b$$

$$=\frac{3c}{4};(1)$$

$$Gakopoulos' Lemma:$$

$$\left(\frac{1}{AF}-\frac{1}{AB}\right)+\left(\frac{1}{AE}-\frac{1}{AC}\right)=\frac{BC}{AB \cdot AC}$$

$$\frac{1}{c-8}-\frac{1}{c}+\frac{1}{b-6}-\frac{1}{b}=\frac{a}{bc}$$

$$(1)\Rightarrow a=\frac{14c}{c-8}$$

$$Gakopoulos'\,Lemma:$$

$$FE^2=\frac{1}{AF\cdot AE}(AF+AE)^2(AB\cdot AD+AF\cdot AE)-2(AB+AD)(AF+AE)$$

$$EF=14\Rightarrow a=24\Rightarrow c=\frac{96}{5}\Rightarrow b=\frac{72}{5}$$

$$a^2=b^2+c^2-2bc\cdot\cos A$$

$$24^2=\left(\frac{96}{5}\right)^2+\left(\frac{72}{5}\right)^2-2\cdot\frac{96}{5}\cdot\frac{72}{5}\cdot\cos A$$

$$\cos A=0\Rightarrow A=\frac{\pi}{2}$$

BC –diameter, $\frac{A}{2}=\frac{\pi}{4}, D=\frac{\pi}{2}\Rightarrow p=q$ and $2q^2=a^2\Rightarrow q=\frac{a\sqrt{2}}{2}\Rightarrow q=12\sqrt{2}$

$$F_1=[ABC]=\frac{1}{2}bc=\frac{1}{2}\cdot\frac{72}{5}\cdot\frac{96}{5}=\frac{3456}{25}$$

$$F_2=[BDC]=\frac{1}{2}q^2=\frac{1}{2}\left(12\sqrt{2}\right)^2=144$$

$$F=[ABCD]=\frac{3456}{25}+144=\frac{7056}{25}$$

796. In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{1}{\tan\frac{A}{2}+\sqrt{1+\tan^2\frac{A}{2}}}\le\frac{4R+r}{s}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum_{cyc}\sec\frac{A}{2}=\sum_{cyc}\sqrt{1+\tan^2\frac{A}{2}}\overset{CBS}{\le}\sqrt{3}\cdot\sqrt{3+\sum_{cyc}\tan^2\frac{A}{2}}=$$

$$=\sqrt{3}\cdot\sqrt{3+\left(\frac{4R+r}{s}\right)^2-2}=\sqrt{3}\cdot\sqrt{1+\left(\frac{4R+r}{s}\right)^2}$$

Now, we have:

$$\sum_{cyc}\frac{1}{\tan\frac{A}{2}+\sqrt{1+\tan^2\frac{A}{2}}}=\sum_{cyc}\frac{1}{\tan\frac{A}{2}+\sec\frac{A}{2}}=$$

$$=\sum_{cyc}\frac{\sec^2\frac{A}{2}-\tan^2\frac{A}{2}}{\tan\frac{A}{2}+\sec\frac{A}{2}}=\sum_{cyc}\frac{\left(\sec\frac{A}{2}+\tan\frac{A}{2}\right)\left(\sec\frac{A}{2}-\tan\frac{A}{2}\right)}{\tan\frac{A}{2}+\sec\frac{A}{2}}=$$

$$=\sum_{cyc}\left(\sec\frac{A}{2}-\tan\frac{A}{2}\right)=\sum_{cyc}\sec\frac{A}{2}-\frac{4R+r}{s}\le\sqrt{3}\cdot\sqrt{1+\frac{4R+r}{s}}-\frac{4R+r}{s}$$

We need to show:

$$\sqrt{3}\cdot\sqrt{1+\frac{4R+r}{s}}-\frac{4R+r}{s}\le\frac{4R+r}{s},\qquad \sqrt{3}\cdot\sqrt{1+\frac{4R+r}{s}}\le 2\cdot\frac{4R+r}{s}$$

$$3\left(1+\frac{4R+r}{s}\right)\le 4\left(\frac{4R+r}{s}\right)^2,\qquad 3\le\frac{(4R+r)^2}{s^2}\Leftrightarrow s^2\le\frac{(4R+r)^2}{3}\ \ (\text{true!})$$

Therefore,

$$\sum_{cyc}\frac{1}{\tan\frac{A}{2}+\sqrt{1+\tan^2\frac{A}{2}}}\le\frac{4R+r}{s}$$

Note:

1) $$\sum_{cyc}\tan\frac{A}{2}=\frac{4R+r}{s}$$

2) $$\sum_{cyc}\tan^2\frac{A}{2}=\left(\frac{4R+r}{s}\right)^2-2$$

3) $$\frac{1}{n}\sum_{k=1}^{n}x_k^m\le\left(\frac{1}{n}\sum_{k=1}^{n}x_k\right)^m$$

797. In $\Delta ABC, T$ –Toricelli's point, holds:

$$TA^{2n}+TB^{2n}+TC^{2n}\ge 3(2r)^{2n}, n\in\mathbb{N}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

Let $TA=x; TB=y; TC=z$, then:

$$F=[ABC]=[ATB]+BTC]+[CTA]=$$

$$=\frac{xy}{2}\sin\frac{2\pi}{3}+\frac{yz}{2}\sin\frac{2\pi}{3}+\frac{zx}{2}\sin\frac{2\pi}{3}=\frac{\sqrt{3}}{4}(xy+yz+zx)$$

$$\Rightarrow xy+yz+zx=\frac{4F}{\sqrt{3}};\qquad (1)$$

$$TA^{2n}+TB^{2n}+TC^{2n}=x^{2n}+y^{2n}+z^{2n}\ge$$

$$\ge\frac{3}{3^{2n}}(x+y+z)^{2n}=\frac{3}{3^{2n}}[(x+y+z)^2]^n\ge$$

$$\ge\frac{3}{3^{2n}}[3(xy+yz+zx)]^n=\frac{3}{3^{2n}}\cdot 3^n(xy+yz+zx)^n=$$

$$=\frac{3}{3^{2n}}\cdot 3^n\left(\frac{4F}{\sqrt{3}}\right)^n=\frac{3}{3^{2n}}\cdot 3^n\left(\frac{4rs}{\sqrt{3}}\right)^n\ge\frac{3}{3^{2n}}\cdot 3^n\left(\frac{4r\cdot 3\sqrt{3}r}{\sqrt{3}}\right)^n=3(2r)^{2n}$$

798. Prove or disprove: In Δ ABC: $n_a+r_a\ge 2g_a$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

We consider an isosceles Δ ABC with $b=c$

$$\text{Now, Stewart's theorem}\Rightarrow b^2(s-c)+c^2(s-b)\overset{(i)}{=}an_a^2+a(s-b)(s-c)\text{ and}$$

$$b^2(s-b)+c^2(s-c)\overset{(ii)}{=}ag_a^2+a(s-b)(s-c)\text{ and (i)}+\text{(i)}\Rightarrow$$

$$(b^2+c^2)(2s-b-c)=an_a^2+ag_a^2+2a(s-b)(s-c)$$

$$\Rightarrow 2a(b^2+c^2)=2a(n_a^2+g_a^2)+a(a+b-c)(c+a-b)$$

$$\Rightarrow 2(b^2+c^2)=2(n_a^2+g_a^2)+a^2-(b-c)^2$$

$$\Rightarrow 2(b^2+c^2)-a^2+(b-c)^2=2(n_a^2+g_a^2)$$

$$\Rightarrow 4m_a^2+(b-c)^2=2(n_a^2+g_a^2)\Rightarrow(b-c)^2+4s(s-a)+(b-c)^2=2(n_a^2+g_a^2)$$

$$\Rightarrow n_a^2+g_a^2\overset{(*)}{=}(b-c)^2+2s(s-a)$$

$$\text{Also, Stewart's theorem}\Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$$

$$\Rightarrow s(b^2+c^2)-bc(2s-a)=an_a^2+a(s^2-s(2s-a)+bc)\Rightarrow s(b^2+c^2)-2sbc$$

$$=an_a^2+a(as-s^2)\Rightarrow s(b^2+c^2-a^2-2bc)=an_a^2-as^2$$

$$\Rightarrow an_a^2=as^2+s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}=as^2-\frac{4as(s-b)(s-c)}{a}$$

$$\Rightarrow n_a^2\overset{(**)}{=}s^2-\frac{4s(s-b)(s-c)}{a}$$

$$\therefore (*),(**)\Rightarrow g_a^2=(b-c)^2+2s(s-a)-s^2+\frac{4s(s-b)(s-c)}{a}$$

$$=(b-c)^2+2s(s-a)-s^2+\frac{s(a^2-(b-c)^2)}{a}$$

$$= (b-c)^2 - \frac{s(b-c)^2}{a} + s(s-a) = s(s-a)\ (\because b=c) \Rightarrow g_a^2 \stackrel{(\bullet)}{=} s(s-a)$$

$$\text{Again, (i)} - \text{(ii)} \Rightarrow an_a^2 + a(s-b)(s-c) - ag_a^2 - a(s-b)(s-c)$$

$$= (s-c)(b^2-c^2) - (s-b)(b^2-c^2) \Rightarrow a(n_a^2 - g_a^2) = (b^2-c^2)(b-c)$$

$$= (b+c)(b-c)^2 = 0\ (\because b=c) \Rightarrow n_a = g_a$$

$$\therefore n_a + r_a - 2g_a = r_a - g_a \stackrel{?}{\geq} 0 \stackrel{\text{via }(\bullet)}{\Leftrightarrow} \frac{s(s-a)(s-b)(s-c)}{(s-a)^2} \stackrel{?}{\geq} s(s-a)$$

$$\Leftrightarrow \frac{a^2-(b-c)^2}{4} \stackrel{?}{\geq} (s-a)^2 \Leftrightarrow a^2 \stackrel{?}{\geq} 4(s-a)^2\ (\because b=c) \Leftrightarrow a \stackrel{?}{\geq} b+c-a$$

$$\Leftrightarrow 2a \stackrel{?}{\geq} 2b \Leftrightarrow a \stackrel{?}{\geq} b, \text{but if } B = C = 75^\circ\ (\text{and } A = 30^\circ),$$

$$\text{then}: a - b = 2R(\sin 30^\circ - \sin 75^\circ) < 0 \Rightarrow a < b \Rightarrow n_a + r_a - 2g_a < 0$$

$$\therefore n_a + r_a < 2g_a \text{ for } A = 30^\circ \text{ and } B = C = 75^\circ$$

$$\therefore n_a + r_a \geq 2g_a \text{ is not always true (Done)}$$

799. In ΔABC the following relationship holds:

$$6 \leq \sum_{cyc} \frac{a}{b+c} \csc^2 \frac{A}{2} \leq \frac{3R}{r}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have:

$$\sum_{cyc} \frac{a}{b+c} \csc^2 \frac{A}{2} = \sum_{cyc} \frac{abc}{(b+c)(s-b)(s-c)} \stackrel{\text{Hölder}}{\geq} \frac{3^3 . abc}{\sum_{cyc}(b+c) . \sum_{cyc}(s-b)(s-c)} =$$

$$= \frac{27 . 4Rsr}{4s . r(4R+r)} = \frac{27R}{4R+r} = 6 + \frac{3(R-2r)}{4R+r} \stackrel{\text{Euler}}{\geq} 6.$$

Also we have:

$$\sum_{cyc} \frac{a}{b+c} \csc^2 \frac{A}{2} = \sum_{cyc} \frac{abc}{(b+c)(s-b)(s-c)} \stackrel{HM-AM}{\leq} \sum_{cyc} \frac{a(b+c)}{4(s-b)(s-c)} =$$

$$= \frac{1}{4sr^2}\sum_{cyc} a(b+c)(s-a) = \frac{1}{4sr^2}.3abc = \frac{3R}{r}.$$

Therefore: $6 \le \sum_{cyc} \frac{a}{b+c}\csc^2\frac{A}{2} \le \frac{3R}{r}.$

Solution 2 by Tapas Das-India

$$\frac{a}{b+c}\csc^2\frac{A}{2} = \frac{2R\sin A}{2R\sin B + 2R\sin C}\cdot\frac{1}{\sin^2\frac{A}{2}} =$$

$$= \frac{\sin A}{\sin B + \sin C}\cdot\frac{1}{\sin^2\frac{A}{2}} = \frac{2\sin\frac{A}{2}\cos\frac{A}{2}}{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}\cdot\frac{1}{\sin^2\frac{A}{2}} =$$

$$= \frac{\cos\frac{A}{2}}{\sin\frac{A}{2}\sin\left(\frac{\pi}{2}-\frac{A}{2}\right)\cos\frac{B-C}{2}} = \frac{\cos\frac{A}{2}}{\sin\frac{A}{2}\cos\frac{A}{2}\cos\frac{B-C}{2}} =$$

$$= \frac{1}{\sin\frac{A}{2}\cos\frac{B-C}{2}} = \frac{2}{2\cos\frac{B-C}{2}\cos\frac{B+C}{2}} = \frac{2}{\cos B + \cos C}$$

$$\sum_{cyc}\frac{a}{b+c}\csc^2\frac{A}{2} = \sum_{cyc}\frac{2}{\cos B + \cos C} \ge 2\cdot\frac{9}{2\sum\cos A} = \frac{9}{1+\frac{r}{R}}$$

$$\sum_{cyc}\frac{a}{b+c}\csc^2\frac{A}{2} = \sum_{cyc}\frac{abc}{(b+c)(s-b)(s-c)} =$$

$$= \sum_{cyc}\frac{abc(s-a)}{(b+c)(s-a)(s-b)(s-c)} = abc\sum_{cyc}\frac{s-a}{(b+c)sr^2} =$$

$$= \frac{abc}{sr^2}\sum_{cyc}\frac{s-a}{b+c} = \frac{4Rrs}{sr^2}\sum_{cyc}\frac{s-a}{b+a} =$$

$$= \frac{4R}{r}\sum_{cyc}\left(\frac{s}{b+c}-\frac{a}{b+c}\right) = \frac{4R}{r}\sum_{cyc}\frac{1}{2}\left(1+\frac{a}{b+c}\right) =$$

$$= \frac{4R}{r}\left(\frac{3}{2}-\frac{1}{2}\sum_{cyc}\frac{a}{b+c}\right) \le \frac{4R}{r}\cdot\frac{3}{4} = \frac{3R}{r}$$

800.

$$I-\text{incenter in } \Delta ABC.\ \text{Prove that}:$$

$$\frac{AI+4R+r_b+r_c+r}{3m_a+m_b+m_c+h_a-r}\le\sqrt{\frac{R}{2r}}$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}=\sqrt{\frac{a.2(s-a)}{8Rr}}\overset{AM-GM}{\le}\frac{a+2(s-a)}{2\sqrt{8Rr}}=\frac{b+c}{4\sqrt{2Rr}}\quad(1)$$

$$Now,\ AI=\frac{r}{\sin\frac{A}{2}}=\frac{4Rr\cos\frac{A}{2}}{a}\overset{(1)}{\le}\sqrt{\frac{Rr}{2}}.\frac{b+c}{a}=\sqrt{\frac{Rr}{2}}.\left(\frac{2s}{a}-1\right)=\sqrt{\frac{R}{2r}}(h_a-r)\quad(2)$$

$$Also,\ r_b+r_c=sr\left(\frac{1}{s-b}+\frac{1}{s-c}\right)=\frac{sr.a}{(s-b)(s-c)}=\frac{a(s-a)}{r}$$

$$=\frac{abc}{sr}.\cos^2\frac{A}{2}\overset{(1)}{\le}4R\cos\frac{A}{2}.\frac{b+c}{4\sqrt{2Rr}}=$$

$$=2\sqrt{\frac{R}{2r}}.\frac{b+c}{2}\cos\frac{A}{2}\overset{Lascu}{\le}2\sqrt{\frac{R}{2r}}.m_a.\quad Then:\ r_b+r_c\le 2\sqrt{\frac{R}{2r}}.m_a\ (3)$$

$$Thus,\ 4R+r=\sum_{cyc}\frac{r_b+r_c}{2}\overset{(3)}{\le}\sum_{cyc}\sqrt{\frac{R}{2r}}.m_a=\sqrt{\frac{R}{2r}}(m_a+m_b+m_c)\quad(4)$$

Adding the inequalities (2), (3) *and* (4) *we get*:

$$AI+4R+r_b+r_c+r\le\sqrt{\frac{R}{2r}}(3m_a+m_b+m_c+h_a-r).$$

$$Therefore,\ \frac{AI+4R+r_b+r_c+r}{3m_a+m_b+m_c+h_a-r}\le\sqrt{\frac{R}{2r}}.$$

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru